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Research article

The orthogonal polynomials method using Gegenbauer polynomials to solve mixed integral equations with a Carleman kernel

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Abstract: The orthogonal polynomials approach with Gegenbauer polynomials is an effective tool for analyzing mixed integral equations (MIEs) due to their orthogonality qualities. This article reviewed recent breakthroughs in the use of Gegenbauer polynomials to solve mixed integral problems. Previous authors studied the problem with a continuous kernel that combined both Volterra (V) and Fredholm (F) components; however, in this paper, we focused on a singular Carleman kernel. The kernel of FI was measured with respect to position in the space $L_2[-1,1]$, while the kernel of VI was considered as a function of time in the space $C[0,T], T < 1$. The existence of a unique solution was discussed in $L_2[-1,1] \times C[0,T]$ space. The solution and its error stability were both investigated and commented on. Finally, numerical examples were reviewed, and their estimated errors were assessed using Maple (2022) software.

Keywords: orthogonal method; Carleman function; variable separation; Gegenbauer polynomials; algebraic system; stability of error
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1. Introduction

Several problems in astrophysics, including linear and nonlinear elasticity and engineering crack problems, lead to an integral equation of the first and second kinds of Fredholm integral equations (FIEs), which deal with problems having boundary conditions, while problems having initial conditions are described by the Volterra integral equation (VIE); see Popov [1] and Aleksandrovsk and Kovalenko [2]. After using Kiren's method, many spectral relationships for FIEs with discontinuous kernels were obtained (Mkhitaryan and Abdou [3]). For Carleman kernel and its importance in the nonlinear theory of plasticity, see Artinian [4]. Alalyani et al. [5] dealt with the solution of the third kind of mixed integro-differential equations with displacement using orthogonal polynomials. Gegenbauer polynomials, also known as ultraspherical polynomials, constitute a family of orthogonal polynomials that have found widespread applications in various mathematical disciplines. In recent years, researchers have increasingly turned to these polynomials for solving mixed integral equations **(**MIEs**),** which refers to problems with a Fredholm kernel in position and Volterra in time, and often arise in mathematical modeling across different fields. Using the polynomial method, Abdou and Khamis [6] were able to solve a first type F-VIE with a Carleman kernel. El-Gindy et al. [7] used the shifted Gegenbauer polynomials and Tau method to present numerical solutions to multi-order fractional differential equations. Atta [8] applied the shifted Gegenbauer polynomials to solve the time fractional cable problems. Nasr and Abdel-Aty [9] used the degenerate method to solve a V-FIE. Mirzaee and Samadyar [10] represented the Bernstein collocation method for solving 2D-mixed Volterra-Fredholm integral equations. Alhazmi et al. [11] used the Lerch polynomial method to solve MIEs, which had a strongly singular kernel. More details on several approaches for solving integral equations can be found in [12–17].

In this work, we discuss innovative approaches and algorithms employed to solve MIEs and show how the Gegenbauer polynomials contribute to the efficiency and accuracy of these methods. Consider the following MIEs of two types: V-FIE and F-VIE, respectively.

$$\mu\Phi(u,t)-\lambda\int_0^t\int_{-1}^1 k(|y-u|)f(t,s)\Phi(y,s)dyds=F(u,t), \tag{1a}$$

$$\mu\Phi(u,t)-\lambda\int_0^t f(t,s)\Phi(u,s)ds-\lambda\int_{-1}^1 k(|y-u|)\Phi(y,t)dy=H(u,t), \tag{1b}$$

with the dynamical condition

$$\int_{-1}^1 \Phi(u,t)du=P(t), \qquad t\in[0,T], \qquad T<1. \tag{2}$$

Condition (2) is of particular importance in applied sciences, as all the unknown functions during the integration period do not exceed the pressure exerted on the bodies and are changing with time.

Here, the function $f(t,s)\in C[0,T]^2$, $T<1$. The singular kernel $k(|y-u|)$ takes the Carleman function form. The constants μ and λ have several physical meanings. The given function

$$F(u,t)\in L_2[-1,1]\times C[0,T],$$

while function $\Phi(u,t)$ will be determined in the same space of the function $F(u,t)$.

The paper is structured as follows: Using the fixed-point theorem and under certain conditions in Section 2, the existence of a unique solution is established. In Section 3, the convergence and the error stability of the solution are discussed. In Section 4, we use the separation variables technique to obtain a Fredholm equation of the second type with the Carleman kernel. In Section 5, the orthogonal polynomials method and Gegenbauer polynomials are used to convert the system of FIEs with time parameters to an algebraic system, where its convergence is considered in Section 6. The numerical results are given in Section 7. Finally, Section 8 presents our conclusions from the present research study.

2. Existence of a unique solution for the MIE

Consider the following assumptions:

(a) The kernel $k(|y-u|)$ in $L_2[-1,1]$ space in which its discontinuity condition is

$$\left[\int_{-1}^{1}\int_{-1}^{1} k^2(|y-u|)\,dudy\right]^{\frac{1}{2}} = \alpha, \qquad (\alpha \text{ is a constant}).$$

(b) For $t,s \in [0,T], T<1$, the time function $f(t,s) \in C[0,T]$ and satisfies

$$\|f(t,s)\| \le \beta, \qquad (\beta \text{ is a constant}).$$

(c) The given function $F(u,t) \in L_2[-1,1] \times C[0,T]$ is well-defined and its norm

$$\|F\| = \max_{0\le t\le T}\left|\int_0^t \left[\int_{-1}^{1} F^2(u,s)du\right]^{\frac{1}{2}} ds\right| = M, \text{ where } M \text{ is a constant.}$$

To discuss the existence of a unique solution for Eq (1a), we write it in the integral operator form as

$$\boldsymbol{\chi}\Phi(u,t) = \frac{1}{\mu}[F(u,t) + \boldsymbol{K}\Phi], \tag{3}$$

where

$$\boldsymbol{K}\Phi = \lambda\int_0^t\int_{-1}^{1} k(|y-u|)f(t,s)\Phi(y,s)dyds. \tag{4}$$

Theorem 1 (Existence and uniqueness for V-FIE). There exists a unique solution for the MIE (1a) under the condition

$$T\alpha\beta|\lambda| < |\mu|, \qquad T<1. \tag{5}$$

Proof. To demonstrate this theorem, we use the following results:

Lemma 1. Under the condition of Theorem 1, $\boldsymbol{\chi}$ is a bounded operator.

Proof. The normality of Eq (3) leads to

$$\|\boldsymbol{\chi}\Phi\| \le \frac{1}{|\mu|}[\|F\| + \|\boldsymbol{K}\Phi\|],$$
$$\|\boldsymbol{K}\Phi\| = |\lambda| \left\| \int_0^t \int_{-1}^1 f(t,s)k(|x-y|)\Phi(y,s)dyds \right\|. \tag{6}$$

The conditions (a), (b), and the Cauchy-Schwarz inequality lead to

$$\|\boldsymbol{K}\Phi\| \le |\lambda|\|f\| \left[\int_{-1}^{1} \int_{-1}^{1} k^2(|u-y|)\,dxdy \right]^{\frac{1}{2}} T\|\Phi\| \le |\lambda|\alpha\beta T\|\Phi\|. \tag{7}$$

Hence, (6) becomes

$$\|\boldsymbol{\chi}\Phi\| \le \frac{1}{|\mu|}[M + |\lambda|\alpha\beta T\|\Phi\|]. \tag{8}$$

Therefore, the operator $\boldsymbol{\chi}$ transforms the ball $S_r \subset L_2[-1,1] \times C[0,T]$ into itself, where

$$r = \frac{\delta}{1-\rho}, \qquad \delta = \frac{M}{\mu}, \qquad and\ \varrho = \frac{T\alpha\beta|\lambda|}{|\mu|}.$$

Since $r > 0$, under the hypothesis condition $T\alpha\beta|\lambda| < |\mu|$, the operator $\boldsymbol{K}$ is bounded and, accordingly, $\boldsymbol{\chi}$ is bounded.

Lemma 2. In the space of integration, the operator $\boldsymbol{\chi}$ is a contraction.

Proof. Let Φ_1, Φ_2 be two different solutions of Eq (1a); hence, formula (3), once using (4), leads to

$$\|\boldsymbol{\chi}\Phi_1 - \boldsymbol{\chi}\Phi_2\| \le \frac{T}{|\mu|}[|\lambda|\alpha\beta\|\Phi_1 - \Phi_2\|]. \tag{9}$$

So, we have the continuity of $\boldsymbol{\chi}$.

Furthermore, under the condition $\alpha\beta|\lambda|T < |\mu|$, $\boldsymbol{\chi}$ is a contraction mapping.

By the fixed-point theorem, since $\boldsymbol{\chi}$ is a bounded and continuous operator, and moreover, it is a contraction mapping, then the solution of Eq (1a) has a unique solution.

Theorem 2 (Existence and uniqueness for F-VIE) (without proof). There exists a unique solution for the MIE (1b) under the condition

$$(\alpha + \beta T)|\lambda| < |\mu|.$$

3. The convergence and the error stability of the solution

3.1. The solution convergence

To study the solution behavior of Eq (1a), we construct the sequence

$$\{\Phi_0(u,t), \Phi_1(u,t), \Phi_2(u,t), \cdots, \Phi_{n-1}(u,t), \Phi_{\mathrm{n}}(u,t), \cdots\}.$$

Hence, consider a specific equation

$$\mu\Phi_n(u,t) = F(x,t) + \lambda\int_0^t\int_{-1}^1 k(|y-u|)f(t,s)\Phi_{n-1}(y,s)dyds,$$
$$\Phi_0 = \frac{F(u,t)}{\mu}. \tag{10}$$

We define

$$\Psi_n = \Phi_n - \Phi_{n-1}, \qquad \Psi_0(u,t) = \Phi_0, \tag{11}$$

to establish

$$\Phi_{\mathrm{n}}(x,t) = \sum_{i=0}^{n} \Psi_i.$$

In view of (11), we can adapt Eq (10) to take the form

$$\mu\Psi_{\mathrm{n}}(u,t) = \lambda\int_0^t\int_{-1}^1 k(|y-u|)f(t,s)\Psi_{n-1}(y,s)dyds. \tag{12}$$

Theorem 3 (Convergence of the solution for V-FIE). A sequence $\{\Psi_{\mathrm{n}}\}_{\mathrm{n}=0}^{\infty}$ of the solution to (12), under condition (5), is uniformly convergent.

Proof. Formula (12), when using the Cauchy-Schwarz inequality, yields

$$|\mu|\|\Psi_n(u,t)\| \le |\lambda|\left|\int_0^t\int_{-1}^1 k(|y-u|)f(t,s)dyds\right|\|\Psi_{n-1}(u,t)\|. \tag{13}$$

Then, by using assumptions (a)–(c), we have

$$|\mu|\|\Psi_n(u,t)\| \le |\lambda|\alpha\beta T\|\Psi_{n-1}(u,t)\|. \tag{14}$$

Therefore,

$$|\mu|\|\Psi_n\| \le |\lambda|\alpha\beta T\|\Psi_{n-1}\|. \tag{15}$$

Hence, with the use of (11), we get

$$\|\Psi_n\| \le \varrho^n\|F\|, \qquad \varrho = \frac{T\alpha\beta|\lambda|}{|\mu|}. \tag{16}$$

So, $\Phi_{\mathrm{n}}(x,t) = \sum_{i=0}^{n}\Psi_i$ is uniformly convergent, provided that $\varrho < 1$.

If $n \to \infty$, then $\Phi_{\mathrm{n}}(x,t) \to \Phi(x,t)$, i.e., $\Phi(x,t)$ is uniformly convergent.

As an approximation of Eq (1a), we have

$$\mu\Phi_{\mathrm{n}}(u,t) = F_n(u,t) + \lambda\int_0^t\int_{-1}^1 k(|y-u|)f(t,s)\Phi_{\mathrm{n}}(y,s)dyds, \tag{17}$$

where $F_n(u,t) \to F(u,t)$ as $n \to \infty$.

By considering Eq (1a), we get the equation of the error as follows:

Let the error function be defined as

$$\Re_n(x,t) = \Phi(x,t) - \Phi_n(x,t).$$

Hence, we get

$$\mu\Re_n(u,t) - \lambda\int_0^t\int_{-1}^1 k(|y-u|)f(t,s)\Re_n(y,s)dyds = H_n(u,t), \tag{18}$$

$$(H_n(u,t) = F - F_n).$$

Theorem 4 (Convergence of the error for V-FIE). A sequence $\{\Re_n\}$ of error for Eq (18) is uniformly convergent under condition (5).

Proof. After constructing the sequence of errors $\{\Re_0(u,t), \Re_1(u,t), \cdots, \Re_{n-1}(u,t), \Re_n(u,t), \cdots\}$, and considering

$$\mathbb{R}_n = \Re_n - \Re_{n-1},\ \mathbb{R}_0(x,t) = \frac{H_0(u,t)}{\mu},\ \Re_n(u,t) = \sum_{i=0}^{n}\mathbb{R}_i, \tag{19}$$

then by using assumptions (a)–(c), we get

$$|\mu|\|\mathbb{R}_n\| \le |\lambda|\alpha\beta T\|\mathbb{R}_{n-1}\|. \tag{20}$$

By induction,

$$\|\mathbb{R}_n\| \le \varrho^n\|f\|, \quad \varrho = \frac{T\alpha\beta|\lambda|}{|\mu|}. \tag{21}$$

So, under the inequality $\varrho < 1$, $\mathbb{R}_n(x,t)$ is convergent.

If $n \to \infty$, $\Re_n(u,t) \to \Re(u,t)$, then the error function $\Re(u,t)$ is convergent.

3.2. Examples

3.2.1. Consider the V-FIE

$$\mu\Phi(x,t) - \lambda\int_0^t\int_{-1}^1 ts|x-y|^{-\nu}\Phi(y,s)dyds = F(x,t),$$

where

$$\alpha = \left[\int_{-1}^1\int_{-1}^1 |y-u|^{-2\nu}\, dudy\right]^{\frac{1}{2}}$$

and $\beta = T^2$. So, there exist numerical solutions under the convergence condition $\frac{\lambda\alpha\beta T}{\mu} < 1$. The convergence condition for a given $\mu = 1$ and various values of $\nu, \lambda,$ and T are displayed in Table 1:

Table 1. Convergence condition for $\mu = 1$ and different values of $\nu, \lambda,$ and T.

T	ν	α	λ	Convergence condition
0.005	0.07	2.130	0.018	4.79×10^{-9}
			0.700	1.80×10^{-7}
0.09			0.018	2.70×10^{-5}
			0.700	0.001
0.7			0.018	0.013
			0.700	0.510
0.005	0.4	4.375	0.018	9.83×10^{-9}
			0.700	3.80×10^{-7}
0.09			0.018	5.70×10^{-5}
			0.700	0.002
0.7			0.018	0.027
			0.700	1.050

From Table 1, there is a fast convergence to the solution whenever we have decreasing $\nu, \lambda,$ and T, and for example, there is a divergence when $\nu = 0.4$, $T = 0.7$, and $\lambda = 0.7$.

3.2.2. Consider the F-VIE

$$\mu\Phi(x,t) - \lambda \int_0^t ts\Phi(x,s)dy - \lambda \int_{-1}^1 |x-y|^{-\nu}\Phi(y,t)dy = F(x,t).$$

In this example, there exist numerical solutions under the convergence condition $\frac{|\lambda|(\alpha+\beta T)}{|\mu|} < 1$. The convergence condition for a given $\mu = 1$ and various values of $\nu, \lambda,$ and T are displayed in Table 2:

Table 2. Convergence condition for $\mu = 1$ and different values of $\nu, \lambda,$ and T.

T	ν	α	λ	Convergence condition
0.005	0.07	2.130	0.018	0.0383
			0.100	0.2131
0.09			0.018	0.0384
			0.100	0.2130
0.7			0.018	0.0445
			0.100	0.2473
0.005	0.4	4.375	0.018	0.0788
			0.100	0.4375
0.09			0.018	0.0788
			0.100	0.4376
0.7			0.018	0.0849
			0.100	0.4718

From Table 2, there is a straightforward time effect, while when ν and λ decrease, there is rapid convergence.

4. The separation variables technique

Some researchers have solved the integral equations at zero time using the earlier technique; see, for example, [9]. In other studies, time and position may be separated using the Laplace or Fourier transforms; however, this method has drawbacks when trying to identify inverse transformers. Another technique is to divide the time into periods to create an entire set of integral equations that are especially applicable to the position.

The approach of separating variables using explicit functions in position and time makes the discussion of the time impact more comprehensible.

Assume the unknown and given functions, respectively, take the form

$$\Phi(u,t) = A(t)\psi(u), \quad F(u,t) = g(u)B(t). \tag{22}$$

Hence, when using formula (22), Eq (1a) yields

$$\mu^*\psi(u) - \lambda^* \int_{-1}^{1} k(|y-u|)\psi(y)dy = g(u), \tag{23}$$

where

$$\lambda^* = \frac{\lambda}{B(t)} \int_0^t f(t,s)A(s)d\tau\,,\ \mu^* = \frac{A(t)}{B(t)}. \tag{24}$$

In all previous research, the solution to MIEs cannot be discussed in view of time. Also, μ^* determines the kind of the integral equation. If $\mu^* = 0$, we have an IE of the first kind, while if $\mu^* = \text{constant} \neq 0$, we have an IE of the second kind. A third kind of IE can be obtained if $\mu^* = \mu^*(u)$. The significance of the separation method came from obtaining a quadratic FIE with a time-related coefficient. In this case, the time can be computed explicitly at any point.

5. The orthogonal polynomials method and Gegenbauer polynomials

The solution of Eq (23) will be discussed using Gegenbauer polynomials of the order $\frac{v}{2}$, which is

$$C_n^{\left(\frac{v}{2}\right)}(u) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k \Gamma\left(n-k+\frac{v}{2}\right)(2u)^{n-2k}}{\Gamma\left(\frac{v}{2}\right) k!\,(n-2k)!}.$$

So, we write the unknown function $\psi(x)$ and the given function $g(x)$ in the following forms:

$$\psi(u) = \sum_{n=0}^{\infty} b_n (1-u^2)^{\frac{v-1}{2}} C_n^{\left(\frac{v}{2}\right)}(u), \qquad (b_n \text{ are unknown constants}). \tag{25a}$$

$$g(u)=\sum_{n=0}^{\infty} g_n(1-u^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(u),$$
$$g_n=\frac{n!\left(n+\frac{\upsilon}{2}\right)\Gamma^2\left(\frac{\upsilon}{2}\right)}{\pi 2^{1-\upsilon}\Gamma(n+\upsilon)}\int_{-1}^{1} g(u) C_n^{\left(\frac{\upsilon}{2}\right)}(u)\,du. \tag{25b}$$

Now, consider the following relationships (see [18]).

a) Spectral relationship:

$$\int_{-1}^{1}\frac{(1-y^2)^{\frac{\upsilon-1}{2}}}{|u-y|^{\nu}} C_n^{\left(\frac{\upsilon}{2}\right)}(y)dy=\frac{\pi\Gamma(n+\upsilon)}{\Gamma(\upsilon)\Gamma(n+1)\cos\left(\frac{\pi\upsilon}{2}\right)} C_n^{\left(\frac{\upsilon}{2}\right)}(u). \tag{26a}$$

b) Orthogonal relationship:

$$\int_{-1}^{1}(1-y^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(y) C_m^{\left(\frac{\upsilon}{2}\right)}(y)dy=\frac{\pi 2^{1-\upsilon}\Gamma(n+\upsilon)}{n!\left(n+\frac{\upsilon}{2}\right)\Gamma^2\left(\frac{\upsilon}{2}\right)}\delta_{m,n}. \tag{26b}$$

In (25a), $b_n, n\geq 0$ are called the eigenvalues of the unknown function $\psi(x)$. The function $(1-x^2)^{\frac{\upsilon-1}{2}}$ is the weight function of Gegenbauer polynomials of order $\left(\frac{\upsilon}{2}\right)$.

Truncate formula (25a) as an approximate solution to take the form

$$\psi_N(u)=\sum_{n=0}^{N} b_n(1-u^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(u),\quad \lim_{N\to\infty}\psi_N(u)=\psi(u). \tag{27}$$

So, Eq (24) takes the form

$$\mu^*\sum_{n=0}^{N} b_n(1-u^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(u)-\lambda^*\int_{-1}^{1}\sum_{n=0}^{N} b_n|y-u|^{-\upsilon}(1-y^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(y)\,dy$$
$$=\sum_{n=0}^{N} g_n(1-u^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(u). \tag{28}$$

By using Eq (26a), we get

$$\mu^*\sum_{n=0}^{N} b_n(1-u^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(u)-\lambda^*\sum_{n=0}^{N} b_n\frac{\pi\Gamma(n+\upsilon)}{\Gamma(\upsilon)\Gamma(n+1)\cos\left(\frac{\pi\upsilon}{2}\right)} C_n^{\left(\frac{\upsilon}{2}\right)}(u)$$
$$=\sum_{n=0}^{N} g_n(1-u^2)^{\frac{\upsilon-1}{2}} C_n^{\left(\frac{\upsilon}{2}\right)}(u). \tag{29}$$

Multiplying (29) by $C_m^{\left(\frac{v}{2}\right)}(u)$ and integrating it with respect to u, $u \in [-1,1]$, we get

$$\begin{aligned} &\mu^* \sum_{n=0}^{N} b_n \frac{\pi 2^{1-v}\Gamma(n+v)}{n!\left(n+\frac{v}{2}\right)\Gamma^2\left(\frac{v}{2}\right)} \\ &- \lambda^* \sum_{n=0}^{N} b_n \frac{\pi\Gamma(n+v)}{\Gamma(v)\Gamma(n+1)\cos\left(\frac{\pi v}{2}\right)} \int_{-1}^{1} C_n^{\left(\frac{v}{2}\right)}(u) C_m^{\left(\frac{v}{2}\right)}(u)du \\ &= \sum_{n=0}^{N} g_n \frac{\pi\, 2^{1-v}\Gamma(n+v)}{n!\left(n+\frac{v}{2}\right)\Gamma^2\left(\frac{v}{2}\right)}. \end{aligned} \tag{30}$$

By using the Gegenbauer relations (see [18]),

$$\begin{aligned} &C_m^v(u) C_n^v(u) \\ &= \sum_{k=|m-n|}^{m+n} \frac{\left[\dfrac{(1-(k+m+n)\bmod 2)(k+v)k!\,\Gamma\left(\frac{1}{2}(-k+m+n+2v)\right)}{\Gamma\left(\frac{1}{2}(k+m-n+2v)\right)\Gamma\left(\frac{1}{2}(k-m+n+2v)\right)\Gamma\left(\frac{1}{2}(k+m+n+4v)\right)}\right]}{\left[\dfrac{\Gamma\left(\frac{1}{2}(-k+m+n+2)\right)\Gamma\left(\frac{1}{2}(k+m-n+2)\right)}{\Gamma\left(\frac{1}{2}(k-m+n+2)\right)\Gamma\left(\frac{1}{2}(k+m+n+2v+2)\right)\Gamma(k+2v)\Gamma^2(v)}\right]} C_k^v(u), \end{aligned} \tag{31a}$$

$$\int_{-1}^{1} C_n^v(x)\,dx = \frac{C_{n+1}^{v-1}(1) - C_{n+1}^{v-1}(-1)}{2(v-1)}, \tag{31b}$$

$$C_n^v(1) = \frac{\Gamma(2v+n)}{\Gamma(2v)\Gamma(n+1)}, \tag{31c}$$

$$C_n^v(-1) = \frac{\Gamma(2v+n)}{\Gamma(2v)\Gamma(n+1)} \cos\big(\pi(v+n)\big)\sec(\pi v), \tag{31d}$$

and we have the following linear algebraic system (LAS):

$$\begin{aligned} &\mu^* b_n - \frac{\lambda^*\left(n+\frac{v}{2}\right)\Gamma^2\left(\frac{v}{2}\right)}{2^{1-v}(v-2)\Gamma(v)\cos\left(\frac{\pi v}{2}\right)} \sum_{k=|m-n|}^{m+n} b_m \chi_{k,n,m} \left(\frac{\Gamma(v+k-1)}{\Gamma(v-2)\Gamma(k+2)}\Big\{1\right. \\ &\left. - \cos\left(\frac{\pi v}{2}+\pi k\right)\sec\left(\frac{\pi v}{2}-\pi\right)\Big\}\right) = g_n, \end{aligned} \tag{32}$$

where $n = 0, 1, 2, 3, \cdots, N$ and $\chi_{k,n,m}$ is given by

$$\chi_{k,n,m} = \left[\begin{array}{c}(1-(k+m+n) \bmod 2)\left(k+\frac{v}{2}\right)k!\,\Gamma\left(\frac{1}{2}(-k+m+n+v)\right)\Gamma\left(\frac{1}{2}(k+m-n+v)\right) \\ \Gamma\left(\frac{1}{2}(k-m+n+v)\right)\Gamma\left(\frac{1}{2}(k+m+n+2v)\right)\end{array}\right]$$
$$/\left[\begin{array}{c}\Gamma\left(\frac{1}{2}(-k+m+n+2)\right)\Gamma\left(\frac{1}{2}(k+m-n+2)\right)\Gamma\left(\frac{1}{2}(k-m+n+2)\right) \\ \Gamma\left(\frac{1}{2}(k+m+n+v+2)\right)\Gamma(k+v)\Gamma^2\left(\frac{v}{2}\right)\end{array}\right]. \tag{33}$$

6. The convergence of the algebraic system

We write the LAS (32) in the operator form

$$\bar{Q}b_n = \frac{1}{\mu^*}g_n + \frac{1}{\mu^*}Qb_m, \quad Qb_m = \lambda^*\omega(n,v)\sum_{k=|m-n|}^{m+n} b_m C(k,v)\chi_{k,n,m}, \tag{34}$$

where

(a) $\omega(n,v) = \dfrac{\left(n+\frac{v}{2}\right)\Gamma^2\left(\frac{v}{2}\right)}{2^{1-v}(v-2)\Gamma(v)\cos\left(\frac{\pi v}{2}\right)}$,

(b) $C(k,v) = \left(\dfrac{\Gamma(v+k-1)}{\Gamma(v-2)\Gamma(k+2)}\left\{1-\cos\left(\frac{\pi v}{2}+\pi k\right)\sec\left(\frac{\pi v}{2}-\pi\right)\right\}\right)$.

To prove the convergence of (34), we assume

(I) $\|\chi_{k,n,m}\| = \max\limits_{N}\sum\limits_{k=|m-n|}^{m+n}|\chi_{k,n,m}| = \beta_1, \qquad N = \begin{cases}\max(m+n), \\ \min|m-n|.\end{cases}$

(II) $|g_n| = \gamma$,

(III) $\|\omega(n,v)\| = \max\limits_{n}|\omega(n,v)| = \delta_1$,

(IV) $\|C(k,v)\| = \max\limits_{N}\sum\limits_{k=|m-n|}^{m+n}|C(k,v)| = \delta_2$.

Then, we state the following theorem.

Theorem 5. The LAS (32) or (34) is convergent in l^∞ space, under the above assumptions, and has a unique solution under the condition:

$$\beta_1\delta_1\delta_2\lambda^* < \mu^*. \tag{35}$$

Proof. From Eq (34), we have

$$|Qb_m| \le |\lambda^*||\omega(n,v)| \sum_{k=|m-n|}^{m+n} \left|b_m C(k,v)\chi_{k,n,m}\right|.$$

The above inequality takes the form

$$\|Qb_m\| \le \max_n |\omega(n,v)|\, |\lambda^*| \left(\max_{m,n} \sum_{k=|m-n|}^{m+n} \left|\chi_{k,n,m}\right|\right)\left(\max_{m,n} \sum_{k=|m-n|}^{m+n} |C(k,v)|\right)|b_m|.$$

Hence, we have

$$\|Qb_m\| \le \beta_1\delta_1\delta_2|\lambda^*||b_m|. \tag{36}$$

Finally, we get

$$\|\bar{Q}b_n\| \le \frac{1}{|\mu^*|}\{\gamma + \beta_1\delta_1\delta_2|\lambda^*||b_m|\}. \tag{37}$$

Formula (36) leads to the convergence of the linear algebraic system, while inequality (37) leads to the uniqueness of the system under the given condition (35).

An illustrative example

Consider the V-FIE

$$\mu\Phi(x,t) - \lambda\int_0^t\int_{-1}^1 ts|x-y|^{-\nu}\Phi(y,s)dyds = F(x,t).$$

By using formula (24), we have

$$\mu^*\psi(x) - \lambda^*\int_{-1}^1 |x-y|^{-\nu}\psi(y)dy = g(x).$$

If $A(t) = B(t)$, we have

$$|\lambda^*| = |\lambda|T^3,\ |\mu^*| = 1.$$

The convergent approximate solution can be obtained under the condition $\frac{\beta_1\delta_1\delta_2|\lambda^*|}{|\mu^*|} < 1$, which can be shown in Table 3.

Table 3. Convergence condition for different values of $\nu, \lambda,$ and N with $T = 0.7$.

ν	n	N	β_1	δ_1	δ_2	λ	λ^*	Convergence condition
0.07	4	8	0.0265	62.963	5.0×10^{-4}	0.018	0.006	5.0×10^{-6}
						0.700	0.240	2.0×10^{-4}
	8	16	0.0177	125.381	7.0×10^{-5}	0.018	0.006	9.0×10^{-7}
						0.700	0.240	3.0×10^{-5}
	16	32	0.0133	250.215	1.0×10^{-5}	0.018	0.006	2.0×10^{-7}
						0.700	0.240	8.0×10^{-6}
0.4	4	8	0.4794	20.340	0.004	0.018	0.006	2.0×10^{-4}
						0.700	0.240	0.008
	8	16	0.5266	39.711	6.0×10^{-4}	0.018	0.006	7.0×10^{-5}
						0.700	0.240	0.003
	16	32	0.6241	78.453	1.0×10^{-4}	0.018	0.006	3.0×10^{-5}
						0.700	0.240	0.001

From Table 3, we deduce that there is fast convergence whenever N increases, while there is slow convergence when the values of ν and λ increase.

7. Numerical results

In this section, we present some examples to demonstrate the accuracy and applicability of the presented techniques by considering the requirements for the existence of a solution and its numerical convergence, shown by Tables 1–3.

Example 1. Consider the V-FIE

$$\mu\Phi(x,t) - \lambda \int_0^t \int_{-1}^1 ts|x-y|^{-\nu}\Phi(y,s)dyds = F(x,t), \qquad 0 \le t \le T < 1. \tag{38}$$

Here, $F(x,t)$ is given by sitting $\Phi(x,t) = x^2t^2$ as an exact solution, and its error function is given by $\Re_N = |\Phi(x,t) - \Phi_N(x,t)|$. Under the assumption $\frac{\lambda\alpha\beta T}{\mu} < 1$ for some values of λ and for the given values $\nu = 0.07$ and $T = 0.09$, shown in Table 1, mean errors and their rate of convergence for different values of N are represented in Table 4.

Table 4. Mean errors and their rate of convergence.

λ	N	Mean error	Convergence rate
0.018	4	9.02×10^{-4}	2.25
	8	1.91×10^{-4}	1.32
	16	7.60×10^{-5}	1.84
	32	2.13×10^{-5}	---
0.7	4	9.03×10^{-4}	2.07
	8	2.15×10^{-4}	1.49
	16	7.64×10^{-5}	1.84
	32	2.15×10^{-5}	---

From Table 4, the error decreases with increasing values of N, and its approximate numerical solution is stable with increasing values of λ under the convergence condition.

Example 2. Consider the V-FIE

$$\Phi(x,t) - 0.18\int_0^t \int_{-1}^1 t^2 s^2 |x-y|^{-\nu}\Phi(y,s)dyds = F(x,t), \tag{39}$$

where $F(x,t)$ is specified by setting $\Phi(x,t) = (x^2 + x^5)(0.005 + 0.03t + 0.7t^3)$ as an accurate solution.

Table 5. The solution $\Phi(x,t)$ and its corresponding errors where $N = 16$ for different values of time and υ.

T	x	$\upsilon = 0.07$		$\upsilon = 0.4$	
		$\Phi(x,t)$	Error $(\Re_N)$	$\Phi(x,t)$	Error $(\Re_N)$
	–0.8	0.0016164	0.0000079	0.0016204	0.0000119
	–0.4	0.0007548	0.0000164	0.0007523	0.0000189
0.005	0.0	–0.0000122	0.0000122	– 0.0000141	0.0000141
	0.4	0.0008726	0.0000041	0.0008738	0.0000029
	0.8	0.0050392	0.0000555	0.0050360	0.0000523
	–0.8	0.0025769	0.0000126	0.0025833	0.0000191
	–0.4	0.0012033	0.0000262	0.0011993	0.0000302
0.09	0.0	–0.0000195	0.0000195	–0.0000224	0.0000224
	0.4	0.0013911	0.0000066	0.0013930	0.0000047
	0.8	0.0080335	0.0000886	0.0080284	0.0000834
	–0.8	0.0825486	0.0005598	0.0824931	0.0006153
	–0.4	0.0380344	0.0018165	0.03770211	0.0021488
0.7	0.0	–0.0015983	0.0015983	–0.0018176	0.0018176
	0.4	0.0440846	0.0012165	0.0437332	0.0015679
	0.8	0.2592969	0.0017963	0.2579618	0.0004611

In Table 5, the errors are presented for different values of ν, demonstrating that the errors increase with time T and are stable.

By taking $\upsilon = 0.4$, for example, the error increases with time and decreases by increasing the number of iterations; see Figures 1–4.

Figure 5 represents the approximate solution $\Phi(x,t)$ of Example 2 in 3-dimensional space.

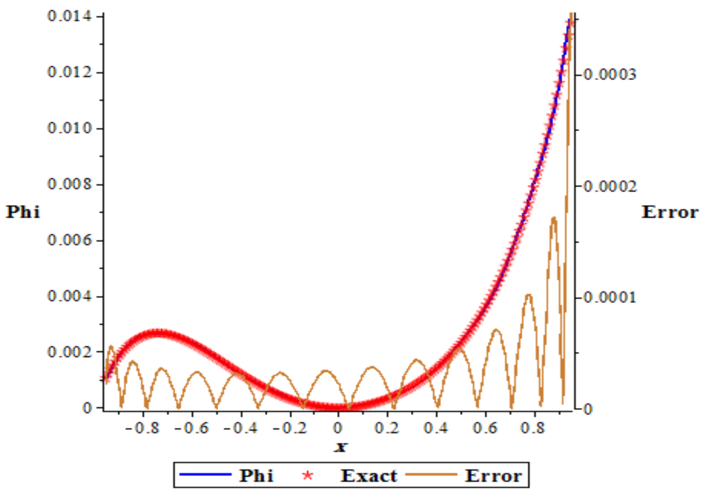

Figure 1. The error of Example 2, where $N = 16, T = 0.09$.

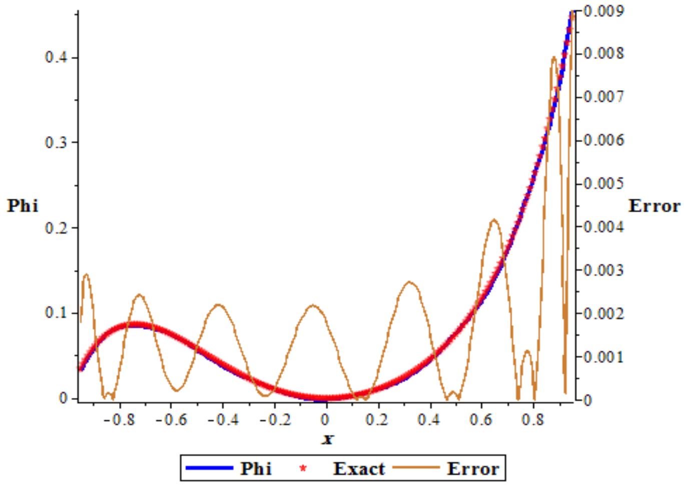

Figure 2. The error of Example 2, where $N = 16, T = 0.7$.

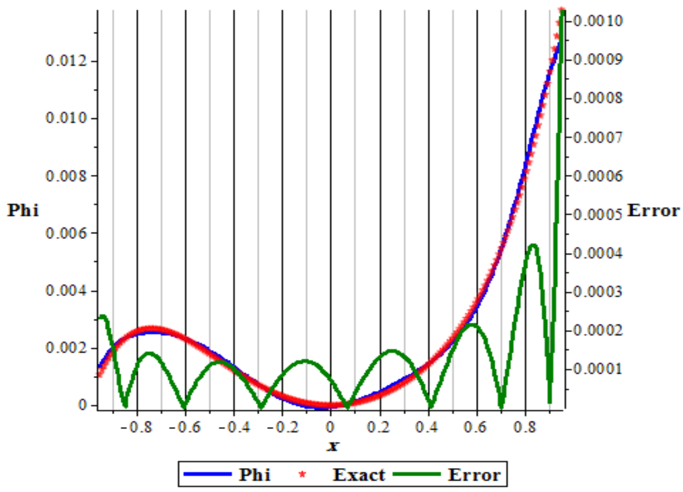

Figure 3. The error of Example 2, where $N = 8, T = 0.09$.

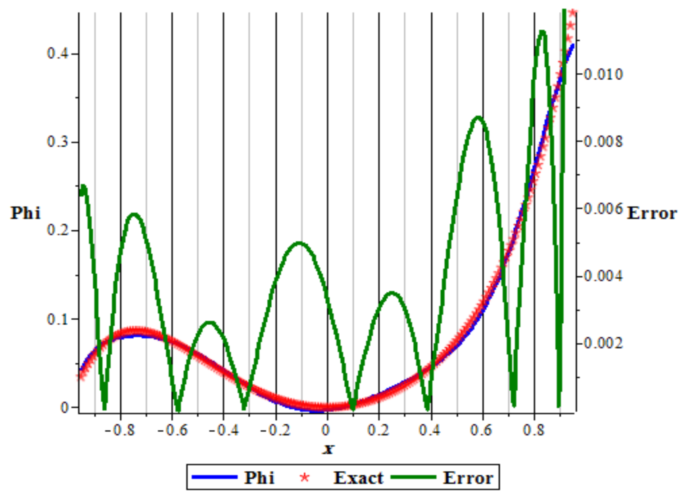

Figure 4. The error of Example 2, where $N = 8, T = 0.7$.

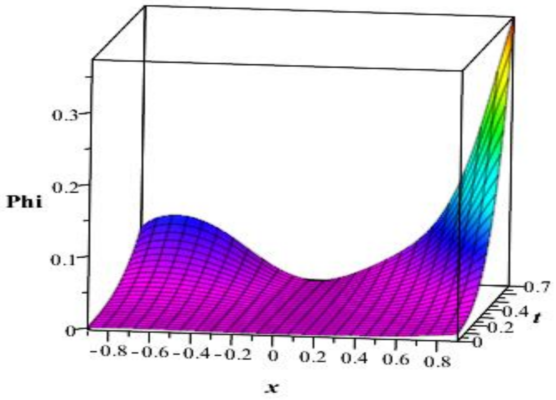

Figure 5. The approximate solution $\Phi(x,t)$ of Example 2, where $N = 8$.

Example 3. Consider the F-VIE

$$\mu\Phi(x,t) - \lambda\int_0^t t^2 s^2 \Phi(x,s)ds - \lambda\int_{-1}^{1} |x-y|^{-\nu}\Phi(y,t)dy = F(x,t). \tag{40}$$

Figures 6–11 show the solution and the associated errors for the F-VIE with a Carleman kernel. We noticed that the errors declined as the values of ν decreased.

Figure 12 represents the approximate solution $\Phi(x,t)$ of Example 3 in 3-dimensional space

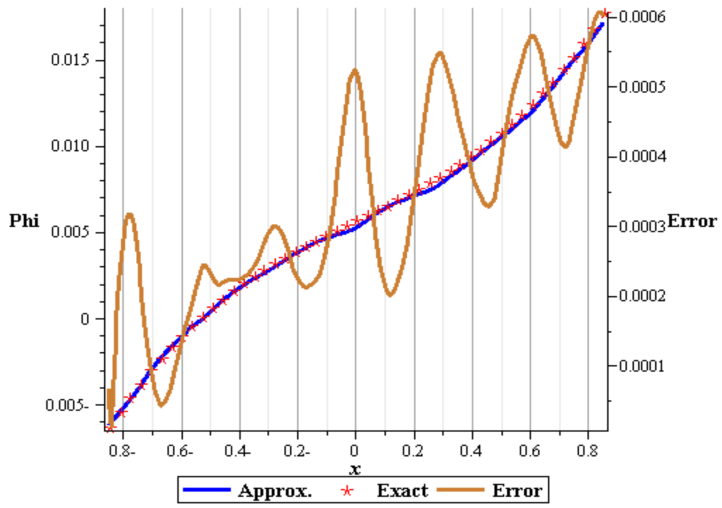

Figure 6. The error of Example 3, where $N = 16, T = 0.09$, and $\nu = 0.4$.

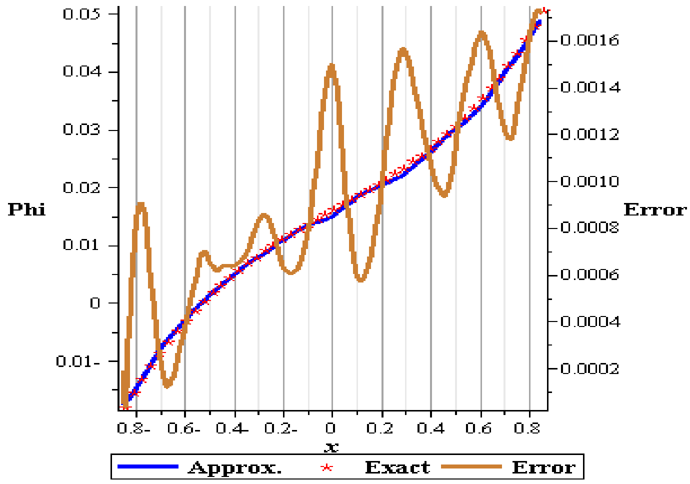

Figure 7. The error of Example 3, where $N = 16, T = 0.25$, and $\nu = 0.4$.

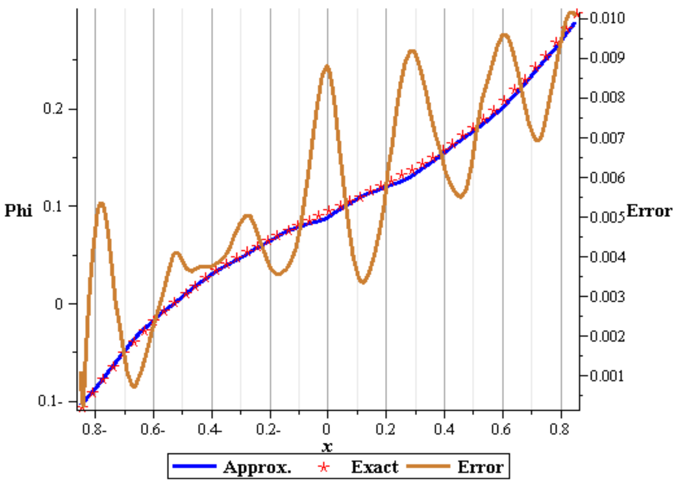

Figure 8. The error of Example 3, where $N = 16, T = 0.55$, and $\nu = 0.4$.

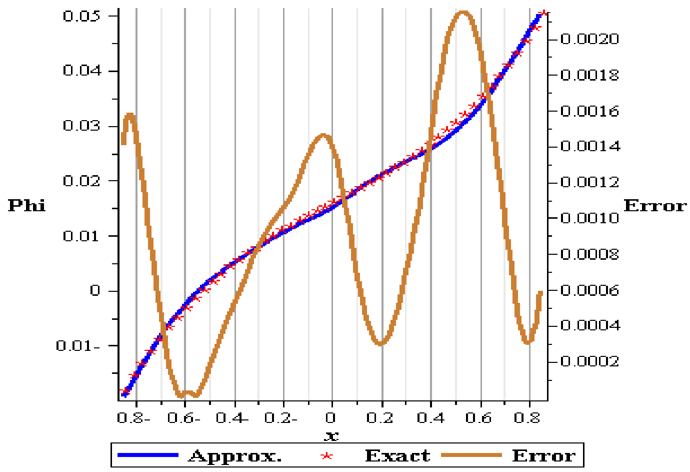

Figure 9. The error of Example 3, where $N = 8, T = 0.25$, and $\nu = 0.4$.

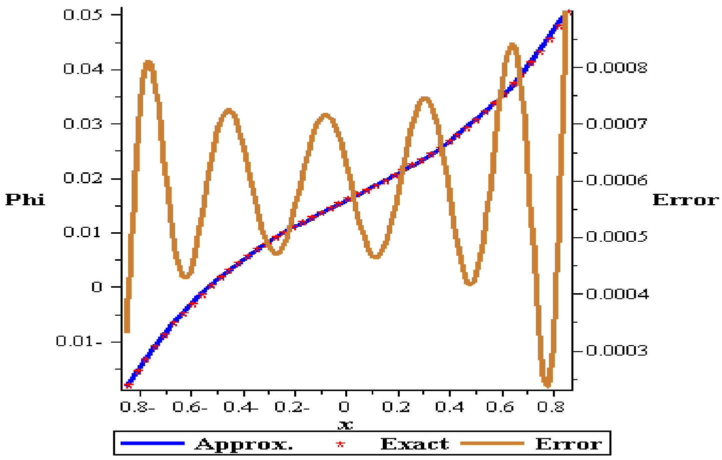

Figure 10. The error of Example 3, where $N = 16, T = 0.25$, and $\nu = 0.007$.

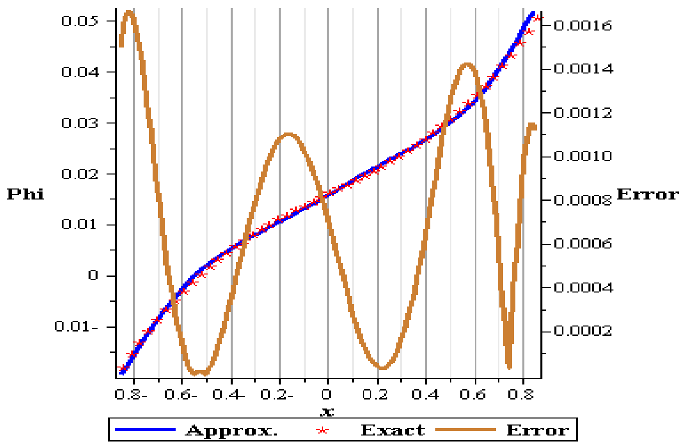

Figure 11. The error of Example 3, where $N = 8, T = 0.25$, and $\nu = 0.007$.

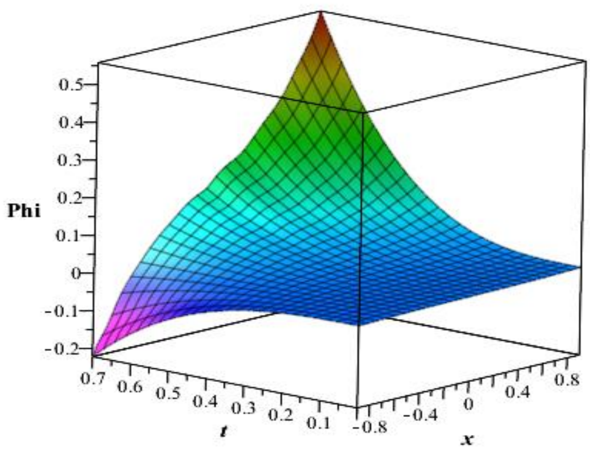

Figure 12. The approximate solution $\Phi(x, t)$ of Example 3, where $N = 8$.

8. Conclusions

This study aimed to deepen our understanding of the role that Gegenbauer polynomials play in solving mixed integral equations. By combining a rigorous exploration of mathematical principles with insights from recent references, this study aimed to contribute to the ongoing discourse in the field and provide a valuable resource for researchers and experts seeking the potential of Gegenbauer polynomials in the solution of MIEs. From the previous results, we conclude the following:

The separation variables technique is a strategy that assists in solving the scientific deficiencies of previous approaches, as it allows researchers to control the time required to solve the problem in a specific way.

The method of separation of variables was used in this research to transform the mixed integral equation in position and time into an integral equation in position and with coefficients in time. Furthermore, spectral relationships can be derived, which helps in solving many mathematical physics problems.

Using the orthogonal polynomials technique and certain special functions, we may quickly express that the solution is a linear relationship between the eigenvalues and the eigenfunctions.

In Example 2, we considered a V-FIE with a Carleman kernel for different values of ν and time. We observed that the errors increased with time and were extremely stable for different values of ν (see Table 5).

By increasing the iteration number N, the errors decreased, which can be observed in Figures 1–4, while the approximate solution $\Phi(x,t)$ of Example 2 appears in Figure 5.

In Example 3, we numerically presented the solution of a F-VIE with a Carleman kernel. The solution and its corresponding errors are displayed in Figures 6–11, and we observed that by decreasing the values of ν, the errors decreased. The approximate solution $\Phi(x,t)$ is also illustrated in Figure 12.

Author contributions

Conceptualization, M.A.A.; Methodology, A.A. and M.B.; Software, A.A. and M.B.; Validation, M.A.A.; Formal analysis, A.A., M.A.A., and M.B.; Resources, A.A. and M.B.; Writing - original draft, A.A. and M.B.; Writing - review & editing, A.A., M.A.A., and M.B.; Project administration, A.A.; Funding acquisition, A.A. and M.B. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the publication of this article.

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