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Research article

On the minimum distances of binary optimal LCD codes with dimension 5

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Abstract: Let $d_a(n,5)$ and $d_l(n,5)$ be the minimum weights of optimal binary $[n,5]$ linear codes and linear complementary dual (LCD) codes, respectively. This article aims to investigate $d_l(n,5)$ of some families of binary $[n,5]$ LCD codes when $n = 31s + t \geq 14$ with s integer and $t \in \{2,8,10,12,14,16,18\}$. By determining the defining vectors of optimal linear codes and discussing their reduced codes, we classify optimal linear codes and calculate their hull dimensions. Thus, the non-existence of these classes of binary $[n,5,d_a(n,5)]$ LCD codes is verified, and we further derive that $d_l(n,5) = d_a(n,5) - 1$ for $t \neq 16$ and $d_l(n,5) = 16s + 6 = d_a(n,5) - 2$ for $t = 16$. Combining them with known results on optimal LCD codes, $d_l(n,5)$ of all $[n,5]$ LCD codes are completely determined.

Keywords: optimal code; LCD code; hull dimension; defining vector; reduced code
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1. Introduction

Let F_2^n be the n-dimensional row vector space over the binary field F_2. An binary $[n,k]$ linear code is a k-dimensional subspace of F_2^n. The weight $w(X)$ of a vector $X \in F_2^n$ is the number of its nonzero coordinates. If the minimum weight of nonzero vectors in an $[n,k]$ code is d, then d is called the minimum distance of C and the code C is denoted as $[n,k,d]$. C is optimal if d can meet the largest value for n,k, which is denoted as $[n,k,d_a(n,k)]$ or $[n,k,d_a]$. Two binary codes C and C' are equivalent if one can be obtained from the other by permuting the coordinates [1]. They are denoted as $C \cong C'$. A matrix whose rows form a basis of C is called a generator matrix of this code.

The dual code $C^\perp$ of C is defined as $C^\perp = \{X \in F_2^n \mid X \cdot Y = 0 \text{ for all } Y \in C\}$. A code C is *self-orthogonal* (SO) if $C \subseteq C^\perp$. The hull of a linear code C was defined as $Hu(C) = C^\perp \cap C$ in [2], and was called a radical code of C in the nomenclature of classical groups in [3]. Define $h(C) = \dim Hu(C)$ as the *hull dimension* of C, and $h([n,k,d]) = \min\{h(C) \mid C \text{ is a binary } [n,k,d] \text{ code}\}$.

If $Hu(C) = \{\mathbf{0}\}$ (or $h(C) = 0$), then C is an LCD code [4]. LCD cyclic codes were introduced by

Massey [4] and gave an optimal linear coding solution for the two-user binary adder channel. Carlet et al. showed that LCD codes can be used to fight against side-channel attacks [5]. In recent years, much work has been done on the properties and construction of LCD codes [5–20]. It has been shown in [6] that any code over F_q is equivalent to some LCD code for $q \geq 4$, which motivates people to study binary and ternary LCD codes. In this paper, we focus on the hull dimensions of binary optimal linear codes and LCD codes.

It is an important problem to determine the largest minimum weight $d_l(n,k)$ among all LCD codes for n, k. Recently, constructions of optimal LCD codes with short lengths or low dimensions have been discussed, and low and upper bounds for $d_l(n,k)$ have been established [5–20]. If $n \leq 24$ and $1 \leq k \leq n$, $d_l(n,k)$ were determined, and for $k \leq n \leq 40$, most of $d_l(n,k)$ were given in [6–15]. For $n \leq 50$ and $k < 13$, Ref. [17] obtained $d_l(n,k)$ by exhaustive search. If $k \leq 4$, all $d_l(n,k)$ were determined in [8–12]. As for $k = 5$, $d_l(n,5)$ was partially determined in [11–13] except for $n = 31s + t \geq 40$ when $t \in \{2, 8, 10, 12, 14, 16, 18\}$. In [15], Li *et al.* introduced the *reduced code* of a linear code and developed some new approaches to determine upper bounds on optimal linear codes.

Let C be an $[n,k,d]$ linear code with generator matrix G and parity-check matrix H, C is LCD if and only if the matrix GG^T or HH^T is invertible [4]. Thus, to prove C is not LCD, one only needs to verify $h(C) = k - (rank(GG^T)) \geq 1$ or $h(C) = n - k - (rank(HH^T)) \geq 1$. If all $[n,k,d]$ linear codes have hull dimensions greater than 1, that is to say that $h([n,k,d]) \geq 1$, then there is no LCD code with minimum distance d for given n, k.

In [19], Li *et al.* introduced two concepts called the *defining vector* and the *weight vector* of an $[n,5,d]$ linear code with a given generator matrix, and further established relations among parameters of this code, its defining vector, and weight vector. They changed the classification problem of binary optimal self-orthogonal codes into solving the system of linear equations. Further research on defining vectors and weight vectors of optimal linear codes and their applications was made in [20]. The classifications of all optimal $[n,k]$ codes with $k \leq 4$ and some optimal $[n,k]$ codes with $k \geq 5$ were determined [20].

Inspired by Refs. [15,19,20], we will show all optimal $[n,5]$ codes are not LCD for $n = 31s + t \geq 14$ and $t \in \{2, 8, 10, 12, 14, 16, 18\}$. Set $k = 5$ and $N = 31$. Denote $L = (l_1, l_2, \cdots, l_{31})$ as the defining vector of an $[n,5,d_a]$ code with generator matrix G (for details see Section 2), and let $l_{max} = max_{1 \leq i \leq N}\{l_i\}$, $l_{min} = min_{1 \leq i \leq N}\{l_i\}$. The main techniques used in this manuscript can be briefly described as follows:

(1) From the parameters of all $[n,5,d_a]$ codes, estimate l_{max} and l_{min} according to Ref. [20].

(2) According to l_{max} and l_{min} , one can first analyze the following two items:

i) Whether C has a reduced code $\mathcal{D}$ with a hull dimension greater than 2.

ii) Whether C is equivalent to a code that is not LCD.

(3) If l_{max} and l_{min} do not meet any of (2), determine all such L's and all $[n,5,d_a]$ codes with defining vectors L's, classify $[n,5,d_a]$ codes, and calculate their hull dimensions and weight enumerators.

For details, see Definition 1, Lemmas 2 and 5. One can clearly understand the above three steps according to the proof of Lemma 5. If $h(\mathcal{D}) \geq 2$, then $h(C) \geq 1$, it follows that C is not LCD. For all $[n,5,d_a]$ codes, if all of their hull dimensions are greater than 1, then one can know any $[n,5,d_a]$ code is not LCD.

Our main conclusion is given by Theorem 1.

Theorem 1. *If s is an integer, $t \in \{2, 8, 10, 12, 14, 16, 18\}$ and $n = 31s + t \geq 14$, then an optimal*

$[n, 5, d_a(n, 5)]$ *linear code is not LCD, and we further have*

$$d_l(n,5) = \begin{cases} d_a - 1 & \text{if } t \in \{2, 8, 10, 12, 14, 18\}; \\ d_a - 2 & \text{if } t = 16. \end{cases}$$

Combining it with the results of Refs. [7–14,17,19] on optimal LCD codes, one can completely determine $d_l(n, 5)$ for all $n \geq 5$, which is shown in Table 1 and Theorem 2.

Table 1. Minimum distances of optimal binary $[n, 5]$ LCD codes with $n = 31s + t \geq 14$.

n	$31s$	$31s+1$	$31s+2$	$31s+3$	$31s+4$	$31s+5$	$31s+6$
d_a	$16s$	$16s$	$16s$	$16s$	$16s$	$16s+1$	$16s+2$
d_l	$16s-2$	$16s-1$	$16s-1$	$16s$	$16s$	$16s+1$	$16s+1$
n	$31s+7$	$31s+8$	$31s+9$	$31s+10$	$31s+11$	$31s+12$	$31s+13$
d_a	$16s+2$	$16s+3$	$16s+4$	$16s+4$	$16s+4$	$16s+5$	$16s+6$
d_l	$16s+2$	$16s+2$	$16s+3$	$16s+3$	$16s+4$	$16s+4$	$16s+5$
n	$31s+14$	$31s+15$	$31s+16$	$31s+17$	$31s+18$	$31s+19$	$31s+20$
d_a	$16s+6$	$16s+7$	$16s+8$	$16s+8$	$16s+8$	$16s+8$	$16s+9$
d_l	$16s+5$	$16s+6$	$16s+6$	$16s+7$	$16s+7$	$16s+8$	$16s+9$
n	$31s+21$	$31s+22$	$31s+23$	$31s+24$	$31s+25$	$31s+26$	$31s+27$
d_a	$16s+10$	$16s+10$	$16s+11$	$16s+12$	$16s+12$	$16s+12$	$16s+13$
d_l	$16s+9$	$16s+10$	$16s+10$	$16s+11$	$16s+11$	$16s+12$	$16s+12$
n	$31s+28$	$31s+29$	$31s+30$				
d_a	$16s+14$	$16s+14$	$16s+15$				
d_l	$16s+13$	$16s+13$	$16s+14$				

*Note: d_a and d_l denote the minimum weights of optimal binary $[n, 5]$ linear codes and LCD codes, respectively.

Theorem 2. *If $n = 31s + t \geq 5$, then there are optimal LCD codes as follows:*

(1) ([7,8,17]) If $5 \leq n \leq 13$ and $n \neq 6, 10$, then there is an optimal LCD $[n, 5, d_a(n, 5)]$ code, while $n = 6, 10$, an optimal $[n, 5, d_a(n, 5) - 1]$ LCD code exists.

(2) ([9–13,17]) If $t = 3, 4, 5, 7, 11, 19, 20, 22, 26$, $n = 31s + t \geq 14$, then there is an optimal $[n, 5, d_a(n, 5)]$ LCD code.

(3) If $t \neq 0, 3, 4, 5, 7, 11, 16, 19, 20, 22, 26$ and $n = 31s + t \geq 14$, then there is an optimal $[n, 5, d_a(n, 5) - 1]$ LCD code according to Refs. [9–13,17] and Theorem 1 above.

(4) If $t = 0, 16$ and $n = 31s + t \geq 14$, there is an optimal $[n, 5, d_a(n, 5) - 2]$ LCD code according to Ref. [19] and Theorem 1 above.

Remark 1. *From Ref. [18], it is easy to know all optimal [n,5] codes can achieve the Griesmer bound for $14 \leq n \leq 256$. For $n > 256$, the length n can be denoted as $n = 31s+t$, where $s \geq 7$ and $31 \leq t \leq 61$ are integers. By the juxtaposition of s simplex codes [31,5,16] and an optimal $[t, 5, d_a(t, 5)]$ code, one can easily obtain all optimal $[n, 5, d_a(n, 5)]$ linear codes with $d_a(n, 5)$ achieving the Griesmer bound for $n > 256$. That is to say, any $d_a(n, 5)$ can be obtained by the Griesmer bound for all lengths $n \geq 14$. It naturally follows that $d_l(n, 5)$ can be denoted by $d_a(n, 5)$ for some code lengths in Theorems 1 and 2.*

The rest of this paper is organized as follows: In Section 2, some definitions, notations, and basic results about optimal LCD codes are given. The proof of the main result, Theorem 1, is provided in Section 3. Section 4 gives conclusions and discussions.

2. Preliminaries

In this section, some concepts and notations will be given for later use [19,20]. The all-one vector and zero vector of length n are defined as $\mathbf{1_n}=(1,1,\cdots,1)_{1\times n}$ and $\mathbf{0_n}=(0,0,...,0)_{1\times n}$, respectively. Let $iG=(G,G,\cdots,G)$ be the juxtaposition of i copies of G for given matrix G, then the juxtaposition of i copies of C can be denoted as iC with generator matrix iG. In this article, we consider linear codes without zero coordinates and matrices without zero columns.

Let $N=2^k-1$, consider

$$S_2=\begin{pmatrix}101\\011\end{pmatrix}, S_3=\begin{pmatrix}S_2 & \mathbf{0}_2^T & S_2\\ \mathbf{0}_3 & 1 & \mathbf{1}_3\end{pmatrix},\cdots, S_{k+1}=\begin{pmatrix}S_k & \mathbf{0}_k^T & S_k\\ \mathbf{0}_{2^k-1} & 1 & \mathbf{1}_{2^k-1}\end{pmatrix}.$$

The matrix S_k generates the k-dimensional simplex code $\mathcal{S}_k=[2^k-1,k,2^{k-1}]$. Let $\boldsymbol{\alpha}_i$ be the i-th column of S_k for $1\le i\le N$. The last 2^k-2^m columns of S_k form a matrix $M_{k,m}$ for $1\le m\le k-1$, $M_{k,m}$ generates the k-dimensional $\mathcal{MD}_{k,m}=[2^k-2^m,k,2^{k-1}-2^{m-1}]$ MacDonald code [21]. Simplex codes $\mathcal{S}_k$ and MacDonald codes $\mathcal{MD}_{k,m}$ for $k\ge 4$ will be used to discuss the hull dimensions of some optimal codes.

Let $G=G_{k\times n}$ be a generator matrix of C. If there are l_i copies of $\boldsymbol{\alpha}_i$ in G for $1\le i\le N$, we then denote G as $G=(l_1\boldsymbol{\alpha}_1,\cdots,l_N\boldsymbol{\alpha}_N)$ for short, and call $L=(l_1,\cdots,l_N)$ the *defining vector* of C with generator matrix G. Let l_{j_l} $(1\le l\le t)$ be different coordinates of $L=(l_1,l_2,\cdots,l_N)$ with $l_{j_1}<l_{j_2}<\cdots<l_{j_t}$ in ascending order by the number of equal l_{j_l}. If there are m_l entries equal to l_{j_l}, we say L is of *type* $]\!](l_{j_1})_{m_1}\mid\cdots\mid(l_{j_t})_{m_t}]\!]$. For example, a code with defining vector $L_1=(3,1,1,3,1,3,1)$ is an SO code, this can be derived from the type $]\!](1)_4\mid(3)_3]\!]$ of L_1, and $L_2=(s+1,s-1,s,s,s+1,s-1,s+1)$ is of type $]\!](s-1)_2\mid(s)_2\mid(s+1)_3]\!]$.

Some properties of an $[n,k,d]$ code can be characterized by its defining vectors. Relations among these objects are connected by some matrices P_k and Q_k derived from the simplex code $\mathcal{S}_k$ [19]. On the other hand, if an $[n,k,d_a]$ code is optimal, we can determine all defining vectors whose corresponding codes have such parameters by solving linear equations [19].

Let J_k be the $(2^k-1)\times(2^k-1)$ all-one matrix, and P_2 be a $(2^2-1)\times(2^2-1)$ matrix whose rows are the non-zero codewords of $\mathcal{S}_2$. Using the recursive method, one can construct

$$P_2=\begin{pmatrix}101\\011\\110\end{pmatrix}, P_3=\begin{pmatrix}P_2 & 0 & P_2\\ \mathbf{0_3} & 1 & \mathbf{1_3}\\ P_2 & \mathbf{1_3^T} & Q_2\end{pmatrix},\cdots, P_{k+1}=\begin{pmatrix}P_k & \mathbf{0^T_{2^k-1}} & P_k\\ \mathbf{0_{2^k-1}} & 1 & \mathbf{1_{2^k-1}}\\ P_k & \mathbf{1^T_{2^k-1}} & Q_k\end{pmatrix},$$

where $Q_k=J_k-P_k$ for $k\ge 2$. Then, the seven rows of P_3 are just the seven nonzero vectors of the simplex code $\mathcal{S}_3=[7,3,4]$. For $k\ge 3$, then the matrix formed by nonzero codewords of $(k+1)$-dimensional simplex code can be obtained from P_k. Each row of P_k has (2^{k-1})'s ones and $(2^{k-1}-1)$'s zeros. Hence, each row of Q_k has $(2^{k-1}-1)$'s ones and (2^{k-1})'s zeros. According to Ref. [20], P_k and Q_k are symmetric matrices, and the matrix P_k is invertible over the rational field and

$$P_k^{-1}=\frac{1}{2^{k-1}}[J_k-2Q_k].$$

If C has a generator matrix $G=(l_1\boldsymbol{\alpha}_1,\cdots,l_N\boldsymbol{\alpha}_N)$, the minimum distance d of C and its codeword weights can be determined by its defining vector $L=(l_1,\cdots,l_N)$. Let $W^T=P_kL^T$, then $W=$

$(w_1, w_2, \cdots, w_N)$ is a vector formed by weights of 2^k-1 nonzero codewords of C, and $d = min_{1\le i\le 2^k-1}\{w_i\}$ is the distance of C. W is called the *weight vector* of C [19,20].

Suppose $W = d\mathbf{1}_{2^k-1} + \Lambda$, where $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ with $\lambda_i = w_i - d \ge 0$ and at least one $\lambda_i = 0$. Denote $\sigma = \lambda_1 + \lambda_2 + \cdots + \lambda_N$, then $\sigma = 2^{k-1}n - d(2^k - 1)$ from $W^T = P_k L^T$.

For an $[n, k, d]$ code, to determine all defining vectors, one can solve the system of linear equations

$$\begin{cases} P_k^{-1} = \frac{1}{2^{k-1}}[J_k - 2Q_k] \\ W^T = P_k L^T \\ W = d\mathbf{1}_{2^k-1} + \Lambda \\ \Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) \\ \sigma = \lambda_1 + \lambda_2 + \cdots + \lambda_N \end{cases}$$

and then obtain this equation

$$L^T = P_k^{-1} W^T = \frac{1}{2^{k-1}}[(d + \sigma)\mathbf{1}^T_{2^k-1} - 2Q_k\Lambda^T]. \qquad (\star)$$

By determining all nonnegative integer solutions L of Eq $(\star)$ for given σ, one can obtain all $[n, k, d]$ codes and their weight distributions using MATLAB [22]. The process of solving the linear equations was simplified in [19,20], and the uniqueness of some optimal codes was derived from the following known conclusions:

Proposition 3. *([19] Theorem 1.1) Suppose $k \ge 3$, $s \ge 1$, $1 \le t \le 2^k - 2$ and $n = (2^k - 1)s + t$. Then every binary $[n, k, d]$ code with $d \ge (2^{k-1})s$ and without zero coordinates is equivalent to a code with generator matrix $G = ((s - c(k, s, t))S_k | B)$, where $c(k, s, t) \le min\{s, t\}$ is a function of k, s, and t, and B has $(2^k - 1)c(k, s, t) + t$ columns.*

Notation 1. *For $s \ge 0$, $n = 31s + t \ge 14$ with $t \in \{2, 8, 10, 12, 14, 16, 18\}$, one can check that an optimal $[n, 5, d_a(n, 5)]$ linear code without zero coordinates is equivalent to a code with generator matrix $G = ((s - c(k, s, t))S_k | B)$, where $c(k, s, t) \le 2$ and B has $(2^k - 1)c(k, s, t) + t$ columns. To determine all nonnegative integer solutions L of the system of linear equations for given $\sigma = 2^{k-1}n - d(2^k - 1)$, one only needs to determine all nonnegative integer solutions for fixed lengths $n' = (2^k - 1)c(k, s, t) + t$ (see Section 3 for details).*

Lemma 1. *Let $s \ge 1$, $k \ge 4$, $1 \le m \le k - 1$, $N = 2^k - 1$. Then the following holds:*
1) ([19] Corollary 2.2) Every $[sN, k, s2^{k-1}]$ code is equivalent to the SO code with generator matrix sS_k.
2) ([20] Theorem 2) Each $[n, k, d_a] = [sN + 2^k - 2^m, k, s2^{k-1} + 2^{k-1} - 2^{m-1}]$ code is equivalent to the code $\mathcal{MD}_s(k, m)$, the juxtaposition of $s\mathcal{S}_k$ and a $\mathcal{MD}(k, m)$ code.

Hence, if $m = 1, 2$, and ≥ 3, then $h([n, k, d_a]) = k - 1, k - 2, k$, respectively.

Notice that $h([n, k, d_a])$ can be estimated from extended codes or codes of low dimensions in [15]. We will introduce some results referring to $h([n, k, d_a])$. Since Ref. [15] is a conference report that has not been published, the detailed proofs of two lemmas will be given for readability.

Definition 1. *Let C be an $[n, k, d]$ code with generator matrix G and C_1 be an $[n - m, k - 1, \ge d]$ code with generator matrix G_1. Suppose $U_{1,n-m}$ is a matrix of 1 row and $n - m$ columns. Define $0_{k-1,m}$ as the* *zero matrix with $k-1$ rows and m columns. If*

$$G = \begin{pmatrix} \mathbf{1}_m & U_{1,n-m} \\ 0_{k-1,m} & G_1 \end{pmatrix},$$

then C_1 is called a reduced code of C.

Lemma 2. *If C_1 is a reduced code of C and $h(C_1) = r \geq 2$, then $h(C) \geq r-1 \geq 1$ and C is not LCD.*

Proof. If C is an $[n, k, d]$ code and $G = \begin{pmatrix} \mathbf{1}_m & U_{1,n-m} \\ 0_{k-1,m} & G_1 \end{pmatrix}$.

Let G_1 and G be the generator matrices of C_1 and C, respectively. One can calculate that

$$GG^T = \begin{pmatrix} \mathbf{1}_m\mathbf{1}_m^T + UU^T & UG_1^T \\ G_1U^T & G_1G_1^T \end{pmatrix}.$$

Since $h(C_1) = r \geq 2$, we have $R(G_1G_1^{\perp}) = k-h(C_1) = k-r \leq k-2$ and then $R(GG^T) \leq R(G_1G_1^T)+1 = k-1$. It naturally follows that $h(C) = k - R(GG^T) \geq 1$ and C is not an LCD code, the lemma holds. □

Lemma 3. *Let C be an $[n, k, d]$ code with d odd. If C^e is an extended code of C and $h(C^e) = r \geq 2$, then C^e is an $[n+1, k, d+1]$ code, $h(C) \geq r-1 \geq 1$, and C is not LCD.*

Proof. If C^e is an extended code of C and $H^e = \begin{pmatrix} \mathbf{1}_n & 1 \\ H_{n-k-1,n} & \mathbf{0}_{n-k-1}^T \end{pmatrix}$.

Let H^e and H be the parity-check matrices of C_e and C, respectively. We have

$$H^e(H^e)^T = \begin{pmatrix} \mathbf{1}_n\mathbf{1}_n^T + 1 & \mathbf{1}_nH^T \\ H\mathbf{1}_n^T & HH^T \end{pmatrix}.$$

When d is odd, one can know C^e is an $[n+1, k, d+1]$ code from Ref. [1]. Since $h(C_e) = r \geq 2$, we have $R(H^e(H^e)^T) = n+1-k-h(C_e) = n+1-k-r \leq n-k-1$, and then $R(HH^T) \leq R(H^e(H^e)^T) = n-k-1$. Then one can infer that $h(C) = n-k-R(HH^T) \geq 1$ and C is not LCD. The lemma holds. □

3. The proof of Theorem 1

In this section, Theorem 1 will be proved by showing $h([31s+t, 5, d_a-1]) \geq 1$ for $t = 16$ and $h([31s+t, 5, d_a]) \geq 1$ for $t \in \{2, 8, 10, 12, 14, 16, 18\}$. In the rest of this section, let C be an $[n, k, d]$ code. Fix $k = 5$ and $N = 31$, and let $L = (l_1, l_2, \cdots, l_N)$ be the defining vector of C with generator matrix G. Set $l_{max} = max_{1\leq i\leq N}\{l_i\}$ and $l_{min} = min_{1\leq i\leq N}\{l_i\}$. We will use some results from Section 2 to calculate $h(C)$. Our discussions are presented in four subsections. The first subsection verifies $h([31s+t, 5, d_a]) \geq 1$ for $t \in \{2, 8, 12, 16\}$, while the other subsections prove $h([31s+t, 5, d_a]) \geq 1$ for $t = 10, 14$, and 18, respectively.

3.1. $h([32s+2, 5, d_a]) \geq 1$ and $h([32s+t, 5, d_a]) \geq 2$ for $t = 8, 12, 16$

Lemma 4. *If $s \geq 1$, a $[31s+2, 5, 16s]$ code has hull dimension $h \geq 1$ and a $[31s+9, 5, 16s+4]$ code has hull dimension $h \geq 3$.*

Proof. A $[31s + 2, 5, 16s]$ code has a reduced code $[30s + 1, 4, 16s]$, which can give a reduced code $[28s, 3, 16s] = [7 \times 4s, 3, 4 \times 4s]$. Notice its self-orthogonality, one can infer $h([31s + 2, 5, 16s]) \geq 1$.

A $[31s + 9, 5, 16s + 4]$ code has a reduced code $[30s + 8, 4, 16s + 4] = [15 \times 2s + 8, 4, 8 \times 2s + 4]$. It follows that a $[30s + 8, 4, 16s + 4]$ code is SO, and then $h([31s + 9, 5, 16s + 4]) \geq 3$. □

For clarity, the following example is given to show the process of finding L and calculating $h([n, 5, d_a])$.

Example 1. *Let $s \geq 1$ and C be an optimal $[31s + 13, 5, 16s + 6]$ code. One can check $\sigma = 2^4 + 6$ and $s - 1 \leq l_i \leq s + 1$ for $L = (l_1, l_2, \cdots, l_N)$. According to Ref. [16], there is no $[13, 5, 6]$ code, thus $l_{max} = s + 1$ and $l_{min} = s - 1$. Hence, $L = (s - 1)\mathbf{1_N} + L'$, where L' is the defining vector of a $[44, 5, 22]$ code with given generator matrix. We can assume the type of L' is $]\!](0)_a \mid (1)_b \mid (2)_c]\!]$, where $a \geq 1$, $a + b + c = 31$, and $b + 2c = 44$. From Eq ($\star$), one can obtain*

$$(L')^T = \frac{1}{16}[12 \cdot \mathbf{1^T_{2^k-1}} - 2Q_k\Lambda^T]. \quad (\star')$$

By solving Eq ($\star'$), we get all possible L' and L. There are a total of 4805 solutions; these (L')'s can be divided into two groups, one group has 3720 solutions, and the other has 1085 solutions. Using Magma [23], one can check that all (L')'s in the same group give equivalent codes. Hence, there are altogether two inequivalent $[31s + 13, 5, 16s + 6]$ codes. More details of $h([31s + 13, 5, 16s + 6])$ and weight enumerators of inequivalent $[31s + 13, 5, 16s + 6]$ codes are given in the following lemma.

Lemma 5. *If $s \geq 1$, then a $[31s + 13, 5, 16s + 6]$ code and a $[31s + 17, 5, 16s + 8]$ code both have hull dimension $h \geq 3$.*

Proof. Case 1. Let $n = 31s + 13$ and $d = 16s + 6$. Then one can check $\sigma = 2^4 + 6$ and $s - 1 \leq l_i \leq s + 1$ for $1 \leq i \leq N$. Since there is no $[13, 5, 6]$ code, L may have $l_{max} = s + 1$ and $l_{min} = s - 1$, which implies $L = (s - 1)\mathbf{1_N} + L'$, where L' is the defining vector of a $[44, 5, 22]$ code with a given generator matrix. In this case, C is the juxtaposition of $(s - 1)\mathcal{S}_5$ and a $[44, 5, 22]$ code. Suppose L is of type $]\!](s - 1)_a \mid (s)_b \mid (s + 1)_c]\!]$ with $a \geq 1$. By solving Eq ($\star$), one can obtain that L' is one of the following two types $]\!](0)_a \mid (1)_b \mid (2)_c]\!]$:

L'_1: $]\!](0)_1 \mid (1)_{16} \mid (2)_{14}]\!]$; L'_2: $]\!](0)_3 \mid (1)_{12} \mid (2)_{16}]\!]$.

There are 3720 solutions (L')'s that are of type L'_1, all these 3720 defining vectors give equivalent $[44, 5, 22]$ codes. They are equivalent to a code with defining vector $L'_{1,1}$, where $L'_{1,1}$ = (1111101111111112222222222212122). One can check that the corresponding code C has $h = h(C) = 3$ and weight enumerator $1 + 23y^{16s+6} + 7y^{16s+8} + y^{16s+14}$.

There are 1085 solutions (L')'s that are of type L'_2, all these 1085 defining vectors give equivalent $[44, 5, 22]$ codes. They are equivalent to a code with defining vector $L'_{2,1}$, where $L'_{2,1}$ = (1111101111010112222222222222222). Similarly, one can further calculate that the corresponding code C has hull dimension $h = 3$ and weight enumerator $1 + 24y^{16s+6} + 6y^{16s+8} + y^{16s+16}$.

Summarizing previous discussions, we have $h([31s + 13, 5, 16s + 6]) = 3$.

Case 2. Let $n = 31s + 17$ and $d = 16s + 8$. It is easy to check $\sigma = 2^4 + 8$ and $s - 1 \leq l_i \leq s + 2$ for $1 \leq i \leq N$. Thus, L may be one of the following types:

(1) $l_{max} = s + 2$; (2) $l_{max} = s + 1$ and $l_{min} = s$; (3) $l_{max} = s + 1$ and $l_{min} = s - 1$.

If $l_{max} = s + 2$, then C has a reduced code $[30s + 15, 4, 16s + 8] = [15m, 4, 8m]$ where $m = 2s + 1$. It is easy to know a $[30s + 15, 4, 16s + 8]$ code is SO, and one can further deduce that $h(C) \geq 3$.

If $l_{max} = s + 1$ and $l_{min} = s$, then $L = s\mathbf{1_N} + L_0$, where L_0 is the defining vector of a projective $[17, 5, 8]$ code with the given generator matrix. In this case, C is the juxtaposition of $s\mathcal{S}_5$ and a projective $[17, 5, 8]$ code. According to Ref. [24], a $[17, 5, 8]$ code is unique, and its hull dimension $h = 4$.

If $l_{max} = s+1$ and $l_{min} = s-1$, then $L = (s-1)\mathbf{1_N} + L'$, where L' is the defining vector of a $[48, 5, 24]$ code with given generator matrix. In this case C is the juxtaposition of $(s-1)\mathcal{S}_5$ and a $[48, 5, 24]$ code. Suppose L is of type $]\!](s-1)_a \mid (s)_b \mid (s+1)_c]\!]$ with $a \geq 1$. By solving Eq $(\star)$, we obtain the following types $]\!](0)_a \mid (1)_b \mid (2)_c]\!]$ of L':

L'_1: $]\!](0)_1 \mid (1)_{12} \mid (2)_{18}]\!]$; L'_2: $]\!](0)_3 \mid (1)_8 \mid (2)_{20}]\!]$; L'_3: $]\!](0)_7 \mid (1)_0 \mid (2)_{24}]\!]$.

There are altogether two classes of inequivalent $[48, 5, 24]$ codes with defining vectors of type $]\!](0)_1 \mid (1)_{12} \mid (2)_{18}]\!]$. Denote their defining vectors as $L'_{1,i}$ $(i = 1, 2)$, respectively. Then the corresponding codes $\mathcal{D}$ have h and weight enumerators as follows:

$L'_{1,1}$= (2201111211212212222222221122112), $h = 5$, $1 + 24y^{16s+8} + 6y^{16s+12}$;

$L'_{1,2}$= (2202112211212212222211221121221), $h = 3$, $1 + 24y^{16s+8} + 8y^{16s+10} + y^{16s+16}$.

There are a class of $[48, 5, 24]$ codes with defining vectors of type $]\!](0)_3 \mid (1)_8 \mid (2)_{20}]\!]$ and a class of $[48, 5, 24]$ codes with defining vectors of type $]\!](0)_7 \mid (1)_0 \mid (2)_{24}]\!]$, respectively. Denote their defining vectors as L'_j $(j = 3, 4)$. Then the corresponding codes $\mathcal{D}$ have hull dimensions h and weight enumerators as follows:

L'_3= (2202002211212212222222221121221), $h = 5$, $1 + 26y^{16s+8} + 4y^{16s+12} + y^{16s+16}$;

L'_4= (2202002200202202222222222222222), $h = 5$, $1 + 28y^{16s+8} + 3y^{16s+16}$.

Summarizing previous discussions, we have $h([31s + 17, 5, 16s + 8]) \geq 3$. □

From the previous lemmas, one can derive the following conclusion:

Lemma 6. *The codes* $[31s + 8, 5, 16s + 3]$, $[31s + 12, 5, 16s + 5]$ *and* $[31s + 16, 5, 16s + 7]$ *all have hull dimension* $h \geq 2$*, hence they are not LCD codes.*

Combining with known results on $[n, 5]$ LCD codes of lengths $n = 8, 9, 12, 13, 16, 33$, we can obtain that $[31s + t, 5, 16s + d_t]$ are optimal LCD codes, where $d_t = -1, 2, 3, 4, 5, 6$ for $t = 2, 8, 9, 12, 13, 16$, respectively.

Thus, Theorem 1 holds for the cases of $t = 2, 8, 12, 16$.

3.2. $h([31s + 10, 5, 16s + 4]) \geq 1$

In this subsection, set $n = 31s + 10$ and $d = 16s + 4$. It is easy to check $\sigma = 2 \times 2^4 + 4$ and $s - 2 \leq l_i \leq s + 2$ for $1 \leq i \leq N$. Thus, L may be one of the following types:

(1) $l_{max} = s + 2$; (2) $l_{max} = s + 1$ and $l_{min} = s$;

(3) $l_{max} = s + 1$ and $l_{min} = s - 1$; (4) $l_{max} = s + 1$ and $l_{min} = s - 2$.

If $l_{max} = s + 2$, then C has a reduced $[30s + 8, 4, 16s + 4]$ SO code. Hence, in this case, one can deduce that $h(C) \geq 3$ and C is not LCD.

If $l_{max} = s+1$ and $l_{min} = s$, then $L = s\mathbf{1_N} + L_0$, where L_0 is the defining vector of a projective $[10, 5, 4]$ code with the given generator matrix. In this case, C is the juxtaposition of $s\mathcal{S}_5$ and a $[10, 5, 4]$ code. According to Ref. [10], a $[10, 5, 4]$ code is not an LCD code. Hence C is also not LCD.

For verifying Cases (3) and (4), two additional lemmas to determine $h(C)$ are provided as follows:

Lemma 7. *If L satisfies $l_{max} = s + 1$ and $l_{min} = s - 1$, then $h(C) \geq 1$ and C is not an LCD code.*

Proof. If $l_{max} = s+1$ and $l_{min} = s-1$, then $s \geq 1$ and $L = (s-1)\mathbf{1_N} + L'$, where L' is the defining vector of a $[41, 5, 20]$ code with the given generator matrix. In this case, C is the juxtaposition of $(s-1)\mathcal{S}_5$ and a $[41, 5, 20]$ code. Suppose L is of type $\llbracket (s-1)_a \mid (s)_b \mid (s+1)_c \rrbracket$ with $a \geq 1$. By solving Eq ($\star$), we obtain the following types of L':

L'_1: $\llbracket (0)_1 \mid (1)_{19} \mid (2)_{11} \rrbracket$; L'_2: $\llbracket (0)_2 \mid (1)_{17} \mid (2)_{12} \rrbracket$; L'_3: $\llbracket (0)_3 \mid (1)_{15} \mid (2)_{13} \rrbracket$;
L'_4: $\llbracket (0)_4 \mid (1)_{13} \mid (2)_{14} \rrbracket$; L'_5: $\llbracket (0)_5 \mid (1)_{11} \mid (2)_{15} \rrbracket$; L'_6: $\llbracket (0)_6 \mid (1)_9 \mid (2)_{16} \rrbracket$;
L'_7: $\llbracket (0)_7 \mid (1)_7 \mid (2)_{17} \rrbracket$.

There are nineteen classes of inequivalent $[41, 5, 20]$ codes with defining vectors of the above seven types; all these codes have hull dimension $h \geq 1$, hence $h([31s+10, 5, 16s+4]) \geq 1$ when L satisfying $l_{max} = s+1$ and $l_{min} = s-1$. For the defining vectors $L'_{i,j}$ of these inequivalent $[41, 5, 20]$ codes, $h(C)$, and weight enumerators of their corresponding $[31s+10, 5, 16s+4]$ codes, one can refer to Table 2. □

Table 2. 19 inequivalent $[31s+10, 5, 16s+4]$ codes.

Type of defining vector of L': $\llbracket (0)_1 \mid (1)_{19} \mid (2)_{11} \rrbracket$		
Defining vector	h	Weight enumerator of C
(2212121201212112211111121111112)	3	$1 + 18y^{16s+4} + 8y^{16s+6} + 5y^{16s+8}$
(2212112201212112211111121111121)	1	$1 + 17y^{16s+4} + 11y^{16s+6} + 2y^{16s+8} + y^{16s+10}$
(2212111201212112211112121111112)	4	$1 + 12y^{16s+4} + 14y^{16s+5} + 3y^{16s+8} + 2y^{16s+9}$
Type of defining vector: $\llbracket (0)_2 \mid (1)_{17} \mid (2)_{12} \rrbracket$		
(0111111222222211122222101111111)	1	$1 + 17y^{16s+4} + 12y^{16s+6} + y^{16s+8} + y^{16s+12}$
(2202112122021121111122111111221)	4	$1 + 11y^{16s+4} + 16y^{16s+5} + 3y^{16s+8} + y^{16s+12}$
(2222111201212112210112121111112)	3	$1 + 19y^{16s+4} + 7y^{16s+6} + 4y^{16s+8} + y^{16s+10}$
(2222111201212112210111221111121)	1	$1 + 18y^{16s+4} + 10y^{16s+6} + y^{16s+8} + 2y^{16s+10}$
Type of defining vector: $\llbracket (0)_3 \mid (1)_{15} \mid (2)_{13} \rrbracket$		
(2222021201212112210121121111112)	5	$1 + 22y^{16s+4} + 9y^{16s+8}$
(2122211202121111021221102122111)	1	$1 + 19y^{16s+4} + 9y^{16s+6} + 3y^{16s+10}$
(2202112200212112221111121121121)	3	$1 + 19y^{16s+4} + 8y^{16s+6} + 3y^{16s+8} + y^{16s+12}$
(0111111212222222112222222001111111)	4	$1 + 12y^{16s+4} + 15y^{16s+5} + 3y^{16s+8} + y^{16s+13}$
(2202112200212212221111121121111)	4	$1+13y^{16s+4}+14y^{16s+5}+y^{16s+8}+2y^{16s+9}+y^{16s+12}$
Type of defining vector: $\llbracket (0)_4 \mid (1)_{13} \mid (2)_{14} \rrbracket$		
(2021212202121211011212102121212)	1	$1 + 18y^{16s+4} + 11y^{16s+6} + y^{16s+8} + y^{16s+14}$
(2202212200212212221101121121111)	3	$1+20y^{16s+4}+7y^{16s+6} + 2y^{16s+8}+y^{16s+10}+y^{16s+12}$
Type of defining vector: $\llbracket (0)_5 \mid (1)_{11} \mid (2)_{15} \rrbracket$		
(2202221201212112221100221021121)	5	$1 + 23y^{16s+4} + 7y^{16s+8} + y^{16s+12}$
Type of defining vector: $\llbracket (0)_6 \mid (1)_9 \mid (2)_{16} \rrbracket$		
(1222201102222110022220101222211)	1	$1 + 18y^{16s+4} + 12y^{16s+8} + y^{16s+16}$
(1102222111022221001222210012222)	4	$1 + 12y^{16s+4} + 16y^{16s+6} + 2y^{16s+8} + y^{16s+16}$
Type of defining vector: $\llbracket (0)_7 \mid (1)_7 \mid (2)_{17} \rrbracket$		
(2202002200202212221211221121220)	3	$1 + 20y^{16s+4} + 8y^{16s+6} + 2y^{16s+8} + y^{16s+16}$
(2202002200202202221211221121221)	4	$1 + 14y^{16s+4} + 14y^{16s+5} + 2y^{16s+9} + y^{16s+16}$

Lemma 8. *If L meets $l_{max} = s + 1$ and $l_{min} = s - 2$, then $h(C) \geq 3$ and C is not an LCD code.*

Proof. If $l_{max} = s + 1$ and $l_{min} = s - 2$, then $s \geq 2$ and $L = (s-2)\mathbf{1_N} + L''$, where L'' is a defining vector of a $[72, 5, 36]$ code with the given generator matrix. In this case, C is the juxtaposition of $(s-2)\mathcal{S}_5$ and a $[72, 5, 36]$ code. Suppose L is of type $\rrbracket (s-2)_a \mid (s-1)_b \mid (s)_c \mid (s+1)_d \rrbracket$ with $a \geq 1$. By solving Eq ($\star$), we obtain the following six types of L'':

$L''_{1,0}$: $\rrbracket (0)_1 \mid (1)_0 \mid (2)_{18} \mid (3)_{12} \rrbracket$; $L''_{1,2}$: $\rrbracket (0)_1 \mid (1)_2 \mid (2)_{14} \mid (3)_{14} \rrbracket$;
$L''_{1,4}$: $\rrbracket (0)_1 \mid (1)_4 \mid (2)_{10} \mid (3)_{16} \rrbracket$; $L''_{1,6}$: $\rrbracket (0)_1 \mid (1)_6 \mid (2)_6 \mid (3)_{18} \rrbracket$;
$L''_{3,4}$: $\rrbracket (0)_3 \mid (1)_4 \mid (2)_4 \mid (3)_{20} \rrbracket$; $L''_{7,0}$: $\rrbracket (0)_7 \mid (1)_0 \mid (2)_0 \mid (3)_{24} \rrbracket$.

There are thirteen classes of inequivalent $[72, 5, 36]$ codes with defining vectors of the above types, seven classes have hull dimension $h = 5$, and six classes have hull dimension $h = 3$, thus all these codes have hull dimensions greater than 3 and $h([31s+10, 5, 16s+4]) \geq 3$ when L satisfies $l_{max} = s+1$ and $l_{min} = s - 2$. For details of the defining vectors $L''_{i,j}$ of these inequivalent $[72, 5, 36]$ codes, $h(C)$, and weight enumerators of their corresponding $[31s + 10, 5, 16s + 4]$ codes, see Table 3.

Table 3. 13 inequivalent $[31s + 10, 5, 16s + 4]$ codes.

Defining vector	h	Weight enumerator of C
Type of defining vector of L'': $\rrbracket (0)_1 \mid (1)_0 \mid (2)_{18} \mid (3)_{12} \rrbracket$		
(3323232332222222033232323233222222)	5	$1 + 22y^{16s+4} + 9y^{16s+8}$
(3323232332222222033232323233222222)	3	$1 + 20y^{16s+4} + 6y^{16s+6} + 3y^{16s+8} + 2y^{16s+10}$
(3323232332222222033232323233222222)	3	$1 + 19y^{16s+4} + 8y^{16s+6} + 3y^{16s+8} + y^{16s+12}$
Type of defining vector: $\rrbracket (0)_1 \mid (1)_2 \mid (2)_{14} \mid (3)_{14} \rrbracket$		
(3222203333232332122222133323233)	5	$1 + 23y^{16s+4} + 7y^{16s+8} + y^{16s+12}$
(3323213233031232332322223322223)	3	$1 + 20y^{16s+4} + 7y^{16s+6} + 3y^{16s+8} + y^{16s+14}$
(3333222333022232331222313323222)	3	$1{+}21y^{16s+4}{+}6y^{16s+6}{+}y^{16s+8} + 2y^{16s+10}{+}y^{16s+12}$
Type of defining vector: $\rrbracket (0)_1 \mid (1)_4 \mid (2)_{10} \mid (3)_{16} \rrbracket$		
(333303332121332331312322323222)	5	$1 + 24y^{16s+4} + 5y^{16s+8} + 2y^{16s+12}$
(3323203332131232332322323313123)	3	$1 + 20y^{16s+4} + 8y^{16s+6} + 2y^{16s+8} + y^{16s+16}$
Type of defining vector: $\rrbracket (0)_1 \mid (1)_6 \mid (2)_6 \mid (3)_{18} \rrbracket$		
(333303233131232331312323313123)	5	$1 + 24y^{16s+4} + 6y^{16s+8} + y^{16s+16}$
(3313103333232332113132133323233)	5	$1 + 20y^{16s+4} + 7y^{16s+6} + 3y^{16s+8} + y^{16s+14}$
(3333123333032132331321313323123)	3	$1 + 22y^{16s+4} + 6y^{16s+6} + 2y^{16s+10} + y^{16s+16}$
Type of defining vector: $\rrbracket (0)_3 \mid (1)_4 \mid (2)_4 \mid (3)_{20} \rrbracket$		
(3333033333030332331312313323213)	5	$1 + 26y^{16s+4} + 2y^{16s+8} + 2y^{16s+12} + y^{16s+16}$
Type of defining vector: $\rrbracket (0)_7 \mid (1)_0 \mid (2)_0 \mid (3)_{24} \rrbracket$		
(3333303333030330333330333303033)	5	$1 + 28y^{16s+4} + 3y^{16s+16}$

Summarizing the above, we have shown $h([31s + 10, 5, 16s + 4]) \geq 1$ for all $s \geq 1$. □

3.3. $h([31s + 14, 5, 16s + 6]) \geq 1$

In this subsection, let $n = 31s + 14$ and $d = 16s + 6$. It is easy to check for this code, $\sigma = 2 \times 2^4 + 6$ and $s - 2 \leq l_i \leq s + 2$ for $1 \leq i \leq N$. Thus, L may have the following types:

(1) $l_{max} = s + 2$;
(2) $l_{max} = s + 1$ and $l_{min} = s$;
(3) $l_{max} = s + 1$ and $l_{min} = s - 1$;
(4) $l_{max} = s + 1$ and $l_{min} = s - 2$.

If $l_{max} = s + 2$, then C has a reduced code $[30s + 12, 4, 16s + 6]$ with hull dimension $h = 2$. Thus, in this case, one can deduce that $h(C) \geq 1$, and C is not an LCD code.

If $l_{max} = s+1$ and $l_{min} = s$, then $L = s\mathbf{1_N}+L_0$, where L_0 is the defining vector of a projective $[14, 5, 6]$ code with the given generator matrix. In this case, C is the juxtaposition of $s\mathcal{S}_5$ and a $[14, 5, 6]$ code. According to Refs. [10,11], one can know a $[14, 5, 6]$ code is not LCD, and then neither is C.

Lemma 9. *If L satisfies $l_{max} = s + 1$ and $l_{min} = s - 1$, then $h(C) \geq 1$ and C is not an LCD code.*

Proof. If $l_{max} = s + 1$ and $l_{min} = s - 1$, then $L = (s - 1)\mathbf{1_N} + L'$, where L' is the defining vector of a $[45, 5, 22]$ code with the given generator matrix. In this case, C is the juxtaposition of $(s - 1)\mathcal{S}_5$ and a $[45, 5, 22]$ code. Suppose L is of type $]\!](s - 1)_a \mid (s)_b \mid (s + 1)_c]\!]$ with $a \geq 1$. By solving Eq ($\star$), we obtain the following seven types $]\!](0)_a \mid (1)_b \mid (2)_c]\!]$ of L':

L'_1: $]\!](0)_1 \mid (1)_{15} \mid (2)_{15}]\!]$; L'_2: $]\!](0)_2 \mid (1)_{13} \mid (2)_{16}]\!]$; L'_3: $]\!](0)_3 \mid (1)_{11} \mid (2)_{17}]\!]$; L'_4: $]\!](0)_4 \mid (1)_9 \mid (2)_{18}]\!]$;
L'_5: $]\!](0)_5 \mid (1)_7 \mid (2)_{19}]\!]$; L'_6: $]\!](0)_6 \mid (1)_5 \mid (2)_{20}]\!]$; L'_7: $]\!](0)_7 \mid (1)_3 \mid (2)_{21}]\!]$.

There are twenty-one classes of inequivalent $[45, 5, 22]$ codes with defining vectors of the above seven types. And all these codes have hull dimension $h \geq 1$, hence $h([31s + 14, 5, 16s + 6]) \geq 1$ when L satisfies $l_{max} = s + 1$ and $l_{min} = s - 1$. For details of the defining vectors $L'_{i,j}$ of these inequivalent $[45, 5, 22]$ codes, $h(C)$, and weight enumerators of their corresponding $[31s + 14, 5, 16s + 6]$ codes, one can refer to Table 4.

Table 4. 21 inequivalent $[31s+14, 5, 16s+6]$ codes.

Type of defining vector of L': $\llbracket (0)_1 \mid (1)_{15} \mid (2)_{15} \rrbracket$		
Defining vector	h	Weight enumerator of C
(2212112122212112011212212112121221)	5	$1+15y^{16s+6}+15y^{16s+8}+y^{16s+14}$
(2212112211211111222021122112121221)	3	$1+15y^{16s+6}+15y^{16s+8}+y^{16s+14}$
(2211111211112112221221221122022)	3	$1+18y^{16s+6}+7y^{16s+8}+6y^{16s+10}$
(2111112121222220111111221122222)	1	$1+17y^{16s+6}+10y^{16s+8}+3y^{16s+10}+y^{16s+12}$
(1111111222222221222222201111111)	3	$1+8y^{16s+6}+15y^{16s+7}+7y^{16s+8}+y^{16s+15}$
(2212112211211212220211221121211)	2	$1+11y^{16s+6}+12y^{16s+6}+3y^{16s+8}+4y^{16s+9}+y^{16s+14}$
Type of defining vector: $\llbracket (0)_2 \mid (1)_{13} \mid (2)_{16} \rrbracket$		
(1111112122222220011111221222222)	1	$1+18y^{16s+6}+10y^{16s+8}+2y^{16s+10}+y^{16s+14}$
(1211112122122220012111221122222)	3	$1+19y^{16s+6}+6y^{16s+8}+5y^{16s+10}+y^{16s+12}$
(1112222111122220111222201112222)	3	$1+8y^{16s+6}+16y^{16s+7}+6y^{16s+8}+y^{16s+16}$
(1111122121222220011112221122222)	1	$1+18y^{16s+6}+9y^{16s+8}+2y^{16s+10}+2y^{16s+12}$
Type of defining vector: $\llbracket (0)_3 \mid (1)_{11} \mid (2)_{17} \rrbracket$		
(1112222111122220011222220112222)	3	$1+18y^{16s+6}+10y^{16s+8}+2y^{16s+10}+y^{16s+14}$
(2111120122222220111112021222222)	1	$1+18y^{16s+6}+7y^{16s+8}+4y^{16s+10}+2y^{16s+14}$
(1111122122122220002112221122222)	1	$1+20y^{16s+6}+5y^{16s+8}+4y^{16s+10}+2y^{16s+12}$
(2111222122022210111122221202221)	3	$1+20y^{16s+6}+8y^{16s+8}+y^{16s+10}+2y^{16s+12}$
(2212102211212212220201221121221)	2	$1+14y^{16s+6}+10y^{16s+7}+2y^{16s+8}+4y^{16s+9}+y^{16s+16}$
Type of defining vector: $\llbracket (0)_4 \mid (1)_9 \mid (2)_{18} \rrbracket$		
(1112222121022220011222221102222)	1	$1+18y^{16s+6}+10y^{16s+8}+2y^{16s+10}+y^{16s+16}$
(1211022122122220012102221122222)	3	$1+21y^{16s+6}+4y^{16s+8}+3y^{16s+10}+3y^{16s+12}$
(1101222122122220001122221122222)	3	$1+20y^{16s+6}+6y^{16s+8}+3y^{16s+10}+y^{16s+12}+y^{16s+14}$
Type of defining vector: $\llbracket (0)_5 \mid (1)_7 \mid (2)_{19} \rrbracket$		
(1112222112022220002222220112222)	3	$1+20y^{16s+6}+6y^{16s+8}+4y^{16s+10}+y^{16s+16}$
Type of defining vector: $\llbracket (0)_6 \mid (1)_5 \mid (2)_{20} \rrbracket$		
(1122222120022220012222221002222)	1	$1+20y^{16s+6}+8y^{16s+8}+2y^{16s+12}+y^{16s+16}$
Type of defining vector: $\llbracket (0)_7 \mid (1)_3 \mid (2)_{21} \rrbracket$		
(2222220122002220122222021200222)	3	$1+21y^{16s+6}+7y^{16s+8}+3y^{16s+14}$

□

Lemma 10. *If L has $l_{max} = s+1$ and $l_{min} = s-2$, then $h(C) \geq 3$, and C is not an LCD code.*

Proof. If $l_{max} = s+1$ and $l_{min} = s-2$, then $s \geq 2$ and $L = (s-2)\mathbf{1_N} + L''$, where L'' is the defining vector of a $[76, 5, 38]$ code with given generator matrix. In this case, C is the juxtaposition of $(s-2)\mathcal{S}_5$ and a $[76, 5, 38]$ code. Suppose L is of type $\llbracket (s-2)_a \mid (s-1)_b \mid (s)_c \mid (s+1)_d \rrbracket$ with $a \geq 1$. By solving Eq $(\star)$, we obtain the following types $\llbracket (0)_a \mid (1)_b \mid (2)_c \mid (3)_d \rrbracket$ of L'':

$L''_{1,0}$: $\llbracket (0)_1 \mid (1)_0 \mid (2)_{14} \mid (3)_{16} \rrbracket$; $L''_{1,2}$: $\llbracket (0)_1 \mid (2)_2 \mid (2)_{10} \mid (3)_{18} \rrbracket$;
$L''_{1,4}$: $\llbracket (0)_1 \mid (1)_4 \mid (2)_6 \mid (3)_{20} \rrbracket$; $L''_{1,6}$: $\llbracket (0)_1 \mid (1)_6 \mid (2)_2 \mid (3)_{22} \rrbracket$;
$L''_{3,0}$: $\llbracket (0)_3 \mid (1)_0 \mid (2)_8 \mid (3)_{20} \rrbracket$; $L''_{3,4}$: $\llbracket (0)_3 \mid (1)_4 \mid (2)_0 \mid (3)_{24} \rrbracket$.

There are ten classes of inequivalent $[76, 5, 38]$ codes with the defining vectors of the above six types. And all these codes have hull dimension $h \geq 3$, hence $h([31s + 14, 5, 16s + 6]) \geq 3$ when L satisfies $l_{max} = s + 1$ and $l_{min} = s - 2$. For the defining vectors $L''_{i,j}$ of these inequivalent $[76, 5, 38]$ codes, $h(C)$, and their weight enumerators of $[31s + 14, 5, 16s + 6]$ codes, see Table 5.

Table 5. 10 inequivalent $[31s + 14, 5, 16s + 6]$ codes.

Type of defining vector of L'': $]\!](0)_1 \mid (1)_0 \mid (2)_{14} \mid (3)_{16}]\!]$		
Defining vector	h	Weight enumerator of C
(3323223233232223233232230332322 3)	5	$1 + 16y^{16s+6} + 14y^{16s+8} + y^{16s+16}$
(3323223233322232330322333323222)	3	$1 + 19y^{16s+6} + 7y^{16s+8} + 4y^{16s+10} + y^{16s+14}$
(3323223323332222323303223333233222)	3	$1 + 20y^{16s+6} + 5y^{16s+8} + 4y^{16s+10} + 2y^{16s+12}$
Type of defining vector: $]\!](0)_1 \mid (1)_2 \mid (2)_{10} \mid (3)_{18}]\!]$		
(3323223233332132330322333323123)	3	$1 + 20y^{16s+6} + 6y^{16s+8} + 4y^{16s+10} + y^{16s+16}$
(3323313323333222233031333323222 3)	3	$1 + 22y^{16s+6} + 3y^{16s+8} + 2y^{16s+10} + 4y^{16s+12}$
(3323213233323232330312333323232)	3	$1+21y^{16s+6}+5y^{16s+8}+2y^{16s+10}+2y^{16s+12}+y^{16s+14}$
Type of defining vector: $]\!](0)_1 \mid (1)_4 \mid (2)_2 \mid (3)_{20}]\!]$		
(3323113233332332330311333323323)	3	$1+22y^{16s+6}+4y^{16s+8}+2y^{16s+10}+2y^{16s+12}+y^{16s+16}$
Type of defining vector: $]\!](0)_1 \mid (1)_6 \mid (2)_2 \mid (3)_{22}]\!]$		
(3323203333131333331313313333313)	3	$1 + 22y^{16s+6} + 6y^{16s+8} + 2y^{16s+14} + y^{16s+16}$
Type of defining vector: $]\!](0)_3 \mid (1)_0 \mid (2)_8 \mid (3)_{20}]\!]$		
(3323203233333232330302333323233)	3	$1 + 24y^{16s+6} + 2y^{16s+8} + 4y^{16s+12} + y^{16s+16}$
Type of defining vector: $]\!](0)_3 \mid (1)_4 \mid (2)_0 \mid (3)_{24}]\!]$		
(3333303333131333330303313333313)	3	$1 + 24y^{16s+6} + 4y^{16s+8} + 3y^{16s+16}$

Summarizing the above, we have shown $h([31s + 14, 5, 16s + 6]) \geq 1$ for all $s \geq 1$. □

3.4. $h([31s + 18, 5, 16s + 8]) \geq 1$

In this subsection, fix $n = 31s + 18$ and $d = 16s + 8$. It is easy to check $\sigma = 2 \times 2^4 + 8$ and $s - 2 \leq l_i \leq s + 3$ for $1 \leq i \leq N$. Thus, L may have the following types:

(1) $l_{max} = s + 3$; (2) $l_{max} = s + 2$; (3) $l_{max} = s + 1$ and $l_{min} = s$;

(4) $l_{max} = s + 1$ and $l_{min} = s - 1$; (5) $l_{max} = s + 1$ and $l_{min} = s - 2$.

If $l_{max} = s + 3$, then C has a reduced $[30s + 15, 4, 16s + 8]$ SO code. It naturally follows that $h(C) \geq 3$ and C is not LCD.

If $l_{max} = s + 2$, then C has a reduced code $[30s + 16, 4, 16s + 8]= [15m + 1, 4, 8m]$ for $m = 2s + 1$, which is a code with hull dimension $h \geq 2$, and one can deduce that $h(C) \geq 1$ and C is not LCD.

If $l_{max} = s + 1$ and $l_{min} = s$, then $L = s\mathbf{1_N} + L_0$, where L_0 is the defining vector of a projective $[18, 5, 8]$ code. In this case, C is the juxtaposition of sS_5 and an $[18, 5, 8]$ code. According to [10,11], an $[18, 5, 8]$ code is not LCD and $h([18, 5, 8]) \geq 1$, hence $h(C) \geq 1$ and C is not LCD.

For L satisfying Cases (4) or (5), we use the following two lemmas to verify $h(C) \geq 1$.

Lemma 11. *If L meets $l_{max} = s + 1$ and $l_{min} = s - 2$, then $h(C) \geq 1$ and C is not an LCD code.*

Proof. If $l_{max} = s + 1$ and $l_{min} = s - 2$, then $L = (s-1)\mathbf{1_N} + L'$, where L' is the defining vector of a $[49, 5, 24]$ code with the given generator matrix. In this case, C is the juxtaposition of $(s-1)\mathcal{S}_5$ and a $[49, 5, 24]$ code. By solving Eq ($\star$), we obtain the following types of L':

L'_1: $\llbracket(0)_1 \mid (1)_{11} \mid (2)_{19}\rrbracket$; L'_2: $\llbracket(0)_2 \mid (1)_9 \mid (2)_{20}\rrbracket$; L'_3: $\llbracket(0)_3 \mid (1)_7 \mid (2)_{21}\rrbracket$;

L'_4: $\llbracket(0)_4 \mid (1)_5 \mid (2)_{22}\rrbracket$; L'_5: $\llbracket(0)_6 \mid (1)_1 \mid (2)_{24}\rrbracket$.

There are fifteen classes of inequivalent $[49, 5, 24]$ codes with defining vectors of the above five types, all these codes have $h \geq 1$, hence $h([31s + 18, 5, 16s + 8]) \geq 1$ when L satisfies $l_{max} = s + 1$ and $l_{min} = s - 2$. For details of the defining vectors $L'_{i,j}$ of these inequivalent $[49, 5, 24]$ codes, $h(C)$, and their weight enumerators of $[31s + 18, 5, 16s + 8]$ codes, see Table 6.

Table 6. 15 inequivalent $[31s + 18, 5, 16s + 8]$ codes.

Type of defining vector of L': $\llbracket(0)_1 \mid (1)_{11} \mid (2)_{19}\rrbracket$		
Defining vector	h	Weight enumerator of C
(2212112122121122222121022222121 22)	3	$1 + 15y^{16s+8} + 16y^{16s+10} + y^{16s+16}$
(2212121211122122222121022222211 22)	3	$1 + 17y^{16s+8} + 8y^{16s+10} + 6y^{16s+12}$
(2212121211221122222121022222112 22)	1	$1 + 16y^{16s+8} + 11y^{16s+10} + 3y^{16s+12} + y^{16s+14}$
(2212122211122122222121022222211 21)	1	$1 + 11y^{16s+8} + 14y^{16s+9} + 4y^{16s+12} + 2y^{16s+13}$
(2212112122121222222121122222121 02)	2	$1{+}10y^{16s+8}{+}12y^{16s+9}{+}4y^{16s+10}{+}4y^{16s+11}{+}y^{16s+16}$
Type of defining vector: $\llbracket(0)_2 \mid (1)_9 \mid (2)_{20}\rrbracket$		
(2212112122122222222121022222120 12)	1	$1 + 16y^{16s+8} + 12y^{16s+10} + 2y^{16s+12} + y^{16s+16}$
(2212122122121222222121022222121 02)	4	$1 + 10y^{16s+8} + 16y^{16s+9} + 4y^{16s+12} + y^{16s+16}$
(2212212212122212222120022212112 2)	3	$1 + 18y^{16s+8} + 7y^{16s+10} + 5y^{16s+12} + y^{16s+14}$
(2212212211122222222120022222211 21)	1	$1 + 17y^{16s+8} + 10y^{16s+10} + 2y^{16s+12} + 2y^{16s+14}$
Type of defining vector: $\llbracket(0)_3 \mid (1)_7 \mid (2)_{21}\rrbracket$		
221211221121201222022222222222022)	5	$(1 + 21y^{16s+8} + 10y^{16s+12}$
(2222201212222122220202222111122)	3	$1 + 18y^{16s+8} + 8y^{16s+10} + 4y^{16s+12} + y^{16s+16}$
(2222202212122212222020122212112 2)	1	$1 + 10y^{16s+8} + 9y^{16s+10} + y^{16s+12} + 3y^{16s+14}$
(2222202212121212222020222212121 2)	4	$1{+}12y^{16s+8}{+}14y^{16s+9}{+}2y^{16s+12}{+}2y^{16s+13}{+}y^{16s+16}$
Type of defining vector: $\llbracket(0)_4 \mid (1)_5 \mid (2)_{22}\rrbracket$		
(2212212122222022222120022222220 12)	1	$1 + 18y^{16s+8} + 10y^{16s+10} + 2y^{16s+14} + y^{16s+16}$
Type of defining vector: $\llbracket(0)_6 \mid (1)_1 \mid (2)_{24}\rrbracket$		
(2222202122222022222020222220202 2)	4	$1 + 12y^{16s+8} + 16y^{16s+9} + 3y^{16s+16}$

□

Lemma 12. *If L has $l_{max} = s + 1$ and $l_{min} = s - 2$, then $h(C) \geq 3$, and C is not an LCD code.*

Proof. If $l_{max} = s+1$ and $l_{min} = s-2$, then $s \geq 2$ and $L = (s-2)\mathbf{1_N} + L''$, where L'' is the defining vector of an $[80, 5, 40]$ code with the given generator matrix. In this case, C is the juxtaposition of $(s-2)\mathcal{S}_5$ and an $[80, 5, 40]$ code. Suppose L is of type $\llbracket(s-2)_a \mid (s-1)_b \mid (s)_c \mid (s+1)_d\rrbracket$ with $a \geq 1$. By solving Eq ($\star$), we obtain the following types of L'':

$L''_{1,0}$: $\llbracket(0)_1 \mid (1)_0 \mid (2)_{10} \mid (3)_{20}\rrbracket$; $L''_{1,2}$: $\llbracket(0)_1 \mid (2)_2 \mid (2)_6 \mid (3)_{22}\rrbracket$;

$L''_{1,4}$: $\llbracket(0)_1 \mid (1)_4 \mid (2)_2 \mid (3)_{24}\rrbracket$; $L''_{3,0}$: $\llbracket(0)_3 \mid (1)_0 \mid (2)_4 \mid (3)_{24}\rrbracket$.

There are seven classes of inequivalent $[80, 5, 40]$ codes with defining vectors of the above four types; all these codes have $h \geq 3$, hence $h([31s + 18, 5, 16s + 8]) \geq 3$ when L satisfying $l_{max} = s + 1$ and $l_{min} = s - 2$. For details of the defining vectors $L''_{i,j}$ of these inequivalent $[80, 5, 40]$ codes, and $h(C)$ and weight enumerators of their corresponding $[31s + 18, 5, 16s + 8]$ codes, see Table 7.

Table 7. 7 inequivalent $[31s + 18, 5, 16s + 8]$ codes.

Type of defining vector L'': $⟧(0)_1 \mid (1)_0 \mid (2)_{10} \mid (3)_{20}⟧$		
Defining vector	h	Weight enumerator of C
(3333233323333222333033233323223)	5	$1 + 21y^{16s+8} + 10y^{16s+12}$
(3332332323333223333023323323223)	3	$1 + 18y^{16s+8} + 8y^{16s+10} + 4y^{16s+12} + y^{16s+16}$
(3323233322233333332320333332232)	3	$1 + 19y^{16s+6} + 6y^{16s+10} + 4y^{16s+12} + 2y^{16s+14}$
Type of defining vector: $⟧(0)_1 \mid (1)_2 \mid (2)_6 \mid (3)_{22}⟧$		
(3323203322232332333131333333333)	5	$1 + 22y^{16s+8} + 8y^{16s+12} + y^{16s+16}$
(3333303323232323331313333232333)	3	$1+20y^{16s+8}+6y^{16s+10}+2y^{16s+12}+2y^{16s+14}+y^{16s+16}$
Type of defining vector: $⟧(0)_1 \mid (1)_4 \mid (2)_2 \mid (3)_{24}⟧$		
(3333331323333313333013333233133)	3	$1 + 21y^{16s+8} + 8y^{16s+10} + 3y^{16s+16}$
Type of defining vector: $⟧(0)_3 \mid (1)_0 \mid (2)_4 \mid (3)_{24}⟧$		
(3333303323333323330303333232333)	5	$1 + 24y^{16s+8} + 4y^{16s+12} + 3y^{16s+16}$

Summarizing the above, we have shown $h([31s + 18, 5, 16s + 8]) \geq 1$ for all $s \geq 1$.

□

4. Conclusions

Combining with known results on optimal LCD codes, the minimum distances of all binary optimal LCD codes of dimension 5 have been wiped out in this manuscript. More precisely, we have determined the minimum distances of optimal $[n, 5]$ LCD codes with $n = 31s + t \geq 14$ and $t \in \{2, 8, 10, 12, 14, 16, 18\}$, which have not been systematically investigated in the literature. By the methods of reduced codes, classifying optimal linear codes and calculating the hull dimension of C, one may further study the classification of optimal linear codes, and determine the minimum distances of optimal LCD codes with higher dimensions.

Author contributions

Yang Liu, Ruihu Li, Qiang Fu and Hao Song: Writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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