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Research article

Metallic deformation on para-Sasaki-like para-Norden manifold

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Abstract: The main goal of this paper is to define the concept of metallic deformation through a relation between the metallic structure and paracontact structure on an almost paracontact para-Norden manifold. A Riemannian connection is obtained on a metallically deformed para-Sasaki-like para-Norden manifold. A φ-connection is obtained via the Riemannian connection on a metallically deformed para-Sasaki-like para-Norden manifold. The curvature tensors, Ricci tensors, scalar curvatures, and $*$-scalar curvatures are investigated with respect to the Riemannian connection and the φ-connection. Finally, an example is given of a metallically deformed 3-dimensional para-Sasaki-like para-Norden manifold.

Keywords: para-Sasaki-like para-Norden manifold; paracontact structure; metallic deformation
Mathematics Subject Classification: 53C15, 53C25, 53C50

1. Introduction

The notion of almost paracontact paracomplex Riemannian manifold was introduced in the studies by Manev and collaborators, specifically in [1, 2]. Various topics have been addressed on almost paracontact paracomplex Riemannian manifolds, commonly abbreviated as apcpcR manifolds [3–6].

Spinadel pioneered the notion of the metallic means family [7, 8]. The quadratic equation $x^2 - px - q = 0$ has two eigenvalues $\sigma_{p,q}$ and $\sigma^*_{p,q}$. Within this family, all members are characterized by positive quadratic irrational numbers $\sigma_{p,q}$. Building upon the inspiration drawn from the metallic means family, Crasmareanu and Hretcanu introduced a metallic structure on the manifold M [9]. This structure is defined by a $(1,1)$-tensor field Ψ on manifold M, with the crucial property that $\Psi^2 = p\Psi + qI$, where p and q belong to the set of positive integers. In accordance with certain values taken by p and q, special cases of metallic structures emerge. These special cases of metallic structures are golden structure, silver structure, bronze structure, copper structure, nickel structure. Investigating metallic structures and their special cases on Riemannian manifolds is one of the current topics in differential geometry. Gezer and Karaman [10] studied an integrability condition and curvature properties for metallic Riemannian structures. Turanli et al. [11] constructed metallic Kähler and nearly metallic Kähler structures on Riemannian manifolds. Cayir [12] applied the Tachibana and Vishnevskii operators to vertical and horizontal lifts with respect to the metallic Riemannian structure on $(1,1)$-tensor bundle. Hretcanu and Blaga [13] described the warped product bi-slant, warped product semi-slant and warped product hemi-slant submanifolds in locally metallic Riemannian manifolds. Khan and De [14] showed that the r-lift of the metallic structure in the tangent bundle of order r is a metallic structure. Ahmad et al. [15] investigated some interesting results on bi-slant lightlike submanifolds of golden semi-Riemannian manifolds. Şahin et al. [16] reached some results on Norden golden manifold having a constant sectional curvature. Özkan et al. [17] investigated integrability and parallelism of silver structure in tangent bundle. Akpınar [18] studied an integrability condition and curvature properties for bronze Riemannian structures. For details on finding several classical eigenvalues and remarkable eigenproblems, we refer reader to [19, 20].

Some studies have been conducted on para-Sasakian manifolds and Lorentzian para-Sasakian manifolds based on the relation between metallic structure and paracontact structure [21, 22]. The notion of metallic deformation has been addressed through the relation between the metallic structure and the almost contact structure in [23].

The main goal of this study is to define the notion of metallic deformation through the relation between the metallic structure and paracontact structure on almost paracontact para-Norden manifolds (referred to as almost paracontact paracomplex Riemannian manifolds in [2–6]). The progress of the study has been structured as follows: In Section 2, some fundamental information for almost paracontact para-Norden manifolds is given. In Section 3, the metallic deformation is defined through a relation between the metallic structure and paracontact structure on an almost paracontact para-Norden manifold. In Section 4, the curvature tensors, Ricci tensors, scalar curvatures, and $*$-scalar curvatures are computed with respect to the Riemannian connection and the φ-connection on metallically deformed para-Sasaki-like para-Norden manifolds. In Section 5, an example is given of a metallically deformed 3-dimensional para-Sasaki-like para-Norden manifold.

2. Almost paracontact para-Norden manifolds

A $(2n+1)$-dimensional differentiable manifold (M, φ, ξ, η) is referred to as an almost paracontact paracomplex manifold if it is endowed with an almost paracontact structure (φ, ξ, η), comprising a $(1,1)$-tensor field φ, a Reeb vector field ξ, and its dual 1-form η. The almost paracontact structure (φ, ξ, η) satisfies the following conditions:

$$\begin{gathered}\varphi^2 = I - \eta \otimes \xi, \quad \varphi\xi = 0, \\ \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad tr\, \varphi = 0,\end{gathered} \tag{2.1}$$

where I is the identity transformation on the tangent bundle TM. The manifold $(M, \varphi, \xi, \eta, g)$ is called an almost paracontact para-Norden manifold equipped with a para-Norden metric g relative to (φ, ξ, η) determined by

$$g(\varphi x, w) = g(x, \varphi w), \tag{2.2}$$

or equivalently

$$g(\varphi x, \varphi w) = g(x, w) - \eta(x)\eta(w), \tag{2.3}$$

for any smooth vector fields x, w on M, i.e. $x, w \in \chi(M)$ [2,24]. The almost paracontact para-Norden manifold is briefly called the apcpN manifold. As a result, the following equations are obtained:

$$g(x,\xi) = \eta(x), \qquad g(\xi,\xi) = 1, \qquad \eta(\nabla_x \xi) = 0, \tag{2.4}$$

where ∇ denotes the Riemannian connection of g. From here onwards, x, w, z are arbitrary vector fields from $\chi(M)$ or vectors in TM at a fixed point of M. The basis $\{e_0 = \xi,\ e_1, ..., e_n,\ e_{n+1} = \varphi e_1,\ e_{2n} = \varphi e_n\}$ is an orthonormal basis on the structure (φ, ξ, η, g) with

$$g\left(e_i, e_j\right) = \delta_{ij}, \qquad i, j = 0, 1, ..., 2n. \tag{2.5}$$

The metric $\widehat{g}$ is an associated metric of g and defined on $(M, \varphi, \xi, \eta, g)$ by

$$\widehat{g}(x, w) = g(x, \varphi w) + \eta(x)\eta(w). \tag{2.6}$$

The associated metric $\widehat{g}$ is an indefinite metric with signature $(n+1, n)$ and compatible with $(M, \varphi, \xi, \eta, g)$ in a manner analogous to that of g [2]. The apcpN manifolds are classified in [1]. This classification comprises eleven fundamental classes denoted as $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{11}$. The eleven fundamental classes are based on the $(0,3)$-tensor field F determined by

$$F(x, w, z) = g((\nabla_x \varphi) w, z), \tag{2.7}$$

and has the properties

$$F(x,w,z) = F(x,z,w) = -F(x,\varphi w,\varphi z) + \eta(w) F(x,\xi,z) + \eta(z) F(x,w,\xi), \tag{2.8}$$
$$(\nabla_x \eta)(w) = g(\nabla_x \xi, w) = -F(x, \varphi w, \xi).$$

3. Metallic deformation

Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact para-Norden manifold (apcpN manifold). We construct a metallic structure on an apcpN manifold M.

Proposition 3.1. *The* $(1,1)$*-tensor field* Ψ *defined by*

$$\Psi = \frac{p}{2} I - \left(\frac{p - 2\sigma}{2}\right)(\varphi + \eta \otimes \xi), \tag{3.1}$$

is a metallic structure on an apcpN manifold, where p and q are positive integers.

Proof. (3.1) is written for $x \in \chi(M)$ as

$$\Psi x = \frac{p}{2} x - \left(\frac{p - 2\sigma}{2}\right)(\varphi x + \eta(x)\xi).$$

In order for Ψ to be a metallic structure on apcpN manifold, it must satisfy the equation $\Psi^2 x = p\Psi x + qx$ [9]. By utilizing (2.1) and (3.1), the equation

$$\Psi^2 x = \frac{p}{2}\Psi x - \left(\frac{p - 2\sigma}{2}\right)(\Psi(\varphi x) + \eta(x)\Psi\xi)$$

is written. Considering the equation $\sigma^2 = \sigma p + q$ for the eigenvalue σ, $\Psi^2 x = p\Psi x + qx$ is obtained for every $x \in \chi(M)$. Hence, the proof is concluded. □

Proposition 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be an apcpN manifold, and Ψ is given as* (3.1). *In this way, the following equality is satisfied:*

$$g(\Psi x, \Psi w) = \frac{\sigma_*^2 + \sigma^2}{2} g(x, w) - \frac{\sigma_*^2 - \sigma^2}{2} \widehat{g}(x, w). \tag{3.2}$$

Proof. By utilizing (2.1), (2.2), and (3.1), the equality

$$g(\Psi x, \Psi w) = \frac{\left((p-\sigma)^2 + \sigma^2\right)}{2} g(x, w) - \frac{p(p-2\sigma)}{2} g(\varphi x, w) - \frac{p(p-2\sigma)}{2} \eta(x)\eta(w),$$

is written. Using (2.6) and (3.1), the above equality is rearranged as

$$g(\Psi x, \Psi w) = \frac{\left((p-\sigma)^2 + \sigma^2\right)}{2} g(x, w) - \frac{p(p-2\sigma)}{2} \widehat{g}(x, w).$$

Considering $\sigma + \sigma_* = p$, (3.2) is obtained. □

Considering Propositions 3.1 and 3.2, a change in the structure tensors can be generated in

$$\widetilde{\varphi} = \varphi, \qquad \widetilde{\xi} = \frac{1}{\sigma}\xi, \qquad \widetilde{\eta} = \sigma\eta, \tag{3.3}$$

$$\begin{aligned} \widetilde{g}(x, w) &= g(\Psi x, \Psi w) = \frac{\sigma_*^2 + \sigma^2}{2} g(x, w) - \frac{\sigma_*^2 - \sigma^2}{2} \widehat{g}(x, w) \\ &= \frac{\sigma_*^2 + \sigma^2}{2} g(x, w) - \frac{\sigma_*^2 - \sigma^2}{2} g(\varphi x, w) - \frac{\sigma_*^2 - \sigma^2}{2} \eta(x)\eta(w). \end{aligned}$$

This is called a metallic deformation on the apcpN manifold. Thus, $\left(M, \widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ is also an apcpN manifold.

4. Metallic deformation on para-Sasaki-like para-Norden manifolds

Definition 4.1. *An apcpN manifold $(M, \varphi, \xi, \eta, g)$ is called a para-Sasaki-like para-Norden manifold if the structure tensors (φ, ξ, η, g) satisfy the following conditions for $x, w, z \in H = \ker(\eta)$:*

$$\begin{aligned} F(x, w, z) &= F(\xi, w, z) = F(\xi, \xi, z) = 0, \\ F(x, w, \xi) &= -g(x, w). \end{aligned} \tag{4.1}$$

The class of para-Sasaki-like para-Norden manifolds is defined and examined in [3]. This particular subclass of the examined manifolds is determined by:

$$\begin{aligned} (\nabla_x \varphi) w &= -g(x, w)\xi - \eta(w)x + 2\eta(x)\eta(w)\xi \\ &= -g(\varphi x, \varphi w)\xi - \eta(w)\varphi^2 x. \end{aligned} \tag{4.2}$$

In this section, we focus on the para-Sasaki-like para-Norden manifolds. These manifolds have also been called para-Sasaki-like Riemannian manifolds and para-Sasaki-like Riemannian Π-manifolds in [3–5]. In [3], the following identities are proved:

$$\begin{array}{ll} \nabla_x \xi = \varphi x, & (\nabla_x \eta) w = g(x, \varphi w), \\ R(x, w)\xi = -\eta(w)x + \eta(x)w, & R(\xi, w)\xi = \varphi^2 w, \\ Ric(x, \xi) = -2n\eta(x), & Ric(\xi, \xi) = -2n, \end{array} \tag{4.3}$$

where R and Ric denote the curvature tensor and the Ricci tensor, respectively.

The distribution $H = \ker(\eta)$ is a $2n$-dimensional paracontact distribution of a para-Sasaki-like para-Norden manifold equipped with an almost paracomplex structure $P = \varphi_{|H}$ and a metric $h = g_{|H}$, are the restrictions of φ and g on paracontact distribution H, respectively [3]. The metric h is pure according to P, as follow:

$$h(Px, w) = h(x, Pw), \tag{4.4}$$

or equivalently

$$h(Px, Pw) = h(x, w). \tag{4.5}$$

Such a metric is known as the almost product Riemannian metric [25, 26], the real part of the paracomplex Riemannian metric [27], the para-B-metric, or the para-Norden metric [28, 29]. Remember that an almost paracomplex manifold of dimension $2n$, equipped with a para-Norden metric h satisfying (4.4), is called an almost para-Norden manifold. On a para-Sasaki-like para-Norden manifold, (4.2) is written in form $F(x, y, z)$ as

$$\begin{aligned} F(x, w, z) &= g((\nabla_x\varphi) w, z) \\ &= -g(x, w)\eta(z) - \eta(w) g(x, z) + 2\eta(x)\eta(w)\eta(z). \end{aligned}$$

The $(0, 3)$-tensor field F becomes zero on the paracontact distribution $H = \ker(\eta)$ of a para-Sasaki-like para-Norden manifold. So, ${}^h\nabla_x\varphi_{|H} = 0$ for every $x, w, z \in H$. When the almost paracomplex structure is parallel with respect to the Riemannian connection of the metric h, an almost para-Norden manifold is known as a para-Kahler-Norden manifold [27, 29].

Theorem 4.1. *If $\left(M, \widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ is a metallically deformed para-Sasaki-like para-Norden manifold, then the relation between the Riemannian connections $\widetilde{\nabla}$ of the metrics $\widetilde{g}$ and ∇ of the metric g is given by*

$$\widetilde{\nabla}_x w = \nabla_x w + \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} g(\varphi x, \varphi w)\xi - \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} g(\varphi x, w)\xi. \tag{4.6}$$

Proof. Utilizing the general Kozsul formula, the equation

$$\begin{aligned} 2\widetilde{g}\left(\widetilde{\nabla}_x w, z\right) = {} & x\widetilde{g}(w, z) + w\widetilde{g}(z, x) - z\widetilde{g}(x, w) \\ & + \widetilde{g}([x, w], z) + \widetilde{g}([z, x], w) + \widetilde{g}([z, w], x) \end{aligned}$$

is written on a metallically deformed para-Sasaki-like para-Norden manifold. Considering (2.2), (2.4), (3.3) and (4.3), the relation between $\widetilde{\nabla}$ and ∇ is reached. □

Proposition 4.1. *Let $\left(M, \widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ be a metallically deformed para-Sasaki-like para-Norden manifold. The following equalities hold:*

$$\begin{aligned} &\widetilde{\nabla}_x\widetilde{\xi} = \tfrac{1}{\sigma}\varphi x, \\ &\left(\widetilde{\nabla}_x\widetilde{\eta}\right) w = \tfrac{\sigma_*^2+\sigma^2}{2\sigma} g(\varphi x, w) - \tfrac{\sigma_*^2-\sigma^2}{2\sigma} g(\varphi x, \varphi w), \\ &\left(\widetilde{\nabla}_x\widetilde{\varphi}\right) w = (\nabla_x\varphi) w + \tfrac{\sigma_*^2-\sigma^2}{2\sigma^2} g(\varphi x, w)\xi - \tfrac{\sigma_*^2-\sigma^2}{2\sigma^2} g(\varphi x, \varphi w)\xi. \end{aligned} \tag{4.7}$$

Proof. Using (3.3) and (4.3), the assertions in (4.7) is directly obtain. □

Taking into account (4.6), the $(0,3)$-tensor field $\widetilde{F}$ has the following form:

$$\widetilde{F}(x,w,z) = \frac{\sigma_*^2+\sigma^2}{2}F(x,w,z)+\frac{\sigma_*^2-\sigma^2}{2}g(\varphi x,z)\eta(w)+\frac{\sigma_*^2-\sigma^2}{2}g(\varphi x,w)\eta(z). \tag{4.8}$$

It is seen that the Riemannian connections $\widetilde{\nabla}$ and ∇ coincide for $x,w\in H$. Moreover, $\widetilde{F}(x,w,z) = \frac{\sigma_*^2+\sigma^2}{2}F(x,w,z)$ holds for $x,w\in H$. Considering Definition 4.1 and Eq (4.8), a metallically deformed para-Sasaki-like para-Norden manifold is a para-Sasaki-like para-Norden manifold.

The curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ is defined as follows:

$$\widetilde{R}(x,w)z=\widetilde{\nabla}_x\widetilde{\nabla}_w z-\widetilde{\nabla}_w\widetilde{\nabla}_x z-\widetilde{\nabla}_{[x,w]}z.$$

Taking into account (4.3) and (4.6), the following relation between the corresponding curvature tensors $\widetilde{R}$ and R of the Riemannian connections $\widetilde{\nabla}$ and ∇, respectively, is obtained:

$$\begin{aligned}\widetilde{R}(x,w)z = {} & R(x,w)z+\frac{\sigma_*^2-\sigma^2}{2\sigma^2}\{g(\varphi x,\varphi z)\eta(w)\xi+g(w,\varphi z)\eta(x)\xi+g(\varphi w,\varphi z)\varphi x+g(\varphi x,z)\varphi w\}\\ & -\frac{\sigma_*^2-\sigma^2}{2\sigma^2}\{g(\varphi w,\varphi z)\eta(x)\xi+g(x,\varphi z)\eta(w)\xi+g(\varphi x,\varphi z)\varphi w+g(\varphi w,z)\varphi x\}.\end{aligned} \tag{4.9}$$

On the apcpN manifold, the Ricci tensor Ric, the scalar curvature $scal$, and the $*$-scalar curvature $scal^*$ are defined as usual traces of the $(0,4)$-type curvature tensor $R(x,w,z,y)=g(R(x,w)z,y)$,

$$\begin{aligned}Ric(x,w) &= \sum_{i=0}^{2n}R(e_i,x,w,e_i),\\ scal &= \sum_{i=0}^{2n}Ric(e_i,e_i),\\ scal^* &= \sum_{i=0}^{2n}Ric(e_i,\varphi e_i),\end{aligned}$$

with respect to an arbitrary orthonormal basis $\{e_0,...,\ e_{2n}\}$ of its tangent space [3]. On account of (4.9), the Ricci tensor $\widetilde{Ric}$, the scalar curvature tensor $\widetilde{scal}$, and $*$-scalar curvature tensor $\widetilde{scal}^*$ are obtained on a metallically deformed para-Sasaki-like para-Norden manifold as

$$\widetilde{Ric}(x,w)=Ric(x,w), \tag{4.10}$$

$$\widetilde{scal}=\frac{\sigma^2+\sigma_*^2}{2\sigma^2\sigma_*^2}scal-\frac{\sigma^2-\sigma_*^2}{2\sigma^2\sigma_*^2}(scal^*-2n), \tag{4.11}$$

$$\widetilde{scal}^*=\frac{\sigma^2+\sigma_*^2}{2\sigma^2\sigma_*^2}scal^*-\frac{\sigma^2-\sigma_*^2}{2\sigma^2\sigma_*^2}(scal+2n). \tag{4.12}$$

Let us consider the almost paracontact structure $\left(\widetilde{\varphi},\widetilde{\xi},\widetilde{\eta}\right)$ defined by (3.3) and the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{g}$ given in Theorem 4.1. On a metallically deformed para-Sasaki-like para-Norden manifold $\left(M,\widetilde{\varphi},\widetilde{\xi},\widetilde{\eta},\widetilde{g}\right)$, a $(1,2)$-tensor field can be defined as

$$\widetilde{S}(x,w)=\frac{1}{2}\left\{\left(\widetilde{\nabla}_{\varphi w}\varphi\right)x+\varphi\left(\left(\widetilde{\nabla}_w\varphi\right)x\right)-\varphi\left(\left(\widetilde{\nabla}_x\varphi\right)w\right)\right\}, \tag{4.13}$$

for all $x, w \in \chi(M)$. Then, the linear connection

$$\overline{\nabla}_x w = \widetilde{\nabla}_x w - \widetilde{S}(x, w) \tag{4.14}$$

is an almost paracontact connection on $\left(M, \widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$. We will refer to the connection $\overline{\nabla}$ as the φ-connection on a metallically deformed para-Sasaki-like para-Norden manifold. An almost product connection has been obtained through a similar method on almost product manifolds [30].

Theorem 4.2. *Let* $\left(M, \widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ *be a metallically deformed para-Sasaki-like para-Norden manifold. Then the* φ*-connection* $\overline{\nabla}$ *constructed by the Riemannian connection* $\widetilde{\nabla}$ *and the almost paracontact structure* $\left(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}\right)$ *is as follows:*

$$\overline{\nabla}_x w = \nabla_x w + \frac{\sigma_*^2 - \sigma^2}{4\sigma^2} g(\varphi x, \varphi w)\xi - \frac{\sigma_*^2 - 3\sigma^2}{4\sigma^2} g(\varphi x, w)\xi + \eta(x)\varphi w - \frac{1}{2}\eta(w)\varphi x. \tag{4.15}$$

Proof. Considering (4.7)

$$(\widetilde{\nabla}_{\varphi w}\varphi)x = (\nabla_{\varphi w}\varphi)x + \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} g(\varphi x, \varphi w)\xi - \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} g(\varphi x, w)\xi \tag{4.16}$$

is written. Using (2.2), (3.3), (4.7), and (4.16),

$$\widetilde{S}(x, w) = \frac{\sigma_*^2 - \sigma^2}{4\sigma^2} g(\varphi x, \varphi w)\xi - \frac{\sigma_*^2 + \sigma^2}{4\sigma^2} g(\varphi x, w)\xi - \eta(x)\varphi w + \frac{1}{2}\eta(w)\varphi w \tag{4.17}$$

is obtained. Considering Theorem 4.1 and Proposition 4.1, we directly obtain (4.15) from (4.14). □

The torsion tensor $\overline{T}$ of $\overline{\nabla}$ is defined as follows:

$$\begin{aligned} \overline{T}(x, w) &= \overline{\nabla}_x w - \overline{\nabla}_w x - [x, w] \\ &= -\frac{3}{2}\{\eta(w)\varphi x - \eta(x)\varphi w\}. \end{aligned}$$

Golab [31] defined and studied quarter-symmetric linear connections in a differentiable manifold. A linear connection ∇ is called a quarter-symmetric connection if its torsion tensor T satisfies

$$T(x, w) = \eta(w)\varphi x - \eta(x)\varphi w. \tag{4.18}$$

When we obtain the torsion tensor $\overline{T}$ of $\overline{\nabla}$, we observe that the connection $\overline{\nabla}$ is a quarter-symmetric connection on a metallically deformed para-Sasaki-like para-Norden manifold. Taking into account (4.3) and (4.15), the following relation between the corresponding curvature tensors $\overline{R}$ and R of the Riemannian connections $\overline{\nabla}$ and ∇, respectively, is obtained:

$$\begin{aligned} \overline{R}(x, w)z &= R(x, w)z + \frac{\sigma_*^2 - \sigma^2}{8\sigma^2}\{g(\varphi w, \varphi z)\varphi x - g(\varphi x, \varphi z)\varphi w\} \\ &\quad + \frac{\sigma_*^2 - 7\sigma^2}{8\sigma^2}\{g(\varphi x, z)\varphi w - g(\varphi w, z)\varphi x\} + \eta(x)g(w, z)\xi - \eta(w)g(x, z)\xi. \end{aligned} \tag{4.19}$$

The Ricci tensor $\overline{Ric}$, the scalar curvature $\overline{scal}$, and the $*$-scalar curvature $\overline{scal}^*$ are obtained on a metallically deformed para-Sasaki-like para-Norden manifold as

$$\overline{Ric}\,(x,w) = Ric\,(x,w) + \frac{\sigma_*^2+\sigma^2}{8\sigma^2} g\,(\varphi x,\varphi w) - \frac{\sigma_*^2-\sigma^2}{8\sigma^2} g\,(x,\varphi w), \tag{4.20}$$

$$\overline{scal} = \frac{\sigma_*^2+\sigma^2}{2\sigma_*^2\sigma^2} scal - \frac{\sigma^2-\sigma_*^2}{2\sigma_*^2\sigma^2} scal^* + \frac{2\sigma^2-\sigma_*^2}{4\sigma_*^2\sigma^2} 2n, \tag{4.21}$$

$$\overline{scal}^* = \frac{\sigma_*^2+\sigma^2}{2\sigma_*^2\sigma^2} scal^* - \frac{\sigma^2-\sigma_*^2}{2\sigma_*^2\sigma^2} \{scal + 2n\}. \tag{4.22}$$

5. An example

Consider a 3-dimensional real connected Lie group denoted by L. Then, the Lie group L has a basis of left-invariant vector fields $\{e_0,\ e_1,\ e_2\}$ with associated Lie algebra determined as follows:

$$[e_0,\ e_1] = -e_2, \quad [e_0,\ e_2] = -e_1, \quad [e_1,\ e_2] = 0. \tag{5.1}$$

The Lie group L is endowed with an almost paracontact para-Norden structure (φ,ξ,η,g) as follows:

$$\begin{aligned} \xi &= e_0, \quad \varphi e_0 = 0, \quad \varphi e_1 = e_2, \quad \varphi e_2 = e_1, \\ g\,(e_0,e_0) &= g\,(e_1,e_1) = g\,(e_2,e_2) = 1, \quad g\left(e_i,e_j\right) = 0,\ i,j \in \{0,1,2\},\ i \neq j. \end{aligned} \tag{5.2}$$

In [6], it is proved that the solvable Lie group corresponding to the Lie algebra defined by (5.1) and equipped with the almost paracontact para-Norden structure (φ,ξ,η,g) from (5.2) is a para-Sasaki-like para-Norden manifold. Moreover, the basic components of ∇ and R are obtained. The components of the Riemannian connection ∇ are determined via the Kozsul formula as follows:

$$\begin{aligned} &\nabla_{e_0} e_1 = \nabla_{e_0} e_2 = 0, \\ &\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -e_0, \\ &\nabla_{e_1} e_0 = e_2, \\ &\nabla_{e_2} e_0 = e_1, \\ &\nabla_{e_0} e_0 = \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0. \end{aligned}$$

On a para-Sasaki-like para-Norden manifold, the non-zero components of the fundamental tensor F are obtained in the following way:

$$F_{101} = F_{110} = F_{202} = F_{220} = -1.$$

On a para-Sasaki-like para-Norden manifold, the components of the curvature tensor R corresponding to the Riemannian connection ∇ are expressed by:

$$\begin{aligned} &R\,(e_0,e_0)\,e_0 = 0, && R\,(e_0,e_2)\,e_1 = 0, \\ &R\,(e_0,e_1)\,e_0 = e_1, && R\,(e_0,e_2)\,e_2 = -e_0, \\ &R\,(e_0,e_1)\,e_1 = -e_0, && R\,(e_1,e_2)\,e_0 = 0, \\ &R\,(e_0,e_1)\,e_2 = 0, && R\,(e_1,e_2)\,e_1 = -e_2, \\ &R\,(e_0,e_2)\,e_0 = e_2, && R\,(e_1,e_2)\,e_2 = e_1. \end{aligned}$$

The non-zero component of Ricci tensor Ric, the scalar curvature $scal$, and the $*$-scalar tensor $scal^*$ are expressed by:

$$Ric(e_0, e_0) = -2,$$
$$scal = -2,$$
$$scal^* = 0.$$

On a metallically deformed para-Sasaki-like para-Norden manifold, the components of the Riemannian connection $\widetilde{\nabla}$ are obtained in the following way:

$$\begin{array}{lll} \widetilde{\nabla}_{e_0} e_0 = 0, & \widetilde{\nabla}_{e_0} e_1 = 0, & \widetilde{\nabla}_{e_0} e_2 = 0, \\ \widetilde{\nabla}_{e_1} e_0 = e_2, & \widetilde{\nabla}_{e_1} e_1 = \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} e_0, & \widetilde{\nabla}_{e_1} e_2 = -\frac{\sigma_*^2 + \sigma^2}{2\sigma^2} e_0, \\ \widetilde{\nabla}_{e_2} e_0 = e_1, & \widetilde{\nabla}_{e_2} e_1 = -\frac{\sigma_*^2 + \sigma^2}{2\sigma^2} e_0, & \widetilde{\nabla}_{e_2} e_2 = \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} e_0. \end{array}$$

On a metallically deformed para-Sasaki-like para-Norden manifold, the non-zero components of the fundamental tensor $\widetilde{F}$ are obtained in the following way:

$$\widetilde{F}_{101} = \widetilde{F}_{110} = \widetilde{F}_{202} = \widetilde{F}_{220} = -\frac{\sigma_*^2 + \sigma^2}{2},$$
$$\widetilde{F}_{102} = \widetilde{F}_{201} = \widetilde{F}_{210} = \widetilde{F}_{120} = \frac{\sigma_*^2 - \sigma^2}{2}.$$

The manifold $(L, \varphi, \xi, \eta, g)$ is the para-Sasaki-like para-Norden manifold since it satisfies (4.1). Hence, the manifold $\left(L, \widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g}\right)$ is also a para-Sasaki-like para-Norden manifold. If $x = a_0 e_0 + a_1 e_1 + a_2 e_2$, $y = b_0 e_0 + b_1 e_1 + b_2 e_2$, then the metric $\widetilde{g}$ is given by

$$\widetilde{g}(x, y) = \sigma^2 a_0 b_0 + \frac{\sigma_*^2 + \sigma^2}{2}(a_1 b_1 + a_2 b_2) - \frac{\sigma_*^2 - \sigma^2}{2}(a_1 b_2 + a_2 b_1).$$

On a metallically deformed para-Sasaki-like para-Norden manifold, the components of the curvature tensor $\widetilde{R}$ corresponding to the Riemannian connection $\widetilde{\nabla}$ are obtained in the following way:

$$\begin{array}{ll} \widetilde{R}(e_0, e_0) e_0 = 0, & \widetilde{R}(e_0, e_2) e_1 = \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} e_0, \\ \widetilde{R}(e_0, e_1) e_0 = e_1, & \widetilde{R}(e_0, e_2) e_2 = -\frac{\sigma_*^2 + \sigma^2}{2\sigma^2} e_0, \\ \widetilde{R}(e_0, e_1) e_1 = -\frac{\sigma_*^2 + \sigma^2}{2\sigma^2} e_0, & \widetilde{R}(e_1, e_2) e_0 = 0, \\ \widetilde{R}(e_0, e_1) e_2 = \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} e_0, & \widetilde{R}(e_1, e_2) e_1 = -\frac{\sigma_*^2 - \sigma^2}{2\sigma^2} e_1 - \frac{\sigma_*^2 + \sigma^2}{2\sigma^2} e_2, \\ \widetilde{R}(e_0, e_2) e_0 = e_2, & \widetilde{R}(e_1, e_2) e_2 = \frac{\sigma_*^2 + \sigma^2}{2\sigma^2} e_1 + \frac{\sigma_*^2 - \sigma^2}{2\sigma^2} e_2. \end{array}$$

On a metallically deformed para-Sasaki-like para-Norden manifold, the non-zero component of the Ricci tensor $\widetilde{Ric}$ is obtained in the following way:

$$\widetilde{Ric}(e_0, e_0) = -2.$$

On a metallically deformed para-Sasaki-like para-Norden manifold, the scalar tensor $\widetilde{scal}$ and $*$-scalar tensor $\widetilde{scal}^*$ are obtained in the following way

$$\widetilde{scal} = -\frac{2}{\sigma^2},$$
$$\widetilde{scal}^* = 0.$$

On a metallically deformed para-Sasaki-like para-Norden manifold, the components of $\varphi-$connection $\overline{\nabla}$ are obtained in the following way:

$$\begin{array}{lll} \overline{\nabla}_{e_0}e_0 = 0, & \overline{\nabla}_{e_1}e_0 = \frac{1}{2}e_2, & \overline{\nabla}_{e_2}e_0 = \frac{1}{2}e_1, \\ \overline{\nabla}_{e_0}e_1 = e_2, & \overline{\nabla}_{e_1}e_1 = \frac{\sigma_*^2-\sigma^2}{4\sigma^2}e_0, & \overline{\nabla}_{e_2}e_1 = -\frac{\sigma_*^2+\sigma^2}{4\sigma^2}e_0, \\ \overline{\nabla}_{e_0}e_2 = e_1, & \overline{\nabla}_{e_1}e_2 = -\frac{\sigma_*^2+\sigma^2}{4\sigma^2}e_0, & \overline{\nabla}_{e_2}e_2 = \frac{\sigma_*^2-\sigma^2}{4\sigma^2}e_0. \end{array}$$

The non-zero components of the curvature tensor $\overline{R}$ corresponding to the φ-connection $\overline{\nabla}$ are obtained in the following way:

$$\begin{array}{ll} \overline{R}(e_0,e_1)e_0 = e_1, & \overline{R}(e_1,e_2)e_1 = -\frac{\sigma_*^2-\sigma^2}{8\sigma^2}e_1 - \frac{\sigma_*^2+\sigma^2}{8\sigma^2}e_2, \\ \overline{R}(e_0,e_2)e_0 = e_2, & \overline{R}(e_1,e_2)e_2 = \frac{\sigma_*^2+\sigma^2}{8\sigma^2}e_1 + \frac{\sigma_*^2-\sigma^2}{8\sigma^2}e_2. \end{array}$$

The non-zero components of the Ricci tensor $\overline{Ric}$ are obtained in the following way:

$$\begin{array}{lll} \overline{Ric}(e_0,e_0) = -2, & \overline{Ric}(e_1,e_1) = \dfrac{\sigma_*^2+\sigma^2}{8\sigma^2}, & \overline{Ric}(e_2,e_1) = -\dfrac{\sigma_*^2-\sigma^2}{8\sigma^2}, \\ \overline{Ric}(e_1,e_2) = -\dfrac{\sigma_*^2-\sigma^2}{8\sigma^2}, & \overline{Ric}(e_2,e_2) = \dfrac{\sigma_*^2+\sigma^2}{8\sigma^2}. & \end{array}$$

The scalar curvature $\overline{scal}$ and $*$-scalar curvature $\overline{scal}^*$ are obtained in the following way:

$$\overline{scal} = -\frac{3}{2\sigma^2},$$
$$\overline{scal}^* = 0.$$

The non-zero components of the torsion tensor $\overline{T}$ corresponding to the $\varphi-$connection $\overline{\nabla}$ are expressed by:

$$\overline{T}(e_0,e_1) = \tfrac{3}{2}e_2, \quad \overline{T}(e_0,e_2) = \tfrac{3}{2}e_1.$$

6. Conclusions

In this paper, we define a metallic deformation on almost paracontact para-Norden manifolds by establishing the relationship between the metallic structure and the paracontact structure. We obtain the Riemannian connection on metallically deformed para-Sasaki-like para-Norden manifolds and compute the curvature tensor, Ricci tensor, scalar curvature, $*$-scalar curvature, and φ-connection based on this Riemannian connection. Finally, we give an example based on the results of the paper.

Author contributions

Rabia Cakan Akpınar and Esen Kemer Kansu: Writing, review, editing of this paper. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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