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Research article

A robust regional eigenvalue assignment problem using rank-one control for undamped gyroscopic systems

Binxin He[1] **and Hao Liu**[1,2,*]

[1] School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China

[2] Shenzhen Research Institute, Nanjing University of Aeronautics and Astronautics, Shenzhen 518063, China

* **Correspondence:** Email: hliu@nuaa.edu.cn.

Abstract: Considering the advantages of economic benefit and cost reduction by using rank-one control, we investigated the problem of robust regional eigenvalue assignment using rank-one control for undamped gyroscopic systems. Based on the orthogonality relation, we presented a method for solving partial eigenvalue assignment problems to reassign partial undesired eigenvalues accurately. Since it is difficult to achieve robust control by assigning desired eigenvalues to precise positions with rank-one control, we assigned eigenvalues within specified regions to provide the necessary freedom. According to the sensitivity analysis theories, we derived the sensitivity of closed-loop eigenvalues to parameter perturbations to measure robustness and proposed a numerical algorithm for solving robust regional eigenvalue assignment problems so that the closed-loop eigenvalues were insensitive to parameter perturbations. Numerical experiments demonstrated the effectiveness of our method.

Keywords: undamped gyroscopic systems; regional eigenvalue assignment; rank-one control; robustness
Mathematics Subject Classification: 65F18, 93C15

1. Introduction

The undamped gyroscopic system can be discretized into the following equation by finite element techniques

$$M\ddot{z}(t) + G\dot{z}(t) + Kz(t) = 0, \tag{1.1}$$

where $M, G, K \in \mathbb{R}^{n\times n}$ are system matrices, respectively representing the mass, gyroscopic, and stiffness matrix. In general, M is supposed to be symmetric positive definite, G is skew-symmetric, and K is symmetric nonsingular. Such systems derive from elastic structures such as generator rotors and satellite solar panels [1].

To avoid undesirable effects of resonance, caused by a few bad eigenvalues of the system, one needs to reassign those few bad eigenvalues while leaving the rest unchanged. This problem is known as the partial eigenvalue assignment problem in control theory. We consider applying an external control force to alter the dynamics characteristic of the system [2]. Rank-one control refers to one independent control force [3], which is also known as single-input control. It is easier to achieve for its relatively simpler structure, easier maintenance, fewer parameters, and failure tolerance [4]. Considering its advantage of reliability, economic benefit, and cost reduction, rank-one control has been widely studied by researchers and applied for aerial vehicles and satellite control system design [5,6]. Dantas et al. [7] presented a novel approach to design rank-one (single-input) feedback controllers for second-order systems with a time delay. The feedback gains are computed by combining the receptance modeling with classical frequency response methods of control design. Then Dantas et al. [8] presented a partial pole assignment approach for second-order systems with a time delay. The method adopts the versatile system receptance for designing rank-one state-feedback controllers. Richiedei et al. [3] proposed a novel method for antiresonance assignment through active control. The method relies on the unit-rank output feedback control technique to shift one antiresonance. More details can be found in [9, 10].

Integrating rank-one control into the system (1.1) yields

$$M\ddot{z}(t) + G\dot{z}(t) + Kz(t) = bu(t) + p(t), \tag{1.2}$$

in which $b \in \mathbb{R}^n$ is the actuator distribution vector and $p(t) \in \mathbb{R}^n$ is the external force vector [11, 12]. The applied control input $u(t)$ takes the following form

$$u(t) = f^T\dot{z}(t) + g^T z(t), \tag{1.3}$$

where $f, g \in \mathbb{R}^n$ are feedback control vectors. The result of transforming the system (1.2) in the frequency domain can be stated as

$$\left[s^2M + sG + K - b(sf + g)^T\right] x(s) = p(s) \tag{1.4}$$

with any arbitrary complex variable s [13]. The closed-loop transfer function is given by

$$\hat{H}(s) = \left[s^2M + sG + K - b(sf + g)^T\right]^{-1}. \tag{1.5}$$

Define $H(s) = \left(s^2M + sG + K\right)^{-1}$, also named as the receptance matrix in engineering, which is easy to determine for symmetric systems in practice but difficult to measure accurately for asymmetric systems [12]. By applying the Sherman-Morrison formula [14, 15], (1.5) can be written as follows

$$\hat{H}(s) = H(s) + \frac{H(s)\, b(sf + g)^T H(s)}{1 - (sf + g)^T H(s)\, b}. \tag{1.6}$$

The denominator polynomial of (1.6) is known as the characteristic polynomial of the closed-loop system. Based on (1.6), it is obvious that the closed-loop eigenvalues are roots of the following characteristic equation [16]

$$1 - (sf + g)^T H(s)\, b = 0. \tag{1.7}$$

Thus, the partial eigenvalue assignment problem can be mathematically stated as:

PEAP. *Given the system matrices M, G, K, and the control vector b, let the partial undesired eigenvalues $\{\lambda_k\}_{k=1}^{p}$ be altered to the self-conjugate set $\{\mu_k\}_{k=1}^{p}$, given the corresponding right eigenvectors $\{x_k\}_{k=1}^{p}$ of open-loop eigenvalues $\{\lambda_k\}_{k=1}^{p}$, and find the feedback control vectors $f, g \in \mathbb{R}^n$ such that the closed-loop eigenvalues are the desired ones $\{\mu_k\}_{k=1}^{p}$ satisfying (1.7) while keeping the rest of the eigenvalues $\{\lambda_k\}_{k=p+1}^{2n}$ unchanged.*

Ram and Mottershead [17] first proposed the receptance method for solving pole assignment problems with rank-one control. Furthermore, they extended the partial pole assignment problem to a multi-input control condition [18]. More details can be seen in [19–21]. Liu et al. [22] proposed a multi-step method for solving partial eigenvalue assignment problems in undamped gyroscopic systems. These methods need to use system matrices, which can be readily obtained by finite element techniques and adopted in engineering practice [23, 24].

In theory, the solution to the partial eigenvalue assignment problem can be determined uniquely such that the closed-loop eigenvalues are the desired ones while keeping the rest of the eigenvalues unchanged. However, the system matrices in (1.2) rely on structure parameters in some cases. The assigned closed-loop eigenvalues will deviate from the desired ones when perturbations appear in the system parameters. The robustness of closed-loop systems (i.e., the robust eigenvalue assignment problem) is a matter that deserves additional research. We know that it is difficult to allow robust control to assign desired eigenvalues to precise positions by using rank-one control since there is no more freedom. To solve the robust eigenvalue assignment problem, we intend to assign eigenvalues within specified regions rather than at precise positions. This regional assignment is also usually convenient and economical in engineering practice and provides the necessary freedom for finding robust control vectors so that the closed-loop system is robust. The robust regional eigenvalue assignment problem can be stated as:

RREAP. *Find the optimal desired closed-loop eigenvalues $\{\tilde{\mu}_k\}_{k=1}^{p}$ subject to the specified regions and corresponding robust control vectors f_{rob}, g_{rob} such that the closed-loop eigenvalues are as insensitive to parameter perturbations as possible.*

Traditionally, the robustness of the closed-loop system can be measured by the condition number of the closed-loop eigenvector matrix $\kappa(Y) = \|Y\|_F \left\|Y^{-1}\right\|_F$. Brahama and Datta [25] minimized the expression $cond_F(Y) = \left\|\left(Y^H Y - I\right)^2\right\|_F^2$ instead. Bai et al. [26] provided a new cost function for solving the robust partial quadratic eigenvalue assignment with a time delay by using receptance and system matrices. Lu and Bai [27] proposed a modified gradient-based method for solving this problem based on [26]. More details can be seen in [28, 29]. The robustness of the closed-loop system depends on the conditioning of the closed-loop eigenvectors. The technique requires knowledge of the complete spectrum and the associated eigenvectors for its implementation [25]. However, this measurement of robustness does not consider the structure of the control system [30]. In practice, the structure parameters that affect the stability of an undamped gyroscopic system are often known. Hence, we define the measurement of robustness by deriving the partial derivatives of the eigenvalues with respect to the structure parameters, which is more practical. Based on the sensitivity analysis theories, we can assure the robustness of the closed-loop system through minimizing the partial derivatives of the eigenvalues with respect to the structure parameters.

In this paper, we first present a method based on the orthogonality relation for solving partial eigenvalue assignment problems through rank-one control, which is beneficial to reduce control costs. The sensitivity of eigenvalues to parameter perturbations is derived to measure the robustness of the closed-loop system. Then we find the optimal closed-loop eigenvalues and corresponding robust control vectors f_{rob}, g_{rob} by assigning the desired eigenvalues to the specified regions such that the measurement of robustness is as small as possible. Based on these analyses, we present a numerical algorithm to solve robust regional eigenvalue assignment problems using rank-one control for undamped gyroscopic systems.

The construction of our paper is as follows: In Section 2, some notations, assumptions, and lemmas of the orthogonality relation of undamped gyroscopic systems are given, and a method based on the orthogonality relation for solving partial eigenvalue assignment problems using rank-one control is proposed. In Section 3, we derive the sensitivity of the closed-loop eigenvalues to parameter perturbations to measure the robustness of the closed-loop system. Based on these theoretical results, we propose a numerical algorithm for solving robust regional eigenvalue assignment problems using rank-one control for undamped gyroscopic systems. In Section 4, we give some examples to verify the practicability of our method.

2. The partial eigenvalue assignment problem

2.1. Preliminaries

We first make the following notations and assumptions. Let A^T be the transpose of matrix A. Let $\{\lambda_k\}_{k=1}^{2n}$ be the open-loop eigenvalues satisfying $\{\lambda_k\}_{k=1}^{p} \cap \{\lambda_k\}_{k=p+1}^{2n} = \emptyset$. Suppose that the eigenvalues $\{\lambda_k\}_{k=1}^{p}$ are assigned to the self-conjugate set $\{\mu_k\}_{k=1}^{p}$ and $\{\lambda_k\}_{k=1}^{p} \cap \{\mu_k\}_{k=1}^{p} = \emptyset$. Define

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p), X_1 = [x_1, \ldots, x_p],$$

where X_1 is the corresponding right eigenvector matrix of Λ_1. We also set

$$\Lambda_2 = \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_{2n}), \Lambda_c = \mathrm{diag}(\mu_1, \ldots, \mu_p)$$

as the matrices, the diagonal elements of which are the eigenvalues kept unchanged and to be assigned, respectively.

$$X_2 = [x_{p+1}, \ldots, x_{2n}], X_c = \left[x_{c1}, \ldots, x_{cp}\right]$$

are the corresponding right eigenvector matrices of Λ_2 and Λ_c.

Then, we give the lemma related to the orthogonality relation of undamped gyroscopic systems [31].

Lemma 2.1. *Let the open-loop eigenvalues $\{\lambda_k\}_{k=1}^{2n}$ be located in disjoint sets $\{\lambda_k\}_{k=1}^{p}$ and $\{\lambda_k\}_{k=p+1}^{2n}$, i.e., $\{\lambda_k\}_{k=1}^{p} \cap \{\lambda_k\}_{k=p+1}^{2n} = \emptyset$. Then*

$$\Lambda_1 X_1^T M X_2 \Lambda_2 + X_1^T K X_2 = 0. \tag{2.1}$$

2.2. The solution to the partial eigenvalue assignment problem

As mentioned above, the closed-loop receptance matrix can be written as

$$\hat{H}(s) = H(s) + \frac{H(s)\, b (sf + g)^T H(s)}{1 - (sf + g)^T H(s)\, b}, \tag{2.2}$$

and the characteristic equation of the closed-loop system is

$$1 - (sf + g)^T H(s) b = 0.$$

It can be seen that the desired eigenvalue μ_k of the closed-loop system should satisfy

$$\left(\mu_k f^T + g^T\right) H(\mu_k) b = 1, k = 1, \ldots, p. \tag{2.3}$$

From (2.2), we can see that when

$$\left(\lambda_l f^T + g^T\right) H(\lambda_l) = 0, l = p+1, \ldots, 2n,$$

a particular eigenvalue λ_l can make the closed-loop receptance matrix equal to the open-loop receptance matrix, i.e., $\hat{H}(\lambda_l) = H(\lambda_l)$ [32]. Post-multiplying it by b implies that

$$\left(\lambda_l f^T + g^T\right) H(\lambda_l) b = 0, l = p+1, \ldots, 2n. \tag{2.4}$$

Denote

$$r_k = H(s_k) b, k = 1, \ldots, 2n,$$

and with the combined Eqs (2.3) and (2.5), the solution to the PEAP can be derived as

$$Ak = h, \tag{2.5}$$

where

$$A = \begin{bmatrix} \mu_1 r_1^T & r_1^T \\ \vdots & \vdots \\ \mu_p r_p^T & r_p^T \\ \lambda_{p+1} r_{p+1}^T & r_{p+1}^T \\ \vdots & \vdots \\ \lambda_{2n} r_{2n}^T & r_{2n}^T \end{bmatrix} \in \mathbb{C}^{2n \times 2n}, h = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, k = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{2.6}$$

Based on the above analysis, it can be seen that this method can assign the expected eigenvalues and keep the other eigenvalues unchanged by using the receptance matrix, but it needs to know all of the eigenvalues of the open-loop pencil. It is difficult to know all of the eigenvalues of the open-loop pencil in engineering applications. Even if we know all of the eigenvalues, if there are some unchanged eigenvalues of multiplicity m in the last $2n - p$ eigenvalues, the coefficient matrix A in Eq (2.5) will be singular on account of m repeated equations. Hence, we might have to solve singular linear systems [17], which leads to inaccurate assignment results in some cases. In addition, the asymmetric receptance matrix $H(s)$ is difficult to measure accurately in practice for a rotor system. Taking into account these factors, we will think about introducing system matrices for solving the PEAP.

We first consider the unchanged eigenvalues. According to the orthogonality relation of undamped gyroscopic systems, we propose the following theorem:

Theorem 2.2. *Let $M, G, K \in \mathbb{R}^{n\times n}$ be system matrices and $b \in \mathbb{R}^n$ be a control vector, given the self-conjugate eigenvalues $\{\lambda_k\}_{k=1}^p$ and corresponding right eigenvectors $\{x_k\}_{k=1}^p$. Define*

$$f = MX_1\Lambda_1\phi^T, g = KX_1\phi^T, \phi \in \mathbb{C}^{1\times p}, \tag{2.7}$$

then

$$MX_2\Lambda_2^2 + \left(G - bf^T\right)X_2\Lambda_2 + \left(K - bg^T\right)X_2 = 0.$$

Proof. The similar proof can be seen in [22]. □

As previously described, the desired eigenvalues should satisfy the characteristic equation

$$1 - (\mu_k f + g)^T H(\mu_k)\, b = 0, k = 1, \dots, p, \tag{2.8}$$

or, equivalently,

$$r_k^T(\mu_k f + g) = 1, k = 1, \dots, p.$$

Then we rewrite it as

$$W\begin{bmatrix} f \\ g \end{bmatrix} = e, \tag{2.9}$$

where

$$W = \begin{bmatrix} \mu_1 r_1^T & r_1^T \\ \vdots & \vdots \\ \mu_p r_p^T & r_p^T \end{bmatrix}, e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Substituting (2.7) into (2.9) yields

$$W\begin{bmatrix} MX_1\Lambda_1 \\ KX_1 \end{bmatrix}\phi^T = e. \tag{2.10}$$

Define

$$Z = W\begin{bmatrix} MX_1\Lambda_1 \\ KX_1 \end{bmatrix} \triangleq WY, \tag{2.11}$$

and we obtain

$$Z\phi^T = e, \tag{2.12}$$

where $Z \in \mathbb{C}^{p\times p}$. Hence, we can obtain ϕ^T by solving (2.12).

Thus, we give Algorithm 1 for solving the PEAP using rank-one control for undamped gyroscopic systems.

Algorithm 1 The algorithm for solving the PEAP using rank-one control.

Require:

The system matrices $M, G, K \in \mathbb{R}^{n\times n}$ and the control vector $b \in \mathbb{R}^n$;
The eigenvalues to be altered $\{\lambda_k\}_{k=1}^p$ and the corresponding right eigenvectors $\{x_k\}_{k=1}^p$;
The self-conjugate set $\{\mu_k\}_{k=1}^p$.

Ensure:

1: Compute $H(\mu_k) = \left(\mu_k^2 M + \mu_k G + K\right)^{-1}$;
2: Compute $r_k = H(\mu_k)\, b$;
3: Compute $Z = WY$;
4: Solve $Z\phi^T = e$;
5: Compute f, g by (2.7).

Remark 2.3. *In Algorithm 1, we do not need to know all of the eigenvalues of the open-loop pencil. In addition, we do not need to solve Eq* (2.5) *since it might be singular, which may lead to inaccurate assignment results in some cases.*

3. The robust regional eigenvalue assignment problem

Based on the above analysis, we can obtain the unique solution of the PEAP. However, the desired closed-loop eigenvalues will change when perturbations appear in the structure parameters. It is difficult to allow robust control by using rank-one control since there is no more freedom. We usually tend to assign eigenvalues within specified regions rather than at precise positions for solving robust eigenvalue assignment problems by using rank-one control. This regional assignment is also usually convenient and economical in engineering practice and provides the necessary freedom for finding robust control vectors. Then, we need to keep the closed-loop eigenvalues as insensitive to parameter perturbations as possible.

The sensitivity analysis of eigenvalue problems is to study the influence of parameter perturbations in the matrix on the eigenvalues and eigenvectors. Mathematically, the sensitivity can be obtained by differential or difference methods. Based on the sensitivity analysis theories, the sensitivity of the eigenvalues can be explained as the derivatives of the eigenvalues with respect to structure parameters. This implies the rate of change of the eigenvalues with respect to the parameters. Hence, we define the following measurement of robustness

$$J = \left\| \left(\frac{\partial \mu_1}{\partial \omega}, \ldots, \frac{\partial \mu_p}{\partial \omega} \right) \right\|, \tag{3.1}$$

where $\mu_k, k = 1, \ldots, p$ are the desired closed-loop eigenvalues and ω is the structure parameter. We use it to represent the sensitivity of the eigenvalues to structure parameter perturbations. The robustness of the closed-loop system can be assured through minimizing (3.1).

In the contents that follow, we will first derive the partial derivatives of the eigenvalues with respect to the structure parameters. For an undamped gyroscopic system, the system matrix is the function of rotor speed ω so that the vibration frequency will change when perturbations appear in the rotor speed, which results in the deviation of the desired eigenvalues [33]. Hence, we consider a small perturbation of the rotor speed ω in undamped gyroscopic systems to study the robust optimization problem. Introducing rotor speed ω as a structure parameter yields the following original characteristic equation according to (2.8):

$$Q(\omega, \mu_k) = 1 - (\mu_k f + g)^T H(\omega, \mu_k) b = 0, k = 1 \ldots, p \tag{3.2}$$

and

$$H(\omega, \mu_k) = \left[\mu_k^2 M + \mu_k G(\omega) + K(\omega)\right]^{-1}. \tag{3.3}$$

It is evident that a small perturbation $\delta\omega$ causes a corresponding deviation of the original eigenvalue by $\delta\mu_k$. Moreover, the perturbed characteristic equation should still satisfy Eq (3.2) as

$$Q(\omega + \delta\omega, \mu_k + \delta\mu_k) = 1 - \left[(\mu_k + \delta\mu_k) f + g\right]^T H(\omega + \delta\omega, \mu_k + \delta\mu_k) b = 0,$$

when the structure parameters are insensitive to a small perturbation. Based on the first-order Taylor expansion, we can get

$$Q(\omega + \delta\omega, \mu_k + \delta\mu_k) = Q(\omega, \mu_k) + \frac{\partial Q}{\partial \omega}\delta\omega + \frac{\partial Q}{\partial \mu_k}\delta\mu_k = 0, \tag{3.4}$$

where Q meanings $Q(\omega, \mu_k)$. This linear approximation is valid for a small deviation from the nominal values of the structure parameters. Comparing (3.4) with (3.2), we see that

$$\frac{\partial Q}{\partial \omega}\delta\omega + \frac{\partial Q}{\partial \mu_k}\delta\mu_k = 0, \tag{3.5}$$

or, equivalently,

$$\frac{\partial Q}{\partial \omega} + \frac{\partial Q}{\partial \mu_k}\frac{\partial \mu_k}{\partial \omega} = 0. \tag{3.6}$$

Consequently, the partial derivative of the closed-loop eigenvalue μ_k with respect to ω is given as

$$\frac{\partial \mu_k}{\partial \omega} = -\frac{\partial Q}{\partial \omega}\bigg/\frac{\partial Q}{\partial \mu_k}. \tag{3.7}$$

According to Eq (3.2), it is obvious that

$$\frac{\partial Q}{\partial \omega} = -(\mu_k f + g)^T \frac{\partial H(\omega, \mu_k)}{\partial \omega} b \tag{3.8}$$

and

$$\frac{\partial Q}{\partial \mu_k} = -f^T H(\omega, \mu_k) b - (\mu_k f + g)^T \frac{\partial H(\omega, \mu_k)}{\partial \mu_k} b. \tag{3.9}$$

Furthermore, we get

$$\frac{\partial H(\omega, \mu_k)}{\partial \omega} = -H(\omega, \mu_k)\left(\mu_k \frac{\partial G(\omega)}{\partial \omega} + \frac{\partial K(\omega)}{\partial \omega}\right) H(\omega, \mu_k)$$

and

$$\frac{\partial H(\omega, \mu_k)}{\partial \mu_k} = -H(\omega, \mu_k)(2\mu_k M + G(\omega)) H(\omega, \mu_k).$$

Thus, we have

$$\frac{\partial \mu_k}{\partial \omega} = -\frac{(\mu_k f + g)^T H(\omega, \mu_k)\left(\mu_k \frac{\partial G(\omega)}{\partial \omega} + \frac{\partial K(\omega)}{\partial \omega}\right) H(\omega, \mu_k) b}{-f^T H(\omega, \mu_k) b + (\mu_k f + g)^T H(\omega, \mu_k)(2\mu_k M + G(\omega)) H(\omega, \mu_k) b}, k = 1, \ldots, p. \tag{3.10}$$

Based on (3.1) and (3.10), the desired eigenvalues can be assigned robustly when the measurement of robustness (3.1) is as small as possible.

To provide the necessary freedom such that the measurement of robustness (3.1) is minimized, we choose some regions in the complex plane to assign the desired closed-loop eigenvalues. In general, the desired eigenvalues are assigned to the following circular region:

$$(x - x_0)^2 + (y - y_0)^2 \le r^2, \tag{3.11}$$

where (x_0, y_0) is the circle center and r is the radius. When the circular region is in the complex plane, it satisfies $x_0, y_0, r \in R$ and $x_0 < 0, r > 0$. With such a restriction, to solve the RREAP is to find the optimal desired closed-loop eigenvalues $\{\tilde{\mu}_k\}_{k=1}^{p}$ subject to the regions given by (3.11) and the corresponding robust control vectors f_{rob}, g_{rob} such that the measurement of robustness (3.1) is minimized.

Thus, we give a numerical algorithm for solving the RPEAP using rank-one control for undamped gyroscopic systems (see Algorithm 2).

Algorithm 2 The algorithm for solving the RPEAP using rank-one control.

Require:

The system matrices $M, G, K \in \mathbb{R}^{n\times n}$ and control vector $b \in \mathbb{R}^n$;
The circle center (x_0, y_0) and radius r of the specific circular region;
Maximum number of generations T, population size N, and evolution counter $t = 0$;
Crossover probability P_c and mutation probability P_m.

Ensure:

1: Randomly generated N individuals as the initial population $P(t)$ in circular region given by (3.11);
2: Compute $H(\omega, \mu_k)$ by (3.3) for each individual in the initial population $P(t)$;
3: Compute $r_k = H(\omega, \mu_k)\, b$ and $Z = WY$;
4: Solve $Z\phi^T = e$ for ϕ^T and compute the feedback control vectors f, g by (2.7);
5: Compute the sensitivities of the eigenvalues to parameter perturbations $\frac{\partial \mu_k}{\partial \omega}, k = 1, \ldots, p$ by (3.10);
6: Compute the value of the measurement of robustness (3.1) and the fitness function of each individual in population $P(t)$;
7: Apply the roulette wheel selection to the population to generate the next population $P(t+1)$;
8: Apply the one point crossover operator to population $P(t+1)$ with P_c to generate new individuals in population $P(t+1)$;
9: Apply the simple mutation operator to population $P(t+1)$ with P_m to generate new individuals in population $P(t+1)$;
10: If $t > T$, output the optimal solution $\{\tilde{\mu}_k\}_{k=1}^{p}$ and the value of the measurement of robustness (3.1); otherwise, set $t = t + 1$, and go to step 6;
11: Use the optimal solution $\{\tilde{\mu}_k\}_{k=1}^{p}$ to compute the corresponding robust control vectors f_{rob}, g_{rob} by Algorithm 1.

Remark 3.1. *In Algorithm 2, we use a genetic algorithm that is a sufficiently robust optimization method. We use the MATLAB function* ga *to implement the process of a genetic algorithm, which is easy to realize and compute.*

4. Numerical examples

In Section 4, numerical experiments show that our method is practicable. The algorithms are carried out on a personal PC by MATLAB 9.10. For confirmation, we define

$$r1 = \left\| M X_c \Lambda_c^2 + \left(G - b f^T\right) X_c \Lambda_c + \left(K - b g^T\right) X_c \right\|_F$$

and

$$r2 = \left\| MX_2\Lambda_2^2 + \left(G - bf^T\right)X_2\Lambda_2 + \left(K - bg^T\right)X_2 \right\|_F,$$

which represent the residuals of the assigned eigenvalues and the fixed eigenvalues, respectively.

Example 4.1. *Consider spatial oscillations of a particle shown in Figure 1 [2]. Suppose that the ring is rotating with constant angular velocity. We set* $\omega = [\omega_1, \omega_2, \omega_3]^T = [1, 2, 1]^T, \omega_n = 3$, *and* $\gamma = \frac{1}{2}$. *Then*

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 & -2\omega_3 & 2\omega_2 \\ 2\omega_3 & 0 & -2\omega_1 \\ -2\omega_2 & 2\omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 2\omega_n^2 - \omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & 2\omega_n^2\gamma - \omega_1^2 - \omega_3^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & 2\omega_n^2\gamma - \omega_1^2 - \omega_2^2 \end{bmatrix} = \begin{bmatrix} 13 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

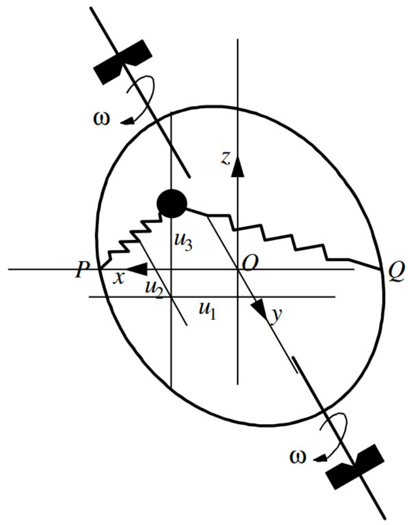

Figure 1. Spatial oscillations of a particle.

The first two eigenvalues $\{\pm 6.0860i\}$ *are reassigned to* $\{-2 \pm 3i\}$ *and the remaining eigenvalues* $\{\pm 3.1895i, \pm 0.8878i\}$ *are kept unchanged. Given the corresponding right eigenvector matrix*

$$X_1 = \begin{bmatrix} 0.7223 & 0.7223 \\ -0.1962 + 0.2933i & -0.1962 - 0.2933i \\ 0.0981 - 0.5867i & 0.0981 + 0.5867i \end{bmatrix},$$

we compute

$$H(\mu_1) = \begin{bmatrix} 0.02571 + 0.0186i & -0.0034 - 0.0010i & 0.0129 + 0.0369i \\ 0.0083 + 0.0289i & 0.0255 + 0.0582i & 0.0084 - 0.0074i \\ -0.0104 - 0.0229i & 0.0026 + 0.0371i & 0.0172 + 0.0359i \end{bmatrix},$$

$$H(\mu_2) = \begin{bmatrix} 0.02571 - 0.0186i & -0.0034 + 0.0010i & 0.0129 - 0.0369i \\ 0.0083 - 0.0289i & 0.0255 - 0.0582i & 0.0084 + 0.0074i \\ -0.0104 + 0.0229i & 0.0026 - 0.0371i & 0.0172 - 0.0359i \end{bmatrix}.$$

According to (2.11), we obtain

$$Z = \begin{bmatrix} 0.4614 - 0.1016i & 1.0031 + 0.6501i \\ 1.0031 - 0.6501i & 0.4614 + 0.1016i \end{bmatrix}.$$

By solving (2.12), the solution ϕ^T *is*

$$\phi^T = \begin{bmatrix} 0.4493 + 0.4550i \\ 0.4493 - 0.4550i \end{bmatrix}.$$

Then we have

$$f = \begin{bmatrix} -4 \\ -0.5178 \\ 3.7517 \end{bmatrix}, g = \begin{bmatrix} 7.9966 \\ -0.9130 \\ 1.5455 \end{bmatrix}.$$

It is found that $r1 = 1.5507 \times 10^{-14}$, *and* $r2 = 1.6043 \times 10^{-14}$.

Example 4.2. *Consider the rigid rotor shown in Figure 2 [34], in which the system matrices are defined as*

$$M = \begin{bmatrix} 30.8113 & 0 & 0 & 0 \\ 0 & 30.8113 & 0 & 0 \\ 0 & 0 & 20.3712 & 0 \\ 0 & 0 & 0 & 20.3712 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0385 \\ 0 & 0 & -0.0385 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.4 & 0 & 0 & 0 \\ 0 & 2.4 & 0 & 0 \\ 0 & 0 & 0.0240 & 0 \\ 0 & 0 & 0 & 0.0240 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 2 \end{bmatrix}.$$

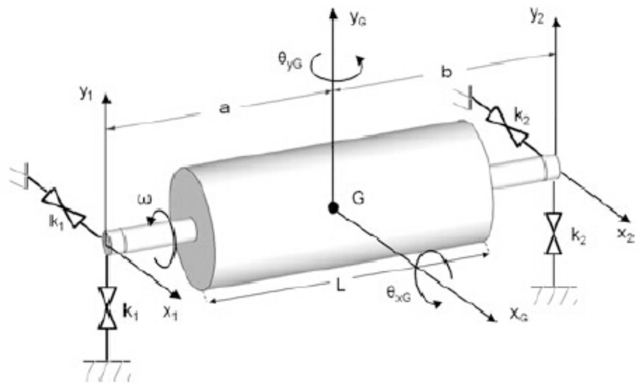

Figure 2. A cylindrical rotor with flexible bearings.

The eigenvalues of the open-loop system are $\{\pm 0.2791i, \pm 0.2791i, \pm 0.0353i, \pm 0.0334i\}$. *We are to alter the eigenvalues* $\{\pm 0.0353i, \pm 0.0334i\}$ *to* $\{-1 \pm 2i, -3 \pm 4i\}$ *and the corresponding right eigenvector matrix* X_1 *is*

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0177 - 0.7069i & -0.0177 + 0.7069i & 0.0214 - 0.7068i & 0.0214 + 0.7068i \\ -0.7069 + 0.0177i & -0.7069 - 0.0177i & 0.7068 + 0.0214i & 0.7068 - 0.0214i \end{bmatrix}.$$

The obtained receptance matrices $H(\mu_1)$, $H(\mu_2)$, $H(\mu_3)$, *and* $H(\mu_4)$ *are*

$$H(\mu_1) = 10^2 \times \begin{bmatrix} -0.9003 - 1.2325i & 0 & 0 & 0 \\ 0 & -0.9003 - 1.2325i & 0 & 0 \\ 0 & 0 & -0.6109 - 0.8148i & -0.0004 + 0.0008i \\ 0 & 0 & 0.0004 - 0.0008i & -0.6109 - 0.8148i \end{bmatrix},$$

$$H(\mu_2) = 10^2 \times \begin{bmatrix} -0.9003 + 1.2325i & 0 & 0 & 0 \\ 0 & -0.9003 + 1.2325i & 0 & 0 \\ 0 & 0 & -0.6109 + 0.8148i & -0.0004 - 0.0008i \\ 0 & 0 & 0.0004 + 0.0008i & -0.6109 + 0.8148i \end{bmatrix},$$

$$H(\mu_3) = 10^2 \times \begin{bmatrix} -2.1328 - 7.3947i & 0 & 0 & 0 \\ 0 & -2.1328 - 7.3947i & 0 & 0 \\ 0 & 0 & -1.4257 - 4.8891i & -0.0012 + 0.0015i \\ 0 & 0 & 0.0012 - 0.0015i & -1.4257 - 4.8891i \end{bmatrix},$$

$$H(\mu_4) = 10^2 \times \begin{bmatrix} -2.1328 + 7.3947i & 0 & 0 & 0 \\ 0 & -2.1328 + 7.3947i & 0 & 0 \\ 0 & 0 & -1.4257 + 4.8891i & -0.0012 - 0.0015i \\ 0 & 0 & 0.0012 + 0.0015i & -1.4257 + 4.8891i \end{bmatrix}.$$

According to (2.11), we have

$$Z = \begin{bmatrix} -0.0552 - 0.0581i & -0.0137 - 0.0812i & -0.0122 - 0.0750i & -0.0547 - 0.0554i \\ -0.0137 + 0.0812i & -0.0552 + 0.0581i & -0.0547 + 0.0554i & -0.0122 + 0.0750i \\ -0.0292 - 0.0212i & -0.0123 - 0.0344i & -0.0111 - 0.0323i & -0.0282 - 0.0199i \\ -0.0123 + 0.0344i & -0.0292 + 0.0212i & -0.0282 + 0.0199i & -0.0111 + 0.0323i \end{bmatrix}.$$

By solving (2.12), the solution ϕ^T *is*

$$\phi^T = 10^7 \times \begin{bmatrix} -2.0819 + 7.2194i \\ -2.0819 - 7.2194i \\ -2.7176 - 7.9358i \\ -2.7176 + 7.9358i \end{bmatrix}.$$

Hence, we have

$$f = 10^8 \times \begin{bmatrix} 0 \\ 0 \\ -0.4314 \\ 1.5099 \end{bmatrix}, \qquad g = 10^5 \times \begin{bmatrix} 0 \\ 0 \\ -2.5293 \\ -1.9544 \end{bmatrix}.$$

Then, we compute $r1 = 2.5620 \times 10^{-8}$*, and* $r2 = 8.8818 \times 10^{-16}$.

Comparing the results of our method with the receptance method shown in Figure 3, we can see that our method can accurately assign the desired eigenvalues better than the receptance method while the remaining eigenvalues are kept unchanged. The reason for this result is that the coefficient matrix A in Eq (2.5) is singular on account of multiple eigenvalues, where the condition number of A is 4.6148×10^{50}*. It indicates that the singularity of coefficient matrix A may lead to an inaccurate assignment.*

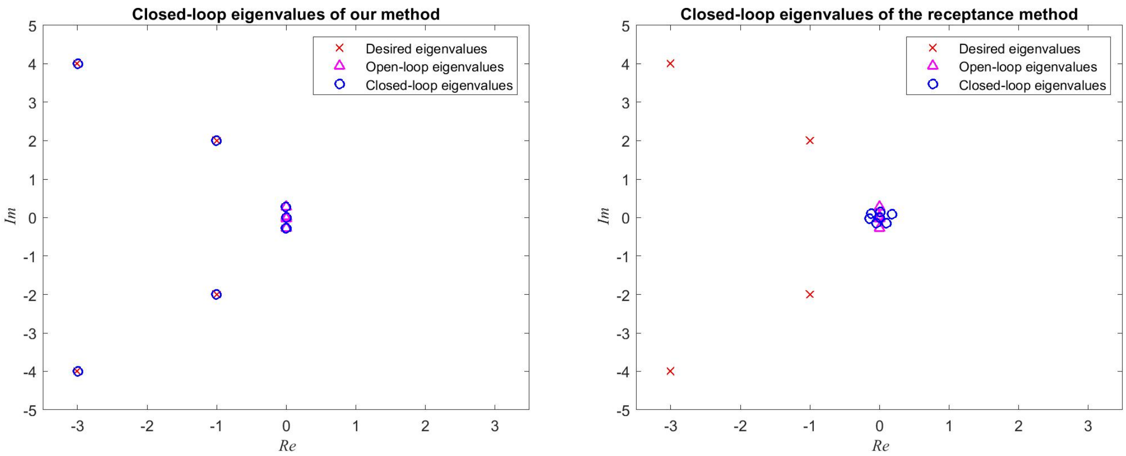

Figure 3. Partial eigenvalue assignment results of Example 4.2.

Example 4.3. *Given the following undamped gyroscopic system,*

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, K = \begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

set $\omega = 0.7$*, then,*

$$G = \omega G_0 = \omega \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2.1 \\ 2.1 & 0 \end{bmatrix}.$$

The given circular regions are

$$(x + 1.6582)^2 + (y \pm 3.1234)^2 \leq 1^2,$$

in which the circle centers are $\{-1.6582 \pm 3.1234\}$ *and the radius is* $r = 1$.

By using Algorithm 1, the eigenvalues $\{\pm 1.7034i\}$ *are assigned to the circle centers* $\{-1.6582 \pm 3.1234\mathrm{i}\}$*. Meanwhile, the other eigenvalues remain unchanged. The feedback vectors* f, g *are*

$$f = \begin{bmatrix} -0.0307 \\ -3.2551 \end{bmatrix}, \qquad g = \begin{bmatrix} -13.2174 \\ 3.2243 \end{bmatrix}.$$

Then, we can compute $r1 = 3.5316 \times 10^{-14}$*, and* $r2 = 2.7144 \times 10^{-14}$.

By using Algorithm 2, the eigenvalues $\{\pm 1.7034i\}$ *are assigned to the given circular regions, and we can obtain the optimal closed-loop eigenvalues* $\tilde{\mu}_{1,2}$ *as* $\{-0.8943 \pm 2.4781i\}$*. The corresponding robust control vectors can be computed as*

$$f_{rob} = \begin{bmatrix} -0.2067 \\ -1.3736 \end{bmatrix}, \qquad g_{rob} = \begin{bmatrix} -5.7622 \\ 2.1456 \end{bmatrix}.$$

Then, we can compute $r1 = 1.7640 \times 10^{-14}$*, and* $r2 = 2.0023 \times 10^{-14}$.

The results of Algorithms 1, 2, and Newton's Method in [28] are given in Table 1. We can see that the measurement of robustness by using Algorithm 2 is smaller than the other compared methods.

Table 1. The results of Example 4.3.

	$\mu_{1,2}$	J(Eq (3.1))	J([28])
Algorithm 1	$-1.6582 \pm 3.1234i$	4.3914	101.4869
Algorithm 2	$-0.8935 \pm 2.4793i$	1.9341	22.7001
Newton's Method	$-1.4924 \pm 3.0525i$	3.9482	82.7390

Then 1000 *samples are taken from a uniform distribution between the variation of* $\pm 20\%$ *on the structure parameter* ω*. The distributions of the desired closed-loop eigenvalues are shown in Figure 4. Note that the circles represent the circle regions to assign the desired closed-loop eigenvalues. We can visually see that the spread of the desired eigenvalues obtained by Algorithm 2 is smaller, which indicates that the robustness of closed-loop system is better when using our method.*

Next, we give the figure with the time response for the feedback vectors f_{rob}, g_{rob} *obtained by Algorithm 2. Comparing the time response of the open-loop and closed-loop system displayed in Figures 5–8, we note that the amplitudes of the closed-loop system are smaller than the open-loop system and tend to transition faster to a stable value, whether the system is put with a perturbation or an additional delay. This implies that our method is efficient for developing control systems for robust regional eigenvalue assignment.*

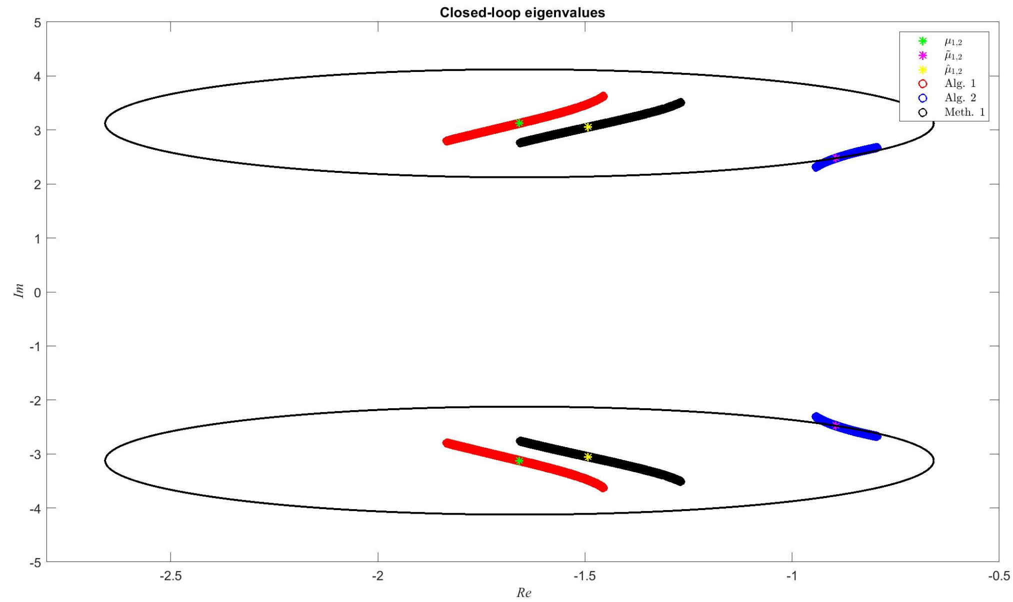

Figure 4. The perturbed desired closed-loop eigenvalues of different methods.

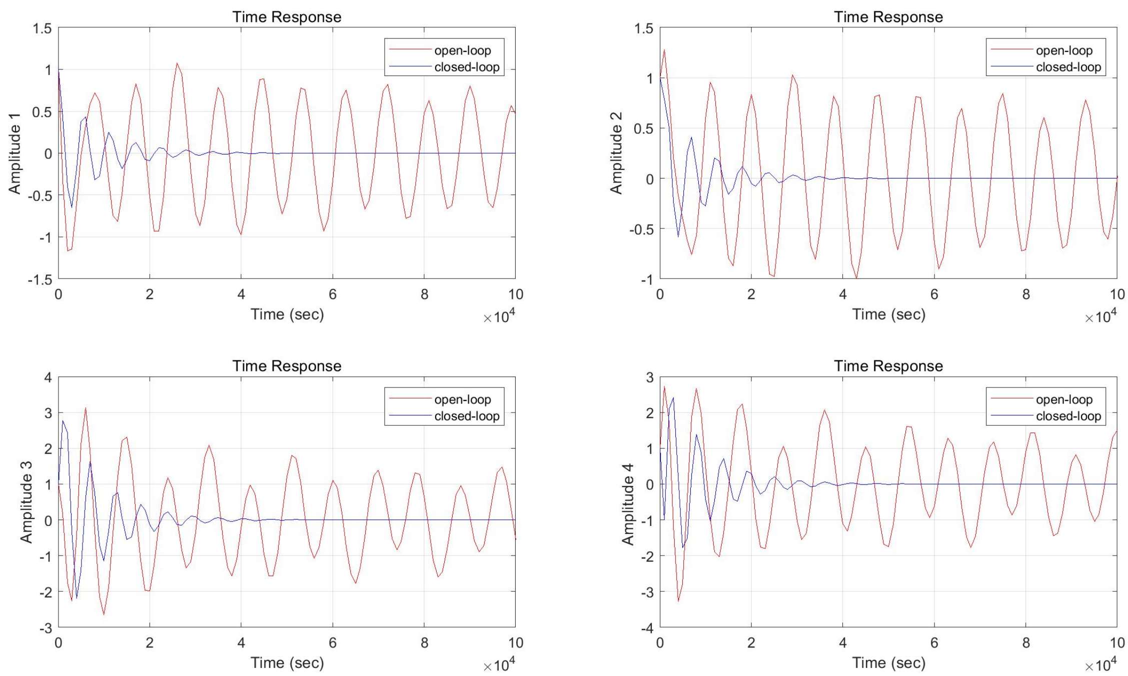

Figure 5. Time response for Example 4.3 on $\omega = 0.7$.

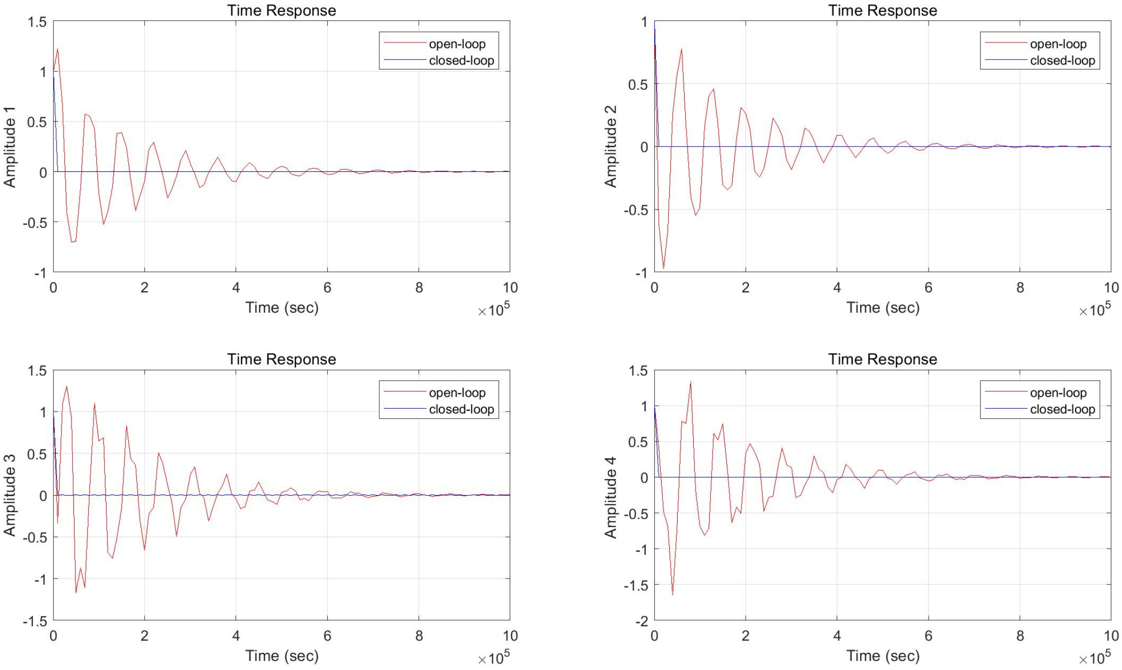

Figure 6. Time response for Example 4.3 on $\omega = 0.66$.

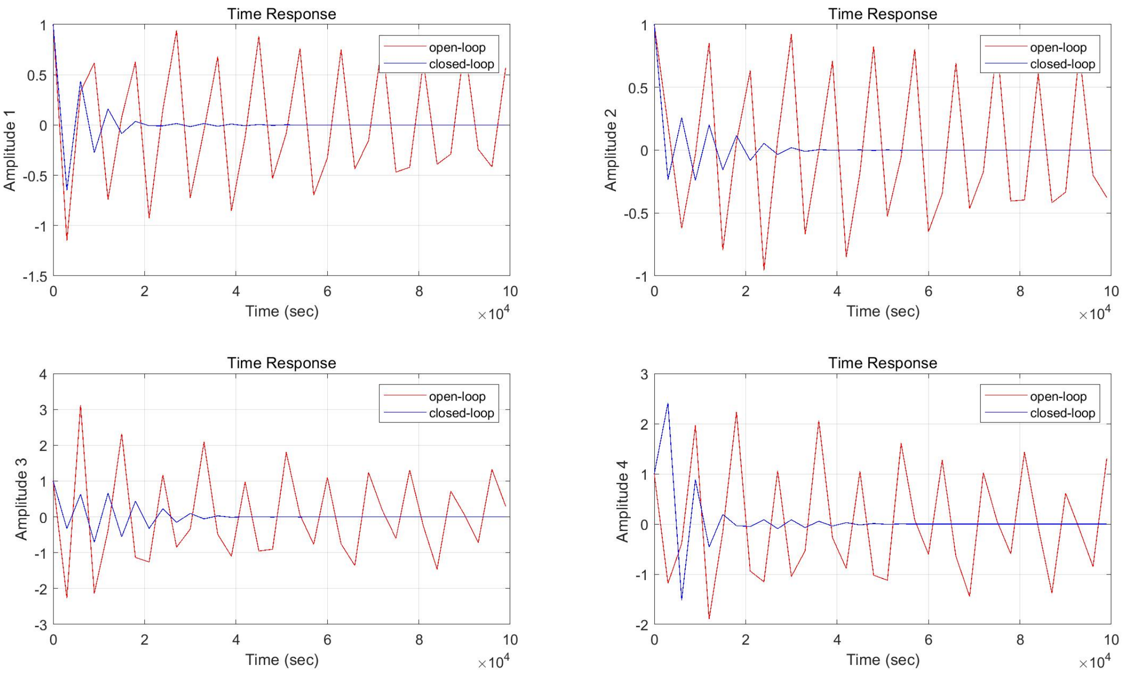

Figure 7. Time response for Example 4.3 with an additional delay on $\omega = 0.7$.

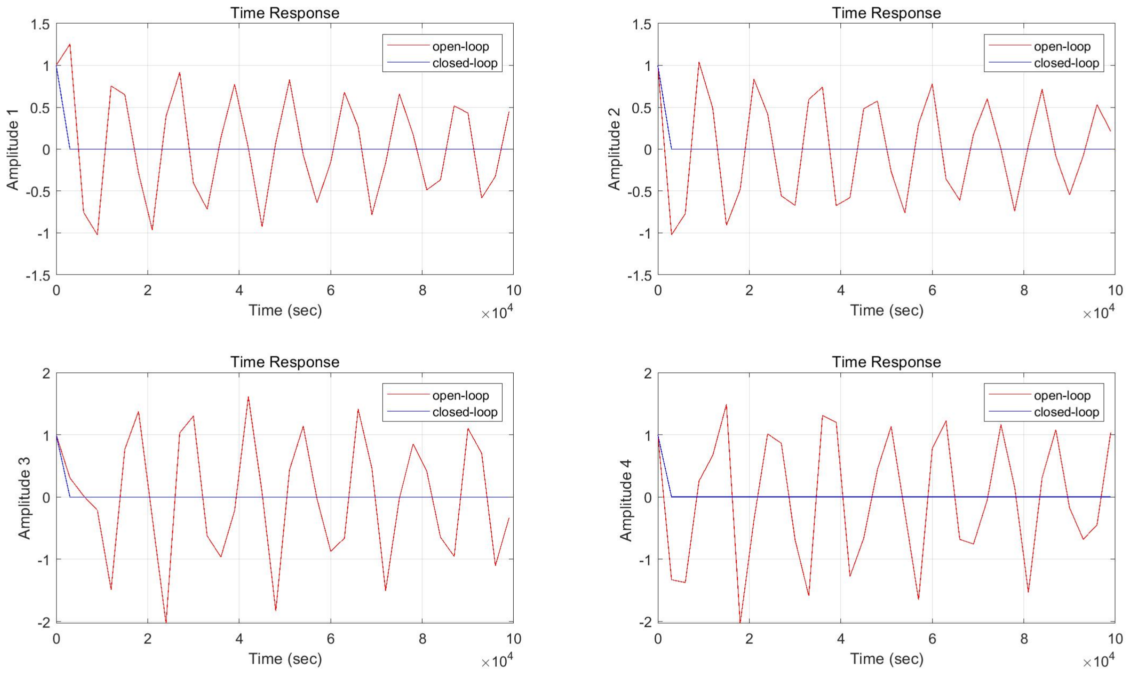

Figure 8. Time response for Example 4.3 with an additional delay on $\omega = 0.66$.

Example 4.4. *Given the undamped gyroscopic system with $M = 2I_n$, set $\omega = 0.9$,*

$$G = \omega G_0 = \omega \begin{bmatrix} 0 & -4 & -4 & \dots & 0 & 0 \\ 4 & 0 & -4 & \dots & 0 & 0 \\ 4 & 4 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -4 \\ 0 & 0 & 0 & \dots & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3.6 & -3.6 & \dots & 0 & 0 \\ 3.6 & 0 & -3.6 & \dots & 0 & 0 \\ 3.6 & 3.6 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -3.6 \\ 0 & 0 & 0 & \dots & 3.6 & 0 \end{bmatrix},$$

$$K = \omega K_0 = \omega \begin{bmatrix} 7 & 4 & 1 & \dots & 0 & 0 \\ 4 & 7 & 4 & \dots & 0 & 0 \\ 1 & 4 & 7 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 7 & 4 \\ 0 & 0 & 0 & \dots & 4 & 7 \end{bmatrix} = \begin{bmatrix} 6.3 & 3.6 & 0.9 & \dots & 0 & 0 \\ 3.6 & 6.3 & 3.6 & \dots & 0 & 0 \\ 0.9 & 3.6 & 6.3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 6.3 & 3.6 \\ 0 & 0 & 0 & \dots & 3.6 & 6.3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 10 \\ \vdots \\ 10 \\ 1 \\ 1 \end{bmatrix}.$$

The last two eigenvalues are reassigned to the circular regions and the other eigenvalues are kept unchanged. We give Table 2 to show the robust regional eigenvalue assignment results of this undamped gyroscopic system when $n = 50, 150, 250$. From Table 2, the measurement of robustness is greatly reduced by using Algorithm 2, which shows the effectiveness of our algorithm.

Table 2. The results of Example 4.4.

n	$\mu_{1,2}$	radius	$\tilde{\mu}_{1,2}$	Algorithm 1	Algorithm 2
50	$-2.0847 \pm 3.9918i$	1	$-2.3119 \pm 3.0180i$	27.4985	0.1168
150	$-2.2530 \pm 4.1488i$	1	$-2.4182 \pm 3.1625i$	21.4072	0.0871
250	$-2.4467 \pm 3.5806i$	1	$-2.2367 \pm 2.6029i$	18.4633	0.0286

5. Conclusions

The robust regional eigenvalue assignment problem using rank-one control for undamped gyroscopic systems is considered in this paper. Based on the orthogonality relation, we find the solution to the partial eigenvalue assignment problem such that partial undesired eigenvalues are reassigned accurately while keeping no spill-over property. On this basis, we assign the desired eigenvalues within specified regions to provide the necessary freedom and derive the sensitivity of closed-loop eigenvalues with respect to parameter perturbations to measure robustness. Furthermore, we propose a numerical algorithm for solving the robust regional eigenvalue assignment problem of undamped gyroscopic systems. Numerical experiments demonstrate that our method is practicable.

Author contributions

Binxin He: Material preparation, Algorithm design, Writing-original draft; Hao Liu: Conceptualization, Methodology, Writing-review and editing. Both of the authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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