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Research article

Improvement of inequalities related to powers of the numerical radius

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Abstract: We presented some improvements of the inequalities involving the numerical radius powers for products and sums of the operators investigated in the Hilbert space. We generalized and improved numerical radius inequalities with a generalization of the mixed Schwarz inequality. Among other things, with the help of a fraction and its power, as well as the introduction of ξ, we provided a very good improvement for the $\omega^r(E)$, for $E \in \mathcal{B}(\mathcal{H}_s)$.

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1. Introduction

In this article, we let $(\mathcal{H}_s; \langle .,. \rangle)$ denote a complex Hilbert space. Now, we recall several concepts and definitions that were investigated in [14]. An arbitrary operator E belongs to $\mathcal{B}(\mathcal{H}_s)$ is called positive, when $\langle Ex, x\rangle \geq 0$, $\forall x \in \mathcal{H}_s$, and surely E is self-adjoint. In other words, E will be positive if, and only if, $E = C^*C$ for several operators, and C belongs to $\mathcal{B}(\mathcal{H}_s)$. We write $E \geq 0$, if E is positive. Note that $|E|$ denotes for the considered positive operator $(E^*E)^{\frac{1}{2}}$.

For the assumed E, the numerical radius is defined as follows:

$$\omega(E) = \sup\{|\langle Ex, x\rangle| : x \in \mathcal{H}_s, \|x\| = 1\}.$$

This is obvious that $\omega(.)$ defines a norm on $\mathcal{B}(\mathcal{H}_s)$, that can be equivalent to the common operator norm $\|.\|$. Note that, $\forall E \in \mathcal{B}(\mathcal{H}_s)$, and the following inequality holds;

$$\frac{1}{2}\|E\| \leq \omega(E) \leq \|E\|. \tag{1.1}$$

More information on the numerical range and the numerical radius of bounded linear operators on complex Hilbert spaces can be found. For example, these applications can be seen in [5] written by Elin et al.

Kittaneh in [10] makes an improvement of the above inequality such that is stated in the following:

$$\begin{aligned}\omega(E) &\leq \frac{1}{2}\||E| + |E^*|\| \\ &\leq \frac{1}{2}(\|E\| + \|E^2\|^{\frac{1}{2}}), \quad \forall E \in \mathcal{B}(\mathcal{H}_s).\end{aligned} \tag{1.2}$$

Similarly, El-Haddad et al. in the research [4] investigated the following inequality as an improvement of the left-hand side of the (1.2) form as follows:

$$\omega^r(E) \leq \frac{1}{2}\left\||E|^{2r\theta} + |E^*|^{2r\nu}\right\|, \quad r \geq 1, \theta + \nu = 1. \tag{1.3}$$

We note that this inequality improves at the end of Section 2. It is our research motivation in this paper.

The purpose of this paper is to describe some new inequalities for numerical radius of the bounded and linear operator in Hilbert space $\mathcal{H}_s$.

Authors of [15] in [15, Corollary 2.1] and [15, Proposition 2.2], respectively, investigated the following sharp inequalities:

$$\begin{aligned}\omega^r(E) &\leq \left\| \int_0^1 (\tau|E| + (1-\tau)|E^*|)^r d\tau \right\| \\ &\leq \frac{1}{2}\left\||E|^r + |E^*|^r\right\|,\end{aligned} \tag{1.4}$$

and

$$\omega^r(F^*E) \leq \left\| \int_0^1 (\tau|E|^2 + (1-\tau)|F|^2)^r d\tau \right\| \leq \frac{1}{2}\left\||E|^{2r} + |F|^{2r}\right\| \tag{1.5}$$

for any $1 \leq r \leq 2$.

As our basic motivation in this research paper in Section 2, we will improve and generalize the left-hand side of inequalities (1.4) and (1.5).

Also, the following inequality has been investigated in [3, Theorem 1], which we try to improve in Section 3:

$$\begin{aligned}&\omega^2(C^*XE) \\ &\leq \frac{1}{2}\left\|X^{\frac{1}{2}}C\right\|^2\Big(\left\|\frac{|X^{\frac{1}{2}}E|^2 + |X^{\frac{1}{2}}F|^2}{2}\right\| + \omega(F^*XE)\Big),\end{aligned}$$

such that X is a positive operator on $\mathcal{H}_s$ and E, F, C in $B(\mathcal{H}_s)$.

In several particular cases, we illustrate our consequence, providing a sharper estimation for the numerical radius than the associated consequence obtained in [3].

2. Results

The essential aim of Section 2 is to give notable improvements of the inequalities (1.2), (1.4), and (1.5). In order to reach our basic purpose, we need to the following lemmas, which will be crucial in our analysis.

Lemma 2.1. *Note that if* $\theta, \nu \geq 0$ *and that real numbers* $m, M \geq 0$ *apply in the inequality* $\max\{\theta, \nu\} \geq M > m \geq \min\{\theta, \nu\}$*, then we have*

$$\frac{M+m}{2\sqrt{Mm}}\sqrt{\theta\nu} \leq \frac{\theta+\nu}{2}. \tag{2.1}$$

Proof. Put $g(x) = \frac{2\sqrt{x}}{1+x}$ on $x \geq \frac{M}{m} \geq 1$. We imply $g(x) \leq g(\frac{M}{m})$, (since $g'(x) = \frac{1-x}{\sqrt{x}(x+1)^2} \leq 0$, for $x \geq 1$), which leads to an interesting result by a simple calculation. □

Now, we introduce the Hermite–Hadamard inequalities for a convex function $h : J \to \mathbb{R}$, which looking for it

$$\begin{aligned} h\Big(\frac{c+d}{2}\Big) &\leq \int_0^1 h(\tau c + (1-\tau)d)d\tau \\ &\leq \frac{h(c)+h(d)}{2}, \end{aligned} \tag{2.2}$$

for each $c, d \in J$, J is a real interval. The following investigation can be an application of Jensen's inequality, which is found in [7].

Lemma 2.2. *Assume that h is a convex function in the real interval J involving* $sp(E)$*, which* $E = E^*$ *is an operator. Therefore, we have*

$$h(\langle Ex, x\rangle) \leq \langle h(E)x, x\rangle \tag{2.3}$$

for every unit vector $x \in \mathcal{H}_s$.

The second lemma in this section follows from the Theorem 2.3 in [1] directly.

Lemma 2.3. *Suppose that* $E, F \in \mathcal{B}(\mathcal{H}_s)$ *are positive operators and h is an increasing nonnegative convex function on the interval* $[0, \infty)$*. Therefore,*

$$||h((1-\alpha)E + \alpha F)|| \leq ||(1-\alpha)h(E) + \alpha h(F)||$$

for every $0 \leq \alpha \leq 1$.

The following inequality is well-known to be a generalized mixed Schwarz inequality:
For $E \in B(\mathcal{H}_s)$, we have

$$\begin{aligned} &|\langle Ex, y\rangle|^2 \\ &\leq \langle |E|^{2\theta}x, x\rangle\langle |E^*|^{2\nu}y, y\rangle \end{aligned} \tag{2.4}$$

for $x, y \in \mathcal{H}_s$ and $\theta + \nu = 1$, (see [11, Lemma 1]).

Next, we present a lemma that will be beneficial in Section 2.

Lemma 2.4. *[15, Proposition 2.1] Suppose that $\Phi : B(\mathcal{H}_s) \to B(\mathcal{H}_s)$ is a positive and unital linear function, $E \in \mathcal{B}(\mathcal{H}_s)$, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing operator and convex function. Hence,*

$$\begin{aligned} &h(\omega^2(\Phi(E))) \\ &\leq \left\| \Phi\Big(\int_0^1 h(\tau|E|^2 + (1-\tau)|E^*|^2) d\tau \Big) \right\|. \end{aligned}$$

In the special case, for every $r \in [1,2]$, it concludes that

$$\omega^{2r}(E) \leq \left\| \int_0^1 (\tau|E|^2 + (1-\tau)|E^*|^2)^r d\tau \right\|,$$

where $\omega(E)$ is the numerical radius of E.

Lemma 2.5. *[10, Lemma 2] If $E, F \in B(\mathcal{H}_s)$ are positive operators, then,*

$$\|E^{\frac{1}{2}} F^{\frac{1}{2}}\| \leq \|EF\|^{\frac{1}{2}}.$$

Kittaneh also proved the effect including a norm inequality for sums of positive operators and that it will be sharper than the triangle inequality, as follows.

Lemma 2.6. *Suppose that $E, F \in B(\mathcal{H}_s)$. Therefore,*

$$\begin{aligned} &\|E + F\| \\ &\leq \frac{1}{2}\Big(\|E\| + \|F\| + \sqrt{(\|E\| + \|F\|)^2 + 4\|E^{\frac{1}{2}} F^{\frac{1}{2}}\|^2} \Big), \end{aligned}$$

(see [10, Lemma 3]), and $|\langle Ex, y\rangle| \leq \langle |E|x, x\rangle^{\frac{1}{2}} \langle |E^*|y, y\rangle^{\frac{1}{2}}$ for every $x, y \in \mathcal{H}_s$, (see [10, Lemma 1]).

Notice the recent definition for the functions t^γ and $(1-\gamma) + \gamma t$, such that $\gamma \in [0,1]$ will end at the operator γ-weighted geometric mean and the operator γ-weighted arithmetic mean, respectively.

In the next theorem, we improve and generalize the inequality (1.4).

Theorem 2.7. *Assume that $E \in B(\mathcal{H}_s)$ is an operator, and also $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing operator and convex function. Suppose that real numbers $m, M \geq 0$ apply one of the conditions, as follows:*

(i) $0 \leq |E|^{2\theta} \leq mI \leq MI \leq |E^*|^{2\nu}$,
(ii) $0 \leq |E^*|^{2\nu} \leq mI \leq MI \leq |E|^{2\theta}$.

Then, for $\theta + \nu = 1$,

$$h(\omega(E)) \leq \frac{\sqrt{Mm}}{M+m} \|h(|E|^{2\theta}) + h(|E^*|^{2\nu})\| \tag{2.5}$$

for every operator $E \in \mathcal{B}(\mathcal{H}_s)$.

Proof. Let $x \in \mathcal{H}_s$ be a unit vector. We know $\frac{2\sqrt{Mm}}{M+m} \leq 1$ and h is a nonnegative operator increasing and convex function (we use the fact if h is an operator convex and increasing function, for $k \leq 1$, we reach: $h(kt) \leq kh(t)$), then for $\theta + \nu = 1$, and by using generalized mixed Schwarz inequality, we have

$$h(|\langle Ex, x\rangle|)$$

$$
\begin{aligned}
&\leq h\Big(\sqrt{\langle|E|^{2\theta}x,x\rangle\langle|E^*|^{2\nu}x,x\rangle}\Big)\\
&\leq h\Big(\frac{\sqrt{Mm}}{M+m}(\langle|E|^{2\theta}x,x\rangle+\langle|E^*|^{2\nu}x,x\rangle)\Big) \qquad \text{(Lemma 2.1)}\\
&\leq \frac{2\sqrt{Mm}}{M+m}h\Big(\frac{\langle|E|^{2\theta}x,x\rangle+\langle|E^*|^{2\nu}x,x\rangle}{2}\Big)\\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(h(\langle|E|^{2\theta}x,x\rangle)+h(\langle|E^*|^{2\nu}x,x\rangle)\Big) \text{ (inequality (2.2))}\\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle h(|E|^{2\theta})x,x\rangle)+\langle h(|E^*|^{2\nu})x,x\rangle)\Big) \text{ (inequality 2.3)}\\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle(h(|E|^{2\theta})+h(|E^*|^{2\nu}))x,x\rangle\Big).
\end{aligned}
$$

We take the supremum over $x \in \mathcal{H}_s$, $\|x\| = 1$ and imply the desired inequality (2.5). □

It should be noted that the map $h(t) = t^r$ (for $r \in [1, 2]$) is an increasing operator and convex function.

Theorem 2.8. *Suppose that $E \in \mathcal{B}(\mathcal{H}_s)$, and h is a positive increasing operator and convex function on real numbers $\mathbb{R}$. Also, we assume that positive operators $|E|^{2\theta}$ and $|E^*|^{2\nu}$ (with $\theta + \nu = 1$) and that real numbers $m, M \geq 0$ apply in the conditions (i) or (ii) as follows:*

(i) $|E|^{2\theta} \leq mI \leq MI \leq |E^*|^{2\nu}$,
(ii) $|E^*|^{2\nu} \leq mI \leq MI \leq |E|^{2\theta}$.

Then, for every $x \in \mathcal{H}_s$, we reach

$$h(\langle Ex, x\rangle) \leq \frac{\sqrt{Mm}}{M+m}\Big[\langle h(|E|^{2\theta})x,x\rangle + \langle h(|E^*|^{2\nu})x,x\rangle\Big]. \tag{2.6}$$

Specifically, for any $r \geq 1$, we reach the following inequality;

$$\omega^r(E) \leq (\frac{\sqrt{Mm}}{M+m})^r\Big\||E|^{2r\theta} + |E^*|^{2r\nu}\Big\|. \tag{2.7}$$

Proof. Assume that $E = U|E|$ is the polar decomposition of E. Using the generalized mixed Schwarz inequality (2.4) in the Hilbert space and Lemma 2.1, and also utilizing the convexity property of the function $h(t) = t^r$ (with $r \geq 1$), we reach

$$
\begin{aligned}
|\langle Ex,x\rangle| &= |\langle|E|^{\theta}x,|E|^{\nu}U^*x\rangle| &(2.8)\\
&\leq \Big\||E|^{\theta}x\Big\|.\Big\||E|^{\nu}U^*x\Big\|\\
&= \langle|E|^{2\theta}x,x\rangle^{\frac{1}{2}}\langle|E^*|^{2\nu}x,x\rangle^{\frac{1}{2}}\\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle|E|^{2\theta}x,x\rangle+\langle|E^*|^{2\nu}x,x\rangle\Big)\\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle|E|^{2\theta}x,x\rangle^r+\langle|E^*|^{2\nu}x,x\rangle^r\Big)^{\frac{1}{r}}, &(2.9)
\end{aligned}
$$

for any $x \in \mathcal{H}_s$. Now, using the Lemma 2.1, we have

$$\begin{aligned}
h\Big[\langle |E|^{2\theta}x, x\rangle^{\frac{1}{2}}\langle |E^*|^{2\nu}x, x\rangle^{\frac{1}{2}}\Big] &\leq h\Big[\frac{2\sqrt{Mm}}{M+m}\Big(\frac{\langle |E|^{2\theta}x, x\rangle + \langle |E^*|^{2\nu}x, x\rangle}{2}\Big)\Big] \\
&\leq \frac{2\sqrt{Mm}}{M+m}h\Big(\frac{\langle |E|^{2\theta}x, x\rangle + \langle |E^*|^{2\nu}x, x\rangle}{2}\Big) \qquad (2.10) \\
&\leq \frac{\sqrt{Mm}}{M+m}h\Big(\langle |E|^{2\theta}x, x\rangle) + h(\langle |E^*|^{2\nu}x, x\rangle\Big) \\
&\leq \frac{\sqrt{Mm}}{M+m}\Big[\langle h(|E|^{2\theta})x, x\rangle + \langle h(|E^*|^{2\nu})x, x\rangle\Big] \qquad (2.11) \\
&\qquad \text{(by inequality 2.3).}
\end{aligned}$$

Note that the inequality (2.10) implies the fact that if h is an increasing operator and convex function and $\gamma = \frac{2\sqrt{Mm}}{M+m} \leq 1$, then $h(\gamma t) \leq \gamma h(t)$. Therefore, by combining inequalities (2.8) and (2.11), we get the desired result (2.6).

Now, by using (2.9), the H$\acute{}$older-McCarthy inequality for the positive operators $|E|^{2\theta}$ and $|E^*|^{2\nu}$, and the convexity property of the function $h(u) = u^r$, (for $r \geq 1$), it implies that

$$\begin{aligned}
&\frac{\sqrt{Mm}}{M+m}\Big(\langle |E|^{2\theta}x, x\rangle^r + \langle |E^*|^{2\nu}x, x\rangle^r\Big)^{\frac{1}{r}} \qquad (2.12) \\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle |E|^{2r\theta}x, x\rangle + \langle |E^*|^{2r\nu}x, x\rangle\Big)^{\frac{1}{r}} \\
&= \frac{\sqrt{Mm}}{M+m}\Big(\langle (|E|^{2r\theta} + |E^*|^{2r\nu})x, x\rangle\Big)^{\frac{1}{r}}
\end{aligned}$$

for all $x \in \mathcal{H}_s$ such that $\|x\| = 1$.

We combine inequalities (2.8) and (2.12) and get

$$|\langle Ex, x\rangle|^r \leq (\frac{\sqrt{Mm}}{M+m})^r\langle (|E|^{2r\theta} + |E^*|^{2r\nu})x, x\rangle,$$

for every unit vector $x \in \mathcal{H}_s$. Because the operator $|E|^{2r\theta} + |E^*|^{2r\nu}$ is self-adjoint, we take the supremum over unit vector $x \in \mathcal{H}_s$, and it results in the desired inequality, as follows:

$$\omega^r(E) \leq (\frac{\sqrt{Mm}}{M+m})^r\Big\||E|^{2r\theta} + |E^*|^{2r\nu}\Big\|.$$

□

This inequality improves (1.3). Also, we can improve this inequality by utilizing the following lemma.

We introduce some improvements of the famous inequalities, such as H$\acute{}$older-McCarthy's inequality, in the following lemma.

Lemma 2.9. *([9, corollary 3.1]) Assume that E is a positive operator on $\mathcal{H}_s$. If $x \in \mathcal{H}_s$ with $\|x\| = 1$, then,*

$$\langle Ex,x\rangle^r \leq \langle E^r x,x\rangle - \langle |E-\langle Ex,x\rangle|^r x,x\rangle, \quad for\, r\geq 2.$$

Proposition 2.10. *Under the assumption of Theorem 2.8, if we apply Lemma 2.9, then we get to the following inequality, which is an improvement of inequality (2.7):*

$$\omega^r(E) \leq (\frac{\sqrt{Mm}}{M+m})^r \left\| |E|^{2r\theta} + |E^*|^{2r\nu} \right\| - \inf_{\|x\|=1} \xi(x),$$

such that

$$\xi(x) = (\frac{\sqrt{Mm}}{M+m})^r \left\langle (\left| |E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle \right|^r + \left| |E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle \right|^r)x,x \right\rangle.$$

Proof. We know $\left||E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle\right|^r + \left||E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle\right|^r$ is a positive operator, then $\left\langle (\left||E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle\right|^r + \left||E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle\right|^r)x,x\right\rangle \geq 0$, and so $\inf \xi(x) \geq 0$, for $x \in \mathcal{H}_s$ with $\|x\| = 1$. From inequalities (2.8) and (2.9), applying Lemma 2.9 for the positive operator $|E|^{2\theta}$ and $|E^*|^{2\nu}$ and convexity property of the function $h(u) = u^r$ (for $r \geq 2$) implies that

$$\begin{aligned}
|\langle Ex,x\rangle| &\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle |E|^{2\theta}x,x\rangle^r + \langle |E^*|^{2\nu}x,x\rangle^r\Big)^{\frac{1}{r}} \\
&\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle |E|^{2r\theta}x,x\rangle + \langle |E^*|^{2r\nu}x,x\rangle \\
&\quad - \left\langle \left||E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle\right|^r x,x\right\rangle - \left\langle \left||E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle\right|^r x,x\right\rangle\Big)^{\frac{1}{r}} \\
&= \frac{\sqrt{Mm}}{M+m}\Big(\left\langle (|E|^{2r\theta} + |E^*|^{2r\nu} - \left||E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle\right|^r - \left||E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle\right|^r)x,x\right\rangle\Big)^{\frac{1}{r}},
\end{aligned}$$

therefore,

$$\begin{aligned}
|\langle Ex,x\rangle|^r &\leq (\frac{\sqrt{Mm}}{M+m})^r \left\langle (|E|^{2r\theta} + |E^*|^{2r\nu})x,x\right\rangle \\
&\quad - (\frac{\sqrt{Mm}}{M+m})^r \left\langle (\left||E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle\right|^r + \left||E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle\right|^r)x,x\right\rangle,
\end{aligned}$$

for every $x \in \mathcal{H}_s$ such that $\|x\| = 1$. Now, we take supremum over unit vector $x \in \mathcal{H}_s$, resulting in the following interest inequality:

$$\omega^r(E) \leq (\frac{\sqrt{Mm}}{M+m})^r \left\| |E|^{2r\theta} + |E^*|^{2r\nu} \right\| - \inf_{\|x\|=1} \xi(x),$$

where

$$\xi(x) = (\frac{\sqrt{Mm}}{M+m})^r \left\langle (\left| |E|^{2\theta} - \langle |E|^{2\theta}x,x\rangle \right|^r + \left| |E^*|^{2\nu} - \langle |E^*|^{2\nu}x,x\rangle \right|^r)x,x \right\rangle.$$

This inequality improves inequality (2.7). □

Theorem 2.11. *Suppose that $E \in B(\mathcal{H}_s)$ and that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing operator and convex function, and assume that the real numbers $m, M \geq 0$ apply in the conditions (i) or (ii) as follows:*

(i) $0 \leq |E|^{2\theta} \leq mI \leq MI \leq F^*|E^*|^{2\nu}F$,

(ii) $0 \leq F^*|E^*|^{2\nu}F \leq mI \leq MI \leq |E|^{2\theta}$.

Hence, for $\theta + \nu = 1$,

$$h(\omega(F^*E)) \leq \frac{2\sqrt{Mm}}{M+m} \sup\Big(\int_0^1 h\Big(\Big\|(t|E|^{2\theta} + (1-t)F^*|E^*|^{2\nu}F)^{\frac{1}{2}}x\Big\|^2\Big)dt\Big), \tag{2.13}$$

for every unit vector $x \in \mathcal{H}_s$.

Proof. Lemma 2.1 and the generalized mixed Schwarz inequality (2.4) imply that

$$\begin{aligned} h(|\langle F^*Ex, x\rangle|) &= h(|\langle Ex, Fx\rangle|) \\ &\leq h(\sqrt{\langle |E|^{2\theta}x, x\rangle\langle F^*|E^*|^{2\nu}Fx, x\rangle}) \\ &\leq h\Big(\frac{\sqrt{Mm}}{M+m}(\langle |E|^{2\theta}x, x\rangle + \langle F^*|E^*|^{2\nu}Fx, x\rangle)\Big) \\ &\leq \frac{2\sqrt{Mm}}{M+m} h\Big(\frac{\langle |E|^{2\theta}x, x\rangle + \langle F^*|E^*|^{2\nu}Fx, x\rangle}{2}\Big). \end{aligned} \tag{2.14}$$

We know the fact that if h is an increasing operator and convex function and $\gamma = \frac{2\sqrt{Mm}}{M+m} \leq 1$, then $h(\gamma t) \leq \gamma h(t)$, so the inequality (2.14) follows from it.

Now, we take $a = \langle |E|^{2\theta}x, x\rangle$ and $b = \langle F^*|E^*|^{2\nu}Fx, x\rangle$ in the inequality (2.2), where $x \in \mathcal{H}_s$, $\|x\| = 1$ is a vector. Therefore,

$$h\Big(\frac{\langle |E|^{2\theta}x, x\rangle + \langle F^*|E^*|^{2\nu}Fx, x\rangle}{2}\Big) \leq \int_0^1 h\Big(\Big\langle (t|E|^{2\theta} + (1-t)F^*|E^*|^{2\nu}F)x, x\Big\rangle\Big)dt. \tag{2.15}$$

Next, combining inequalities (2.14) and (2.15) implies that

$$\begin{aligned} h(|\langle F^*Ex, x\rangle|) &\leq \frac{2\sqrt{Mm}}{M+m} \int_0^1 h\Big(\Big\langle (t|E|^{2\theta} + (1-t)F^*|E^*|^{2\nu}F)x, x\Big\rangle\Big)dt \\ &\leq \frac{2\sqrt{Mm}}{M+m} \int_0^1 h\Big(\Big\|(t|E|^{2\theta} + (1-t)F^*|E^*|^{2\nu}F)x\Big\|\Big)dt \end{aligned}$$

for each unit vector $x \in \mathcal{H}_s$. Then, we take the supremum over unit vector $x \in \mathcal{H}_s$, resulting in the interest inequality (2.13). □

In the next result, by using Lemma 2.4, we try to improve and generalize inequality (1.5).

Corollary 2.12. *Consider the assumption of Theorem 2.11, and take $\varphi = I$ in Lemma 2.4. Hence, for $\theta + \nu = 1$,*

$$h(\omega(F^*E)) \leq \tfrac{2\sqrt{Mm}}{M+m}\Big\|\int_0^1 h(t|E|^{2\theta} + (1-t)F^*|E^*|^{2\nu}F)dt\Big\|.$$

Especially, for every $r \in [1, 2]$, it implies that

$$\omega^r(F^*E)) \leq \tfrac{2\sqrt{Mm}}{M+m}\Big\|\int_0^1 (t|E|^{2\theta} + (1-t)F^*|E^*|^{2\nu}F)^r dt\Big\|.$$

3. Some refinements inequalities related to product operators

In this section, with four operators, we create several new numerical radius inequalities on a Hilbert space. Also, several special cases that generalize and improve upon a previous result are presented.

In the first theorem, as we said, we will improve inequalities related to the product of operators (see [3]).

Theorem 3.1. *Assume that $E, F, C, X \in \mathcal{B}(\mathcal{H}_s)$, and X is a positive operator where $F^*XC = C^*XE$. Suppose that the real numbers $m, M \geq 0$ apply in the conditions (i) or (ii) as follows:*

(i) $E^*XE \leq mI \leq MI \leq F^*XF$,
(ii) $F^*XF \leq mI \leq MI \leq E^*XE$.

Hence,

$$\begin{aligned}&\omega^2(C^*XE)\\&\leq \frac{1}{2}\left\|X^{\frac{1}{2}}C\right\|^2\Big(\Big\|\frac{\sqrt{Mm}}{M+m}(|X^{\frac{1}{2}}E|^2 + |X^{\frac{1}{2}}F|^2)\Big\| + \omega(F^*XE)\Big)\end{aligned}$$

Proof. We use inequality (2.6) of ([3, Theorem 8]), and have

$$\begin{aligned}\frac{1}{2}\Big(\langle E^*XEx, x\rangle^{\frac{1}{2}}\langle F^*XFx, x\rangle^{\frac{1}{2}} + |\langle F^*XEx, x\rangle|\Big)\langle C^*XCx, x\rangle&\\ \geq |\langle x, E^*XCx\rangle\langle x, F^*XCx\rangle|.&\end{aligned} \tag{3.1}$$

for every x in $\mathcal{H}_s$. We have $F^*XC = C^*XE = (E^*XC)^*$. Hence,

$$\begin{aligned}&|\langle x, E^*XCx\rangle\langle x, F^*XCx\rangle|\\&= |\langle x, E^*XCx\rangle\langle x, (E^*XC)^*x\rangle|\\&= |\langle E^*XCx, x\rangle|^2\\&= |\langle C^*XEx, x\rangle|^2\end{aligned} \tag{3.2}$$

for all x in $\mathcal{H}_s$. Using inequality (3.1) and equality (3.2) leads to

$$\begin{aligned}&|\langle C^*XEx, x\rangle|^2\\&\leq \frac{1}{2}\Big[\langle E^*XEx, x\rangle^{\frac{1}{2}}\langle F^*XFx, x\rangle^{\frac{1}{2}} + |\langle F^*XEx, x\rangle|\Big] \times \langle C^*XCx, x\rangle\end{aligned} \tag{3.3}$$

for all x in $\mathcal{H}_s$. Also, utilizing Lemma 2.1, we create

$$\begin{aligned}\langle E^*XEx, x\rangle^{\frac{1}{2}}\langle F^*XFx, x\rangle^{\frac{1}{2}} &\leq \frac{\sqrt{Mm}}{M+m}\Big(\langle E^*XEx, x\rangle + \langle F^*XFx, x\rangle\Big)\\&= \Big\langle\frac{\sqrt{Mm}}{M+m}(E^*XE + F^*XF)x, x\Big\rangle\end{aligned}$$

for all x in $\mathcal{H}_s$. Therefore, by (3.3), we reach

$$|\langle C^*XEx, x\rangle|^2 \leq \frac{1}{2}\Big(\langle\frac{\sqrt{Mm}}{M+m}(E^*XE + F^*XF)x, x\rangle + |\langle F^*XEx, x\rangle|\Big) \times \langle C^*XCx, x\rangle.$$

In particular, because

$$E^*XE = |X^{\frac{1}{2}}E|^2,$$
$$F^*XF = |X^{\frac{1}{2}}F|^2, C^*XC = |X^{\frac{1}{2}}C|^2,$$

we have

$$\begin{aligned}&|\langle C^*XEx, x\rangle|^2\\ &\leq \frac{1}{2}\Big(\langle \frac{\sqrt{Mm}}{M+m}(|X^{\frac{1}{2}}E|^2 + |X^{\frac{1}{2}}F|^2)x, x\rangle + |\langle F^*XEx, x\rangle|\Big) \times \langle |X^{\frac{1}{2}}C|^2x, x\rangle \end{aligned} \tag{3.4}$$

for all x in $\mathcal{H}_s$. Hence, we take the supremum in (3.4) over the unit vector x in $\mathcal{H}_s$, implying the desired inequality, as follows:

$$\omega^2(C^*XE) \leq \frac{1}{2}\Big\|X^{\frac{1}{2}}C\Big\|\Big(\Big\|\frac{\sqrt{Mm}}{M+m}(|X^{\frac{1}{2}}E|^2 + |X^{\frac{1}{2}}F|^2)\Big\| + \omega(F^*XE)\Big).$$

□

Corollary 3.2. *By putting $X = I$ in Theorem 3.1, we get*

$$\begin{aligned}&\omega^2(C^*E)\\ &\leq \frac{1}{2}\|C\|^2\Big(\Big\|\frac{\sqrt{Mm}}{M+m}(|E|^2 + |F|^2)\Big\| + \omega(F^*E)\Big).\end{aligned}$$

Next, using the following lemma leads to obtaining the improved inequalities.

Lemma 3.3. *[13, page 5] Suppose that h on $[a, b]$ is twice differentiable. Assume that h is a convex function such that $h'' \geq \lambda.I$ (where $\lambda := \min_{x\in[a,b]} h(x) > 0$). Hence,*

$$\begin{aligned}&h\Big(\frac{a+b}{2}\Big)\\ &\leq \frac{h(a)+h(b)}{2} - \frac{1}{8}\lambda(b-a)^2\\ &\leq \frac{h(a)+h(b)}{2}.\end{aligned} \tag{3.5}$$

Below is an extension of Furuta's inequality given by Dragomir:

$$|\langle DCFEx, y\rangle|^2 \leq \langle E^*|F|^2Ex, x\rangle\langle D|C^*|^2D^*y, y\rangle, \tag{3.6}$$

for all E, F, C, D in $B(\mathcal{H}_s)$ and every vector $x, y \in \mathcal{H}_s$. The last inequality (3.6) becomes an equality if, and only if, the vectors FEx and C^*D^*y are linearly dependent (see [2]).

Theorem 3.4. *Assume that $E, F, C, D \in B(\mathcal{H}_s)$ are operators and h on $\mathbb{R}$ is an increasing and positive operator and convex function, and notice that h is twice differentiable where $h'' \geq \lambda.I > 0$. Suppose that the real numbers $m, M \geq 0$ apply in the conditions (i) or (ii) as follows:*

(i) $E^*|F|^2E \leq mI \leq MI \leq D|C^*|^2D^*$,

(ii) $D|C^*|^2D^* \le mI \le MI \le E^*|F|^2E$.

Hence, for all x in $\mathcal{H}_s$, results that

$$\begin{aligned} & h(|\langle DCFEx, y\rangle|) \\ & \le \frac{\sqrt{Mm}}{M+m}[\langle h(E^*|F|^2E)x, x\rangle + \langle h(D|C^*|^2D^*)y, y\rangle] \\ & \quad - \frac{1}{8}\lambda(\langle E^*|F|^2Ex, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2, \end{aligned} \tag{3.7}$$

for all x, y in $\mathcal{H}_s$.

Proof. In inequality (3.6), since h is an increasing function, utilizing the convexity and monotocity of h, leads to

$$\begin{aligned} & h(|\langle DCFEx, y\rangle|) \\ & \le h\Big(\langle E^*|F|^2Ex, x\rangle^{\frac{1}{2}}\langle D|C^*|^2D^*y, y\rangle^{\frac{1}{2}}\Big) \\ & \le h\Big(\frac{\sqrt{Mm}}{M+m}(\langle E^*|F|^2Ex, x\rangle + \langle D|C^*|^2D^*y, y\rangle)\Big) \qquad \text{(Lemma 2.1)} \\ & \le \frac{2\sqrt{Mm}}{M+m}h\Big(\frac{\langle E^*|F|^2Ex, x\rangle + \langle D|C^*|^2D^*y, y\rangle}{2}\Big) \\ & \le \frac{2\sqrt{Mm}}{M+m}.\frac{h(\langle E^*|F|^2Ex, x\rangle) + h(\langle D|C^*|^2D^*y, y\rangle)}{2} \qquad \text{(Lemma 3.3)} \\ & \quad - \frac{1}{8}\lambda(\langle E^*|F|^2Ex, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2 \\ & \le \frac{\sqrt{Mm}}{M+m}[\langle h(E^*|F|^2E)x, x\rangle + \langle h(D|C^*|^2D^*)y, y\rangle] \qquad \text{(inequality 2.3)} \\ & \quad - \frac{1}{8}\lambda(\langle E^*|F|^2Ex, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2, \end{aligned} \tag{3.8}$$

for every vectors x, y in $\mathcal{H}_s$, which proves the interest inequality. Note that the inequality (3.8) implies the fact that h is an increasing operator and convex function and $\gamma = \frac{2\sqrt{Mm}}{M+m} \le 1$, then, we have $h(\gamma t) \le \gamma h(t)$. ∎

Corollary 3.5. *Assume that $P \in B(\mathcal{H}_s)$ and h on $\mathbb{R}$ is a positive increasing operator and convex function. Also, let h be twice differentiable where $h'' \ge \lambda.I > 0$. Suppose that the real numbers $m, M \ge 0$ apply in the conditions (i) or (ii) as follows:*

(i) $|P|^{2\theta} \le mI \le MI \le |P^*|^{2\nu}$,
(ii) $|P^*|^{2\nu} \le mI \le MI \le |P|^{2\theta}$.

Then,

$$\begin{aligned} & h\Big(\Big|\Big\langle P|P|^{\theta+\nu-1}x, y\Big\rangle\Big|\Big) \\ & \le \frac{\sqrt{Mm}}{M+m}\Big[\langle h(|P|^{2\theta})x, x\rangle + \langle h(|P^*|^{2\nu})y, y\rangle\Big] - \frac{1}{8}\lambda\Big(\langle |P|^{2\theta}x, x\rangle - \langle |P^*|^{2\nu}y, y\rangle\Big)^2, \end{aligned} \tag{3.9}$$

for each x, y in $\mathcal{H}_s$, and every $0 \le \theta, \nu \le 1$ such that $\theta + \nu \ge 1$.

Proof. We replace D by U, C by $|P|^{\nu}$, F by $1_{\mathcal{H}}$, and E by $|P|^{\theta}$ in (3.7) to conclude that

$$\begin{aligned} DCFE &= U|P|^{\nu}|P|^{\theta} \\ &= U|P||P|^{\theta+\nu-1} = P|P|^{\theta+\nu-1}, \end{aligned}$$

In addition, since $E^*|F|^2E = |P|^{2\theta}$ and $D|C^*|^2D^* = U|P|^{2\nu}U^* = |P|^{2\nu}$, then we achieved the desired inequality (3.9). □

Remark 3.6. Under the assumption of Corollary 3.5, let h be twice differentiable, where $h'' \geq \lambda I > 0$, and by taking $\theta + \nu = 1$ implies the following result:

$$\begin{aligned} & h(\langle Px, y\rangle) \\ & \leq \frac{\sqrt{Mm}}{M+m}\Big[\langle h(|P|^{2\theta})x, x\rangle + \langle h(|P^*|^{2\nu})y, y\rangle\Big] \\ & \quad - \frac{1}{8}\lambda\Big(\langle |P|^{2\theta}x, x\rangle - \langle |P^*|^{2\nu}y, y\rangle\Big)^2. \end{aligned}$$

Lemma 3.7. *([12, Lemma 2.1]) (H'older-McCarthy inequality) Suppose that E in $B(\mathcal{H}_s)$ is a positive operator. Therefore, two inequalities are established as follows:*

$$\langle E^r x, x\rangle \geq \|x\|^{2(1-r)}\langle Ex, x\rangle^r, \qquad for\, x \in \mathcal{H}_s,\ r \geq 1. \tag{3.10}$$

$$\langle E^r x, x\rangle \leq \|x\|^{2(1-r)}\langle Ex, x\rangle^r, \qquad for\, x \in \mathcal{H}_s,\ 0 \leq r \leq 1. \tag{3.11}$$

In addition, in [6, Lemma 2.4] asserts that, for $a, b \geq 0$, $r \neq 0$, and $\theta \in [0, 1]$,

$$a^{\theta}b^{(1-\theta)} \leq (\theta a^r + (1-\theta)b^r)^{\frac{1}{r}}, \qquad for\ r > 0. \tag{3.12}$$

Theorem 3.8. *Assume that E, F in $B(\mathcal{H}_s)$ and $r, r' \geq 1$ and $0 \leq \theta \leq 1$. Suppose that the real numbers $m, M \geq 0$ apply in the conditions (i) or (ii) as follows:*

(i) $|E|^{2\theta} \leq mI \leq MI \leq |F|^{2\theta}$,
(ii) $|F|^{2\theta} \leq mI \leq MI \leq |E|^{2\theta}$.

Hence, for each vector $x \in \mathcal{H}_s$, $\|x\| = 1$,

$$|\langle Ex, x\rangle|^r|\langle Fx, x\rangle|^{r'} \leq \frac{\sqrt{Mm}}{M+m}(\|\theta|E|^{2r} + (1-\theta)|E^*|^{2r} + \theta|F|^{2r'} + (1-\theta)|F^*|^{2r'}\|).$$

Proof. Let x in $\mathcal{H}_s$ be an arbitrary vector, $\|x\| = 1$, hence,

$$\begin{aligned} & \Big|\langle Ex, x\rangle\Big|^r\Big|\langle Fx, x\rangle\Big|^{r'} \\ & \leq (\langle |E|^{2\theta}x, x\rangle\langle |E^*|^{2(1-\theta)}x, x\rangle)^{\frac{r}{2}}(\langle |F|^{2\theta}x, x\rangle\langle |F^*|^{2(1-\theta)}x, x\rangle)^{\frac{r'}{2}} \quad \text{(inequality 2.4)} \\ & \leq \frac{\sqrt{Mm}}{M+m}((\langle |E|^{2\theta}x, x\rangle\langle |E^*|^{2(1-\theta)}x, x\rangle)^r + (\langle |F|^{2\theta}x, x\rangle\langle |F^*|^{2(1-\theta)}x, x\rangle)^{r'}) \quad \text{(Lemma 2.1)} \\ & \leq \frac{\sqrt{Mm}}{M+m}((\langle |E|^{2\theta r}x, x\rangle\langle |E^*|^{2(1-\theta)r}x, x\rangle) + (\langle |F|^{2\theta r'}x, x\rangle\langle |F^*|^{2(1-\theta)r'}x, x\rangle)) \quad \text{(inequality 3.10)} \end{aligned}$$

$$\leq \frac{\sqrt{Mm}}{M+m}(\langle |E|^{2r}x, x\rangle^{\theta}\langle |E^*|^{2r}x, x\rangle^{(1-\theta)} + \langle |F|^{2r'}x, x\rangle^{\theta}\langle |F^*|^{2r'}x, x\rangle^{(1-\theta)}) \quad ((\text{inequality } 3.11))$$
$$\leq \frac{\sqrt{Mm}}{M+m}(\langle (\theta|E|^{2r} + (1-\theta)|E^*|^{2r})x, x\rangle + \langle (\theta|F|^{2r'} + (1-\theta)|F^*|^{2r'})x, x\rangle) \quad (\text{inequality } 3.12)$$
$$\leq \frac{\sqrt{Mm}}{M+m}(\|\theta|E|^{2r} + (1-\theta)|E^*|^{2r} + \theta|F|^{2r'} + (1-\theta)|F^*|^{2r'}\|).$$

□

Therefore, we provide a sharp estimate for the spectral radius of several operators.

Corollary 3.9. *Assume that X in $B(\mathcal{H}_s)$ and $r(X)$ specifies the spectral radius of X. Hence,*

$$r(X) \leq$$
$$\frac{\sqrt[8]{Mm}}{\sqrt[4]{M+m}}\left\| |X^2|^2 + |(X^*)^2|^2 \right\|^{\frac{1}{4}}$$
$$\leq \|X\|.$$

Proof. Put $\theta = \frac{1}{2}$, $E = F = X^2$, and $r = r' = 1$ in Theorem 3.8. We earn

$$\omega(X^2) \leq \frac{\sqrt[4]{Mm}}{\sqrt{M+m}}\left\| |X^2|^2 + |(X^*)^2|^2 \right\|^{\frac{1}{2}}.$$

Therefore, it is obtained that

$$r^2(X) = r(X^2) \leq \omega(X^2) \leq \frac{\sqrt[4]{Mm}}{\sqrt{M+m}}\left\| |X^2|^2 + |(X^*)^2|^2 \right\|^{\frac{1}{2}}.$$

Then,

$$r(X) \leq \frac{\sqrt[8]{Mm}}{\sqrt[4]{M+m}}\left\| |X^2|^2 + |(X^*)^2|^2 \right\|^{\frac{1}{4}} \leq \|X\|.$$

□

4. Conclusions

In this article, we have worked on several numerical radius inequalities, and we have improved and refined them. Using other articles such as [8] and [16] and other new coefficients for improvement and elaboration, we will present new inequalities of numerical radius in new articles.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest in publishing this paper.

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