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Research article

A novel nonzero functional method to extended dissipativity analysis for neural networks with Markovian jumps

Wenlong Xue[1,*], **Yufeng Tian**[2] **and Zhenghong Jin**[3,4]

[1] Internet of Things Department, Henan Institute of Economics and Trade, Zhengzhou 450000, China
[2] College of Automation, Chongqing University, 400044, China
[3] School of Electrical and Electronic Engineering, Nanyang Technological University, 639798, Singapore
[4] College of Control Science and Engineering, Zhejiang University, Hangzhou 310058, China

* **Correspondence:** Email: chanel2048@163.com.

Abstract: This paper explored the topic of extended dissipativity analysis for Markovian jump neural networks (MJNNs) that were influenced by time-varying delays. A distinctive Lyapunov functional, distinguished by a non-zero delay-product types,was presented. This was achieved by combining a Wirtinger-based double integral inequality with a flexible matrix set. This novel methodology addressed the limitations of the slack matrices found in earlier research. As a result, a fresh condition for extended dissipativity in MJNNs was formulated, utilizing an exponential type reciprocally convex inequality in conjunction with the newly introduced nonzero delay-product types. A numerical example was included to demonstrate the effectiveness of the proposed methodology.

Keywords: neural networks; Markovian jumps; extended dissipativity; nonzero delay-product-type functional
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1. Introduction

Neural networks (NNs) are rooted in the emulation of the human brain's intricate architecture. These computational models have been effectively employed across various domains, including medical diagnostics, the financial industry, and signal processing [1, 2], in which the versatility and efficacy underscore the transformative potential of NNs in advancing these fields. Stability holds significant importance as a prerequisite in various applications [3]. Nevertheless, the implementation of (large-scale) neural networks unavoidably introduces time-varying delays (TVDs) arising from finite amplifier switching speeds and inter-neuron communication rates. These delays can potentially lead to instability and oscillations within the NN [4, 5]. Therefore, analyzing the dynamic performance of NNs with TVDs is a meaningful topic [6–8].

Additionally, abrupt parameter changes in the structures of many NNs may occur due to sudden environmental changes, components or interconnection failures. These NNs can exhibit distinct modes that transition or switch unpredictably between each other, and such switching can be determined by a Markov chain, leading to MJNNs [9, 10]. It is noted that, dissipativity, recognized as a critical dynamic performance metric, has been proven to be a potent tool for the analysis of nonlinear systems, particularly when leveraging energy-related input-output characteristics, as highlighted in [11]. While the traditional notion of dissipativity encompasses both passivity and H_∞ performances, it falls short in addressing the $l_2 - l_\infty$ performance. Addressing this lacuna, the seminal work in [12] marked the first stride toward reconciling this gap. Consequently, the concept of "extended dissipativity" emerged, which has been widely applied to practical systems, presenting a more comprehensive performance index [13, 14]. This innovative index effectively amalgamates the features of passivity, traditional dissipativity, H_∞ performance, and the $l_2 - l_\infty$ performance. To further harness the potential and expand the applicability of MJNNs, there's been an increasing scholarly focus on this domain. For example, the delay-product-type (DPT) functional approach is presented to analyze MJNNs [15–17]. Nevertheless, the matrices that were introduced had a constraint imposed upon them, requiring them to be positive definite. This constraint was imposed to guarantee the positive definiteness of the Lyapunov function, which in turn led to certain limitations in the conditions established. In response to this concern, an attempt was made in [18] to mitigate the situation. This involved creating two innovative DPT functionals, aiming to support the extended dissipativity analysis of MJNNs. Nevertheless, it's worth noting that only the DPT Lyapunov functional based on a single integral was developed in [18]. This, in turn, prompted the proposal of an enhanced approach in [19]. In this later work, a double integral-based DPT (DIDPT) functional was introduced. This new functional featured augmented forms of double integrals to provide a more comprehensive solution. The TVDs-dependent components in the DIDPT are efficient for capturing TVDs information but suffer from incomplete zero components, limiting their utilization in system information. This insufficiency motivates the current study.

Moreover, the traditional method for estimating the integral term $-\int_{t-\varsigma}^{t} \dot{\S}^T(s)R\dot{\S}(s)ds$ involves handling the TVDs in the denominators using the traditional reciprocally convex inequality (TRCI) [20]. To improve the conservatism of the TRCI, an improved version known as the improved reciprocally convex inequality (IRCI) was proposed in [21, 22]. Furthermore, to achieve even less conservative results without introducing additional decision variables, the exponential type reciprocally convex inequality (ETRCI) is first introduced in the exponential-type parameters introduced in [23], which quickly approaches the left-hand side of the inequality and also covers [21, 22] as special cases. In the existing works, there have been no related studies exploring the use of ETRCI for reducing conservatism in the analysis of MJNNs, warranting further investigation in this direction.

Building upon the preceding analysis, this paper directs its attention toward the extended dissipativity analysis concerning MJNNs that exhibit TVDs. Through the application of the S-procedure lemma, coupled with the utilization of adaptable matrices, a fresh approach is formulated. Drawing upon the ETRCI and the innovative DIDPT functional, a novel criterion is introduced. This criterion serves to ensure both the stochastic stability and extended dissipativity of MJNNs. The main contributions are as follows:

- By employing the S-procedure lemma and integrating the use of adaptable matrices, we propose the DIDPT functional, which overcomes the zero components in [19]. As a result, the newly formulated DIDPT functional optimally captures the intricate interplay of coupling dynamics among states and TVD, offering a robust solution in the realm of state dynamics.

- Drawing upon the ETRCI and the innovative DIDPT functional, a novel criterion is introduced. This criterion serves to ensure both the stochastic stability and extended dissipativity of MJNNs.

Notation. In this paper, $\mathbb{R}^l$ means the l-dimensional Euclidean space; R^T denotes the transpose of R; '*' represents the symmetric term in linear matrix inequalities (LMIs); He[R] represents $R + R^T$; diag{...} and col[...] denote a block-diagonal matrix and a block-column vector, respectively; Co{...} represents a convex set; $\mathcal{E}[\cdot]$ represents the expectation operator; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ represent the maximum and minimum eigenvalue of X, respectively.

2. Problem formulations

Consider the ensuing MJNNs exhibiting TVDs:

$$\begin{cases} \dot{\S}(t) = -A(r_t)\S(t) + W_0(r_t)\pi(W_2\rho(t)) + W_1(r_t)\pi(W_2\S(t-\sigma(t))) + B_1\rho(t), \\ \dagger(t) = C\S(t) + D_1\S(t-\sigma(t)) + D_2\pi(W_2\S(t)) + B_2\rho(t), \end{cases} \tag{2.1}$$

where the state vector $\S(t) \in \mathbb{R}^n$, the disturbance input $w(t) \in \mathbb{R}^w$, and the output $\dagger(t) \in \mathbb{R}^y$ are defined for the system. The disturbance input $\rho(t)$ belongs to $\mathcal{L}_2[0, +\infty)$. Additionally, the system parameters include the known interconnection weight matrices $A(r_t)$, $W_0(r_t)$, $W_1(r_t)$, B_1, B_2, C, D_1, D_2, and $W_2 = col[W_{21}, W_{22}, \cdots, W_{2n}]$. The Markov chain $r_t(t \geq 0)$, which operates for t greater than or equal to zero, exhibits right continuity and takes on values within a finite set $S = \{1, 2, ..., m\}$. The transition rate matrix (TRM), denoted as Π and defined as $[\tau_{ij}]$, characterizes this chain for $\delta > 0$, $\lim\limits_{\delta \to 0} o(\delta)/\delta = 0$, $\tau_{ij} \geq 0$, $\tau_{ii} = -\sum\limits_{j=1, j\neq i}^{m} \tau_{ij}(j \neq i)$, and

$$Pr\{r_{t+\delta} = j | r_t = i\} = \begin{cases} \tau_{ij}\delta + o(\delta), & j \neq i, \\ 1 + \tau_{ii}\delta + o(\delta), & j = i. \end{cases} \tag{2.2}$$

The TVD $\sigma(t)$ is subject to the following constraints:

$$0 \leq \sigma(t) \leq \varsigma, \quad -\nu \leq \dot{\sigma}(t) \leq \nu, \tag{2.3}$$

where ς and ν are constants. For constants k_l^- and $k_l^+(l = 1, 2, \cdots, n)$, the neuron activation function $\pi(W_2(\S(t)))$ satisfies the following conditions:

$$k_l^- \leq \frac{\pi_l(a_1) - \pi_l(a_2)}{a_1 - a_2} \leq k_l^+, a_1 \neq a_2, \tag{2.4}$$

$$k_l^- \leq \frac{\pi_l(a)}{a} \leq k_l^+, a \neq 0. \tag{2.5}$$

In this paper, a less conservative sufficient condition will be proposed to ensure the extended dissipativity of the MJNNs (2.1). To accomplish this, the subsequent preconditions are established.

From the standpoint of practical application, the extended disspativity can embody more performance indices for physical systems such as H_∞, passivity, dissipativity, and $l_2 - l_\infty$ performance [13, 14]. The definition is given as follows:

Definition 1. *[12] MJNNs (2.1) are considered extended dissipative if there exist prescribed matrices Ψ_1, Ψ_2, Ψ_3, and Ψ_4, satisfying*

$$\Psi_1 = \Psi_1^T \leq 0, \;\; \Psi_3 = \Psi_3^T, \;\; \Psi_4 = \Psi_4^T, \tag{2.6}$$
$$B_2^T \Psi_1 B_2 + He[B_2^T \Psi_2] + \Psi_3 > 0, \tag{2.7}$$
$$(\|\Psi_1\| + \|\Psi_2\|) \cdot \|\Psi_4\| = 0, \tag{2.8}$$

and a scalar ϱ can be found such that the following condition holds:

$$\int_0^T (\dagger^T(t)\Psi_1\dagger(t) + 2\dagger^T(t)\Psi_2\rho(t) + \rho^T(t)\Psi_3\rho(t))dt \geq \rho^T(t)\Psi_4\dagger(t) + \varrho. \tag{2.9}$$

Lemma 1. *[24] Let us denote the set $G = \{g\}$, and introduce the notations: $F(g)$, $X_1(g)$, $X_2(g)$, ..., $X_k(g)$ are defined as functionals or functions. The domain H is specified as*

$$H = \{g \in Z : X_1(g) \geq 0, X_2(g) \geq 0, ..., X_k(g) \geq 0\}.$$

We proceed to examine two specific conditions:
(I) The initial condition states that $F(g) \geq 0$ holds for every $g \in H$.
(II) Additionally, we consider the existence of nonnegative values $\lambda_1 \geq 0, \lambda_2 \geq 0, ..., \lambda_k \geq 0$ such that the inequality holds:

It can be established that condition (II) implies condition (I).

Lemma 2. *[23] For real constants $\beta \in (0, 1)$, $\kappa_1 \geq 0$, $\kappa_2 \geq 0$, and real symmetric positive definite matrices R_1, $R_2 \in \mathbb{R}^{m\times m}$, the matrix inequality below is satisfied in the presence of a real matrix $S \in \mathbb{R}^{m\times m}$:*

$$\begin{bmatrix} \frac{1}{\beta}R_1 & 0 \\ 0 & \frac{1}{1-\beta}R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 + (1-\beta)e^{\kappa_1}T_1 & S \\ * & R_2 + \beta e^{\kappa_2}T_2 \end{bmatrix},$$

where $T_1 = R_1 - SR_2^{-1}S^T$, $T_2 = R_2 - S^T R_1^{-1} S$.

Remark 1. *Different from the unknown slack matrix S in [17], parameters κ_1 and κ_2 are prespecified, which can be freely chosen. Thus, the flexible tuning can lead to less conservatism.*

In this section, our emphasis lies in introducing an adequate criterion to ensure both the stochastic stability and extended dissipativity of the system (2.1).

Let's begin by revisiting the DIDPT functional denoted as $V^*(\S_t)$, which was introduced in [19]. The DIDPT functional is defined as follows:

$$V^*(\S_t) = V_U(\S_t) + V_T(\S_t), \tag{2.10}$$

where

$$\begin{aligned} V_U(\S_t) &= \frac{\sigma(t)}{2}\int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)U_1\dot{\S}(s)dsd\theta + \frac{\varsigma_\sigma(t)}{2}\int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^T(s)U_2\dot{\S}(s)dsd\theta, \\ V_T(\S_t) &= -\frac{\sigma(t)}{\varsigma^2}\chi_4^T(t)T_1\chi_4(t) - \frac{\varsigma_\sigma(t)}{\varsigma^2}\chi_5^T(t)T_2\chi_5(t), \end{aligned}$$

$$T_k = \begin{bmatrix} \frac{3}{2}U_k & 0 & -3U_k \\ * & 3U_k & -6U_k \\ * & * & 18U_k \end{bmatrix}, k = 1, 2,$$

where the positive definite matrices U_1 and U_2 are denoted with appropriate dimensions and

$$\begin{aligned}
\varsigma_\sigma(t) &= \varsigma - \sigma(t), \\
x_1(t) &= \frac{1}{\sigma(t)} \int_{t-\sigma(t)}^{t} \S(s)ds, \\
x_2(t) &= \frac{1}{\varsigma_\sigma(t)} \int_{t-\varsigma}^{t-\sigma(t)} \S(s)ds, \\
x_3(t) &= \frac{1}{\sigma^2(t)} \int_{-\sigma(t)}^{0} \int_{t+\theta}^{t} \S(s)dsd\theta, \\
x_4(t) &= \frac{1}{\varsigma_\sigma^2(t)} \int_{-\varsigma}^{-\sigma(t)} \int_{t+\theta}^{t-\sigma(t)} \S(s)dsd\theta, \\
\chi_1(t) &= col[\S(t), \pi(W_2\S(t)), \dot{\S}(s)], \\
\chi_2(t) &= col[\S(t), \S(t-\sigma(t)), x_1(t)], \\
\chi_3(t) &= col[\S(t-\sigma(t)), \S(t-\varsigma), x_2(t)], \\
\chi_4(t) &= col[\S(t), x_1(t), x_3(t)], \\
\chi_5(t) &= col[x(t-\sigma(t)), x_2(t), x_4(t)], \\
\xi_1(t) &= [\S(t), \S(t-\sigma(t)), \S(t-\varsigma)], \\
\xi_2(t) &= [\pi(W_2\S(t)), \pi(W_2x(t-\sigma(t))), \pi(W_2\S(t-\varsigma))], \\
\xi_3(t) &= [x_1(t), x_2(t), x_3(t), x_4(t)], \\
\xi_4(t) &= [\dot{\S}(t), \dot{\S}(t-\sigma(t)), \dot{\S}(t-\varsigma)], \\
\eta(t) &= col[\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)].
\end{aligned}$$

It should be pointed out that the DIDPT functional constructed in [19] increases the delay information. However, it lacks consideration of the coupling information. This is due to the existence of zero components in T_k, which results in insufficient cross terms among $\S(t)$, $\S(t-\sigma(t))$, $x_1(t)$, and $x_2(t)$. To address this, we use the S-procedure lemma and introduce semi-positive definite matrices $\mathcal{X}_k = \begin{bmatrix} 0 & X_k \\ X_k & 0 \end{bmatrix}$. By deducing G_k, if there exist scalars $\mu_k \geq 0$, we can express it as follows:

$$G_k = T_k - \mu_k \begin{bmatrix} \mathcal{X}_k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}U_k & -\mu_k X_k & -3U_k \\ * & 3U_k & -6U_k \\ * & * & 18U_k \end{bmatrix} \geq 0. \tag{2.11}$$

Then, we have

$$\begin{aligned}
&\frac{\sigma(t)}{2} \int_{-\sigma(t)}^{0} \int_{t+\theta}^{t} \dot{\S}^T(s)U_1\dot{\S}(s)dsd\theta \geq \frac{1}{\sigma(t)}\chi_4^T(t)G_1\chi_4(t) \geq \frac{\sigma(t)}{\varsigma^2}\chi_4^T(t)G_1\chi_4(t), \\
&\frac{\varsigma_\sigma(t)}{2} \int_{-\varsigma}^{-\sigma(t)} \int_{t+\theta}^{t-\sigma(t)} \dot{\S}^T(s)U_2\dot{\S}(s)dsd\theta \geq \frac{1}{\varsigma_\sigma(t)}\chi_5^T(t)G_2\chi_5(t) \geq \frac{\varsigma_\sigma(t)}{\varsigma^2}\chi_5^T(t)G_1\chi_5(t).
\end{aligned}$$

Further, we can construct the following nonzeoro DPT (NDPT) functional.

Proposition 1. *The provided functional $V_0(\S_t)$ is positive definite for the MJNNs (2.1) incorporating (2.3), under the conditions of matrices $U_1 > 0$ and $U_2 > 0$, along with matrices X_1 and X_2 that fulfill (2.11).*

$$V_0(\S_t) = V_U(\S_t) + V_G(\S_t), \tag{2.12}$$

where

$$\begin{aligned} V_U(\S_t) &= \frac{\sigma(t)}{2}\int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)U_1\dot{\S}(s)dsd\theta + \frac{\varsigma_\sigma(t)}{2}\int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^T(s)U_2\dot{\S}(s)dsd\theta, \\ V_G(\S_t) &= -\frac{\sigma(t)}{\varsigma^2}\chi_4^T(t)G_1\chi_4(t) - \frac{\varsigma_\sigma(t)}{\varsigma^2}\chi_5^T(t)G_2\chi_5(t). \end{aligned}$$

Remark 2. *It is shown from [25] that the conservatism of stability criterion can be reduced by increasing the ply of integral terms in the LKF. Thus, a novel NDPT Lyapunov functional with double integrals is introduced in Proposition 1 with the subsequent benefits:*

(i) The novel NDPT functional (2.12) strengthens the connections among the variables $\dot{\sigma}(t)$, $x_3(t)$, and $x_4(t)$, as compared to the existing Lyapunov functionals in [9, 15, 18].

(ii) In addressing the challenge posed by zero components, this paper leverages the double integral condition presented in [5] combined with the S-procedure lemma as outlined in Lemma 1. By establishing a direct relationship between the functional $V_G(\S_t)$ and $V_U(\S_t)$, the introduction of supplementary variables becomes superfluous. Moreover, by incorporating nonintegral components possessing negative attributes, there is a reduction in the prerequisite for matrices to maintain strict positive definiteness. This methodological relaxation not only curtails conservatism but also lessens the computational complexity inherent in the analysis.

(iii) DIDPT functional presented in [19], the innovative NDPT formulation, as defined in Eq (2.12) circumvents the inclusion of zero components. This design choice facilitates the comprehensive integration of additional information among the variables $\S(t)$, $\S(t-\sigma(t))$, $x_1(t)$, and $x_2(t)$. For example, the term $-\mu_k\S^T(t)X_1x_1(t)$ in (2.12) instead of 0 in [19], thereby preserving the connectivity information between $\S^T(t)$ and $x_1(t)$. Consequently, the novel NDPT (2.12) yields less conservative results.

(iv) It is essential to highlight that the number of decision variables (NDVs) plays a pivotal role in the stability analysis, directly influencing the computational complexity. Within the NDPT framework, a delicate balance between conservatism and complexity can be achieved by judiciously selecting different values of μ_k. Opting for nonzero $\mu_k \neq 0$ values aligns with a preference for conservatism, enhancing the robustness of the analysis. Conversely, setting $\mu_k = 0$ to zero caters to computational efficiency, streamlining the process. This flexibility allows for tailored approaches to stability analysis, accommodating diverse research needs and computational resources.

Based on Proposition 1 and Lemma 2, the following theorem provides a sufficient condition for the extended dissipativity of MJNN (2.1).

Theorem 1. *Setting $e_l = [0_{n\times(l-1)n}, I_n, 0_{n\times(14-l)n}], l = 1, 2, \cdots, 14$, the MJNNs (2.1) achieve stochastically stable and extended dissipative if there exist matrices $P_i \in \mathbb{R}^{3n\times 3n} > 0$, $Q_k \in \mathbb{R}^{3n\times 3n} > 0$, $R_k > 0$, $H_k > 0$, $M_k > 0$, $U_k > 0$, $L_i \in \mathbb{R}^{n\times n} \geq> 0$, $N_i \in \mathbb{R}^{n_w\times n_w} > 0$, positive definite diagonal matrices*

Λ_c, Δ_c, $V_k = diag\{v_{k1}, v_{k2}, \cdots, v_{kn}\} \in \mathbb{R}^{3n\times 3n}$, and any matrices X_k satisfying the following conditions, including the inequalities for all $i \in S$, $c = 1,2,3$, and $k = 1,2$, with the provided scalars $\alpha \in (0,1)$, ς, and ν, and given $\mu_k \geq 0$, $\kappa_k \geq 1$, as well as matrices Ψ_1, Ψ_2, Ψ_3, Ψ_4

$$(1-\nu)R_1 + \varsigma H_1 - \nu M_1 > 0, \tag{2.13}$$

$$R_2 - \nu M_2 > 0, H_2 - \frac{\nu}{2}U_1 > 0, H_2 - \frac{\nu}{2}U_2 > 0, \tag{2.14}$$

$$\begin{bmatrix} (\wp_i^{\perp})^T\Omega(0,0)\wp_i^{\perp} & \sqrt{\kappa_1}(\wp_i^{\perp})^T\Pi_{14}^T S \\ * & -\varsigma\mathfrak{R}_2(0,0) \end{bmatrix} < 0, \tag{2.15}$$

$$\begin{bmatrix} (\wp_i^{\perp})^T\Omega(0,\nu)\wp_i^{\perp} & \sqrt{\kappa_1}(\wp_i^{\perp})^T\Pi_{14}^T S \\ * & -\varsigma\mathfrak{R}_2(0,\nu) \end{bmatrix} < 0, \tag{2.16}$$

$$\begin{bmatrix} (\wp_i^{\perp})^T\Omega(\varsigma,0)\wp_i^{\perp} & \sqrt{\kappa_2}(\wp_i^{\perp})^T\Pi_{15}^T S^T \\ * & -\varsigma\mathfrak{R}_2(\varsigma,0) \end{bmatrix} < 0, \tag{2.17}$$

$$\begin{bmatrix} (\wp_i^{\perp})^T\Omega(\varsigma,-\nu)\wp_i^{\perp} & \sqrt{\kappa_2}(\wp_i^{\perp})^T\Pi_{15}^T S^T \\ * & -\varsigma\mathfrak{R}_2(\varsigma,-\nu) \end{bmatrix} < 0, \tag{2.18}$$

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & -C^T\Psi_4 D_2 & -C^T\Psi_4 B_2 \\ * & \Gamma_{22} & -D_1^T\Psi_4 D_2 & -D_1^T\Psi_4 B_2 \\ * & * & \Gamma_{33} & -D_2^T\Psi_4 B_2 \\ * & * & * & \Gamma_{44} \end{bmatrix} > 0, \tag{2.19}$$

where

$$\begin{aligned}
\Omega(\sigma,\dot{\sigma}) &= \sum_{l=1}^{7}\Omega_l, \sigma = \sigma(t), \varsigma_\sigma = \varsigma - \sigma(t), \\
\Omega_1 &= \Pi_1^T\sum_{j\in S}\tau_{ij}P_j\Pi_1 + He[\Pi_1^T P_i\Pi_2], \\
\Omega_2 &= \Pi_3^T Q_1\Pi_3 - \Pi_5^T Q_2\Pi_5 - (1-\dot{d})\Pi_4^T(Q_1 - Q_2)\Pi_4, \\
\Omega_3 &= He[e_4^T(V_1 - V_2)W_2 e_{11} + e_1^T W_2^T(K_2V_2 - K_1V_1)W_2e_{11}], \\
\Omega_4 &= e_{11}^T\Big(\sigma(R_1 + M_1) + \frac{\varsigma^2}{2}T + \frac{\sigma\varsigma}{2}G_1\Big)e_{11} + e_{12}^T\Big((1-\dot{\sigma})(\varsigma_\sigma M_2 - \sigma M_1) \\
&\quad + \varsigma_\sigma(1-\dot{\sigma})R_2 + \varsigma(1-\dot{\sigma})\frac{\varsigma_\sigma}{2}G_2)\Big)e_{12} - \varsigma_\sigma e_{13}^T M_2 e_{13} \\
&\quad - \frac{\dot{\sigma}}{\varsigma}\Big(\Pi_6^T U_1\Pi_6 - \Pi_7^T U_2\Pi_7\Big) - \frac{1}{\varsigma}He[\Pi_6^T\mathcal{M}_1\Pi_8 + \Pi_7^T\mathcal{M}_2\Pi_9] \\
&\quad - \frac{\dot{\sigma}}{\varsigma^2}(\Pi_{10}^T G_1\Pi_{10} - \Pi_{11}^T G_2\Pi_{11}) - \frac{1}{\varsigma^2}He[\Pi_{10}^T G_1\Pi_{12} + \Pi_{11}^T G_2\Pi_{13}], \\
\Omega_5 &= -\frac{1}{\varsigma}\Pi_{16}^T\mathfrak{R}\Pi_{16} - \Pi_{17}^T\mathfrak{T}_1(\sigma,\dot{\sigma})\Pi_{17} - \Pi_{18}^T\mathfrak{T}_2(\sigma,\dot{\sigma})\Pi_{18}, \\
\Omega_6 &= \sum_{l=1}^{3}He[(e_{l+3} - K_1W_2e_l)^T\Lambda_l(K_2W_2e_l - e_{l+3})] \\
&\quad + \sum_{l=1}^{2}He\Big[\Big((e_{l+3} - e_{l+4}) - K_1W_2(e_l - e_{l+1})\Big)^T \times \Delta_l\Big(K_2W_2(e_l - e_{l+1}) - (e_{l+3} - e_{l+4})\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+He\Big[\Big((e_4-e_6)-K_1W_2(e_1-e_3)\Big)^T\times\Delta_3\Big(K_1W_2(e_1-e_3)-(e_4-e_6)\Big)\Big],\\
\Omega_7 &= -e_0^T\Psi_1e_0-He[e_0^T\Psi_2e_{14}]-e_{14}^T\Psi_3e_{14},\\
\Re &= \begin{bmatrix} \bar{\tau}\Re_1(\sigma,\dot{\sigma}) & S \\ * & (1+\frac{\tau e^{\kappa_2}}{\varsigma})\Re_2(\sigma,\dot{\sigma}) \end{bmatrix},\\
\bar{\tau} &= (1+\frac{(\varsigma-\tau)e^{\kappa_1}}{\varsigma}),\\
\Re_1(\sigma,\dot{\sigma}) &= diag\{\Re_1^0(t),3\Re_1^0(t)\},\\
\Re_2(\sigma,\dot{\sigma}) &= diag\{\Re_2^0(t),3\Re_2^0(t)\},\\
\mathfrak{T}_1(\sigma,\dot{\sigma}) &= diag\{2\mathfrak{T}_1^0,4\mathfrak{T}_1^0\},\mathfrak{T}_2(\sigma,\dot{\sigma})=diag\{2\mathfrak{T}_2^0,4\mathfrak{T}_2^0\},\\
\Re_1^0(t) &= (1-\dot{\sigma})R_1+\varsigma_\sigma T+\frac{(1-\dot{\sigma})\sigma}{2}G_1-\dot{\sigma}M_1,\\
\Re_2^0(t) &= R_2+\dot{\sigma}M_2+\frac{\varsigma_\sigma}{2}G_2,\\
\mathfrak{T}_1^0(t) &= H-\frac{\dot{\sigma}}{2}U_1,\mathfrak{T}_2^0(t)=H+\frac{\dot{\sigma}}{2}U_2,\\
K_1 &= diag\{k_1^-,k_2^-,\cdots,k_n^-\},\\
K_2 &= diag\{k_1^+,k_2^+,\cdots,k_n^+\},\\
P_{1i} &= [I,0,0]P_i[I,0,0]^T,\\
\Gamma_{11} &= \alpha P_{1i}-C^T\Psi_4C,\Gamma_{12}=-C^T\Psi_4D_1,\\
\Gamma_{22} &= (1-\alpha)P_{1i}-D_1^T\Psi_4D_1,\\
\Gamma_{33} &= L_i-D_2^T\Psi_4D_2,\Gamma_{44}=N_i-B_2^T\Psi_4B_2,\\
\Pi_1 &= col[e_1,e_2,e_3],\\
\Pi_2 &= col[e_{11},(1-\dot{\sigma})e_{12},e_{13}],\\
\Pi_{l+3} &= col[e_{l+1},e_{l+4},e_{l+11}],l=0,1,2,\\
\Pi_{6+l} &= col[e_{l+1},e_{l+2},e_{l+7}],l=0,1,\\
\Pi_8 &= col[\sigma e_{11},d(1-\dot{\sigma})e_{12},e_1-(1-\dot{\sigma})e_2-\dot{\sigma}e_7],\\
\Pi_9 &= col[\varsigma_\sigma(1-\dot{\sigma})e_{12},\varsigma_\sigma e_{13},(1-\dot{\sigma})e_2-e_3+\dot{\sigma}e_8],\\
\Pi_{l+10} &= col[e_{l+1},e_{l+7},e_{l+9}],l=0,1,\\
\Pi_{12} &= col[\sigma e_{11},e_1-(1-\dot{\sigma})e_2-\dot{\sigma}e_7,e_1-(1-\dot{\sigma})e_7-2\dot{\sigma}e_9],\\
\Pi_{13} &= col[\varsigma_\sigma e_{12},(1-\dot{\sigma})e_2-e_3+\dot{\sigma}e_8,(1-\dot{\sigma})e_2-e_8+2\dot{\sigma}e_{10}],\\
\Pi_{14} &= col[e_1-e_2,e_1+e_2-2e_7],\\
\Pi_{15} &= col[e_2-e_3,e_2+e_3-2e_8],\\
\Pi_{16} &= col[\Pi_{14},\Pi_{15}],\\
\Pi_{17} &= col[e_1-e_7,-\frac{1}{2}e_1-e_7+3e_9],\\
\Pi_{18} &= col[e_2-e_8,-\frac{1}{2}e_2-e_8+3e_{10}],\\
\wp_i &= -A_ie_1+W_{0i}e_4+W_{1i}e_5+B_1e_{14}-e_{11},
\end{aligned}
$$

$$e_0 = Ce_1 + D_1e_2 + D_2e_4 + B_2e_{14}.$$

Proof. Select the subsequent LKF candidate:

$$V(\S_t) = \sum_{l=0}^{6} V_l(\S_t), \tag{2.20}$$

where

$$\begin{aligned}
V_1(\S_t) &= \eta_1^T(t)P_i\eta_1(t),\\
V_2(\S_t) &= \int_{t-\sigma(t)}^{t}\chi_1^T(s)Q_1\chi_1(s)ds + \int_{t-\varsigma}^{t-\sigma(t)}\chi_1^T(s)Q_2\chi_1(s)ds,\\
V_3(\S_t) &= 2\sum_{l=1}^{n}\int_0^{W_{2l}\S_l(t)}[v_{1l}(\pi_l(s) - k_l^- s) + v_{2l}(k_l^+ s - \pi_l(s))]ds,\\
V_4(\S_t) &= \int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)R_1\dot{\S}(s)dsd\theta + \int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^T(s)R_2\dot{\S}(s)dsd\theta,\\
V_5(\S_t) &= \int_{-\varsigma}^{0}\int_{\theta}^{0}\int_{t+u}^{t}\dot{\S}^T(s)H\dot{\S}(s)dsdud\theta,\\
V_6(\S_t) &= \sigma(t)\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)M_1\dot{\S}(s)ds + \varsigma_\sigma(t)\int_{t-\varsigma}^{t-\sigma(t)}\dot{\S}^T(s)M_2\dot{\S}(s)ds\\
&\quad -\frac{\sigma(t)}{\varsigma}\chi_2^T(t)\mathcal{M}_1\chi_2(t) - \frac{\varsigma_\sigma(t)}{\varsigma}\chi_3^T(t)\mathcal{M}_2\chi_3(t).
\end{aligned}$$

The infinitesimal operator along system (2.1) is denoted by $\mathfrak{L}$. One has

$$\sum_{c=1}^{3}\mathfrak{L}V_c(\S_t) = \xi^T(t)(\Omega_1 + \Omega_2 + \Omega_3)\xi(t), \tag{2.21}$$

$$\begin{aligned}
\sum_{l=4}^{5}\mathfrak{L}V_l(\S_t) &= \dot{\S}^T(t)(\sigma(t)R_1 + \frac{\varsigma^2}{2}H)\dot{\S}(t) + \varsigma_\sigma(t)(1-\dot{\sigma}(t))\dot{\S}^T(t-\sigma(t)) \times R_2\dot{\S}(t-\sigma(t))\\
&\quad -(1-\dot{\sigma}(t))\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)R_1\dot{\S}(s)ds - \int_{t-\varsigma}^{t-\sigma(t)}\dot{\S}^T(s)R_2\dot{\S}(s)ds\\
&\quad -\int_{-\varsigma}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)H\dot{\S}(s)dsd\theta,
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
\mathfrak{L}V_6(\S_t) &= \dot{\sigma}(t)\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)M_1\dot{\S}(s)ds - \dot{\sigma}(t)\int_{t-\varsigma}^{t-\sigma(t)}\dot{\S}^T(s)M_2\dot{\S}(s)ds + \sigma(t)\dot{\S}^T(t)M_1\dot{\S}(t) - \varsigma_\sigma(t)\dot{\S}^T(t-h)\\
&\quad \times M_2\dot{\S}(t-\varsigma) - (1-\dot{\sigma}(t))\dot{\S}^T(t-\sigma(t)) \times (\sigma(t)M_1 - \varsigma_\sigma(t)M_2)\dot{\S}(t-\sigma(t)) - \frac{\dot{\sigma}(t)}{\varsigma}\chi_2^T(t)\mathcal{M}_1\chi_2(t)\\
&\quad -\frac{2\sigma(t)}{\varsigma}\chi_2^T(t)\mathcal{M}_1\dot{\chi}_2(t) + \frac{\dot{\sigma}(t)}{\varsigma}\chi_3^T(t)\mathcal{M}_2\chi_3(t) - \frac{2\varsigma_\sigma(t)}{\varsigma}\chi_3^T(t)\mathcal{M}_2\dot{\chi}_3(t),
\end{aligned}\tag{2.23}$$

$$\mathfrak{L}V_0(\S_t) \le \dot{\S}^T(t)\frac{\sigma(t)\varsigma}{2}G_1\dot{\S}(t) + (1-\dot{\sigma}(t))\dot{\S}^T(t-\sigma(t))\frac{\varsigma_\sigma(t)\varsigma}{2}G_2\dot{\S}(t-\sigma(t)) - \frac{\dot{\sigma}(t)}{\varsigma^2}\xi_4^T(t)Z_1\xi_4(t)$$

$$
\begin{aligned}
&-\frac{2\sigma(t)}{\varsigma^2}\xi_4^T(t)Z_1\dot{\xi}_4(t)+\frac{\dot{\sigma}(t)}{\varsigma^2}\xi_5^T(t)Z_2\xi_5(t)-\frac{2\varsigma_\sigma(t)}{\varsigma^2}\xi_5^T(t)Z_2\dot{\xi}_5(t)\\
&-\frac{(1-\dot{\sigma}(t))\sigma(t)}{2}\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)G_1\dot{\S}(s)ds-\frac{\varsigma_\sigma(t)}{2}\int_{t-\varsigma}^{t-\sigma(t)}\dot{\S}^T(s)G_2\dot{\S}(s)ds\\
&+\frac{\dot{\sigma}(t)}{2}\int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)G_1\dot{\S}(s)dsd\theta-\frac{\dot{\sigma}(t)}{2}\int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^T(s)G_2\dot{\S}(s)dsd\theta. \qquad (2.24)
\end{aligned}
$$

Referring to (2.22), to enhance the incorporation of TVDs information, we divide $[0,\varsigma]$ into $[0,\sigma(t)]\cup[\sigma(t),\varsigma]$, resulting in the following expression:

$$
\begin{aligned}
& -\int_{-\varsigma}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)H\dot{\S}(s)dsd\theta\\
= & -\int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)H\dot{\S}(s)dsd\theta\\
& -\varsigma_\sigma(t)\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)H\dot{\S}(s)ds\\
& -\int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^T(s)H\dot{\S}(s)dsd\theta. \qquad (2.25)
\end{aligned}
$$

Combining with (2.22)–(2.25), one has

$$
\begin{aligned}
\sum_{c=4}^{6}\mathfrak{L}V_c(\S_t)+\mathfrak{L}V^*(\S_t) \leq\ & \xi^T(t)\Omega_4\xi(t)-\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)\Re_1^0(t)\dot{\S}(s)ds\\
& -\int_{t-\varsigma}^{t-\sigma(t)}\dot{\S}^T(s)\Re_2^0(t)\dot{\S}(s)ds-\int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^T(s)\mathfrak{T}_1^0(t)\dot{\S}(s)dsd\theta\\
& -\int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^T(s)\mathfrak{T}_2^0(t)\dot{\S}(s)dsd\theta. \qquad (2.26)
\end{aligned}
$$

Taking into account (2.3), (2.13), and (2.14), alongside the presence of positive definite matrices R_k, M_k, U_k, and H_k (for $k=1,2$), we obtain

$$\Re_1^0(t)>0,\Re_2^0(t)>0,\mathfrak{T}_1^0(t)>0,\mathfrak{T}_2^0(t)>0.$$

By utilizing the inequality condition in [4], we can approximate the single integral terms in (2.26) that depend on R, M, U, and H as follows:

$$-\int_{t-\sigma(t)}^{t}\dot{\S}^T(s)\Re_1^0(t)\dot{\S}(s)ds\leq-\frac{1}{\sigma(t)}\xi^T(t)\Pi_{14}^T\mathfrak{R}_1(\sigma(t),\dot{\sigma}(t))\Pi_{14}\xi(t) \qquad (2.27)$$

and

$$-\int_{t-\varsigma}^{t-\sigma(t)}\dot{x}^T(s)\Re_2^0(t)\dot{x}(s)ds\leq-\frac{1}{\varsigma_\sigma(t)}\xi^T(t)\Pi_{15}^T\mathfrak{R}_2(\sigma(t),\dot{\sigma}(t))\Pi_{15}\xi(t). \qquad (2.28)$$

By applying Lemma 2, one has

$$
\begin{aligned}
& -\frac{1}{\sigma(t)}\Pi_{14}^{T}\mathfrak{R}_1(\sigma(t),\dot{\sigma}(t))\Pi_{14}-\frac{1}{\varsigma_{\sigma}(t)}\Pi_{15}^{T}\times\mathfrak{R}_2(\sigma(t),\dot{\sigma}(t))\Pi_{15} \\
\leq\ & -\frac{1}{\varsigma}\begin{bmatrix}\Pi_{14}\\ \Pi_{15}\end{bmatrix}^{T}\begin{bmatrix}\mathfrak{R}_1(\sigma(t),\dot{\sigma}(t))+\mathcal{T}_1 & S\\ * & \mathcal{T}_2\end{bmatrix}\begin{bmatrix}\Pi_{14}\\ \Pi_{15}\end{bmatrix} \\
\leq\ & -\frac{1}{\varsigma}\Pi_{16}^{T}(\mathfrak{R}-\mathfrak{I})\Pi_{16},
\end{aligned}
\tag{2.29}
$$

where

$$
\begin{aligned}
\mathcal{T}_1 &= (1-\frac{e^{\kappa_1}\varsigma_{\sigma}(t)}{\varsigma}\mathcal{T}_{11}),\\
\mathcal{T}_2 &= \mathfrak{R}_2(\sigma(t),\dot{\sigma}(t)+\frac{e^{\kappa_2}\sigma(t)}{\varsigma}\mathcal{T}_{12},\\
\mathcal{T}_{11} &= \mathfrak{R}_1(\sigma(t),\dot{\sigma}(t))-S\mathfrak{R}_2^{-1}(\sigma(t),\dot{\sigma}(t))S^{T},\\
\mathcal{T}_{12} &= \mathfrak{R}_2(\sigma(t),\dot{\sigma}(t))-S^{T}\mathfrak{R}_1^{-1}(\sigma(t),\dot{\sigma}(t))S,\\
\mathfrak{I} &= diag\{\frac{e^{\kappa_1}\varsigma_{\sigma}(t)}{\varsigma}S\mathfrak{R}_2^{-1}(\sigma(t),\dot{\sigma}(t))S^{T},\frac{e^{\kappa_2}\sigma(t)}{\varsigma}S^{T}\mathfrak{R}_1^{-1}(\sigma(t),\dot{\sigma}(t))S\}.
\end{aligned}
$$

By employing the inequality in [5], we can respectively estimate the H- and U-dependent terms in (2.26) as follows:

$$
-\int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}\dot{\S}^{T}(s)\mathfrak{T}_1^{0}(t)\dot{\S}(s)dsd\theta\leq-\xi^{T}(t)\Pi_{17}\mathfrak{T}_1(\sigma(t),\dot{\sigma}(t))\Pi_{17}\xi(t), \tag{2.30}
$$

$$
-\int_{-\varsigma}^{-\sigma(t)}\int_{t+\theta}^{t-\sigma(t)}\dot{\S}^{T}(s)\mathfrak{T}_2^{0}(t)\dot{\S}(s)dsd\theta=-\xi^{T}(t)\Pi_{18}\mathfrak{T}_2(\sigma(t),\dot{\sigma}(t))\Pi_{18}\xi(t). \tag{2.31}
$$

Combining (2.26) and (2.29)–(2.31), we obtain:

$$
\sum_{c=4}^{6}\mathfrak{L}V_c(\S_t)+\mathfrak{L}V_0(\S_t)\leq\xi^{T}(t)(\Omega_4+\Omega_5+\frac{1}{\varsigma}\Pi_{16}^{T}\mathfrak{I}\Pi_{16})\xi(t). \tag{2.32}
$$

By considering (2.4) and (2.5), we can obtain the following inequalities for $a=1,2,3$.

$$
\begin{aligned}
\lambda_a(s) &= 2[\pi(W_2x(s))-K_1W_2\S(s)]^{T}\times\Lambda_c[K_2W_2\S(s)-\pi(W_2\S(s))]\geq 0,\\
\delta_a(s_1,s_2) &= 2[\pi(W_2\S(s_1))-\pi(W_2\S(s_2))-K_1W_2(\S(s_1)-\S(s_2))]^{T}\\
&\quad\times\Delta_c[K_2W_2(\S(s_1)-\S(s_2))-\pi(W_2\S(s_1))+\pi(W_2\S(s_2))]\geq 0.
\end{aligned}
$$

Thus, we can obtain

$$
\begin{aligned}
& \lambda_1(t)+\lambda_2(t-\sigma(t))+\lambda_3(t-\varsigma)\\
=\ & 2[\pi(W_2\rho(s))-K_1W_2\rho(s)]^{T}\times\Lambda_i[K_2W_2\rho(s)-\pi(W_2\rho(s))]\geq 0,\\
& \delta(t,t-\sigma(t))+\delta(t-\sigma(t),t-\varsigma)+\delta(t,t-\varsigma)\\
=\ & 2[\pi(W_2\rho(s_1))-\pi(W_2\rho(s_2))-K_1W_2(\rho(s_1)-\rho(s_2))]^{T}
\end{aligned}
$$

$$\times \quad \Delta_i[K_2 W_2(\rho(s_1) - \rho(s_2)) - \pi(W_2\rho(s_1)) + \pi(W_2\rho(s_2))] \geq 0. \tag{2.33}$$

Further, the cost function can be expressed as follows:

$$\mathfrak{J}_{\mathfrak{T}} = \int_0^T (\dagger^T(t)\Psi_1\dagger(t) + 2\dagger^T(t)\Psi_2\rho(t) + \rho^T(t)\Psi_3\rho(t))dt. \tag{2.34}$$

Considering (2.21) and (2.32)–(2.34), it yields

$$\int_0^T \mathfrak{L}V(\S_t)dt - \mathfrak{J}_{\mathfrak{T}} \leq \int_0^T \xi^T(t)\Upsilon(\sigma(t), \dot{\sigma}(t))\xi(t)dt, \tag{2.35}$$

where

$$\Upsilon(\sigma(t), \dot{\sigma}(t)) = \sum_{l=1}^{7} \Omega_l + \frac{1}{\varsigma}\Pi_{16}^T \mathfrak{I}\Pi_{16}.$$

In the context of the problem, ℓ_l $(l = 1, 2, ..., 4)$ are real matrix combinations that do not depend on $\sigma(t)$ and $\dot{\sigma}(t)$. Therefore, following the approach presented in [26], we have $\Upsilon(\sigma(t), \dot{\sigma}(t)) < 0$.

Considering the condition $\wp_i\xi(t) = 0$, and applying Finsler's Lemma [3], we can deduce

$$(\wp_i^{\perp})^T\Upsilon(\sigma(t), \dot{\sigma}(t))\wp_i^{\perp} < 0 \Rightarrow \xi^T(t)\Upsilon(\sigma(t), \dot{\sigma}(t))\xi(t) < 0.$$

When $\Psi_1 \leq 0$, a nonnegative scalar ϵ can be found, satisfying the subsequent inequality for the case of $w(t) = 0$.

$$\mathfrak{L}V(\S_t) < -\epsilon|\xi(t)|^2.$$

This indicates the stochastic stability of MJNNs (2.1).

From (2.35), one has

$$\mathfrak{J}_{\mathfrak{T}} \geq \int_0^T \mathfrak{L}V(\S_t)dt = V(\S_t) - V(\S_0) \geq \S^T(T)P_{1i}\S(t) - V(\S_0). \tag{2.36}$$

Based on (2.19), it can be observed that $\Gamma_{33} = L_i - D_2^T\Psi_4 D_2 > 0$ and $\Gamma_{44} = N_i - B_2^T\Psi_4 B_2 > 0$. Given that L_i and N_i stand as symmetric positive definite matrices, the condition $\Psi_4 \geq 0$ is assured. Furthermore, it becomes apparent that $\Psi_4 < 0$ is valid for all instances involving matrices $L_i \geq 0$ and $N_i \geq 0$. Regarding the unrestricted parameter Ψ_4, we will explore two distinct scenarios as outlined below:

a: In the case of $\Psi_4 = 0$, by setting $\varrho \leq -V(\S_0)$, we can derive from (2.9) and (2.36) the following result:

$$\mathfrak{J}_{\mathfrak{T}} \geq \varrho. \tag{2.37}$$

b: In the case of $\Psi_4 \neq 0$, based on $\Psi_1 = \Psi_2 = 0$ and $\Psi_3 > 0$, along with (2.36), we obtain the following result:

$$\mathfrak{J}_{\mathfrak{T}} = \int_0^T \rho^T(t)\Psi_3\rho(t)dt \geq 0. \tag{2.38}$$

For $0 \le t \le T$ and $0 \le t - \sigma(t) \le T$, and considering matrices $L_i > 0$, $N_i > 0$, and $\Psi_3 > 0$, we have the following result for all $i \in S$:

$$\begin{aligned}
\mathfrak{J}_{\mathfrak{T}} &\ge \mathcal{J}_t \ge \S^T(t)\mathcal{P}_{0i}\S(t) - V(\S_0) \\
&\ge \S^T(t)\mathcal{P}_{0i}\S(t) - V(\S_0) - \pi^T(W_2\S(t))L_i\pi(W_2\S(t)) - \rho^T(t)N_i\rho(t), &(2.39)\\
\mathfrak{J}_{\mathfrak{T}} &\ge \mathcal{J}_{t-\sigma(t)} \\
&\ge \S^T(t-\sigma(t))\mathcal{P}_{0i}\S(t-\sigma(t)) - V(\S_0) - \pi^T(W_2\S(t))L_i\pi(W_2\S(t)) - \rho^T(t)N_i\rho(t). &(2.40)
\end{aligned}$$

Therefore, by choosing a constant $0 < \alpha < 1$, we obtain

$$\begin{aligned}
\mathfrak{J}_{\mathfrak{T}} \ge\ & (1-\alpha)\S^T(t-\sigma(t))\mathcal{P}_{0i}\S(t-\sigma(t)) + \alpha\S^T(t)\mathcal{P}_{0i}\S(t) - V(\S_0) \\
& -\pi^T(W_2\S(t))L_i\pi(W_2\S(t)) - \rho^T(t)N_i\rho(t). \qquad (2.41)
\end{aligned}$$

Then, one has

$$\begin{aligned}
\dagger^T(t)\Psi_4\dagger(t) =\ & (1-\alpha)\S^T(t-\sigma(t))\mathcal{P}_{0i}\S(t-\sigma(t)) + \alpha\S^T(t)\mathcal{P}_{0i}\S(t) + \rho^T(t)N_i\rho(t) \\
& +\pi^T(W_2\S(t))L_i\pi(W_2\S(t)) - \varphi^T(t)\Theta\varphi(t), \qquad (2.42)
\end{aligned}$$

where

$$\begin{aligned}
\varphi(t) &= col[\S(t), \S(t-\sigma(t)), \pi(W_2\S(t)), \rho(t)], \\
\Theta &= \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & -C^T\Psi_4 D_2 & -C^T\Psi_4 B_2 \\ * & \Gamma_{22} & -D_1^T\Psi_4 D_2 & -D_1^T\Psi_4 B_2 \\ * & * & \Gamma_{33} & -D_2^T\Psi_4 B_2 \\ * & * & * & \Gamma_{44} \end{bmatrix}.
\end{aligned}$$

Combining $\Theta > 0$, (2.41), and (2.42), one has

$$\mathfrak{J}_{\mathfrak{T}} \ge \dagger^T(ts)\Psi_4\dagger(t) - 2\pi^T(W_2\S(t))L_i\pi(W_2\S(t)) - 2\rho^T(t)N_i\rho(t) - V(\S_0). \qquad (2.43)$$

Recalling (2.5), we find that $|\pi(W_2\S(t))| \le \|K_2W_2\| \cdot |\S(t)|$. Now, let's consider two cases for $\Psi_4 \neq 0$:

b(1): For $\Psi_4 > 0$, considering $\rho(t) \in \mathcal{L}_2[0,\infty)$, there exists a constant $\varrho \le 0$ such that

$$\dagger^T(t)\Psi_4\dagger(t) \ge \sup_{0\le t\le T} \dagger^T(t)\Psi_4\dagger(t) + \varrho. \qquad (2.44)$$

Recalling (2.43), one has

$$\mathfrak{J}_{\mathfrak{T}} \ge \sup_{0\le t\le T} \dagger^T(t)\Psi_4\dagger(t) + \varrho. \qquad (2.45)$$

b(2): For $\Psi_4 < 0$, one has

$$\sup_{0\le t\le T} \dagger^T(t)\Psi_4\dagger(t) \le 0. \qquad (2.46)$$

Combining (2.38) and (2.46), one has

$$\mathfrak{J}_{\mathfrak{T}} = \int_0^T \rho^T(s)\Psi_3\rho(s)ds \ge \sup_{0\le t\le T} \dagger^T(t)\Psi_4\dagger(t) + \varrho. \qquad (2.47)$$

Recalling (2.45) and (2.47), we have

$$\mathfrak{J}_{\mathfrak{T}} = \int_0^T \rho^T(s)\Psi_3\rho(s)ds \geq \sup_{0\leq t\leq T} \dagger^T(t)\Psi_4\dagger(t) + \varrho. \tag{2.48}$$

Based on Definition 1, we can conclude the extended dissipativity of system (2.1), which completes the proof. □

Remark 3. *Theorem 1 proposes a new condition with the help of NDPT and ETRCI, ensuring the stochastically stable and extended dissipative behavior of MJNNs (2.1). The advantages of Theorem 1 can be highlighted.*

(i) The condition captures more information about TVDs, involving its derivative in association with $\sigma(t)$, ς, $\dot{\sigma}(t)$, $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$. Additionally, the incorporation of double integrals and the elimination of zero components result in outcomes that are less conservative when contrasted with those presented in [18, 19].

(ii) The ETRCI is introduced to deal with the term $-\int_{t-\varsigma}^{t} \dot{\S}^T(s)R\dot{\S}(s)ds$, which distinguishes it from the approaches in [18,19]. The flexibility of two parameters κ_1 and κ_2 in Theorem 1 allows independent adjustments, leading to better solutions.

(iii) It is noted that extended dissipativity [13, 14] includes strict dissipativity [11] and passivity [27], which enhances the generality of the proposed method. Furthermore, based on the NDPT and ETRCI, the dissipative characteristics are superior compared with some existing works [27, 28].

(iv) The NDPT and ETRCI presented in this paper primarily enhances the interconnectedness between time-delay information and the slack matrices, thereby reducing conservativeness. Consequently, the approach is well-suited for time delayed systems such as delayed semi-Markovian NNs.

Remark 4. *Note that theoretical proof in [25] shows that the conservatism of a stability criterion can be reduced by increasing the ply of integral terms in LKF. Thus, a suitable multiple integral-type LKF can further reduce the conservatism of the stability criterion. Consequently, employing a multiple integral-type LKF can further diminish the conservatism of the stability criterion. Additionally, the literature [8] reveals the introduction of exponentially extended dissipativity, which not only consolidates various performance measures-such as extended dissipative, exponential H_∞, $l_2 - l_\infty$ performance within a cohesive framework, but also delineates both the extended dissipativity and the transient behavior of switched NNs with greater clarity upon establishing the decay rate. Therefore, future research endeavors should focus on the analysis of less conservative exponentially extended dissipativity employing multiple DPT, which promises to yield significant insights.*

3. Example

We consider MJNNs (2.1) with parameters presented in [18]

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, K_1 = D_2 = 0,$$
$$W_{01} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, W_{02} = \begin{bmatrix} -1 & -1 \\ 0.5 & -1 \end{bmatrix},$$

$$W_{11}=\begin{bmatrix}0.88 & 1\\ 1 & 1\end{bmatrix}, W_{12}=\begin{bmatrix}-0.5 & 0.6\\ 0.7 & 0.8\end{bmatrix},$$
$$W_{2}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, B_{1}=\begin{bmatrix}0.0403\\ 0.6771\end{bmatrix}, K_2 = diag\{0.4, 0.8\},$$
$$C=\begin{bmatrix}-0.3775 & -0.2959\end{bmatrix}, D_1=\begin{bmatrix}0.2532 & -0.1684\end{bmatrix}.$$

We choose $\Pi = \begin{bmatrix}-4 & 4\\ 5 & -5\end{bmatrix}$ and $\Psi_l (l = 1, 2, 3, 4)$ from Table 1. For simplicity, we set $\mu = \mu_1 = \mu_2$ and $\kappa = \kappa_1 = \kappa_2$, and consider the following two cases as special cases of extended dissipativity.

Table 1. Extended dissipative performance.

Performance	Ψ_4	Ψ_1	Ψ_2	Ψ_3
H_∞	0	-I	0	γ^2I
$L_2 - L_\infty$	I	0	0	γ^2I

Case 1. The H_∞ performance is investigated by selecting $B_2 = 0.1184$, and Table 2 presents the minimum H_∞ performance of γ and NDVs computed using different techniques.

Table 2. Minimum values γ for $\varsigma = 3$ under H_∞ performance.

Criteria	$\nu = 0.8$	$\nu = 0.9$	NDVs
[18, Theorem 3]	0.8429	2.7968	125
[19, Theorem 1]	0.6113	1.7642	252
Theorem 1($\mu = 0, \kappa = 0.1$)	0.6541	1.7712	252
Theorem 1($\mu = 1, \kappa = 0.1$)	0.5917	1.7504	266
Theorem 1($\mu = 0.4, \kappa = 0.1$)	0.6571	1.8154	266
Theorem 1($\mu = 0.2, \kappa = 0.1$)	0.6897	1.8401	266
Theorem 1($\mu = 0.2, \kappa = 0.3$)	0.6994	1.8577	266

Case 2. The $L_2 - L_\infty$ performance is evaluated under the condition that $B_2 = 0$. Table 3 presents the minimum $L_2 - L_\infty$ performance of γ and NDVs computed using different techniques.

Table 3. Minimum values γ for $\varsigma = 3$ under $L_2 - L_\infty$ performance.

Criteria	$\nu = 0.8$	$\nu = 0.9$	NDVs
[18, Theorem 3]	0.6684	1.3591	125
[19, Theorem 1]	0.4241	0.8744	252
Theorem 1($\mu = 0, \kappa = 0.1$)	0.4271	0.8794	252
Theorem 1($\mu = 2, \kappa = 0.1$)	0.4012	0.8514	266
Theorem 1($\mu = 0.4, \kappa = 0.1$)	0.4475	0.8651	266
Theorem 1($\mu = 0.2, \kappa = 0.1$)	0.4794	0.8841	266
Theorem 1($\mu = 0.2, \kappa = 0.3$)	0.5047	0.8924	266

Based on Cases 1 and 2, the subsequent observations can be made:

(i) Tables 2 and 3 effectively illustrate the adaptable nature of the NDPT approach (2.12), which can be tailored according to distinct values of τ.

(ii) The ETRCI assumes a pivotal role in mitigating excessive caution.

(iii) The proposition presented in this paper regarding MJNNs (2.1) showcases a comparatively diminished level of conservatism in contrast to the outcomes derived in [18, 19].

Although MJNNs have been implemented in various fields such as biomolecules [29], this paper primarily focuses on theoretical methodological innovation. By introducing a novel time-delay product approach, it aims to reduce the conservativeness of time-delay Markovian neural systems. Therefore, extending the method presented in this paper to practical applications is not only meaningful but also constitutes a worthwhile direction for future research.

4. Conclusions

This paper delves into the issue of extended dissipative analysis of MJNNs using the delay-product-type functional method. The approach introduced here involves the creation of a novel NDPT functional, achieved through the utilization of Wirtinger-based double integral inequality combined with the S-procedure lemma. Unlike the approach in [19], this fresh functional overcomes the presence of incomplete components, thereby facilitating enhanced integration of supplementary system information. By melding the exponential type reciprocally convex inequality with the Wirtinger-based integral inequality, we are able to estimate the derivative of the constructed Lyapunov-Krasovskii function. This leads to the derivation of a delay-dependent extended dissipativity condition tailored to address the specificities of delayed MJNNs. The effectiveness of the proposed methodology is demonstrated convincingly through a numerical example.

Author contributions

Wenlong Xue: Writing-original draft; Yufeng Tian: Writing-review & editing; Zhenghong Jin: Validation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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