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Research article

Fast algorithms for nonuniform Chirp-Fourier transform

Yannan Sun* and Wenchao Qian

School of Mathematics Science, Jiangsu University, Zhenjiang, Jiangsu, MO 212013, China

* **Correspondence:** Email: sunyannan@ujs.edu.cn.

Abstract: The Chirp-Fourier transform is one of the most important tools of the modern signal processing. It has been widely used in the fields of ultrasound imaging, parameter estimation, and so on. The key to its application lies in the sampling and fast algorithms. In practical applications, nonuniform sampling can be caused by sampling equipment and other reasons. For the nonuniform sampling, we utilized function approximation and interpolation theory to construct different approximation forms of Chirp-Fourier transform kernel function, and proposed three fast nonuniform Chirp-Fourier transform algorithms. By analyzing the approximation error and the computational complexity of these algorithms, the effectiveness of the proposed algorithms was proved.

Keywords: fast Fourier transform; Chirp-Fourier transform; trigonometric series; approximation theory
Mathematics Subject Classification: 11F20, 11M20

1. Introduction

The majority of digital signal processing theories currently are based on the uniform, ideal sampling signal model. With the development of science and technology, the range of processed signals has become very broad. The digital signals obtained in practical applications are generally nonuniform and nonideal signal sampling sequences. For example, in the process of radar signal acquisition, geographic data acquisition, and medical image signal acquisition, due to the presence of certain interference factors, it is not easy or impossible to obtain completely uniformly sampled signals. Meanwhile, in practical applications, due to the influence of signal receiving equipment and external interference factors, the sampling points of the obtained signal may not necessarily be the sampling points of the original signal. Therefore, nonuniform sampling has become one of the important hotspots in signal processing [1,2]. The Fourier transform (FT), which is very important in the Fourier analysis theory system, is the basis for the traditional signal processing. It has good performance in handling stationary signals. However, it has significant limitations to process nonstationary using FT. The Chirp signal is a common nonstationary signal. Therefore, numerous novel mathematical techniques, including the fractional Fourier transform (FrFT) [2–4], the Chirp-Fourier transform (CFT) [5–8], and the linear canonical transform (LCT) [9–12], have been proposed to process nonstationary signal. Among them, since the CFT has good mathematical qualities and fast algorithms, it is frequently used in nonstationary signal processing.

The application of the CFT relies on its discretization and fast algorithms. Discrete Chirp-Fourier transform (DCFT) was first proposed by Xia in 2000 [5]. However, this method requires that the number of sampling points is prime, and the parameters of frequency modulation must be integers or approximate integers, which has significant limitations in practical applications. Fan et al. derived modified DCFT (modified Chirp-Fourier transform, MDCFT) to improve the aforementioned limitations [6]. On the basis of these, Guo Yong et al. [7] gave a fast method based on CFT for estimating the parameters of Newton's ring. The segmented CFT fast algorithm was proposed by Zong Ying et al. in 2023 [8]. Simulation results demonstrate that the suggested algorithm can realize the fast computing of the local spectrum. Most of the existing CFT algorithms are based on the uniform sampling. For the nonuniform sampling, the above algorithms can't achieve good results. It is important to consider the nonuniform sampling associated with the CFT. Meanwhile there are fewer researches on fast algorithms for nonuniform CFT currently. In order to promote the CFT to be better applied in various fields, we will propose the nonuniform fast Chirp-Fourier transform (NUCFT) algorithm.

The rest of the paper is organized as follows: In Section 2, we give definitions of the CFT, Chirp-Fourier series, and associated lemmas. The main results are derived in Section 3. Finally, the conclusions are presented in Section 4.

2. Preliminary

Definition 2.1. The CFT of a continuous signal $x(t)$ is defined as [13]

$$X_r(u)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi\left(ut+rt^2\right)}dt, \tag{2.1}$$

where $r=-\infty,\cdots,\infty.$

When $r=0,$ the transform degenerates to the FT, i.e., $X_r(u)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ut}dt.$

The inverse of the CFT for a continuous signal $x(t)$ is defined as

$$x(t)=\int_{-\infty}^{\infty}X_r(u)e^{j2\pi\left(ut+rt^2\right)}du. \tag{2.2}$$

By discretizing the time and transform domains, different definitions of the discrete CFT can be obtained [7,14,15]. The most commonly used DCFT is the following.

Definition 2.2. The DCFT of a discrete sequence $x(n)=x(n\Delta t)$ is defined as [16]

$$X_r(m)=X_r(m\Delta u)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi}{N}\left(mn+rn^2\right)}, \tag{2.3}$$

where $m = 0,1,\cdots,N-1$. Δt and Δu are the time and frequency domain sampling intervals for time t and transform domain u, respectively.

The inverse of the DCFT for a discrete sequence $x(n) = x(n\Delta t)$ is defined as

$$x(n) = e^{j\frac{2\pi}{N}\cdot rn^2} \cdot \sum_{m=0}^{N-1} X_r(m) e^{j\frac{2\pi}{N}\cdot mn}. \tag{2.4}$$

The definition of the Chirp-Fourier series for the signal $x(t)$ is obtained as follows.

$$\varphi_{r,k}(t) = \frac{1}{\sqrt{T}} e^{i\left(kt+rt^2\right)}. \tag{2.5}$$

Due to

$$\begin{aligned}(\varphi_{r,k}(t), \varphi_{r,k'}(t)) &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-i(kt+rt^2)} \cdot e^{i(k't+rt^2)} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-i(k-k')t} dt \\ &= \begin{cases} 0, k \neq k', \\ 1, k = k'. \end{cases}\end{aligned} \tag{2.6}$$

The sequence $\left\{\cdots, \varphi_{r,-1}(t), \varphi_{r,0}(t), \varphi_{r,1}(t), \cdots\right\}$ is orthogonal in the interval $[-T/2, T/2]$. Since the CFS only employs finite-length functions and this basis function is an FM function, the CFS of a finite-length signal $x(t)$ is

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} c_k \varphi_{r,k}(t), \tag{2.7}$$

where the coefficient of the CFS is $\mathrm{c}_k = \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} x(t) e^{-i(kt+rt^2)} dt.$

The pertinent formulas that have been applied to derive the NUCFT are given as the following.

Lemma 2.1. For any real c,

$$\int_{-\pi}^{\pi} e^{icx} dx = \frac{2}{c}\sin(c\pi). \tag{2.8}$$

Lemma 2.2. For any integer k,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} dx = \begin{cases} 1, & k = 0, \\ 0, & else. \end{cases} \tag{2.9}$$

Lemma 2.3. [17] For any real $b > 0$ and complex z,

$$\int_{-\infty}^{\infty} e^{-bx^2} \cdot e^{zx} dx = \sqrt{\frac{\pi}{b}} \cdot e^{z^2/4b}. \tag{2.10}$$

Lemma 2.4. [17] For any real $b > 0$ and $a > 0$,

$$\int_a^{\infty} e^{-bx^2} dx < \frac{e^{-ba^2}}{2ba}. \tag{2.11}$$

In the next section, the main results will be derived based on the above facts.

3. Main results

3.1. Exact statement of the problems

In the remainder of this paper, we will operate under the following assumptions.

(1) $\omega = \{\omega_0, \cdots, \omega_N\}$ and $x = \{x_0, \cdots, x_{N-1}\}$ are finite sequences of real numbers.

(2) $\omega_k \in [-N/2, N/2-1]$, for $k = 0, \cdots, N-1$.

(3) $x_i \in [-\pi, \pi]$, for $j = 0, \cdots, N-1$.

(4) $\alpha = \{\alpha_0, \cdots, \alpha_{N-1}\}$, $f = \{f_{-N/2}, \cdots, f_{N/2-1}\}$, $\beta = \{\beta_{-N/2}, \cdots, \beta_{N/2-1}\}$, $g = \{g_0, \cdots, g_{N-1}\}$, $\gamma = \{\gamma_0, \cdots, \gamma_{N-1}\}$, and $h = \{h_0, \cdots, h_{N-1}\}$ are finite sequences of complex numbers.

We will derive three types of NUCFT regarding the sampling uniformity in the time domain and transformation domain. The description of the three problems is presented.

Problem 1: NUCFT(NUCFT-I): Uniform samples and non-integer frequencies:
We consider that given α, to find $f = F(\alpha)$:

$$f_j = F(\alpha)_j = \sum_{k=0}^{N-1} \alpha_k \cdot e^{i(w_k \cdot \frac{2\pi j}{N} + r(\frac{2\pi j}{N})^2)}, \tag{3.1}$$

where $j = -N/2, \cdots, N/2-1$.

Problem 2: NUCFT(NUCFT-II): Nonuniform samples and integer frequencies:
We consider that given β, to find $g = G(\beta)$:

$$g_j = G(\beta)_j = \sum_{k=-N/2}^{N/2-1} \beta_k \cdot e^{i(kx_j + rx_j^2)}, \tag{3.2}$$

where $j = 0, \cdots, N-1$.

Problem 3: NUCFT(NUCFT-III): Nonuniform samples and non-integer frequencies:
We consider that given γ, to find $h = H(\gamma)$:

$$h_j = H(\gamma)_j = \sum_{k=0}^{N-1} \gamma_k \cdot e^{i(w_k x_j + rx_j^2)}. \tag{3.3}$$

The NUCFT algorithms for the above three problems are based on the following principal steps. Any function $e^{i(cx+rx^2)}$ can be represented on any finite interval on the real line using a small number of the terms of $e^{-bx^2} e^{i(kx+rx^2)}$, and this number of the terms of q is independent of the value c. For the efficient calculation (2.12)–(2.14), we approximate each $e^{i(c_m x + rx^2)}$ in terms of q-term CFS, and approximate the value of a CFS at each x_n in terms of values at the nearest q uniformly-spaced nodes.

3.2. Derivation of algorithms

3.2.1. Relevant facts from approximation theory

The main tool of this paper is to conduct a detailed analysis of CFS of $\phi:[-\pi,\pi]\to C$ given by the formula

$$\phi(x)=e^{-bx^2}\cdot e^{i(cx+rx^2)}, \tag{3.4}$$

where $b>\frac{1}{2}$ and c are any real. We present this analysis in the lemmas and theorems of this subsection.

Lemma 3.1. For any real $b>\frac{1}{2},c$ and any integer k,

$$\left|2\int_{\pi}^{\infty}e^{-bx^2}\cos((c-k)x)dx+e^{-b\pi^2}\cdot\int_{-\pi}^{\pi}e^{i(c-k)x}dx\right|<2\pi e^{-b\pi^2}\cdot(1+\frac{1}{\pi^2}). \tag{3.5}$$

Proof. Based on Lemma 2.4, we have

$$\begin{aligned}&\left|2\int_{\pi}^{\infty}e^{-bx^2}\cos((c-k)x)dx+e^{-b\pi^2}\cdot\int_{-\pi}^{\pi}e^{i(c-k)x}dx\right|\\&\le 2\int_{\pi}^{\infty}e^{-bx^2}dx+e^{-b\pi^2}\cdot\int_{-\pi}^{\pi}e^{i(c-k)x}dx\\&<2\pi e^{-b\pi^2}(\frac{1}{\pi^2}+1).\end{aligned} \tag{3.6}$$

□.

Lemma 3.2. For any real $b>\frac{1}{2},c$ and any integer k,

$$\left|2\int_{\pi}^{\infty}e^{-bx^2}\cos((c-k)x)dx+e^{-b\pi^2}\cdot\int_{-\pi}^{\pi}e^{i(c-k)x}dx\right|<\frac{8b\pi e^{-b\pi^2}}{(c-k)^2}\cdot(1+\frac{1}{\pi^2}). \tag{3.7}$$

Proof.

$$\begin{aligned}&2\int_{\pi}^{\infty}e^{-bx^2}\cos((c-k)x)dx\\&=\frac{2}{c-k}\int_{\pi}^{\infty}e^{-bx^2}d\sin((c-k)x)\\&=-\frac{2}{c-k}e^{-b\pi^2}\sin((c-k)\pi)+\frac{4b}{c-k}\int_{\pi}^{\infty}xe^{-bx^2}\sin((c-k)x)dx.\end{aligned} \tag{3.8}$$

Also,

$$
\begin{aligned}
& e^{-b\pi^2} \cdot \int_{-\pi}^{\pi} e^{i(c-k)x} dx \\
& = \frac{e^{-b\pi^2}}{i(c-k)} [e^{i(c-k)\pi} - e^{-i(c-k)\pi}] \\
& = \frac{2e^{-b\pi^2}}{c-k} \sin((c-k)\pi).
\end{aligned} \tag{3.9}
$$

Due to (3.8) and (3.9), we have

$$
\begin{aligned}
& \left| 2\int_{\pi}^{\infty} e^{-bx^2} \cos((c-k)x)dx + \frac{2e^{-b\pi^2}}{c-k} \sin((c-k)\pi) \right| \\
& = \left| \frac{4b}{c-k} \int_{\pi}^{\infty} xe^{-bx^2} \sin((c-k)x)dx \right| \\
& \le \frac{4b}{(c-k)^2} (\pi e^{-b\pi^2} + \int_{\pi}^{\infty} e^{-bx^2} dx + \int_{\pi}^{\infty} x \cdot 2bxe^{-bx^2} dx) \\
& < \frac{8b\pi e^{-b\pi^2}}{(c-k)^2} \cdot (1 + \frac{1}{\pi^2}).
\end{aligned} \tag{3.10}
$$

□.

Theorem 3.3. The function $\phi(x) = e^{-bx^2} e^{i(cx+rx^2)}$ $(x \in (-\pi, \pi))$ can be approximated by the Chirp-Fourier series, and the approximation error can be obtained by the following inequality:

$$
\left| \phi(x) - \sum_{k=-\infty}^{+\infty} \rho_k \cdot e^{i(kx+rx^2)} \right| < e^{-b\pi^2} \cdot (4b + \frac{70}{9}), \tag{3.11}
$$

where $b > \frac{1}{2}$ and c are constants

$$
\rho_k = \frac{1}{2\sqrt{b\pi}} e^{-(c-k)^2/4b}, k = -\infty, \dots, +\infty .
$$

Proof. For $x \in (-\pi, \pi)$,

$$
\phi(x) = \sum_{k=-\infty}^{\infty} \sigma_k e^{i(kx+rx^2)}, \tag{3.12}
$$

The σ_k is the k th Chirp-Fourier coefficient for ϕ,

$$
\begin{aligned}
\sigma_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(x)e^{-i(kx+rx^2)}dx \\
&= \frac{1}{2\pi}[\int_{-\infty}^{+\infty}\phi(x)\cdot e^{-i(kx+rx^2)}dx - \int_{-\infty}^{-\pi}\phi(x)e^{-(kx+rx^2)}dx - \int_{\pi}^{+\infty}\phi(x)e^{-i(kx+rx^2)}dx] \\
&= \frac{1}{2\pi}(\sqrt{\frac{\pi}{b}}\cdot e^{-(c-k)^2/4b} + \int_{\infty}^{\pi}e^{-bx^2+i(c-k)x}dx - \int_{\pi}^{\infty}e^{-bx^2+i(c-k)x}dx) \\
&= \rho_k - \frac{1}{\pi}\int_{\pi}^{\infty}e^{-bx^2}\cos((c-k)x)dx,
\end{aligned}
\tag{3.13}
$$

where $\rho_k = \dfrac{1}{2\sqrt{b\pi}}e^{-(c-k)^2/4b}, k = -\infty, \cdots, +\infty.$

Due to (3.13), we have

$$
\left|\sigma_k - \rho_k - \frac{e^{-b\pi^2}}{2\pi}\int_{-\pi}^{\pi}e^{i(cx+rx^2)}\cdot e^{-i(kx+rx^2)}dx\right| < e^{-b\pi^2}(1+\frac{1}{\pi^2}),
\tag{3.14}
$$

$$
\left|\sigma_k - \rho_k - \frac{e^{-b\pi^2}}{2\pi}\int_{-\pi}^{\pi}e^{i(cx+rx^2)}\cdot e^{-i(kx+rx^2)}dx\right| < \frac{4be^{-b\pi^2}}{(c-k)^2}(1+\frac{1}{\pi^2}).
\tag{3.15}
$$

Due to (3.13), (3.14), and (3.15), we obtain

$$
\begin{aligned}
&\left|\phi(x) - \sum_{k=-\infty}^{+\infty}\rho_k\cdot e^{i(kx+rx^2)} - e^{-b\pi^2}\cdot e^{i(cx+rx^2)}\right| \\
&= \left|\sum_{k=-\infty}^{+\infty}\sigma_k\cdot e^{i(kx+rx^2)} - \sum_{k=-\infty}^{+\infty}\rho_k\cdot e^{i(kx+rx^2)} - e^{-b\pi^2}\cdot e^{i(cx+rx^2)}\right| \\
&= \left|\sum_{k=-\infty}^{+\infty}e^{i(kx+rx^2)}\cdot(\sigma_k - \rho_k - \frac{e^{-b\pi^2}}{2\pi}\int_{-\pi}^{\pi}e^{i(cx+rx^2)}\cdot e^{-i(kx+rx^2)})\right| \\
&< \sum_{k,|c-k|\geq 3}\frac{4be^{-b\pi^2}}{(c-k)^2}(1+\frac{1}{\pi^2}) + \sum_{k,|c-k|<3}e^{-b\pi^2}(1+\frac{1}{\pi^2}) \\
&< 4be^{-b\pi^2}\cdot\frac{9}{8}\cdot 2\sum_{k=3}^{\infty}\frac{1}{k^2} + 6e^{-b\pi^2}\cdot\frac{10}{9}.
\end{aligned}
\tag{3.16}
$$

Owing to

$$
\sum_{k=3}^{\infty}\frac{1}{k^2} < \frac{1}{9} + \int_{3}^{\infty}\frac{dx}{x^2} = \frac{1}{9} + \frac{1}{3} = \frac{4}{9},
\tag{3.17}
$$

and substituting (3.17) into (3.16), we have

$$
\left|\phi(x) - \sum_{k=-\infty}^{+\infty}\rho_k\cdot e^{i(kx+rx^2)} - e^{-b\pi^2}\cdot e^{i(cx+rx^2)}\right| < e^{-b\pi^2}\cdot(4b + \frac{60}{9}).
\tag{3.18}
$$

Thus, combined with (3.18), we obtain

$$
\begin{aligned}
&\left|\phi(x)-\sum_{k=-\infty}^{+\infty}\rho_k\cdot e^{i(kx+rx^2)}\right| \\
&\le\left|\phi(x)-\sum_{k=-\infty}^{+\infty}\rho_k\cdot e^{i(kx+rx^2)}-e^{-b\pi^2}\cdot e^{i(cx+rx^2)}\right|+\left|e^{-b\pi^2}\cdot e^{i(cx+rx^2)}\right| \\
&<e^{-b\pi^2}(4b+\frac{70}{9}).
\end{aligned}
\tag{3.19}
$$

□.

According to Theorem 3.3，the functions $e^{-bx^2}e^{i(cx+rx^2)}$ can be approximated by the CFS whose coefficients are given analytically, and the error of the approximation decreases exponentially as b increases. The coefficients ρ_k have a peak at $k=[c]$ ($[\#]$ is the nearest integer to $\#$), and decay exponentially as $k\to\pm\infty$. We keep only the $q+1$ largest coefficients, where the integer q is chosen such as

$$
q\ge 4b\pi, \tag{3.20}
$$

and the following inequality is succeeded to

$$
e^{-(q/2)^2/4b}\le e^{-b\pi^2}. \tag{3.21}
$$

Theorem 3.4 is presented, which contains the truncation error of CFS of $\phi(x)$. Additionally, the functional approximation of $e^{i(cx+rx^2)}$ is further derived.

Theorem 3.4. The function $\phi(x)=e^{-bx^2}e^{i(cx+rx^2)}(x\in(-\pi,\pi))$ can be approximated by the $q+1$ term Chirp-Fourier series, and the truncation error can be obtained by the following inequality:

$$
\left|\phi(x)-\sum_{k=[c]-q/2}^{[c]+q/2-1}\rho_k e^{i(kx+rx^2)}\right|<e^{-b\pi^2}\cdot(4b+9), \tag{3.22}
$$

where $b>\frac{1}{2}$ and c are constants and q is an even integer such that $q\ge 4b\pi.$

$$
\rho_k=\frac{1}{2\sqrt{b\pi}}e^{-(c-k)^2/4b}, k=-\infty,\dots,+\infty .
$$

Proof.

$$
\begin{aligned}
&\left|\phi(x)-\sum_{k=[c]-q/2}^{[c]+q/2-1}\rho_k e^{i(kx+rx^2)}\right| \\
&\le\left|\phi(x)-\sum_{k=-\infty}^{+\infty}\rho_k e^{i(kx+rx^2)}\right|+\left|\sum_{k=[c]+q/2-1}\rho_k e^{i(kx+rx^2)}\right|+\left|\sum_{k=[c]-q/2}\rho_k e^{i(kx+rx^2)}\right|.
\end{aligned}
\tag{3.23}
$$

Owing to

$$
\left|\sum_{k=[c]+q/2-1}\rho_k e^{i(kx+rx^2)}\right|\le\sum_{k=[c]+q/2}^{\infty}\frac{1}{2\sqrt{b\pi}}e^{-(c-k)^2/4b}<\sum_{k=q/2}^{\infty}\frac{e^{-k^2/4b}}{\sqrt{2\pi}}, \tag{3.24}
$$

$$\left|\sum_{k=[c]-q/2}\rho_k e^{i(kx+rx^2)}\right| \le \sum_{k=-\infty}^{[c]-q/2-1}\frac{e^{-(c-k)^2/4b}}{2\sqrt{b\pi}} < \sum_{k=q/2}^{\infty}\frac{e^{-k^2/4b}}{2\sqrt{b\pi}}. \tag{3.25}$$

Due to (3.24), (3.25), and Lemma 2.4, we have

$$\sum_{k=q/2}^{\infty} e^{-k^2/4b} < e^{-(q/2)^2 4b} + \int_{q/2}^{\infty} e^{-x^2/4b}dx < e^{-(q/2)^2/4b}(1+\frac{4b}{2q/2}), \tag{3.26}$$

and the combination of (3.20), (3.21), and (3.26), we obtain

$$\sum_{k=q/2}^{\infty} e^{-k^2/4b} < e^{-b\pi^2}(1+\frac{1}{\pi}). \tag{3.27}$$

Substituting (3.27) into (3.24) and (3.25), we have

$$\begin{aligned}&\left|\sum_{k=[c]+q/2-1}\rho_k e^{i(kx+rx^2)}\right| + \left|\sum_{k=[c]-q/2}\rho_k e^{i(kx+rx^2)}\right| \\ &< \frac{2e^{-b\pi^2}}{\sqrt{2\pi}}(1+\frac{1}{\pi}) < e^{-b\pi^2}\cdot\frac{10}{9},\end{aligned} \tag{3.28}$$

Substituting (3.19) and (3.28) into (3.23), we obtain

$$\begin{aligned}&\left|\phi(x) - \sum_{k=[c]-q/2}^{[c]+q/2-1}\rho_k e^{i(kx+rx^2)}\right| \\ &< e^{-b\pi^2}(4b+\frac{70}{9}+\frac{10}{9}) \\ &< e^{-b\pi^2}(4b+9).\end{aligned} \tag{3.29}$$

Due to (3.22), we have

$$\left|e^{i(cx+rx^2)} - e^{bx^2}\cdot\sum_{k=[c]-q/2}^{[c]+q/2-1}\rho_k\cdot e^{i(kx+rx^2)}\right| < e^{bx^2}\cdot e^{-b\pi^2}(4b+9) < e^{b\pi^2/m^2}\cdot e^{-b\pi^2}(4b+9). \tag{3.30}$$

Corollary 3.5. For any integer $m \ge 2$, the conditions of Theorem 3.4 are satisfied, and we have

$$\left|e^{i(cx+rx^2)} - e^{bx^2}\cdot\sum_{k=[c]-q/2}^{[c]+q/2-1}\rho_k\cdot e^{i(kx+rx^2)}\right| < e^{b\pi^2/m^2}\cdot e^{-b\pi^2}(4b+9), \tag{3.31}$$

where $x \in [-\frac{\pi}{m}, \frac{\pi}{m}]$.

□.

By extending further $x \in [-\frac{\pi}{m}, \frac{\pi}{m}]$ to $x \in [-d, d]$ in the Corollary 3.5, we obtain the following theorem:

Theorem 3.6. Let $b > \frac{1}{2}, c, d > 0$ be real numbers and $m \ge 2, q \ge 4b\pi$ be integers. Then, for any $x \in [-d, d]$, we have

$$\left| e^{i(cx+rx^2)} - e^{b(x\pi/md)^2} \cdot \sum_{k=[cmd/\pi]-q/2}^{[cmd/\pi]+q/2-1} \rho'_k e^{i(k(x\pi/md)+r(x\pi/md)^2)} \right| < e^{-b\pi^2(1-1/m^2)}(4b+9). \tag{3.32}$$

where

$$\rho'_k = \frac{1}{2\sqrt{b\pi}} e^{-(\frac{mdc}{\pi}-k)^2/4b}.$$

□.

3.2.2. Implementation of algorithms

We provide detailed algorithm descriptions in this subsection.

a) Problem 1: NUCFT(NUCFT-I): Uniform samples and non-integer frequencies
Setting $d = \pi, c = x_j, k = \mu_k + j$ in Theorem 3.6, we obtain that

$$\left| e^{i(w_k x+rx^2)} - e^{b(x/m)^2} \cdot \sum_{j=-q/2}^{q/2-1} P_{jk} \cdot e^{i((\mu_k+j)(x/m)+r(x/m)^2)} \right| < \varepsilon, \tag{3.33}$$

where $m \in Z,\ k = 0,\cdots,N-1, j = -q/2,\cdots,q/2-1, x \in [-\pi,\pi], \varepsilon = e^{-b\pi^2(1-1/m^2)} \cdot (4b+9),$ and P_{jk} is defined as follows:

$$P_{jk} = \frac{1}{2\sqrt{b\pi}} e^{-(\frac{mdc}{\pi}-k)^2/4b}. \tag{3.34}$$

We will denote by μ_k the nearest integer to $m\omega_k$.

Setting $x = \frac{2\pi l}{N}, l \in Z,$ we have

$$\begin{aligned} \tilde{f}_j &= \sum_{k=0}^{N-1} \alpha_k \cdot e^{b(\frac{2\pi l}{mN})^2} \cdot \sum_{j=-q/2}^{q/2-1} P_{jk} \cdot e^{i((\mu_k+j)(\frac{2\pi l}{mN})+r(\frac{2\pi l}{mN})^2)} \\ &= e^{(ir+b)(\frac{2\pi l}{mN})^2} \cdot \sum_{k=0}^{N-1} \alpha_k \cdot \sum_{j=-q/2}^{q/2-1} P_{jk} \cdot e^{i(\mu_k+j)(\frac{2\pi l}{mN})}, \end{aligned} \tag{3.35}$$

so that

$$\tau_l = \sum_{j,k,\mu_k+j=l} \alpha_k \cdot P_{jk}. \tag{3.36}$$

For $k = -\pi,\cdots,\pi$ and by $\{T_j\}$, a set of complex numbers defined by the formula

$$T_j = \sum_{k=-mN/2}^{mN/2-1} \tau_k \cdot e^{ik(\frac{2\pi l}{mN})}, \tag{3.37}$$

for $j = -mN/2,\cdots,mN/2-1.$

Furthermore, $\tilde{f}_j$ will be denoted as

$$\tilde{f}_j = e^{(ir+b)(\frac{2\pi l}{mN})^2} \cdot T_j . \tag{3.38}$$

Combining (3.33)–(3.38), we obtain

$$\left| f_j - \tilde{f}_j \right| \le \varepsilon \cdot \sum_{k=0}^{N-1} \left| \alpha_k \right| , \tag{3.39}$$

where $j = -N/2, \cdots, N/2-1, \{f_j = F(\alpha)_j\}.$

The implementation of NUCFT-I is presented as the following:

Step1: Input parameter $\{\alpha_0, \alpha_1, \cdots, \alpha_{N-1}\}, \{\omega_0, \omega_1, \cdots, \omega_{N-1}\}$, and then choose precision ε, b and $q = [4b\pi]$;

Step2: Compute $\mu_k = [m\omega_k], k = 0,1,2,\cdots, N-1,$ and then calculate P_{jk} and τ_l according to (3.34) and (3.36), respectively;

Step3: According to (3.37), the uniformly sampled Fourier series T_j of τ_k in $[-\pi, \pi]$ is calculated using an inverse fast Fouier transform of length $mN, j = -mN/2, \cdots, mN/2-1.$

Step4: Select the $\{T_j\}$ value at $j = -N/2, \cdots, N/2-1,$ and then get approximate values f_j by calculating $\tilde{f}_j = e^{(ir+b)(\frac{2\pi l}{mN})^2} \cdot T_j .$

b) Problem 2: NUCFT(NUCFT-II): Nonuniform samples and integer frequencies

Setting $d = N/2, c = x_j, k = v_j + k$ in Theorem 3.6, we have

$$\left| e^{ix_j n} - e^{b(2\pi n/mN)^2} \cdot \sum_{l=-q/2}^{q/2-1} Q_{jk} \cdot e^{in(v_j+k)2\pi/mN)} \right| < \varepsilon , \tag{3.40}$$

where $m \in Z, j = 0, \cdots, N-1, k \in [-N/2, N/2], \varepsilon = e^{-b\pi^2(1-1/m^2)} \cdot (4b+9),$ and Q_{jk} is defined as follows:

$$Q_{jk} = \frac{1}{2\sqrt{b\pi}} \cdot e^{-(\frac{x_j mN}{2\pi} - (v_j+k))^2/4b} . \tag{3.41}$$

We will denote by v_j the nearest integer to $x_j mN/2\pi.$

Setting $\omega_k = n, n \in Z,$ we have

$$\begin{aligned} \tilde{g}_j &= \sum_{k=-N/2}^{N/2-1} \beta_k \cdot e^{b(2\pi k/mN)^2} \cdot \sum_{l=-q/2}^{q/2-1} Q_{jk} \cdot e^{in(v_j+k)2\pi/mN} \\ &= \sum_{l=-q/2}^{q/2-1} \sum_{k=-N/2}^{N/2-1} \mu_k \cdot e^{in(v_j+k)2\pi/mN} \cdot Q_{jk} , \end{aligned} \tag{3.42}$$

so that

$$u_k = \beta_k \cdot e^{b(2\pi k/mN)^2} , \tag{3.43}$$

where $j = -N/2, \cdots, N/2-1$.

For $n = -N/2, \cdots, N/2$, a set of complex numbers $\{U_l\}$ is defined by the formula

$$U_l = \sum_{k=-N/2}^{N/2-1} u_k \cdot e^{i2\pi nl/mN}, \tag{3.44}$$

where $l = -mN/2, \cdots, mN/2-1$. Furthermore, $\tilde{g}_j$ will be denoted by the following formula:

$$\tilde{g}_j = e^{b(2\pi j/mN)^2} \cdot U_{v_j+l}. \tag{3.45}$$

Combining (3.40)–(3.45), we obtain

$$\left|g_j - \tilde{g}_j\right| \le \varepsilon \cdot \sum_{k=0}^{N-1} \left|\beta_k\right|, \tag{3.46}$$

where $j = 0, \cdots, N-1, \{g_j = G(\beta)_j\}$.

The implementation of NUCFT-II is given as the following:

Step1: Input parameter $\{\beta_{-N/2}, \cdots, \beta_{N/2-1}\}, \{x_0, x_1, \cdots, x_{N-1}\}$, and then choose precision ε, b and $q = [4b\pi]$;

Step2: Compute $v_j = [x_j mN/2\pi], j = 0, \cdots, N-1$, and then calculate Q_{jk} and μ_k according to (3.41) and (3.43), respectively;

Step3: According to (3.44), the uniformly sampled FT U_l of u_k in $[-N/2, N/2]$ is calculated using an inverse FFT of length $mN, l = -mN/2, \cdots, mN/2-1$.

Step4: Select the $\{U_l\}$ value at $j = 0, \cdots, N-1$, and then get approximate values g_j by calculating $\tilde{g}_j = e^{b(2\pi j/mN)^2} \cdot U_{v_j+l}$.

c) Problem 3: NUCFT(NUCFT-III): Nonuniform samples and non-integer frequencies

Setting $d = N/2, c = t_j/m, k = v_j + l, x = n$ in Theorem 3.6, we have

$$\left| e^{ix_j n/m} - e^{b(2\pi n/mN)^2} \cdot \sum_{l=-q/2}^{q/2-1} R_{jl} \cdot e^{i(v_j+l)\frac{2\pi n}{mN})} \right| < \varepsilon, \tag{3.47}$$

where $m \in Z, j = 0, \cdots, N-1, k \in [-N/2, N/2], \varepsilon = e^{-b\pi^2(1-1/m^2)} \cdot (4b+9)$, and R_{jl} is defined as follows:

$$R_{jl} = \frac{1}{2\sqrt{b\pi}} \cdot e^{-(t_j mN/2\pi - (\eta_j + l))^2/4b}. \tag{3.48}$$

We will denote by η_j the nearest integer to $x_j N/2\pi$.

Due to (3.3) and (3.35), we obtain

$$\tilde{h}_j = e^{(ir+b)(x_j/m)^2} \cdot \sum_{l=-q/2}^{q/2-1} R_{jl} \cdot \sum_{k=0}^{N-1} \gamma_k \cdot \sum_{j=-q/2}^{q/2-1} P_{jk} \cdot e^{(2\pi k/m^2 N)^2} \cdot e^{2\pi ikl/m^2 N}, \tag{3.49}$$

so that

$$v_j = \sum_{j,k,\mu_k+j=l} \gamma_k \cdot R_{jl} \,. \tag{3.50}$$

For $k=-N/2,\cdots,N/2$ and by $\{V_l\}$, a set of complex numbers is defined by the formula

$$V_l = \sum_{k=-mN/2}^{mN/2-1} v_k \cdot e^{(2\pi k/m^2N)^2} \cdot e^{2\pi ikl/m^2N} \,, \tag{3.51}$$

where $l=-m^2N/2,\cdots,m^2N/2-1$.

Furthermore, the $\tilde{h}_j$ will be denoted by the following formula:

$$\tilde{h}_j = e^{(i\gamma+b)(x_j/m)^2} \cdot \sum_{l=-q/2}^{q/2-1} R_{jl} \cdot V_{\eta_j+l} \,. \tag{3.52}$$

Combining (3.47)–(3.52), we have

$$\left|h_j - \tilde{h}_j\right| \le \delta \cdot \sum_{k=0}^{N-1} \left|\gamma_k\right| , \tag{3.53}$$

where $j=0,\cdots,N-1,\{h_j = H(\gamma)_j\}$, and

$$\delta = 2e^{-b\pi^2(1-2/m^2)} \cdot (4b+9). \tag{3.54}$$

The implementation of NUCFT-III is presented as the following:

Step1: Input parameter $\{\gamma_0,\cdots,\gamma_{N-1}\},\{\omega_0,\omega_1,\cdots\omega_{N-1}\}$, and then choose precision ε, b and $q=[4b\pi]$;

Step2: Compute $\eta_j = [x_j N/2\pi], j=0,\cdots,N-1$, and then calculate R_{jl} and v_j according to (3.48) and (3.50), respectively;

Step3: According to (3.51), the uniformly sampled Fourier series V_l of v_k in $[-N/2,N/2]$ is calculated using an inverse FFT of length $m^2N, l=-m^2N/2,\cdots,m^2N/2-1$.

Step4: Select the $\{V_l\}$ value at $j=0,\cdots,N-1$, and then get approximate values h_j by calculating $\tilde{h}_j = e^{b(x_j/m)^2} \cdot e^{i\gamma(x_j/m)^2} \cdot \sum_{l=-q/2}^{q/2-1} R_{jl} \cdot V_{\eta_j+l}$.

3.3. Computation analysis of algorithms

Tables 1–3 display the comparison of the NUCFT computations with the direct computation for the three instances of uniform time but nonuniform transform domain, uniform transform domain but nonuniform time, and nonuniform time and transform domain, respectively.

Table 1. Computational analysis of NUCFT-I.

	Multiplication	Addition
Direct	$4N^2$	$2N^2$
NUCFT-I	$4qN + 0.5mN\log mN + 4N$	$2qN + mN\log mN + 2N$
Computational savings	$4N^2 - 4qN - 0.5mN\log mN - 4N$	$2N^2 - 2qN - mN\log mN - 2N$

Table 2. Computational analysis of NUCFT-II.

	Multiplication	Addition
Direct	$4N^2$	$2N^2$
NUCFT-II	$4qN + 0.5mN\log mN + 4N$	$2qN + mN\log mN + 2N$
Computational savings	$4N^2 - 4qN - 0.5mN\log mN - 4N$	$2N^2 - 2qN - mN\log mN - 2N$

Table 3. Computational analysis of NUCFT-III.

	Multiplication	Addition
Direct	$4N^2$	$2N^2$
NUCFT-III	$4qN + 0.5m^2N\log m^2N + 4N$	$2qN + m^2N\log m^2N + 2N$
Computational savings	$4N^2 - 4qN - 0.5m^2N\log m^2N - 4N$	$2N^2 - 2qN - m^2N\log m^2N - 2N$

The results show that the computational accuracy of the three NUCFT is similar, and the proposed algorithms have lower computational than direct calculation under the condition of guaranteeing the computational accuracy.

4. Conclusions

In this paper, we have described three algorithms for computing DCFT for non-equispaced data, which is based on the interpolation formulae to transform function values from uniform to nonuniform points. The computational complexity of each algorithm is $\mathrm{O}(\mathrm{N}\log\mathrm{N} + \mathrm{N}\log(1/\varepsilon))$, where N is the data length and ε is the computational accuracy. It shows that the derived approach is effective for computing nonuniform DCFT. The proposed algorithms can be viewed as generalizations of DCFT, and will have a broad range of applications in many branches of mathematics, science, and engineering. One of the specific applications is the inverse synthetic aperture radar (ISAR) imaging. In the signal extraction of ISAR imaging processing, the nonuniform rotation of the target can lead to nonuniformity of the processed data and also result in phase errors in the extracted signal. However, nonuniform CFT can process nonuniform data according to actual needs, so the algorithm proposed in this article can be used for signal extraction in ISAR to eliminate phase errors caused by nonuniform rotation. In future research, we will further investigate the application of the proposed algorithm.

Author contributions

Yannan Sun: Supervision, Resources, Conceptualization, Funding acquisition, Writing - review & editing. Wenchao Qian: Formal analysis, Methodology, Writing – original draft, Writing – review & editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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