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Research article

The method of judging satisfactory consistency of linguistic judgment matrix based on adjacency matrix and 3-loop matrix

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Abstract: Language phrases are an effective way to express uncertain pieces of information, and easily conforms to the language habits of decision makers to describe the evaluation of things. The consistency judgment of a linguistic judgment matrices is the key to analytic hierarchy process (AHP). If a linguistic judgment matrix has a satisfactory consistency, then the rank of the decision schemes can be determined. In this study, the comparison relation between the decision schemes is first represented by a directed graph. The preference relation matrix of the linguistic judgment matrix is the adjacency matrix of the directed graph. We can use the $n-1$st power of the preference relation to judge the linguistic judgment matrix whether has a satisfactory consistency. The method is utilized if there is one and only one element in the $n-1$st power of the preference relation, and the element 1 is not on the main diagonal. Then the linguistic judgment matrix has a satisfactory consistency. If there are illogical judgments, the decision schemes that form a 3-loop can be identified and expressed through the second-order sub-matrix of the preference relation matrix. The feasibility of this theory can be verified through examples. The corresponding schemes for illogical judgments are represented in spatial coordinate system.

Keywords: adjacency matrix; linguistic judgment matrix; satisfactory consistency; 3-loop matrix

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1. Introduction

A decision analysis is the process in which decision-makers select one or more decision options by analyzing alternative decision schemes based on their values, individual preferences, and cognitive structure. In general, a basic decision-making process includes identifying decision problems, determining the decision objectives, and selecting the decision schemes. At first, certain mathematical models are used to solve simple decision-making problems. With the complexity of social life, the uncertainty of economic activities, and the increase of unknown factors in the decision-making process, it has become increasingly difficult for decision experts to solely rely on simple mathematical models to solve a problem. Based on this situation and the universality of decision-making problems in people's daily lives, finding more effective ways to express decision-making information is an inevitable requirement to develop decision-making science and human activities.

In the process of developing decision theory, people's understanding of problems has broken down their cognition. There are many uncertain concepts in people's thinking, such as if the weather should be good, if the plan should be good at present, and if the mood should be good. The objects described by these uncertain concepts cannot be simply described by "yes" or "no". Due to the ambiguity of the membership objects of these concepts and their importance in the decision-making process, Zadeh [1] put forward the concept of a fuzzy set in 1965 and a series of theoretical have been proposed, which solved the problem of ambiguous membership relations. In fact, when expressing uncertain information, people tend to prefer language expression. Although language variables are not as precise as traditional numerical variables, they are relatively close to natural language and human cognitive habits. Therefore, this natural language variable that expresses decision information through qualitative means is an effective tool to express uncertain information. In some complex decision-making environments, decision experts may exhibit hesitation between several decision values, therefor relying on a single linguistic term cannot accurately express the expert's hesitation. In view of this, Torra [2] proposed hesitant fuzzy set (HFSs), which allowed decision experts to have multiple membership degrees when evaluating decision information. However, in the actual decision-making process, the personal knowledge and experience of a decision expert made different membership degrees have different degrees of importance. The proposal of a probability language technical terminology (PLTSs) solved this problem well. In the decision-making process, the probability information corresponding to each membership degree was given, which effectively expressed the importance between different membership degrees [3]. Because of the uncertainty of things, it is difficult for experts to express their views in certain terms. The uncertainty language technical terminology (ULTs) [4], the hesitant fuzzy language set [5] (HFLTSs), and the extended hesitant fuzzy language set (EHFLTSs) can all be effective tools for decision experts to express uncertain information. However, in the process of expressing uncertain information, decision makers cannot clearly express the range of decision information. In fact, decision makers are accustomed to expressing themselves using language modifiers, such as "very good weather, very good mood, possibly effective scheme, etc.", in which "very" and "possibly" are language modifiers. Zadeh [6] proposed the concept of a language correction set, and transformed the initial fuzzy set by modifying the shape of the membership function.

Many achievements have been made in determining the satisfactory consistency of linguistic judgment matrices [7–13]. From the current research results, the definition of a satisfactory consistency of linguistic judgment matrices is mature, though there are relatively few definitions that directly determine what a satisfactory consistency is. Lian et al. [7] proposed a multi-granularity language reasoning method, which mainly dealt with incomplete information. Chen et al. [8] interpreted a proportional hesitant fuzzy linguistic term set (PHFLTS) from the perspective of T2 fuzzy logic and proposed a new PHFLTS encoding method based on conceptualization. Zhang et al. [9] proposed a solution method for group consensus decision-making problems based on a multiplication language and an adaptive consistency model of fuzzy information granulation. Dong et. al. [10] stated that decision makers were comfortable to provide the evaluation information with either imprecision or linguistic evaluation variables, which were intuitive and flexible approaches to describe a decision maker's fuzziness and qualitative evaluation information. Liu, Song, and Yang [11] proposed a prospect cross-efficiency (PCE) model to determine the sorting order of decision making units (DMUs). Jin et al. [12] stated that decision makers not only focused on the utility results derived by the alternatives they chose, but also focused on the utility results that may be rewarded if they choose other alternatives, while they avoided choosing alternatives that they would regret. Therefore, as an important behavioral decision-making theory, regret theory can be utilized to deal with the behavior of regret aversion. Wang et al. [13] investigated an interval type-2 fuzzy multi-attribute decision making method by combining the regret theory and a projection model.

In recent years, other consistency determination methods for linguistic judgment matrices have also achieved some results [14–22]. Based on the generalized distance measure of hesitant fuzzy language, Guo et al. [14] defined a compatibility measure of the hesitant fuzzy language preference relationships to handle consistency problems. Grogelj et al. [15] discussed the acceptability of the judgment matrix based on the decision matrix provided by each decision-maker. Wu et al [17] discovered a method that derived an acceptable consistency pairwise comparison matrix (PCM) (consistent ratio (CR) CR<0.1), which eliminated all logical errors and made the departure between the modified PCM and the original PCM even smaller than existed studies. Morente-Molinera et al. [18] presented a new multi-criteria group decision-making method suitable for non-static frameworks, which allowed experts to use the preferred terms of the preferred language label set through the use of multi granularity fuzzy language modeling. Zhang et al. [19] introduced the definition of the intuitionistic fuzzy linguistic preference relation (IFLPR) and its transitive properties, and provided a group decision-making method based on genetic algorithm and incomplete information. Li et al. [20] proposed a consistency driven method using personalized individual semantics (PISS) to manage distributed language preference relationships (DLPRS). This method can estimate unknown elements in incomplete DLPRS and obtain personalized numerical meanings of language expression for decision-makers, thus ensuring optimal consistency between incomplete DLPRS and unknown elements. Mishra et al. [21] provided a weighted aggregated sum product assessment (WASPAS)based judgment method on the basis of the interval valued intuitionistic fuzzy set (IVIFS). In the process of calculating weights, the decision experts' weights and criteria weights were calculated on the basis of interval valued intuitionistic fuzzy information measures. Su et al. [22] studied the use of a Pairwise comparison algorithm to build a multi-criteria decision model with incomplete verbal preference relations. On the basis of the concept of interval valued hesitant fermatean fuzzy sets (IVHFFS), Arunodaya et al. [23] proposed the aggregation operator (AO) to aggregate interval valued hesitant fermatean fuzzy information, and discussed some properties of the operator in detail. Deveci et al. [24] proposed a new entropy based weighted aggregate product assessment (WASPAS) method and a multi criteria decision-making method combined with interval type 2 hesitant Fuzzy set (IT2HFS), and tested it with a specific case of an all service operator in Türkiye. Deveci et al. [25] used the Delphi method based on an interval type 2 Fuzzy set to rank the indicators that affect the location of vehicle crushing facilities. Wang et al. [26] extended the bonferroni mean(BM)operator to the language term with a weakening hedging (LTWH) environment and proposed the LTWHBM operator to describe uncertainty in HCW management. Wu et al. [27] introduced a geometric language scale characterized by the proportional relationship between progressive levels and their corresponding fuzzy meanings. An aggregation method based on the geometric linguistic scale is proposed to deal with decision matrices with a direct linguistic evaluation, a complete Pairwise comparison, and a partial Pairwise comparison. Yang et al. [28] proposed an enhanced large-scale, group decision-making method that combined the proportional hesitant fuzzy language technical terminology (PHFLTS) and the cumulative prospect theory (CPT). Strauch et al [29] proposed a model based on the hesitant fuzzy linguistic term sets with the quality function deployment technique (HFLTS-QFD) method to support the formulation of the sustainable supplier development programs (SSDP), which combined the hesitant fuzzy language technical terminology with a quality function deployment technology. Fan et al. [30] explored a consensus model based on limited trust propagation, which considered an individuals' attitudes towards modifying preference relationships in a social network environment with uncertain preferential information.

Wu [17] noted that a logical consistency was quite similar to an ordinal consistency (or transitivity) in [31] and a weak consistency in [32]. If $x_i \succ x_j \succ x_k$, then $x_i \succ x_k$ should be satisfied. However, if $x_k \succ x_i$ when $x_i \succ x_j \succ x_k$, then the preference judgments are called logically inconsistent i.e., intra-sensitivity. Therefore, the logical inconsistency can be defined as $x_i \succ x_j \succ x_k \succ x_i$, which represents a directed circuit. Actually, logical consistency requires the decision maker to be consistent for each pairwise comparison, which may be unrealistic in real-life applications, especially for a higher order PCM. Some researchers [33] considered logical consistency as the acceptable consistent level, while others [34,35] considered numerical consistency as the acceptable consistent level. In our opinion, both ideas are one-sided. Generally speaking, logical consistency and acceptable numerical consistency do not have the necessary relationships. However, in most cases, a logically inconsistent PCM is usually unacceptable in a numerical consistency.

Gou [14] provided an adaptive consensus model based on fuzzy information granulation (fuzzy IG). First, a granular representation of linguistic terms is concerned with the triangular fuzzy formation of a family of information granules over the given analytical hierarchy process (AHP) numerical scales. On this basis, the individual consistency and group consensus measure indices using the fuzzy granulation technique are constructed. Then, the optimal cut-off points of the fuzzy information granules are obtained by establishing a multi-objective optimization model together with a multi-objective particle swarm optimization (MOPSO) algorithm. A novel group consensus decision-making approach is proposed where the consensus reaching process (CRP) is achieved by adaptively adjusting the individual preferences through the optimization of the cut-off points. After conflict elimination, the obtained group preference gives the ranking of the alternatives.

However, most of these studies use either a consistent ratio (CR) or directed graphs to either determine the satisfactory consistency of the language judgment matrix, or directly study methods to improve the consistency. Few researchers have expressed illogical decision-making schemes, where logical consistency and acceptable numerical consistency were discussed. The acceptable consistency level was the criterion for judgment [17]. An adaptive consensus model was established to judge the consistency [14]. The consistency discussed in this paper mainly manifests in the transitivity of the superiority and the inferiority relationships between the decision schemes, that is a judgment matrix with a satisfactory consistency can determine the ranking of schemes, that is, if $x_i \succ x_j \succ x_k$, then $x_i \succ x_k$.

This paper mainly discusses the method of judging the satisfactory consistency of the linguistic judgment matrix based on the adjacency matrix. The preference relation matrix of the linguistic judgment matrix is regarded as a directed graph. The preference relation matrix $n-1$st power is used to judge whether the linguistic judgment matrix has a satisfactory consistency. If only one element in the preference relation matrix $n-1$st power, which is not in the main diagonal, is 1, then the linguistic judgment matrix has a satisfactory consistency. If the linguistic judgment matrix does not have a satisfactory consistency, then we can determine whether the second-order sub-matrices of the preference relationship matrix are 3-loop matrices to find the illogical judgment. The illogical decision schemes can be represented by a software.

The remainder of this paper is organized as follows:

First, we start with notations, definitions, theorems of the preference matrix, and a adjacency matrix (in Section 2). In Section 3, we present the idea and process of the consistency method, including a satisfactory consistency based on the adjacency matrix, and the representation of illogical judgments. Numerical examples are provided in Section 4 to illustrate and compare the proposed model. The paper is summarized and concluded in Section 5.

2. Mathematical concepts

$I=\{1,2,\cdots,n\}$ and $U=\{0,1,2,\cdots,T\}$ are two sets, where T is an even number. The preference information of a pairwise comparison given by the decision makers can be described by a matrix $\boldsymbol{P}=(p_{ij})_{n\times n}$. The objects in the matrix are selected from the linguistic term set $S=\{s_i|i\in U\}$ as the evaluation results of x_i and x_j. The number of objects in the matrix is called a granularity of the linguistic term set. For example, a linguistic term set with a 13 granularity can be described as follows $S=\{$ s_0=DD =absolute difference, s_1=VHD =quite poor, s_2=HD =very poor, $s_3=MD$ =weak, s_4=LD =Poor, s_5=VLD =slightly poor, s_6=AS =equivalent, s_7=VLP =slightly better, s_8=LP =better， s_9=MP =good, s_{10}=HP =very good, s_{11}=VHP =quite good, $s_{12}=DP$ =absolutely good$\}$.

$S^L=\{s_0,s_1,\cdots,s_{\frac{T}{2}-1}\}$, $S^U=\{s_{\frac{T}{2}+1},\cdots,s_T\}$, $S^L_{\frac{T}{2}}=\{s_0,s_1,\cdots,s_{\frac{T}{2}}\}$, $S^U_{\frac{T}{2}}=\{s_{\frac{T}{2}},\cdots,s_T\}$ are four linguistic term sets.

Definition 1. [36–38] A linguistic judgment matrix on the finite object set $X=\{x_1,x_2,\cdots,x_n\}$ for the linguistic term set $S=\{s_i|i\in U\}$ is defined as $P=(p_{ij})_{n\times n}$, where

$$p_{ij}\in S;\, p_{ii}=s_{\frac{T}{2}};\, p_{ij}=s_k,\, p_{ji}=neg(s_k)\,; \tag{1}$$

for all $i,j\in I$.

Definition 2. In the linguistic judgment matrix $P=(p_{ij})_{n\times n}$, if

$$p_{ij}=s_{\frac{T}{2}},$$

Then x_i and x_j are called equivalent objects and are denoted as $x_i\sim x_j$.

Table 1. Mathematical symbol.

Mathematical	Symbol significance
$x_i\sim x_j$	x_i equivalent to x_j
$x_i\succ x_j$	x_i superior to x_j
$x_i\prec x_j$	x_j superior to x_i

In this paper, we mainly discuss the situation where there is no equivalence between the decision schemes.

Definition 3. In the linguistic judgment matrix $P=(p_{ij})_{n\times n}$, if the dominance relation of the decision schemes is transitive and there is no loop phenomenon except for the equivalent schemes, then the linguistic judgment matrix $P=(p_{ij})_{n\times n}$ is said to have a satisfactory consistency.

Definition 4. In the linguistic judgment matrix $P=(p_{ij})_{n\times n}$, if the degree of the superiority between the decision schemes is also transitive, that is, the degree to which x_i is superior to x_k is equal to the degree to which x_i is superior to x_j plus the degree to which x_j is superior to x_k, then the linguistic judgment matrix $P=(p_{ij})_{n\times n}$ is said to have a complete consistency.

Definition 5. If there is a phenomenon of $x_1 \succ x_2 \succ x_3 \succ x_1$ in the comparison results between the decision schemes, then the comparison result of x_1, x_2, x_3 which is further called an illogical judgment $x_1 \succ x_2 \succ x_3 \succ x_1$ is called a 3-loop formed by x_1, x_2, x_3.

If the linguistic judgment matrix has a satisfactory consistency, then the dominance relation of the decision schemes is transitive. If the linguistic judgment matrix has a complete consistency, then the degree of the dominance relation is transitive. The linguistic judgment matrix provided by the decision makers has a higher requirement of a complete consistency than a satisfactory consistency. A satisfactory consistency requires that there is a dominance relation between the decision schemes, while a complete consistency requires that the degree of a dominance relation between the decision schemes is reflected.

Definition 6. $Q=(q_{ij})_{n\times n}$ is called a preference relation matrix of a linguistic judgment matrix $P=(p_{ij})_{n\times n}$, where:

$$q_{ij}=\begin{cases}1 & if\ p_{ij}\in S^{U}\\ 0 & if\ p_{ij}\in S^{L}\end{cases}. \tag{2}$$

For example, a linguistic judgment matrix given by a decision-maker using a language phrase evaluation set with a granularity of 7 is

$$P=\begin{pmatrix} s_4 & s_2 & s_5 & s_{5.5}\\ s_6 & s_4 & s_6 & s_3\\ s_3 & s_2 & s_4 & s_6\\ s_{2.5} & s_5 & s_2 & s_4 \end{pmatrix},$$

the corresponding preference relation matrix is

$$Q=\begin{pmatrix} 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In the preference relation, 1 signals that the scheme corresponding to the row is superior to the scheme corresponding to the column. 0 signals that the scheme corresponding to the row is inferior to the scheme corresponding to the column, but the degree of the dominance relation is not given. The comparison result between the scheme and itself is represented by 0; this does not mean that it is inferior to itself, but rather that it does not compare itself with itself. To obtain the ranking of the decision schemes and participate in the group decision-making, the provided linguistic judgment matrix must have a satisfactory consistency.

Directed graphs. $D=(V,E)$ is a directed graph, where V is a set of vertices and E is a set of ordered pairs (u,v) of distinct vertices called arcs. Note that we do not allow loops and multiple edges or arcs.

Adjacency matrix. Given a directed graph $D=(V,E)$, $V=\{v_1,v_2,\cdots,v_n\}$, where a_{ij} is the number of edges adjacent to v_j from v_i, $A=(a_{ij})_{n\times n}$ is called the adjacency matrix of D.

The comparison relation is represented by a directed graph, where the scheme is a vertex in the directed graph and the comparison relation between the schemes is represented by an edge. If scheme x_i is better than scheme x_j, then there is an edge starting from x_i and ending from x_j. This way, the comparison results of all schemes can be represented by a directed graph. The pairwise comparison of all schemes is a single comparison and no comparison is made to itself. The directed graph corresponding to the comparison relationship is a simple graph, without rings and parallel edges. Moreover the adjacency matrix corresponding to the directed graph represents the dominance relation of the schemes. In the adjacency matrix, $a_{ij}^{(1)}=1$ means that x_i is better than x_j. Therefore, the elements in the adjacency matrix are only 0 and 1, and the elements on the main diagonal are all 0. From the above analysis, we can see that the preference relation matrix of the linguistic judgment matrix is the corresponding adjacency matrix when the judgment matrix is regarded as a directed graph.

Let $A=(a_{ij})_{n\times n}$ be the adjacency matrix of the directed graph D, $V=\{v_1,v_2,\cdots,v_n\}$, then the element $a_{ij}^{(l)}$ in $A^l=(a_{ij}^{(l)})(l\geq 1)$ is the number of paths with length l from v_i to v_j, and $\sum_{i=1}^{n}\sum_{i=1}^{n}a_{ij}^{(l)}$ is the total number of paths (including loop) with the length l in D, where $\sum_{i=1}^{n}a_{ii}^{(l)}$ is the number of loop with the length l in D. According to the definition of a satisfactory consistency of the linguistic judgment matrix, if the linguistic judgment matrix has a satisfactory consistency, then the order of dominance relation of the schemes can be obtained. There is no illogical judgment between the schemes, and the element $a_{ij}^{(l)}$ in $A^l=(a_{ij}^{(l)})(l\geq 1)$ is the number of paths with a length of l from v_i to v_j. If the linguistic judgment matrix has a satisfactory consistency, then $x_{(1)}\succ x_{(2)}\succ\cdots\succ x_{(n)}$ (the order of dominance relation of the schemes) can be obtained. Then, there is only one path from $x_{(1)}$ to $x_{(n)}$ and the only element in $A^l=(a_{ij}^{l})(l\geq 1)$ is 1 which is not on the main diagonal, while all other elements are 0.

Definition 7. [17] For a given pairwise comparison matrix $P=(p_{ij})_{n\times n}$, if CR<0.1, then P is said to have acceptable numerical consistency.

Definition 8. [17] For a given pairwise comparison matrix $P=(p_{ij})_{n\times n}$, we say that P has reached acceptable consistent level, if and only if P has an acceptable numerical consistency and its preference matrix R is logically consistent.

Definition 9. If the advantages and disadvantages of each scheme have transitivity, and there is no cyclic phenomenon except for the equivalent schemes, then the judgment matrix $P=(p_{ij})_{n\times n}$ is said to have a satisfactory consistency.

Some researchers consider logical consistency as the acceptable consistent level, while others consider numerical consistency as the acceptable consistent level. In our opinion, both ideas are one-sided. Generally speaking, logical consistency and acceptable numerical consistency do not have necessary relationships. However, in most cases, a logically inconsistent pairwise comparison matrix is usually unacceptable in numerical consistency. In this article, the satisfactory consistency mainly refers to the transitivity of relationship between the advantages and disadvantages of the decision-making schemes. In other words, based on the decision matrix provided by the decision-maker, the order of superiority and inferiority among decision schemes can be determined.

3. A method for determining satisfactory consistency of linguistic judgment matrix

Theorem 2. The sufficient and necessary condition for a linguistic judgment matrix to have a satisfactory consistency is that there is one and only one element in its preference relation matrix, $n-1$ power $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$, and that element 1 is not on the main diagonal.

Proof. **Necessity** According to the definition of a satisfactory consistency, if the linguistic judgment matrix has a satisfactory consistency, there is a relation between the advantages and disadvantages $x_{(1)} \succ x_{(2)} \succ \cdots \succ x_{(n)}$ (the order of schemes) all of the schemes. $x_{(i)} \succ x_{(j)}$ represents that the decision scheme $x_{(i)}$ is superior to decision scheme $x_{(j)}$. There is no illogical judgment in the comparison results between schemes. In the directed graph corresponding to the linguistic judgment matrix, there exists a directed edge with $x_{(i)}$ as the starting point and $x_{(j)}$ as the ending point. The comparison relation is represented by a directed graph. Then there is only one path with the length of $n-1$ from the best scheme to the worst scheme in the directed graph. There is only element is 1 in the preference relation matrix $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$ of the linguistic judgment matrix, and that element is not on the main diagonal. If the element of the main diagonal is 1, then it means that the comparison result is a loop. In the comparison results, the elements of the starting and ending points cannot be the same.

Sufficiency If there is one and only one element that is not on the main diagonal, which is 1, then the $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$ and the other elements are 0. Consider the comparison relationship of the decision schemes as a directed graph; then, the preference relationship matrix of the linguistic judgment matrix is the adjacency matrix after the comparison relationship of the decision schemes as a directed graph. The element $a_{ij}^{(1)}$ in $A^l=(a_{ij}^l)(l\geq 1)$ is the number of paths with the length l from x_i to x_j. One and only one element in $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$ that is not on the main diagonal is 1, which means that there is a path with length $n-1$ in the directed graph corresponding to the linguistic judgment matrix. In the directed graph, the scheme corresponding to the start point of the edge is better than the scheme corresponding to the end point of the edge. In the path with the length $n-1$, the scheme corresponding to the start point is the best, the scheme corresponding to the end point is the worst, and the middle is the order of the schemes. Then, the ranking of all the decision schemes can be obtained.

From the above theorem, if the linguistic judgment matrix has a satisfactory consistency, then there is only one element that is not on the main diagonal, namely 1 in $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$. If multiple elements are 1, then the judgment matrix does not have a satisfactory consistency. The following are the cases where multiple elements are 1.

(1) Multiple 1 appear in positions that are not on the main diagonal

If there are multiple 1’s in $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$ of the preference relation matrix, then one 1 corresponds to one path and multiple 1’s indicate multiple paths. According to the meaning expressed by the $n-1$

power of the adjacency matrix of a digraph, a multiple dominance relation appears in comparison results. Whether there are equivalent schemes, the comparison results of the schemes are not unique. If the linguistic judgment matrix has a satisfactory consistency, then there is only one order of dominance relation between decision schemes. If multiple 1's correspond to multiple orders of the dominance relation, then it indicates that the linguistic judgment matrix does not have satisfactory consistency and there must be an illogical phenomena in the judgment process. It is necessary to identify the illogical judgments and correct them to obtain a judgment matrix with a satisfactory consistency.

(2) 1 appears on the main diagonal

1 appears on the main diagonal of $Q^{n-1}=(q_{ij}^{n-1})_{n\times n}$, which means that a loop with a length of $n-1$ appears in the directed graph corresponding to the linguistic judgment matrix and one of the schemes in the dominance relation cannot appear twice. A loop means that the starting point and the end point of the comparison are the same, which obviously does not conform to the actual comparison situation. This indicates that the judgment is unreasonable in the comparison process. Then, the comparison result of the schemes is a loop. If 1 appears on the main diagonal of the preference relation matrix and is less than $n-1$ power, then the corresponding comparison result of the schemes is also a loop. For example, if 1 appears on the main diagonal $Q^3=(q_{ij}^3)_{n\times n}$, the comparison result of four schemes is a loop.

Assuming

$$Q=\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

is the preference relation matrix of a certain linguistic judgment matrix, then

$$Q^2=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$Q^2\times Q=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this example, there are multiple elements of 1 in $Q^3=(q_{ij}^3)_{n\times n}$. Therefor the judgment matrix does not have satisfactory consistency and cannot obtain the dominance relation of the schemes.

In the above example, there are four decision schemes. Using enumeration methods, we can obtain the following paths with a length of 3: $x_1\to x_4\to x_3\to x_2$, $x_1\to x_2\to x_4\to x_3$, $x_2\to x_4\to x_3\to x_2$, $x_4\to x_3\to x_2\to x_4$, and $x_1\to x_3\to x_2\to x_4$. The starting point and ending point of are the same in $x_2\to x_4\to x_3\to x_2$ and $x_4\to x_3\to x_2\to x_4$, which are called loops in graph theory. In this way, the element with 1 on the main diagonal of $Q^3=(q_{ij}^3)_{n\times n}$ represents a loop. For example, the loop corresponding to q_{22}^3 in $Q^3=(q_{ij}^3)_{n\times n}$ is $x_2\succ x_4\succ x_3\succ x_2$. Additionally, it can be inferred from the result of matrix multiplication that in Q^3, the 1 in the second row and second column corresponds to x_2 in

Q^3 and the element at this position is obtained by multiplying and adding the second row in Q^2 and the second column in Q ,that is $q_{21}^2 q_{12} + q_{22}^2 q_{22} + q_{23}^2 q_{32} + q_{24}^2 q_{42} = q_{22}^3$, where $q_{23}^2 = 1, q_{32} = 1$ and the other elements are 0, then, $q_{23}^2 q_{32} = 1$, so in the comparison relationship, the element before x_2 is x_3. From $q_{23}^2 = 1$ and $q_{23}^2 = 1 = q_{21}q_{13} + q_{22}q_{23} + q_{23}q_{33} + q_{24}q_{43}$, only $q_{24}q_{43}$ is 1 and the other elements are 0. Therefor it can be concluded that the element before x_3 in the comparison relationship is x_4, so the comparison relationship corresponding to the loop is $x_2 \succ x_4 \succ x_3 \succ x_2$.

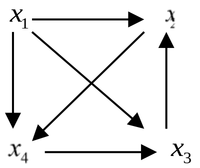

Figure 1. Directed graph corresponding to comparison relations.

The elements of the matrix obtained by multiplying $Q = (q_{ij})_{n \times n}$ may have values greater than 1, such as:

$$Q = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ x_1 & 0 & 1 & 1 & 1 \\ x_2 & 0 & 0 & 1 & 1 \\ x_3 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 1 & 0 \end{matrix} \quad Q^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad q_{13}^2 = 2.$$

There are two comparison results for the three decision schemes, including decision schemes x_1, x_3, which can be obtained by multiplying the matrix by rule $q_{13}^2 = q_{11}q_{13} + q_{12}q_{23} + q_{13}q_{33} + q_{14}q_{43}$ and $q_{12} = 1, q_{23} = 1, q_{14} = 1, q_{43} = 1$. The comparison including the decision schemes x_1, x_3 are $x_1 \succ x_2 \succ x_3$ and $x_1 \succ x_4 \succ x_3$.

$$Q^2 \times Q = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

It can be concluded that the linguistic judgment matrix has a satisfactory consistency. In the process of determining whether the linguistic judgment matrix has a satisfactory consistency, the following theorem can also be used.

Theorem 3. The necessary condition for a linguistic judgment matrix $P = (p_{ij})_{n \times n}$ to have satisfactory consistency is that the sum of the elements in $Q^m (1 < m < n)$ is C_n^{m+1}.

Proof. If the linguistic judgment matrix $P = (p_{ij})_{n \times n}$ has a satisfactory consistency, then the comparison relation between the schemes can be $x_{(1)} \succ x_{(2)} \succ \cdots \succ x_{(n)}$. The comparison relation formed by $m+1$ elements can be judged from the elements of $Q^m = (q_{ij}^m)_{n \times n}$. The number of comparison relations formed by $m+1$ elements is to select $m+1$ elements from n, that is, C_n^{m+1}.

If the sum of $Q^m = (q_{ij}^m)_{n \times n} (1 < m < n)$ elements is not C_n^{m+1} during the calculation of Q^{n-1}, then the linguistic judgment matrix $P = (p_{ij})_{n \times n}$ must not have a satisfactory consistency.

The following is a general method for finding the corresponding elements of a cycle.

Step 1: In Q^{n-1}, if $q_{ii}^{n-1} = a$ and $a = 1$, then there is one loop, and if a is not equal to 1, then there are multiple loops;

Step 2: $q_{ii}^{n-1}=1=q_{i1}^{n-2}q_{1i}+q_{i2}^{n-2}q_{2i}+\cdots+q_{in}^{n-2}q_{ni}$,

(1) Assuming $q_{ik}^{n-2}q_{kj}=1$ and all other parts are 0, it can be concluded that the comparison scheme before x_j is x_k, and $x_k \succ x_j$;

(2) If $q_{ii}^{n-1}=a$ and part of $q_{i1}^{n-2}q_{1i}+q_{i2}^{n-2}q_{2i}+\cdots+q_{in}^{n-2}q_{ni}$ is a;

If $q_{ik}^{n-2}q_{kj}=a$,then at least one of q_{ik}^{n-2} and q_{kj} is greater than 1; then, q_{ik}^{n-2} must be greater than 1 and $q_{ik}^{n-2}=a$, go to step 3;

(3) If $q_{ii}^{n-1}=a$ and the sum of several parts is a, then the part with a value of 1 shall be calculated according to step 2 (1), and the part with a value of not 1 shall be calculated according to step 2 (2);

Step 3: (1) Assuming that $q_{ik}^{n-2}=1=q_{i1}^{n-3}q_{1k}+q_{i2}^{n-3}q_{2k}+\cdots+q_{in}^{n-3}q_{nk}$, $q_{il}^{n-3}q_{lk}=1$, and all other terms are 0, the comparison scheme before x_k is x_l, and $x_l \succ x_k \succ x_j$;

(2) If a portion of $q_{ik}^{n-2}=a$ and $q_{ik}^{n-2}=a=q_{i1}^{n-3}q_{1k}+q_{i2}^{n-3}q_{2k}+\cdots+q_{in}^{n-3}q_{nk}$ is a. If $q_{il}^{n-3}q_{lk}=a$, then at least one of q_{il}^{n-3} and q_{lk} is greater than 1, then, q_{il}^{n-3} must be greater than 1 and $q_{il}^{n-3}=a$, go to step 4;

(3) If the sum of several parts is 1, then the part with a value of 1 shall be calculated according to step 2 (1), and the part with a value of not 1 shall be calculated according to step 2 (2);

Step 4: Repeat the above process until: $q_{im}^{2}=1=q_{i1}q_{1m}+q_{i2}q_{2m}+\cdots+q_{in}q_{nm}$. Assuming that $q_{ip}q_{pm}=1$ and all other terms are 0, the comparison scheme before x_m is x_p, and $x_p \succ x_m$, and the final cycle obtained is $x_i \succ x_p \succ x_m \succ \cdots \succ x_l \succ x_k \succ x_i$.

The method of the dominance relation of the decision schemes can also be derived in $\boldsymbol{Q}^{n-1}=(q_{ij}^{n-1})_{n\times n}$.

Step 1: If $q_{ij}^{n-1}=1$, then the optimal scheme is x_i and the worst scheme is x_j;

Step 2: $q_{ij}^{n-1}=1=q_{i1}^{n-2}q_{1j}+q_{i2}^{n-2}q_{2j}+\cdots+q_{in}^{n-2}q_{nj}$, then there is only one option that is not 0 and is 1. Assuming $q_{ip}^{n-2}q_{pj}=1$, then the decision scheme that ranks before x_j is x_p;

Step 3: Assuming $q_{ip}^{n-2}q_{pj}=1$ in Step 2, it can be inferred that neither q_{ip}^{n-2} nor q_{ip}^{n-2} is 0, and $q_{ip}^{n-2}=q_{i1}^{n-3}q_{1p}+q_{i2}^{n-3}q_{2p}+\cdots+q_{in}^{n-3}q_{np}\neq 0$. Assuming $q_{io}^{n-3}q_{op}\neq 0$, the decision scheme in front of x_p is x_0;

Step 4: Assuming $q_{io}^{n-3}q_{op}\neq 0$ in Step 3, it can be inferred that neither q_{io}^{n-3} nor q_{op} is 0, and $q_{io}^{n-3}=q_{i1}^{n-4}q_{1o}+q_{i2}^{n-4}q_{2o}+\cdots+q_{in}^{n-4}q_{no}\neq 0$. Assuming $q_{ik}^{n-4}q_{kp}\neq 0$, the decision scheme in front of x_o is x_k;

Step 5: Repeat the above process until $q_{il}\neq 0$, and the decision scheme after x_i is x_l;

Step 6: Obtain the final decision plan in the following order: $x_i \succ x_l \succ \cdots \succ x_k \succ x_o \succ x_p \succ x_j$.

The following is a method to identify illogical decision schemes using a cyclic matrix.

Definition 10. In the second-order sub-matrix of the preference relationship matrix $Q=(q_{ij})_{n\times n}$ of the linguistic judgment matrix $P=(p_{ij})_{n\times n}$, the following are called a 3-loop,

$$\begin{matrix} & x_k & x_j \\ x_j & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{matrix}$$

$$\begin{array}{c} \\ x_j \\ x_i \end{array}\!\!\begin{array}{c} x_k \quad x_j \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array};\quad \begin{array}{c} \\ x_i \\ x_j \end{array}\!\!\begin{array}{c} x_j \quad x_k \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array};\quad \begin{array}{c} \\ x_i \\ x_j \end{array}\!\!\begin{array}{c} x_k \quad x_j \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array};\quad \begin{array}{c} \\ x_j \\ x_i \end{array}\!\!\begin{array}{c} x_j \quad x_k \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

which corresponds to the loop $x_i \succ x_j \succ x_k \succ x_i$ formed by the schemes of x_i, x_j, x_k.

The loop matrix is a sub matrix of the preference relationship matrix, though the opposite may not necessarily hold true. The comparison relation is same in the loop matrix and the original linguistic judgment matrix $P=(p_{ij})_{n\times n}$. The comparison result of the decision schemes corresponding to the loop matrix is an illogical judgment. Although there are four formal forms mentioned above, it corresponds to the comparison result of the same three decision schemes.

For example, in the 3-loop matrix $\begin{array}{cc} & \begin{array}{cc} x_j & x_k \end{array} \\ \begin{array}{c} x_j \\ x_i \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$, the result of comparing x_i with x_j is $x_i \succ x_j$, the result of comparing x_j with x_k is $x_j \succ x_k$, and the result of comparing x_k with x_i is $x_k \succ x_i$, and the result comparison of x_i, x_j, x_k is $x_i \succ x_j \succ x_k \succ x_i$. In the 3-loop matrix $\begin{array}{cc} & \begin{array}{cc} x_k & x_j \end{array} \\ \begin{array}{c} x_j \\ x_i \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$, the result of comparing x_i with x_j is $x_i \succ x_j$, the result of comparing x_j with x_k is $x_j \succ x_k$, the result of comparing x_k with x_i is $x_k \succ x_i$, and the result comparison of x_i, x_j, x_k is $x_i \succ x_j \succ x_k \succ x_i$. Similarly, in $\begin{array}{cc} & \begin{array}{cc} x_l & x_j \end{array} \\ \begin{array}{c} x_k \\ x_i \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ and $\begin{array}{cc} & \begin{array}{cc} x_j & x_l \end{array} \\ \begin{array}{c} x_k \\ x_i \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$, the result of comparing x_i, x_j, x_k is also $x_i \succ x_j \succ x_k \succ x_i$. This indicates that the illogical judgments obtained by the same three decision schemes have different representations. The reason for the different representations is that a 3-loop matrix can exchange rows and columns. After exchanging rows and columns, the corresponding decision schemes are also exchanged. Although the form has changed, the essence has not changed.

Theorem 4. The necessary and sufficient condition for the linguistic judgment matrix $P = (p_{ij})_{n\times n}$ to not have a satisfactory consistency is that there is at least one loop matrix in the second-order sub matrix of the preference relation matrix $Q = (q_{ij})_{n\times n}$ of the linguistic judgment matrix.

Prove sufficient: If there is a loop matrix in the sub-matrix of the preference relation matrix $Q = (q_{ij})_{n\times n}$, then the dominance relation of the decision scheme corresponding to the loop matrix is a loop, which is an illogical judgment. The ranking decision schemes cannot be obtained, that is, the linguistic judgment matrix $P = (p_{ij})_{n\times n}$ does not have a satisfactory consistency.

Necessity: If the linguistic judgment matrix $P = (p_{ij})_{n\times n}$ does not have a satisfactory consistency, then according to the definition of a satisfactory consistency, there is a phenomenon of an illogical judgment in the linguistic judgment matrix $P = (p_{ij})_{n\times n}$. The illogical judgment is represented by a loop matrix. No matter how many decision schemes are formed, the illogical judgment must include the illogical judgment formed by the three decision schemes. The sub-matrix of the preference relationship matrix $Q = (q_{ij})_{n\times n}$ at least has a 3-loop matrix.

The theorem has been proven. Theorem 4 not only provides a method to determine the satisfactory consistency of the linguistic judgment matrix, but more importantly, illogical decision schemes can be represented by a second-order sub-matrix. The feasibility of this method has been theoretically proven. The key to this method is how to use a second-order sub-matrix to identify illogical decision solutions. Finding illogical decision-making solutions can not only determine whether the linguistic judgment matrix has a satisfactory consistency, but also prepare for improving the consistency.

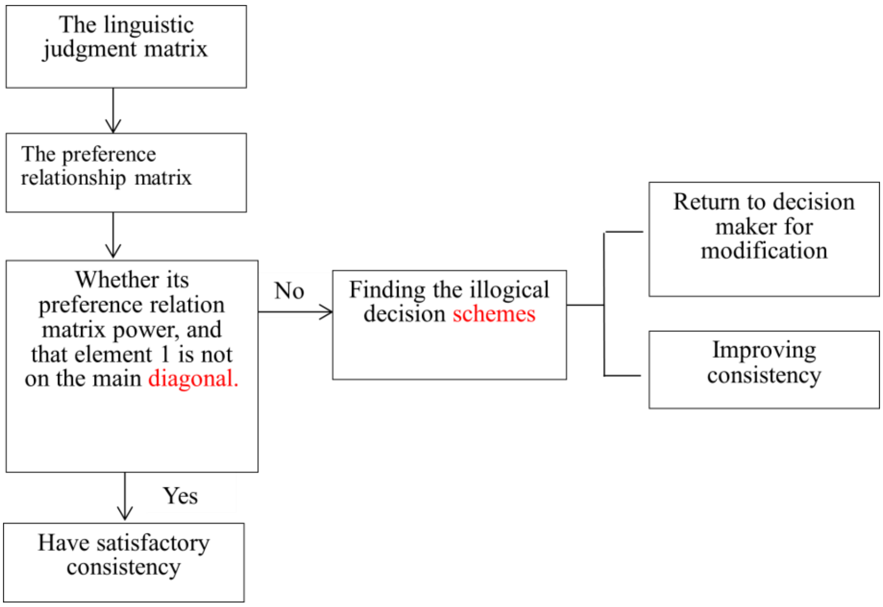

Figure 2. Flow chart of judgment method.

4. Example analysis

Example 1. A teacher gave the following evaluation comparison to his four graduates:

$$P=\begin{pmatrix} AS & LD & MP & HD \\ LP & AS & HP & MP \\ MD & HD & AS & HD \\ HP & MD & HP & AS \end{pmatrix}.$$

Determine whether the linguistic judgment matrix provided by this teacher has a satisfactory consistency. If there is a satisfactory consistency, then provide the rank of the four graduates.

The preference relation matrix can be obtained from formula (2):

$$Q=\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

According to Theorem 4, it can be determined whether the second-order sub-matrix of the preference relationship matrix $\boldsymbol{Q}$ is a 3-loop matrix. For a 4-order matrix, 36 second-order sub matrices need to be judged. Therefore, manually determining whether there is a 3-loop matrix in the second-order sub-matrix is difficult. Therefore, we can use Matlab to find a second-order sub matrix regardless if it is a 3-loop matrix, where the output result is as follows:

Primitive matrix

$$A = \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{matrix}.$$

Sub-matrix

$$B = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

There are 0 sub-matrices in the original matrix. There are 0 sub-matrices with elements located on the main diagonal of the original matrix.

The output results indicate that there is no 3-loop matrix in the second-order sub-matrix of this preference relation matrix $Q = (q_{ij})_{n \times n}$, indicating that the linguistic judgment matrix $P = (p_{ij})_{n \times n}$ has a satisfactory consistency. We can calculate CR=0.0784<0.1 [17]. However, this method requires a large amount of computation. Using an adjacency matrix judgment only requires observing the characteristics of the adjacency matrix. Representing illogical solutions using second-order sub-matrixes only requires observing the output results.

Example 2. Let the decision maker give the linguistic judgment matrix of five alternatives $X = \{x_1, x_2, x_3, x_4, x_5\}$ as follows:

$$P = \begin{pmatrix} AS & VLP & LP & HD & MP \\ VLD & AS & VLD & LP & DVL \\ LD & VLP & AS & HP & HP \\ VHP & LD & MD & AS & VLD \\ MD & VLP & HD & VLP & AS \end{pmatrix};$$

then judge whether P is consistent or not.

The preference relation matrix Q of P is as follows:

$$Q = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 0 & 1 & 1 & 0 & 1 \\ x_2 & 0 & 0 & 0 & 1 & 0 \\ x_3 & 0 & 1 & 0 & 1 & 1 \\ x_4 & 1 & 0 & 0 & 0 & 0 \\ x_5 & 0 & 1 & 0 & 1 & 0 \end{matrix}.$$

Using the Matlab, we can judge a second-order sub-matrix regardless if it is a 3-loop matrix. The output result is as follows:

Primitive matrix

$$A = \begin{matrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{matrix}.$$

Sub-matrix

$$B = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

There are 6 sub-matrices of the original matrix. There are 3 sub-matrixes with elements located on the main diagonal of the original matrix. The position of the first sub-matrix:

a11:(1,2)

a12:(1,4)

a21:(2,2)

a22:(2,4)

The Position of the second sub-matrix:

a11:(1,3)

a12:(1,4)

a21:(3,3)

a22:(3,4)

The Position of the third sub-matrix:

a11:(4,1)

a12:(4,4)

a21:(5,1)

a22:(5,4)

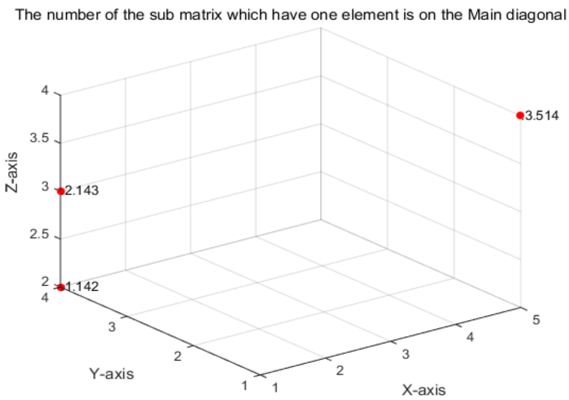

Figure 3. The corresponding schemes for illogical judgment.

From the output results, it can be seen that there are three 3-loop matrices in the second-order sub-matrix, represented as follows:

$$\begin{matrix} & x_2 & x_4 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{matrix},\quad \begin{matrix} & x_3 & x_4 \\ x_1 & 1 & 0 \\ x_3 & 0 & 1 \end{matrix},\quad \begin{matrix} & x_1 & x_4 \\ x_4 & 1 & 0 \\ x_5 & 0 & 1 \end{matrix}.$$

The illogical judgment corresponding to four 3-loop matrices are:

$$\begin{matrix} & x_2 & x_4 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{matrix} : x_1 \succ x_2 \succ x_4 \succ x_1;\quad \begin{matrix} & x_3 & x_4 \\ x_1 & 1 & 0 \\ x_3 & 0 & 1 \end{matrix} : x_1 \succ x_3 \succ x_4 \succ x_1;\quad \begin{matrix} & x_1 & x_4 \\ x_4 & 1 & 0 \\ x_5 & 0 & 1 \end{matrix} : x_4 \succ x_1 \succ x_5 \succ x_4.$$

The illogical judgments are expressed by three-dimensional coordinates as follows:

From Figure 3, it can be seen that there are three, 3-loop matrices: the first 3-loop matrix is formed by x_1, x_2, x_4, the second 3-loop matrix is formed by x_1, x_3, x_4, and the third 3-loop matrix is formed by x_1, x_4, x_5. The loop depends on the comparison relation in the original judgment matrix. However, as long as points appear in the three-dimensional coordinate map, then it indicates that there are illogical judgments in the judgments given by the decision-maker. The linguistic judgment matrix does not have a satisfactory consistency and cannot determine the dominance of the decision schemes.

By definition of the 3- loop matrix, the formation of a 3-loop matrix is as follows:

$$\begin{matrix} & x_j & x_k \\ x_j & 0 & 1 \\ x_i & 1 & 0 \end{matrix}、\quad \begin{matrix} & x_k & x_j \\ x_i & 0 & 1 \\ x_j & 1 & 0 \end{matrix} \text{ and } \begin{matrix} & x_k & x_j \\ x_j & 1 & 0 \\ x_i & 0 & 1 \end{matrix}、\quad \begin{matrix} & x_j & x_k \\ x_i & 1 & 0 \\ x_j & 0 & 1 \end{matrix}.$$

Although their representations are different, the loop they represent are essentially the same.

Let's use $\begin{matrix} & x_j & x_k \\ x_j & 0 & 1 \\ x_i & 1 & 0 \end{matrix}$ to find the 3-loop matrix to find illogical judgments.

Primitive matrix

$$A = \begin{matrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{matrix}.$$

Sub-matrix

$$B = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}.$$

There are 9 sub-matrices of the original matrix. There are 6 sub-matrices with elements located on the main diagonal of the original matrix. The position of the first sub-matrix:

a11:(1,1)

a12:(1,2)

a21:(4,1)

a22:(4,2)

The position of the second sub-matrix:

a11:(1,1)

a12:(1,3)

a21:(4,1)

a22:(4,3)

The position of the third sub-matrix:

a11:(1,1)

a12:(1,5)

a21:(4,1)

a22:(4,5)

The position of forth sub-matrix is as follows:

a11:(1,4)

a12:(1,5)

a21:(5,4)

a22:(5,5)

The position of fifth sub-matrix is as follows:

a11:(2,1)

a12:(2,4)

a21:(4,1)

a22:(4,4)

The position of sixth sub-matrix is as follows:

a11:(3,1)

a12:(3,4)

a21:(4,1)

a22:(4,4)

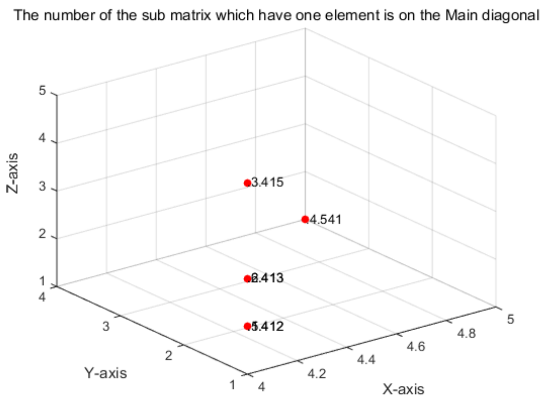

Figure 4. The corresponding schemes for illogical judgment.

The illogical judgments corresponding to the 3-loop matrix using $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are as follows: $x_1 \succ x_2 \succ x_4 \succ x_1$, $x_1 \succ x_3 \succ x_4 \succ x_1$, and $x_4 \succ x_1 \succ x_5 \succ x_4$. Then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to find the illogical judgments corresponding to the 3-loop matrix, as shown in the figure above. The first and fifth points are the same, representing the illogical judgments formed by x_1, x_2, x_4. The second and sixth points represent the illogical judgments formed by x_1, x_3, x_4. The third and fourth points represent the illogical judgments formed by x_1, x_4, x_5, and the illogical judgments present in the linguistic judgment matrix are as follows: $x_1 \succ x_2 \succ x_4 \succ x_1$, $x_1 \succ x_3 \succ x_4 \succ x_1$, and $x_4 \succ x_1 \succ x_5 \succ x_4$. This also verifies that the forms

$$\begin{matrix} & x_j & x_k \\ x_j & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ x_i & & \end{matrix}、\begin{matrix} & x_k & x_j \\ x_i & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ x_j & & \end{matrix} \text{ and } \begin{matrix} & x_k & x_j \\ x_j & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ x_i & & \end{matrix}、\begin{matrix} & x_j & x_k \\ x_i & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ x_j & & \end{matrix}$$

given in the definition of the loop matrix are different, but represent the same illogical judgments.

We can calculate CR=0.1351>0.1 [17]. This linguistic judgment matrix does not have a satisfactory consistency. The satisfactory consistency of the language judgment matrix in example 2 can also be determined using definitions 8 and 9 [9]. The fuzzy gross consistency degree associated and COG [9] can be obtained. However, it was not determined which judgments were illogical. The method used in this paper not only assessed the satisfactory consistency, but also provided illogical schemes. Illogical schemes can either be returned to the decision-makers or improve the consistency.

Example 3. The decision maker makes the following evaluation for the four printers $X = \{x_1, x_2, x_3, x_4\}$, and gives the following preference information:

$$P = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ x_1 & AS & LD & MP & MD \\ x_2 & LP & AS & VLD & MD \\ x_3 & MD & VLP & AS & HD \\ x_4 & MP & MP & HP & AS \end{matrix}.$$

According to the above steps, one can either judge or revise the satisfactory consistency of P.

The preference relation matrix obtained from definition 6 is as follows:

$$Q=(q_{ij})_{n\times n}=\begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ x_1 & 0 & 0 & 1 & 0 \\ x_2 & 1 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ x_4 & 1 & 1 & 1 & 0 \end{matrix}.$$

Using Matlab to edit the program and search for the second-order sub-matrix of a 3-loop matrix, the output result is as follows:

Primitive matrix

$$A=\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{matrix}.$$

Sub-matrix

$$B=\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

All sub-matrices of the original matrix have 1. There is 1 sub matrix with elements located in the main diagonal of the original matrix. The position of first sub-matrix is as follows:

a11:(2,1)

a12:(2,2)

a21:(3,1)

a22:(3,2)

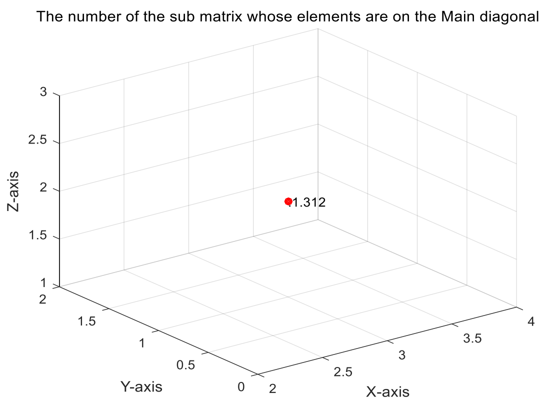

Figure 5. Schemes corresponding to illogical judgment.

Next, we will use the 3-loop matrix

$$\begin{matrix} & x_j & x_k \\ x_j & 0 & 1 \\ x_i & 1 & 0 \end{matrix}$$

to find illogical judgments, and the output results are as follows:

Primitive matrix

$$A = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{matrix}.$$

Sub-matrix

$$B = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}.$$

All sub-matrices of the original matrix have 2. There are 2 sub-matrixes with elements located on the main diagonal of the original matrix. The position of first sub-matrix is as follows::

a11:(1,1)

a12:(1,3)

a21:(2,1)

a22:(2,3)

The position of second sub-matrix is as follows:

a11:(1,2)

a12:(1,3)

a21:(3,2)

a22:(3,3)

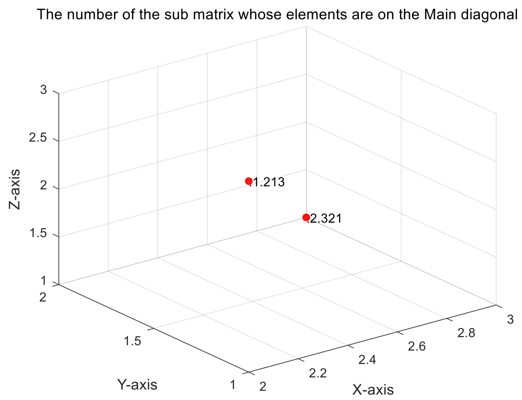

Figure 6. Schemes corresponding to illogical judgments

From the output results, it can be seen that there are two 3-loop matrices in the second-order sub matrix, which are as follows:

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \text{ and } \begin{array}{cc} & \begin{array}{cc} x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_3 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}.$$

The illogical judgments represented are all $x_1 \succ x_3 \succ x_2 \succ x_1$; therefor using different forms of 3-cyclic matrices to find illogical judgments produces the same results.

The linguistic judgment matrix in example 3 does not have satisfactory consistency. Compared with the judgment methods in references [9,14,17], this method does not require a large amount of computation and does not establish a complex model. Satisfactory consistency can be obtained through simple calculation and observation. When there is no satisfactory consistency, the second-order sub-matrix can represent illogical judgments.

5. Conclusions

This article mainly used a directed graph to represent the comparison relation in the linguistic judgment matrix, and used the $n-1$ power of the preference relation to determine whether the linguistic judgment matrix had a satisfactory consistency. If the linguistic judgment matrix did not have satisfactory consistency, then the sub-matrix of the linguistic judgment matrix was used to determine whether it was a 3-loop matrix; then, this was used to find the corresponding decision scheme for illogical judgments, and a program was used to implement the search process. Moreover, this paper studied how to find the illogical judgements formed by the three schemes. The sub-matrix of the linguistic judgment matrix was used to represent the three schemes that were judged to be illogical, and the characteristics of the sub-matrix were analyzed. This method is simple and intuitive. By observing and comparing, the solution that was judged to be illogical was found. This method can be applied to a comprehensive evaluation. The decision-maker gives pairwise comparisons between the decision schemes according to a set of language phrases. A language judgment matrix could be formed by a pairwise comparison. If the language judgment matrix had a satisfactory consistency, then the ranking relation between the decision options could be obtained. If the language judgment matrix did not have a satisfactory consistency, then the second order sub-matrix could be used to represent illogical judgments. Illogical judgments were either returned to the decision-makers or consistency was improved.

Although this article provides a method to determine the satisfactory consistency of the language judgment matrix and identify illogical decision schemes, it did not provide a method for to determine the complete consistency of the linguistic judgment matrix. A satisfactory consistency reflects the transferability of the relationship between the strengths and weaknesses, and further research is needed on the transferability of the degree of strengths and weaknesses and methods to improve the inconsistency.

Author contributions

Fengxia Jin: Conceptualization, Methodology, Writing-review and editing, Writing-original draft preparation; Feng Wang: Software, Project administration; Huatao Chen: Methodology, Writing-review and editing, Project administration; Juan L. G. Guirao: Validation. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All the authors declare that there are no potential conflicts of interest and approval of the submission.

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