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Research article

Hyperbolic Ricci solitons on perfect fluid spacetimes

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Abstract: In the present paper, we investigate perfect fluid spacetimes and perfect fluid generalized Roberston-Walker spacetimes that contain a torse-forming vector field satisfying almost hyperbolic Ricci solitons. We show that the perfect fluid spacetimes that contain a torse-forming vector field satisfy an almost hyperbolic Ricci soliton, and we prove that a perfect fluid generalized Roberston-Walker spacetime satisfying an almost hyperbolic Ricci soliton (g, ζ, ϱ, μ) is an Einstein manifold. Also, we study an almost hyperbolic Ricci soliton (g, V, ϱ, μ) on these spacetimes when V is a conformal vector field, a torse-forming vector field, or a Ricci bi-conformal vector field.

Keywords: perfect fluid spacetimes; hyperbolic Ricci solitons; Einstein manifolds
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1. Introduction

In general relativity and symmetry of spacetime, a perfect fluid energy-momentum tensor is often used to describe the source of the gravitational field. The energy-momentum tensor T plays a key role as a matter content of the spacetime, matter is supposed to have pressure, density, and kinematic and dynamical quantities such as acceleration, speed, vorticity, expansion and shear [1]. In usual cosmological models, the universe's matter content is supposed to act like a perfect fluid. In general relativity, a perfect fluid solution is an exact solution of the Einstein field equation (EFE), where the gravitational field is completely produced by the momentum, mass and stress density of a fluid.

In cosmology and astrophysics, fluid solutions are usually used as stellar models and cosmological models, respectively.

The most important geometric flow is the Ricci flow. In 1982, Hamilton [2] presented the notion of Ricci soliton as a generalization of Einstein metrics and a special solution to Ricci flow. A Ricci soliton on a manifold (pseudo-Riemannian) M is defined by [3,4]

$$\mathcal{L}_V g = -2S - 2\varrho g,$$

where $\mathcal{L}_V$ is the Lie derivative along the potential vector field V, S is the Ricci tensor of g, and ϱ is a real constant. If $\varrho < 0$, $= 0$, or > 0, then the Ricci soliton is said to be shrinking, steady, or expanding, respectively. If $V = \mathrm{grad}\psi$ for some function ψ, then g is named a gradient Ricci soliton. If $\varrho \in C^\infty(M)$, then g is termed an almost Ricci soliton.

The authors Venkatesha and Kumara [5] studied Ricci solitons in perfect fluid spacetime (briefly, PF-spacetime) with a torse-forming vector field (briefly, TFVF). Also, the geometrical structure in a perfect fluid spacetime have been studied by many authors in several ways to a different extent such Blaga [6], Chaubey [7], Siddiqi [8], De et al. [9], Zhang et al. [10] and many others. In 2022, Li et al. [11] studied LP-Kenmotsu manifolds admitting η-Ricci solitons. Moreover, they proved some results for η-Ricci solitons in LP-Kenmotsu manifolds in the spacetime of general relativity. About a decade ago, the authors Arslan et al. [12] investigated some curvature conditions on generalized Robertson-Walker spacetime. In 2017, Mantica and Molinari [13] presented a survey on generalized Robertson-Walker spacetime with main focus on Chen's characterization in terms of a timelike concircular vector and obtained some new results. Very recently, the authors Azami et al. [14] studied left-invariant cross curvature solitons on Lorentzian three-dimensional Lie groups.

Another type of geometric flow is hyperbolic geometric flow, defined by Dai et al. in [15],

$$\frac{\partial^2}{\partial t^2} g = -2S, \qquad g(0) = g_0, \qquad \frac{\partial g}{\partial t}(0) = k_0, \tag{1.1}$$

where k_0 is a symmetric 2-tensor field on M.

Let $(M^n, g(t))$ be a solution of (1.1) on (M, g_0). The self-similar solution of (1.1) is given as follows [16,17]

$$S(g_0) + \varrho\mathcal{L}_V g_0 + (\mathcal{L}_V \circ \mathcal{L}_V) g_0 = \mu g_0,$$

for some constants ϱ and μ. In such a case, we say g_0 is a hyperbolic Ricci soliton (briefly, HRS), and we denote it by $(M, g_0, V, \varrho, \mu)$ (briefly (g_0, V, ϱ, μ)). A HRS is an Einstein metric when V vanishes identically; thus, a HRS is a generalization of the Einstein metric. If $\varrho = \frac{1}{2}$ and V is a 2-Killing vector field [18, 19], i.e., $(\mathcal{L}_V \circ \mathcal{L}_V) g_0 = 0$, then; a HRS is a Ricci soliton. We say that $(M, g_0, \nabla f, \varrho, \mu)$ is a gradient HRS whenever $\nabla f = V$ for some smooth functions $f : M \to \mathbb{R}$.

An almost HRS (or AHRS) is a 4-tuple (g, V, ϱ, μ), where ϱ and μ are two functions and the metric g (pseudo-Riemannian) obeys the equation

$$S + \varrho\mathcal{L}_V g + (\mathcal{L}_V \circ \mathcal{L}_V) g = \mu g, \tag{1.2}$$

where S is concerned with g. In the following, we provide an example of an AHRS that is not Einstein and Ricci soliton.

Example 1.1. *Suppose that* (x, y, z) *is the standard coordinates in* $\mathbb{R}^3$ *and* $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}$. *We consider the linearly independent vector fields*

$$e_1 = z^2 \frac{\partial}{\partial x}, \qquad e_2 = z^2 \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

We define the metric g by

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2\}, \\ -1, & \text{if } i = j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

We define symmetric $(1, 1)$*-tensor* ϕ*, vector field* ξ*, and 1-form* η *on M by*

$$\xi = e_3, \quad \eta(X) = g(X, e_3), \quad \phi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all vector fields X. Note that $\phi(X) = X + \eta(X)\xi$*, and* $\eta(\xi) = -1$. *We obtain*

$[,]$	e_1	e_2	e_3
e_1	0	0	$-\frac{2}{z}e_1$
e_2	0	0	$-\frac{2}{z}e_2$
e_3	$\frac{2}{z}e_1$	$\frac{2}{z}e_2$	0

The Levi-Civita connection ∇ *of M is defined by*

$$\nabla_{e_i} e_j = \begin{pmatrix} -\frac{2}{z}e_3 & 0 & -\frac{2}{z}e_1 \\ 0 & -\frac{2}{z}e_3 & -\frac{2}{z}e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that $\nabla_X \xi = -\frac{2}{z}(X + \eta(X)\xi)$. *The nonvanishing components of the curvature tensor are:*

$$R(e_1, e_2)e_1 = -\frac{4}{z^2}e_2, \; R(e_1, e_2)e_2 = \frac{4}{z^2}e_1, \; R(e_1, e_3)e_1 = -\frac{6}{z^2}e_3,$$
$$R(e_2, e_3)e_2 = -\frac{6}{z^2}e_3, \; R(e_1, e_3)e_3 = -\frac{6}{z^2}e_1, \; R(e_2, e_3)e_3 = -\frac{6}{z^2}e_2.$$

Hence, we get

$$S = \begin{pmatrix} -\frac{2}{z^2} & 0 & 0 \\ 0 & -\frac{2}{z^2} & 0 \\ 0 & 0 & -\frac{12}{z^2} \end{pmatrix} = -\frac{2}{z^2}g - \frac{14}{z^2}\eta \otimes \eta.$$

Then $\mathcal{L}_\xi g = -\frac{4}{z}(g + \eta \otimes \eta)$ *and*

$$(\mathcal{L}_\xi \circ \mathcal{L}_\xi)g = \frac{20}{z^2}(g + \eta \otimes \eta).$$

Therefore, $(M, g, \xi, \frac{3}{z}, \frac{12}{z^2})$ *is an AHRS on manifold M.*

Motivated by the above studies, we study HRSs on PF-spacetimes. The paper is presented as follows: Section 2 is concerned with some fundamental concepts and formulas for PF-spacetimes. In Section 3, we study PF-spacetimes with TFVF satisfied in an AHRS. In Section 4, we investigate how perfect fluid generalized Roberston-Walker spacetimes (or PF-GRW-spacetimes) satisfy an AHRS.

2. Preliminaries

Throughout this section, we assume that the vector fields $\mathcal{X}_1, \mathcal{X}_2$, and $\mathcal{X}_3$ are arbitrary vector fields, unless otherwise noted. The energy-momentum tensor T in PF-spacetime is defined through [20]

$$T(\mathcal{X}_1, \mathcal{X}_2) = \rho g(\mathcal{X}_1, \mathcal{X}_2) + (\rho + \sigma)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2), \tag{2.1}$$

where σ is the energy-density and ρ is the isotropic pressure, and $\eta(\mathcal{X}_1) = g(\mathcal{X}_1, \zeta)$ is 1-form, such that $g(\zeta, \zeta) + 1 = 0$. Let λ and ϵ be the cosmological and gravitational constants, respectively. The field equation governing the perfect fluid motion is Einstein's gravitational equation [20]

$$S(\mathcal{X}_1, \mathcal{X}_2) + (\lambda - \frac{r}{2})g(\mathcal{X}_1, \mathcal{X}_2) = \epsilon T(\mathcal{X}_1, \mathcal{X}_2), \tag{2.2}$$

where r is the scalar curvature of g. From (2.1) and (2.2), we have

$$S(\mathcal{X}_1, \mathcal{X}_2) = (-\lambda + \frac{r}{2} + \epsilon\rho)g(\mathcal{X}_1, \mathcal{X}_2) + \epsilon(\rho + \sigma)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{2.3}$$

Let (M^4, g) be a relativistic PF-spacetime admitting (2.3). Thus, by contracting (2.3) and using $g(\zeta, \zeta) = -1$, we conclude

$$r = \epsilon(\sigma - 3\rho) + 4\lambda. \tag{2.4}$$

Applying (2.4) to (2.3), it follows:

$$S(\mathcal{X}_1, \mathcal{X}_2) = \left(\lambda + \frac{\epsilon(\sigma - \rho)}{2}\right)g(\mathcal{X}_1, \mathcal{X}_2) + \epsilon(\rho + \sigma)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{2.5}$$

This implies

$$Q\mathcal{X}_1 = \left(\lambda + \frac{\epsilon(\sigma - \rho)}{2}\right)\mathcal{X}_1 + \epsilon(\rho + \sigma)\eta(\mathcal{X}_1)\zeta,$$

where Q is the Ricci operator such that $g(Q\mathcal{X}_1, \mathcal{X}_2) = S(\mathcal{X}_1, \mathcal{X}_2)$.

Let (M^n, g) be a Lorentzian manifold. A manifold (M^n, g) $(3 \leq n)$ is termed a generalized Robertson-Walker (GRW) spacetime [21]. If we write M as $M = -I \times_{f^2} M^*$, here $I \subset \mathbb{R}$ is an open interval, $f(> 0) \in C^\infty(M)$, and M^* is a Riemannian $(n-1)$-manifold. If, $n = 4$ and M^* is of constant curvature, then the spacetime becomes the Robertson-Walker (RW) spacetime. Thus, every RW spacetime is a PF-spacetime, where, as in $n = 4$, the GRW spacetime is a PF-spacetime if and only if it is a RW spacetime.

Definition 2.1. *If a vector field $V(\neq 0)$ on a pseudo-Riemannian manifold (M, g) obeys*

$$\nabla_{\mathcal{X}_1} V = h\mathcal{X}_1 + \omega(\mathcal{X}_1)V, \tag{2.6}$$

then it is called torse-forming [22, 23]. Here ∇ is the Levi-Civita connection of g, ω is a 1-form, and $h \in C^\infty(M)$. In this case, V becomes

- *concircular [24, 25] whenever $\omega = 0$ in (2.6),*
- *concurrent [26, 27] if in the equation (2.6), $\omega = 0$ and $h = 1$,*
- *parallel vector field if in equation (2.6), $h = \omega = 0$,*

- *torqued vector field [28] if in equation (2.6), $\omega(V) = 0$.*

Chen [24] presented a simple description of an ($3 \leq$) n-dimensional Lorentzian manifold with a timelike concircular vector field as follows:

Theorem 2.1. *[24] Let M be a Lorentzian n-manifold, $n \geq 3$. Then, M is a GRW spacetime if and only if it has a timelike concicular vector field.*

3. PF-spacetime with TFVF

In this section, we study HRS on PF-spacetimes. Throughout this section, we assume that $\mathcal{X}_1, \mathcal{X}_2$, and $\mathcal{X}_3$ are arbitrary vector fields on a PF-spacetime M, unless otherwise noted.

Now, we suppose that ζ is a TFVF and

$$\nabla_{\mathcal{X}_1}\zeta = \mathcal{X}_1 + \eta(\mathcal{X}_1)\zeta. \tag{3.1}$$

In this case, on a PF-spacetime, we have the following relations:

$$\begin{aligned}
&\nabla_{\zeta}\zeta = 0,\\
&(\nabla_{\mathcal{X}_1}\eta)(\mathcal{X}_2) = g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2),\\
&R(\mathcal{X}_1, \mathcal{X}_2)\zeta = \eta(\mathcal{X}_2)\mathcal{X}_1 - \eta(\mathcal{X}_1)\mathcal{X}_2,\\
&\eta(R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3) = \eta(\mathcal{X}_1)g(\mathcal{X}_2, \mathcal{X}_3) - \eta(\mathcal{X}_2)g(\mathcal{X}_1, \mathcal{X}_3).
\end{aligned}$$

Let ζ be a TFVF on PF-spacetime, (M^4, g) and g admit an AHRS (1.2) such that $V = f\zeta$ for some $f \in C^\infty(M)$. Using (3.1), we obtain

$$\mathcal{L}_{f\zeta}g(\mathcal{X}_1, \mathcal{X}_2) = (\mathcal{X}_1 f)\eta(\mathcal{X}_2) + (\mathcal{X}_2 f)\eta(\mathcal{X}_1) + 2f(g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2)), \tag{3.2}$$

hence

$$\begin{aligned}
&(\mathcal{L}_{f\zeta}(\mathcal{L}_{f\zeta}g))(\mathcal{X}_1, \mathcal{X}_2)\\
= \;& f\zeta\left((\mathcal{X}_1 f)\eta(\mathcal{X}_2) + (\mathcal{X}_2 f)\eta(\mathcal{X}_1) + 2f(g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2))\right)\\
&-\Big(((\mathcal{L}_{f\zeta}\mathcal{X}_1)f)\eta(\mathcal{X}_2) + (\mathcal{X}_2 f)\eta(\mathcal{L}_{f\zeta}\mathcal{X}_1) + 2f(g(\mathcal{L}_{f\zeta}\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{L}_{f\zeta}\mathcal{X}_1)\eta(\mathcal{X}_2))\Big)\\
&-\Big((\mathcal{X}_1 f)\eta(\mathcal{L}_{f\zeta}\mathcal{X}_2) + ((\mathcal{L}_{f\zeta}\mathcal{X}_2)f)\eta(\mathcal{X}_1) + 2f(g(\mathcal{X}_1, \mathcal{L}_{f\zeta}\mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{L}_{f\zeta}\mathcal{X}_2))\Big).
\end{aligned}$$

Applying $V = f\zeta$ and the above equation in (1.2), we infer

$$\begin{aligned}
&S(\mathcal{X}_1, \mathcal{X}_2) + \varrho\left((\mathcal{X}_1 f)\eta(\mathcal{X}_2) + (\mathcal{X}_2 f)\eta(\mathcal{X}_1) + 2f(g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2))\right)\\
&+f\zeta\left((\mathcal{X}_1 f)\eta(\mathcal{X}_2) + (\mathcal{X}_2 f)\eta(\mathcal{X}_1) + 2f(g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2))\right)\\
&-\Big(((\mathcal{L}_{f\zeta}\mathcal{X}_1)f)\eta(\mathcal{X}_2) + (\mathcal{X}_2 f)\eta(\mathcal{L}_{f\zeta}\mathcal{X}_1) + 2f(g(\mathcal{L}_{f\zeta}\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{L}_{f\zeta}\mathcal{X}_1)\eta(\mathcal{X}_2))\Big)\\
&-\Big((\mathcal{X}_1 f)\eta(\mathcal{L}_{f\zeta}\mathcal{X}_2) + ((\mathcal{L}_{f\zeta}\mathcal{X}_2)f)\eta(\mathcal{X}_1) + 2f(g(\mathcal{X}_1, \mathcal{L}_{f\zeta}\mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{L}_{f\zeta}\mathcal{X}_2))\Big)\\
&-\mu g(\mathcal{X}_1, \mathcal{X}_2) = 0,
\end{aligned}$$

which, by plugging $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ and using (2.5), yields

$$\lambda - \frac{\epsilon(\sigma + 3\rho)}{2} + 2f\zeta(\zeta(f)) + 2\varrho\zeta(f) + 4(\zeta(f))^2 - \mu = 0. \tag{3.3}$$

Thus, we state:

Theorem 3.1. *Let ζ be a TFVF on PF-spacetime (M^4, g) and g admit an AHRS (g, V, ϱ, μ) such that $V = f\zeta$ for some $f \in C^\infty(M)$, then the relation* (3.3) *holds.*

Now, let ζ be a TFVF on PF-GRW-spacetime (M^4, g). Then, from (3.2), we have

$$\mathcal{L}_\zeta g(X_1, X_2) = 2(g(X_1, X_2) + \eta(X_1)\eta(X_2)), \tag{3.4}$$

and

$$\begin{aligned}(\mathcal{L}_\zeta(\mathcal{L}_\zeta g))(X_1, X_2) &= 2\zeta\left(g(X_1, X_2) + \eta(X_1)\eta(X_2)\right) \\ &\quad -2\left(g(\mathcal{L}_\zeta X_1, X_2) + \eta(\mathcal{L}_\zeta X_1)\eta(X_2)\right) \\ &\quad -2\left(g(X_1, \mathcal{L}_\zeta X_2) + \eta(X_1)\eta(\mathcal{L}_\zeta X_2)\right).\end{aligned} \tag{3.5}$$

We have

$$g(\mathcal{L}_\zeta X_1, X_2) = g(\nabla_\zeta X_1, X_2) - \eta(X_1)\eta(X_2) - g(X_1, X_2). \tag{3.6}$$

Similarly, we have

$$g(X_1, \mathcal{L}_\zeta X_2) = g(X_1, \nabla_\zeta X_2) - g(X_1, X_2) - \eta(X_1)\eta(X_2).$$

Then

$$g(\mathcal{L}_\zeta X_1, X_2) + g(X_1, \mathcal{L}_\zeta X_2) = \zeta(g(X_1, X_2)) - 2(\eta(X_1)\eta(X_2) + g(X_1, X_2)). \tag{3.7}$$

Since $\nabla_\zeta \zeta = \zeta + \eta(\zeta)\zeta = 0$, using (3.6), we have

$$\eta(\mathcal{L}_\zeta X_1) = g(\mathcal{L}_\zeta X_1, \zeta) = g(\nabla_\zeta X_1, \zeta) = \zeta(g(X_1, \zeta)) = \zeta(\eta(X_1)),$$

similarly, we find

$$\eta(\mathcal{L}_\zeta X_2) = \zeta(\eta(X_2)).$$

Thus,

$$\eta(\mathcal{L}_\zeta X_1)\eta(X_2) + \eta(X_1)\eta(\mathcal{L}_\zeta X_2) = \zeta(\eta(X_1)\eta(X_2)). \tag{3.8}$$

Therefore, applying (3.7) and (3.8) to (3.5), it follows

$$(\mathcal{L}_\zeta(\mathcal{L}_\zeta g))(X_1, X_2) = 4(\eta(X_1)\eta(X_2) + g(X_1, X_2)). \tag{3.9}$$

Using (2.5), (3.4), and (3.9), we get

$$S + \varrho\mathcal{L}_\zeta g + (\mathcal{L}_\zeta \circ \mathcal{L}_\zeta)g - \mu g = \left(\frac{\epsilon(\sigma - \rho) + \varrho}{2} + 2\varrho + 4 - \mu\right)g + (\epsilon(\rho + \sigma) + 2\varrho + 4)\eta \otimes \eta.$$

From the above equation, (M^4, g) admits an AHRS (g, ζ, ϱ, μ).

Hence, we have:

Theorem 3.2. *Let ζ be a TFVF on a PF-spacetime (M^4, g). Then, manifold M satisfies an AHRS $(g, \zeta, -\frac{\epsilon(\sigma+\rho)+4}{2}, \varrho - \frac{\epsilon}{2}(\sigma + 3\rho))$.*

Remark 3.1. *i) Let, in dust-like matter, the energy density be σ and η be the same as defined in (2.1). The energy-momentum of pressure-free fluid spacetime, or dust, is $T(X_1, X_2) = \sigma\eta(X_1)\eta(X_2)$. Hence, if ζ is a TFVF on a dust fluid spacetime M, then M satisfies an AHRS $(g, \zeta, -\frac{\epsilon\sigma+4}{2}, \varrho - \frac{\epsilon}{2}\sigma)$.*

ii) The energy-momentum tensor of dark fluid spacetime $\rho = -\sigma$; is $T(X_1, X_2) = \rho g(X_1, X_2)$. Thus, if ζ is a TFVF on dark fluid spacetime M, then M satisfies an AHRS $(g, \zeta, -2, \varrho - \epsilon\rho)$.

iii) The energy-momentum of a radiation fluid spacetime $\sigma = 3\rho$; is $T(X_1, X_2) = \rho[g(X_1, X_2) + 4\eta(X_1)\eta(X_2)]$. Thus, if ζ is a TFVF on dust fluid spacetime M, then M satisfies an AHRS $(g, \zeta, -2(\epsilon\rho + 1), \varrho - 3\epsilon\rho)$.

Definition 3.1. *On a pseudo-Riemannian manifold (M, g), a vector field V obeying*

$$(\mathcal{L}_V g)(X_1, X_2) = 2hg(X_1, X_2), \tag{3.10}$$

for some $h \in C^\infty(M)$, is named a conformal Killing vector field. The vector field V reduces to a proper, a homothetic, or a Killing vector field when h is not constant, constant, or $h = 0$, respecticvely.

Let ζ be a TFVF on a PF-spacetime (M^4, g), and let V be a conformal Killing vector field satisfying (3.10). Then

$$\begin{aligned} ((\mathcal{L}_V \circ \mathcal{L}_V)g)(X_1, X_2) &= V(\mathcal{L}_V g(X_1, X_2)) - \mathcal{L}_V g(\mathcal{L}_V X_1, X_2) - \mathcal{L}_V g(X_1, \mathcal{L}_V X_2) \\ &= V(2hg(X_1, X_2)) - 2hg(\mathcal{L}_V X_1, X_2) - 2hg(X_1, \mathcal{L}_V X_2) \\ &= 2V(h)g(X_1, X_2) + 2h\mathcal{L}_V g(X_1, X_2) \\ &= (2V(h) + 4h^2)g(X_1, X_2). \end{aligned} \tag{3.11}$$

By inserting (3.11) in Eq (1.2), we deduce

$$S(X_1, X_2) + 2h\varrho g(X_1, X_2) + (2V(h) + 4h^2)g(X_1, X_2) - \mu g(X_1, X_2) = 0. \tag{3.12}$$

We get

$$(-\varrho + \frac{r}{2} + \epsilon\rho + 2\varrho h + 2V(h) + 4h^2 - \mu)g(X_1, X_2) + \epsilon(\sigma + \rho)\eta(X_1)\eta(X_2) = 0. \tag{3.13}$$

We conclude with the following result:

Theorem 3.3. *If ζ is a TFVF on a PF-spacetime (M^4, g) and g satisfies the AHRS (g, V, ϱ, μ), where V is the conformally Killing vector field, then M is Einstein, $\epsilon(\sigma + \rho) = 0$, and*

$$-\varrho + \frac{r}{2} + \epsilon\rho + 2\varrho h + 2V(h) + 4h^2 - \mu = 0.$$

Let ζ be a TFVF on a PF-spacetime (M^4, g) and (g, V, ϱ, μ) be an AHRS such that V is a TFVF and satisfied in (2.6). Then

$$\mathcal{L}_V g(X_1, X_2) = 2hg(X_1, X_2) + \omega(X_1)g(V, X_2) + \omega(W)g(V, X_1), \tag{3.14}$$

and

$$
\begin{aligned}
(\mathcal{L}_V(\mathcal{L}_V g))(X_1, X_2) &= V\,(2hg(X_1, X_2) + \omega(X_1)g(V, X_2) + \omega(X_2)g(V, X_1)) \\
&\quad -2hg(\mathcal{L}_V X_1, X_2) - \omega(\mathcal{L}_V X_1)g(V, X_2) - \omega(X_2)g(V, \mathcal{L}_V X_1) \\
&\quad -2hg(X_1, \mathcal{L}_V X_2) - \omega(X_1)g(V, \mathcal{L}_V X_2) - \omega(\mathcal{L}_V X_2)g(V, X_1). \qquad (3.15)
\end{aligned}
$$

On the other hand,

$$
g(\mathcal{L}_V X_1, X_2) = g(\nabla_V X_1, X_2) - hg(X_1, X_2) - \omega(X_2)g(V, X_1),
$$

similarly,

$$
g(X_1, \mathcal{L}_V X_2) = g(X_1, \nabla_V X_2) - hg(X_1, X_2) - \omega(X_1)g(V, X_2).
$$

Thus,

$$
g(\mathcal{L}_V X_1, X_2) + g(X_1, \mathcal{L}_V X_2) = V(g(X_1, X_2)) - 2hg(X_1, X_2) - \omega(X_2)g(V, X_1) - \omega(X_1)g(V, X_2).
$$

Also, we have

$$
\begin{aligned}
\omega(\mathcal{L}_V X_1) &= \omega(\nabla_V X_1 - \nabla_{X_1} V) = \omega(\nabla_V X_1 - hX_1 - \omega(X_1)V) \\
&= \omega(\nabla_V X_1) - h\omega(X_1) - \omega(X_1)\omega(V),
\end{aligned}
$$

similarly,

$$
\omega(\mathcal{L}_V X_2) = \omega(\nabla_V X_2) - h\omega(X_2) - \omega(X_2)\omega(V).
$$

Therefore, applying the above equations to (3.15), we infer

$$
\begin{aligned}
(\mathcal{L}_V(\mathcal{L}_V g))(X_1, X_2) &= (2V(h) + 4h^2)g(X_1, X_2) + V\,(\omega(X_1)g(V, X_2) + \omega(X_2)g(V, X_1)) \\
&\quad +4h\omega(X_1)g(V, X_2) + 4h\omega(W)g(V, X_1) \\
&\quad -\omega(\nabla_V X_1)g(V, X_2) + 2\omega(X_1)\omega(V)g(V, X_2) - \omega(X_2)g(\nabla_V X_1, V) \\
&\quad +2\omega(X_2)\omega(V)g(V, X_1) - \omega(\nabla_V X_2)g(V, X_1) - \omega(X_1)g(\nabla_V X_2, V). \quad (3.16)
\end{aligned}
$$

Putting $X_1 = X_2 = \zeta$ in (3.14) and (3.16), it follows

$$
\mathcal{L}_V g(\zeta, \zeta) = -2h + 2\omega(\zeta)\eta(V) \qquad (3.17)
$$

and

$$
\begin{aligned}
(\mathcal{L}_V(\mathcal{L}_V g))(\zeta, \zeta) &= -(2V(h) + 4h^2) + V\,(2\omega(\zeta)\eta(V)) + 8h\omega(\zeta)\eta(V) + 4\omega(\zeta)\omega(V)\eta(V) \\
&\quad -2\omega(\zeta)\eta(V) - 4(\eta(V))^2\omega(\zeta) - 2\omega(\zeta)|V|^2. \qquad (3.18)
\end{aligned}
$$

Applying (3.17) and (3.18) to (1.2), we arrive at

$$
\begin{aligned}
&\frac{\epsilon(\sigma + 3\rho)}{2} + \mu + \varrho(-1 - 2h + 2\omega(\zeta)\eta(V)) - (2V(h) + 4h^2) + V\,(2\omega(\zeta)\eta(V)) \\
&+8h\omega(\zeta)\eta(V) + 4\omega(\zeta)\omega(V)\eta(V) - 2\omega(\zeta)\eta(V) - 4(\eta(V))^2\omega(\zeta) - 2\omega(\zeta)|V|^2 = 0. \qquad (3.19)
\end{aligned}
$$

Thus, we conclude with the following result:

Theorem 3.4. *If ζ is a TFVF on a PF-spacetime (M^4, g) and the metric g satisfies an AHRS (g, V, ϱ, μ) such that V is TFVF and satisfies* (2.6)*, then the relation* (3.19) *holds.*

Corollary 3.1. *If ζ is a TFVF on a PF-spacetime (M^4, g) and the metric g satisfies an AHRS (g, V, ϱ, μ) such that V is the concircular vector field, that is, $\nabla_{X_1} V = hX_1$ for all vector field X_1, then*

$$\frac{\epsilon(\sigma + 3\rho)}{2} + \mu - (1 + 2h)\varrho - 2V(h) - 4h^2 = 0.$$

Garcia-Parrado and Senovilla [29] introduced bi-conformal vector fields; later, this notion was defined by De et al. in [30] as follows:

Definition 3.2. *If a vector field X_1 on a Riemannian manifold (M, g) obeys*

$$(\mathcal{L}_{X_1} g)(X_2, X_3) = \alpha g(X_2, X_3) + \beta S(X_2, X_3), \tag{3.20}$$

and

$$(\mathcal{L}_{X_1} S)(X_2, X_3) = \alpha S(X_2, X_3) + \beta g(X_2, X_3), \tag{3.21}$$

for some smooth functions $\alpha(\neq 0)$ and $\beta(\neq 0)$, then it is called Ricci bi-conformal vector field.

Let ζ be a TFVF on PF-spacetime (M^4, g), and g satisfies the AHRS (g, V, ϱ, μ) such that V is the Ricci bi-conformal vector field and satisfies (3.20) and (3.21). We get

$$\mathcal{L}_V(\mathcal{L}_V g) = (\alpha^2 + \beta^2 + V(\alpha))g + (2\alpha\beta + V(\beta))S. \tag{3.22}$$

Inserting (3.20) and (3.22) in (1.2), we have

$$(1 + \varrho\beta + 2\alpha\beta + V(\beta))S(X_1, X_2) + (\varrho\alpha - \mu + \alpha^2 + \beta^2 + V(\alpha))g(X_1, X_2) = 0. \tag{3.23}$$

Substituting $X_1 = X_2 = \zeta$ in (3.23), we arrive at

$$(1 + \varrho\beta + 2\alpha\beta + V(\beta))(-\varrho + \epsilon(\sigma + \rho)) - \frac{\epsilon(\sigma - \rho)}{2} + (\varrho\alpha - \mu + \alpha^2 + \beta^2 + V(\alpha)) = 0, \tag{3.24}$$

and

$$(1 + \varrho\beta + 2\alpha\beta + V(\beta))\left(S(X_1, X_2) + \left(\varrho - \frac{\epsilon(\sigma + 3\rho)}{2}\right)g(X_1, X_2)\right) = 0. \tag{3.25}$$

Set $F = 1 + \varrho\beta + 2\alpha\beta + V(\beta)$ and $G = \varrho\alpha - \mu + \alpha^2 + \beta^2 + V(\alpha)$.

Taking the Lie derivative of (3.23) and using (3.20) and (3.21), we deduce

$$(\alpha F + \beta G + V(F))S(X_1, X_2) + (\alpha G + \beta F + V(G))g(X_1, X_2) = 0. \tag{3.26}$$

Inserting $X_1 = X_2 = \zeta$ in the Eq (3.26), we conclude

$$(\alpha F + \beta G + V(F))\left(-\varrho + \frac{\epsilon(\sigma + 3\rho)}{2}\right) + \alpha G + \beta F + V(G) = 0. \tag{3.27}$$

Using (3.24) and (4.12), we infer

$$F\left(\beta - \beta\left(-\varrho + \frac{\epsilon(\sigma + 3\rho)}{2}\right) - V\left(-\varrho + \frac{\epsilon(\sigma + 3\rho)}{2}\right)\right) = 0.$$

If $F \neq 0$ then (3.25) yields that M is an Einstein manifold and $r = 4\left(-\varrho + \frac{\epsilon(\sigma+3\rho)}{2}\right)$, where $-\varrho + \frac{\epsilon(\sigma+3\rho)}{2}$ is constant. If $F = 0$, then $G = 0$. Therefore, we have:

Theorem 3.5. *Suppose that ζ is a TFVF on PF-spacetime (M^4, g) and g satisfies the HRS (g, V, ϱ, μ), where V is the Ricci bi-conformal vector field and satisfies* (3.20) *and* (3.21). *Then PF-spacetime is Einstein,*

$$\beta - \beta\left(-\varrho + \frac{\epsilon(\sigma + 3\rho)}{2}\right) - V\left(-\varrho + \frac{\epsilon(\sigma + 3\rho)}{2}\right) = 0,$$

and $r = 4\left(-\varrho + \frac{\epsilon(\sigma+3\rho)}{2}\right)$ *or* $\varrho = -\frac{1}{\beta}(1 + 2\alpha\beta + V(\beta))$ *and* $\mu = -\frac{\alpha}{\beta}(1 + V(\beta)) - \alpha^2 + \beta^2 + V(\alpha)$.

4. PF-GRW-spacetime

In this section, we study HRS on PF-GRW-spacetimes. Throughout this section, we assume that the vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ are arbitrary vector fields on PF-GRW-spacetime M, unless otherwise noted. Now, we consider the velocity vector field ζ of the PF-GRW-spacetime (M^4, g) as a concircular vector field, i.e.,

$$\nabla_{\mathcal{X}_1}\zeta = \psi\mathcal{X}_1. \tag{4.1}$$

In this case, we have

$$R(\mathcal{X}_1, \mathcal{X}_2)\zeta = (\mathcal{X}_1\psi)\mathcal{X}_2 - (\mathcal{X}_2\psi)\mathcal{X}_1. \tag{4.2}$$

Executing contraction over $\mathcal{X}_1$, we obtain

$$S(\mathcal{X}_1, \zeta) = -3(\mathcal{X}_1\psi),$$

and

$$\mathcal{X}_2 f = -\frac{1}{3}\left(\varrho - \frac{\epsilon(\sigma + 3\rho)}{2}\right)\eta(\mathcal{X}_2). \tag{4.3}$$

Applying (4.3) to (4.2), we get

$$R(\mathcal{X}_1, \mathcal{X}_2)\zeta = -\frac{1}{3}\left(\varrho - \frac{\epsilon(\sigma + 3\rho)}{2}\right)(\eta(\mathcal{X}_1)\mathcal{X}_2 - \eta(\mathcal{X}_2)\mathcal{X}_1).$$

Now, let ζ be a concircular vector field on a PF-GRW-spacetime (M^4, g). Then, we obtain

$$\mathcal{L}_\zeta g(\mathcal{X}_1, \mathcal{X}_2) = 2\psi g(\mathcal{X}_1, \mathcal{X}_2), \tag{4.4}$$

and

$$\begin{aligned}(\mathcal{L}_\zeta(\mathcal{L}_\zeta g))(\mathcal{X}_1, \mathcal{X}_2) &= 2\zeta\left(\psi g(\mathcal{X}_1, \mathcal{X}_2)\right) - 2\psi g(\mathcal{L}_\zeta\mathcal{X}_1, \mathcal{X}_2) - 2\psi g(\mathcal{X}_1, \mathcal{L}_\zeta\mathcal{X}_2)\\ &= 2\zeta(\psi)g(\mathcal{X}_1, \mathcal{X}_2) + 2\psi\mathcal{L}_\zeta g(\mathcal{X}_1, \mathcal{X}_2)\\ &= (2\zeta(\psi) + 4\psi^2)g(\mathcal{X}_1, \mathcal{X}_2).\end{aligned} \tag{4.5}$$

Using (4.4) and (4.5), we conclude

$$\varrho\mathcal{L}_\zeta g(\mathcal{X}_1, \mathcal{X}_2) + (\mathcal{L}_\zeta(\mathcal{L}_\zeta g))(\mathcal{X}_1, \mathcal{X}_2) = \left(2\varrho\psi + 2\zeta(\psi) + 4\psi^2\right)g(\mathcal{X}_1, \mathcal{X}_2).$$

Then Eq (1.2) implies that

$$S(\mathcal{X}_1, \mathcal{X}_2) = (-2\psi\varrho - 2V(\psi) - 4\psi^2 + \mu)g(\mathcal{X}_1, \mathcal{X}_2).$$

Thus, we can state:

Theorem 4.1. *If a PF-GRW-spacetime* (M^4, g) *with concircular vector field* ζ *satisfies an AHRS* (g, ζ, ϱ, μ)*, then* M *is an Einstein spacetime.*

Now, suppose that (M^4, g) with concircular vector field ζ admits the AHRS (1.2) and $V = \varphi\zeta$ for some function φ on M. Using (4.1), we have

$$\mathcal{L}_{\varphi\zeta} g(\mathcal{X}_1, \mathcal{X}_2) = (\mathcal{X}_1\varphi)\eta(\mathcal{X}_2) + (\mathcal{X}_2\varphi)\eta(\mathcal{X}_1) + 2\varphi\psi g(\mathcal{X}_1, \mathcal{X}_2),$$

and

$$\begin{aligned}(\mathcal{L}_{\varphi\zeta}(\mathcal{L}_{\varphi\zeta} g))(\mathcal{X}_1, \mathcal{X}_2) \quad = \quad & \varphi\zeta\left((\mathcal{X}_1\varphi)\eta(\mathcal{X}_2) + (\mathcal{X}_2\varphi)\eta(\mathcal{X}_1) + 2\varphi\psi g(\mathcal{X}_1, \mathcal{X}_2)\right) \\ & - \Big(((\mathcal{L}_{\varphi\zeta}\mathcal{X}_1)\varphi)\eta(\mathcal{X}_2) + (\mathcal{X}_2\varphi)\eta(\mathcal{L}_{\varphi\zeta}\mathcal{X}_1) + 2\varphi\psi g(\mathcal{L}_{\varphi\zeta}\mathcal{X}_1, \mathcal{X}_2)\Big) \\ & - \Big((\mathcal{X}_1\varphi)\eta(\mathcal{L}_{\varphi\zeta}\mathcal{X}_2) + ((\mathcal{L}_{\varphi\zeta}\mathcal{X}_2)\varphi)\eta(\mathcal{X}_1) + 2\varphi\psi g(\mathcal{X}_1, \mathcal{L}_{\varphi\zeta}\mathcal{X}_2)\Big).\end{aligned}$$

Inserting $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ in the above equations, we get

$$\mathcal{L}_{\varphi\zeta} g(\zeta, \zeta) = -2(\zeta\varphi) - 2\varphi\psi,$$

and

$$(\mathcal{L}_{\varphi\zeta}(\mathcal{L}_{\varphi\zeta} g))(\zeta, \zeta) = -2\varphi\zeta\left((\zeta\varphi) + \varphi\psi\right) - 4\left((\zeta\varphi)^2 + \varphi\psi(\zeta\varphi)\right).$$

Applying $V = \varphi\zeta$ and the above equations in (1.2), we infer

$$3\zeta(\psi) + 2\varrho\left((\zeta\varphi) + \varphi\psi\right) + 2\varphi\zeta\left((\zeta\varphi) + \varphi\psi\right) + 4\left((\zeta\varphi)^2 + \varphi\psi(\zeta\varphi)\right) - \mu = 0. \tag{4.6}$$

Therefore, this leads to:

Theorem 4.2. *Let* ζ *be a concircular vector field on PF-GRW-spacetime* (M^4, g) *and* (g, V, ϱ, μ) *be an AHRS such that* $V = \varphi\zeta$*, then the identity* (4.6) *holds.*

Let V be a conformal Killing vector field on a PF-GRW-spacetime (M^4, g) with concircular vector field ζ. Then, by inserting (3.11) in (1.2), we obtain (3.13). Plugging $\mathcal{X}_2 = \zeta$ in (3.13), we infer

$$3\mathcal{X}_1(\psi) = (-2h\varrho - 2V(h) - 4h^2 + \mu)\eta(\mathcal{X}_1). \tag{4.7}$$

Also, by inserting $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ in (3.13), we infer

$$3\zeta(\psi) + 2h\varrho + 2V(h) + 4h^2 - \mu = 0. \tag{4.8}$$

Also, from (4.7), we get

$$3g(\nabla\psi, \mathcal{X}_1) = (-2h\varrho - 2V(h) - 4h^2 + \mu)g(\xi, \mathcal{X}_1).$$

Since $\mathcal{X}_1$ is an arbitrary vector field, we conclude

$$3\nabla\psi = (-2h\varrho - 2V(h) - 4h^2 + \mu)\xi. \tag{4.9}$$

We conclude the following:

Theorem 4.3. *If the metric g of a PF-GRW-spacetime (M^4, g) with concircular vector field ζ satisfies the AHRS (g, V, ϱ, μ) such that V is the conformally Killing vector field, then spacetime is Einstein and the identities (4.8) and (4.9) are true.*

Let ζ be a concircular vector field on a PF-GRW-spacetime (M^4, g), and (g, V, ϱ, μ) be an AHRS such that V is a TFVF and satisfies (2.6). Then, putting $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ in (3.14) and (3.16), it follows that

$$\mathcal{L}_V g(\zeta, \zeta) = -2h + 2\omega(\zeta)\eta(V) \tag{4.10}$$

and

$$\begin{aligned}(\mathcal{L}_V(\mathcal{L}_V g))(\zeta, \zeta) &= -(2V(h) + 4h^2) + V(2\omega(\zeta)\eta(V)) + 8h\omega(\zeta)\eta(V) \\ &\quad +4\omega(\zeta)\omega(V)\eta(V) - 2\psi\omega(V)\eta(V) - 2\psi\omega(\zeta)|V|^2.\end{aligned}$$

Applying (4.10) and (4.11) to (1.2), we arrive at

$$\begin{aligned}&-3\zeta(\psi) + \mu + \varrho(-2h + 2\omega(\zeta)\eta(V)) - (2V(h) + 4h^2) + V(2\omega(\zeta)\eta(V)) \\ &+8h\omega(\zeta)\eta(V) + 4\omega(\zeta)\omega(V)\eta(V) - 2\psi\omega(V)\eta(V) - 2\psi\omega(\zeta)|V|^2 = 0.\end{aligned} \tag{4.11}$$

Therefore, we have:

Theorem 4.4. *If the metric g of a PF-GRW-spacetime (M^4, g) with concircular vector field ζ satisfies the AHRS (g, V, ϱ, μ) such that V is a TFVF and satisfies* (2.6)*, then the Eq* (4.11) *holds.*

Let ζ be a concircular vector field on PF-GRW-spacetime (M^4, g) and g satisfies the AHRS (g, V, ϱ, μ) such that V is the Ricci bi-conformal vector field and satisfies (3.20) and (3.21). Substituting $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ in (3.23), we arrive at

$$(1 + \varrho\beta + 2\alpha\beta + V(\beta))(-3\zeta(\psi)) + (\varrho\alpha - \mu + \alpha^2 + \beta^2 + V(\alpha)) = 0,$$

and

$$(1 + \varrho\beta + 2\alpha\beta + V(\beta))(S(\mathcal{X}_1, \mathcal{X}_2) - (-3\zeta(\psi))g(\mathcal{X}_1, \mathcal{X}_2)) = 0.$$

Putting $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ in (3.26), we conclude

$$(\alpha F + \beta G + V(F))(-3\zeta(\psi)) + \alpha G + \beta F + V(G) = 0. \tag{4.12}$$

Using (3.24) and (4.12), we infer

$$F(\beta - \beta(-3\zeta(\psi)) - V(-3\zeta(\psi))) = 0.$$

If $F \neq 0$ then the identity (3.25) yields M is an Einstein manifold, and $r = -12\zeta(\psi)$, where $\zeta(\psi)$ is a constant. If $F = 0$, then $G = 0$. Therefore, by using (4.3), we conclude:

Theorem 4.5. *Suppose that ζ is a concircular vector field on a PF-GRW-spacetime (M^4, g) and the metric g satisfies the HRS (g, V, ϱ, μ) such that V is the Ricci bi-conformal vector field and satisfies* (3.20) *and* (3.21)*. Then, M is an Einstein manifold,*

$$\beta - \beta\left(\varrho - \frac{\epsilon(\sigma + 3\rho)}{2}\right) - V\left(\varrho - \frac{\epsilon(\sigma + 3\rho)}{2}\right) = 0,$$

and $r = -12\zeta(\psi)$ or $\varrho = -\frac{1}{\beta}(1 + 2\alpha\beta + V(\beta))$ and $\mu = -\frac{\alpha}{\beta}(1 + V(\beta)) - \alpha^2 + \beta^2 + V(\alpha)$.

5. Conclusions

In this paper, we obtain the geometrical conditions and characteristics of HRS to utilize their existence in perfect fluid and GRW spacetimes. We first assume that $(g, f\zeta, \varrho, \mu)$ satisfies AHRS such that ζ is a TFVF on PF-spacetime (M^4, g), and we give a differential equation that the function f admits in it. Then, we show that any PF-spacetime with a TFVF satisfies a HRS. Also, we prove that if any PF-spacetime with TFVF ζ satisfies HRS (g, V, ϱ, μ), where V is the conformal Killing vector field or Ricci bi-conformal vector field, then PF-spacetime is Einstein. Next, we show that a PF-GRW-spacetime with concicular vector field ζ and AHRS $(g, \varphi\zeta, \varrho, \mu)$ is Einstein for $\varphi = 1$, and in general case φ admits in a differential equation. Also, we study a HRS (g, V, ϱ, μ) on PF-GRW-spacetimes when V is a conformal Killing vector field, a TFVF, or a Ricci bi-conformal vector field.

Author contributions

Shahroud Azami: Conceptualization, investigation, methodology; Mehdi Jafari: Investigation, methodology, writing – original draft; Nargis Jamal: Conceptualization, methodology, writing – review & editing; Abdul Haseeb: Conceptualization, investigation, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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