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Research article

On existence results for a class of biharmonic elliptic problems without (AR) condition

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Abstract: In this paper, we study the following biharmonic elliptic equation in $\mathbb{R}^N$:

$$\Delta^2\psi - \Delta\psi + P(x)\psi = g(x,\psi), \quad x \in \mathbb{R}^N,$$

where g and P are periodic in $x_1, \cdots, x_N$, $g(x,\psi)$ is subcritical and odd in ψ. Without assuming the Ambrosetti-Rabinowitz condition, we prove the existence of infinitely many geometrically distinct solutions for this equation, and the existence of ground state solutions is established as well.

Keywords: biharmonic elliptic equation; (AR) condition; critical point theory; Nehari manifold; infinitely many solutions; geometrically distinct solutions
Mathematics Subject Classification: 35J35, 35B38, 35J91

1. Introduction and main results

In the present paper, we consider the following biharmonic elliptic equation with potential:

$$\begin{cases} \Delta^2\psi - \Delta\psi + P(x)\psi = g(x,\psi) & \text{in } \mathbb{R}^N, \\ \psi(x) \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator. We assume that $P(x)$ and $g(x,\psi)$ satisfy the hypotheses below:

(P) $P(x) \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each of x_i, $1 \le i \le N$, and $\inf_{x\in\mathbb{R}^N} P(x) \ge a_0 > 0$.

(g_1) $g(x,t) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is 1-periodic in each of x_i, $1 \le i \le N$, and there exists $a_1 > 0$ such that $|g(x,t)| \le a_1(1 + |t|^{q-1})$ for $2 < q < 2_*$, where $2_* = \frac{2N}{N-4}$ if $N > 4$, $2_* = +\infty$ if $N \le 4$.

(g_2) $\lim\limits_{|t|\to 0} \frac{g(x,t)}{t} = 0$ uniformly for $x \in \mathbb{R}^N$.

(g_3) $\lim\limits_{|t|\to+\infty} \frac{G(x,t)}{t^2} = +\infty$ uniformly for $x \in \mathbb{R}^N$, where $G(x,\psi) = \int_0^\psi g(x,t)dt$.

(g_4) $\frac{g(x,t)}{t}$ is strictly increasing on $(-\infty, 0)$ and on $(0, +\infty)$.

Problem (1.1) is usually used to describe some phenomena appearing in different physical, engineering and other sciences. Over the course of the last decades, plenty of results for the biharmonic elliptic equations have been presented. When $\Omega \subset \mathbb{R}^N (N > 4)$ is a smooth bounded domain, the problem

$$\begin{cases} \Delta^2\psi + a\Delta\psi = g(x,\psi), & x \in \Omega, \\ \psi = \Delta\psi = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

arising in the study of traveling waves in suspension bridges (see for instance, [2, 8, 9, 15]) and the study of the static deflection of an elastic plate in a fluid, has drawn a great deal of attention, see for example, [1, 3, 7, 11–14, 23] and the references therein. Furthermore, biharmonic elliptic problems on the whole space $\mathbb{R}^N$ also attract a lot of attention, see [4, 5, 10, 21, 22, 24, 25]. It is worth noticing that in the paper by Yin and Wu [22], a sequence of high energy solutions to problem (1.1) has been established by using variational methods. Later, based on Rabinowitz's symmetric mountain pass theorem, Ye and Tang extended the results in [21] to a more generic conditions, and obtained similar results. Subsequently, Zhang et al. [24] obtained the existence of infinitely many solutions by applying the genus properties in critical point theory.

Resting on the different assumptions $(g_1)-(g_4)$ from those applied previously, our paper states some new existence results of problem (1.1) and, meanwhile, we do not assume the Ambrosetti-Rabinowitz condition ((AR) in short):

(AR) there is $\mu > 2$ such that $0 < \mu G(x,\psi) \leq g(x,\psi)\psi$ for $\psi \neq 0$ and $x \in \mathbb{R}^N$.

It is noticeable that the (AR) condition is to ensure the boundedness of the Palais-Smale sequences of the corresponding functional, which is very essential in applying the critical point theory. It would be more complicated for this problem without (AR) condition. However, there are numerous functions superlinear at infinity not satisfying the (AR) condition for any $\mu > 2$. Virtually, the (AR) condition implies that $G(x,\psi) \geq c_1|\psi|^\mu - c_2$ for some $c_1, c_2 > 0$. Thus, for example the superlinear function

$$g(x,\psi) = a(x)\psi \ln(1 + |\psi|), \text{ where } a(x) > 0 \text{ is 1-periodic in } x_i, 1 \leq i \leq N,$$

does not satisfy (AR) condition. But it satisfies our conditions $(g_1) - (g_4)$.

To state our results, we need to present some notations. For $y = (y_1, \cdots, y_N) \in \mathbb{Z}^N$, the action of $\mathbb{Z}^N$ on $H^2(\mathbb{R}^N)$ is given by

$$\tau_y\psi(x) = \psi(x - y), \quad y \in \mathbb{Z}^N. \tag{1.3}$$

It follows from (P) and (g_1) that if ψ_0 is a solution of (1.1), then so is $\tau_y\psi_0$ for all $y \in \mathbb{Z}^N$. Set $O(\psi_0) = \{\tau_y\psi_0 : y \in \mathbb{Z}^N\}$. Two solutions ψ_1, ψ_2 of (1.1) are regarded as geometrically distinct if $O(\psi_1) \neq O(\psi_2)$.

The main results of this paper are the following:

Theorem 1.1. *Assume that $P(x)$ satisfies (P), and $g(x,\psi)$ satisfies $(g_1) - (g_4)$. Then the Eq (1.1) has at least one ground state solution.*

In our next result, we verify the existence of infinitely many solutions for (1.1) if $g(x,t)$ is an odd function about t. More specifically, we suppose

(g_5) $g(x,-t) = -g(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Theorem 1.2. *Assume that $P(x)$ satisfies (P), and $g(x,\psi)$ satisfies $(g_1)-(g_5)$. Then problem (1.1) has infinitely many pairs $\pm\psi$ of geometrically distinct solutions.*

The paper is organized as follows. In Section 2, some preliminary results for proving our main results are presented, and the fact that problem (1.1) has a ground state solution is proved. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminary results and proof of Theorem 1.1

First, let us set some notations to be used in this paper. $L^r(\mathbb{R}^N)(1 \le r < +\infty)$ denotes Lebesgue space, the usual norm of $L^r(\mathbb{R}^N)$ is denoted by $\|\cdot\|_r$ for $1 \le r < +\infty$. Let

$$E = \left\{\psi \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta\psi|^2 + |\nabla\psi|^2 + P(x)\psi^2)dx < +\infty\right\},$$

then E is a Hilbert space with the inner product

$$\langle\psi, v\rangle_E = \int_{\mathbb{R}^N} (\Delta\psi\Delta v + \nabla\psi\nabla v + P(x)\psi v)\, dx,$$

and the induced norm is denoted by $\|\psi\| = \sqrt{\langle\psi,\psi\rangle_E}$. Note that the following embedding is continuous:

$$E \hookrightarrow L^r(\mathbb{R}^N) \ \ (2 \le r < 2_*),$$

consequently, for each $r \in [2, 2_*)$, there exists a constant $a_r > 0$ such that

$$\|\psi\|_r \le a_r\|\psi\|, \ \ \forall\ \psi \in E. \tag{2.1}$$

The dual space of a space E will be denoted by E^{-1} and $\mathcal{S}$ is the unit sphere in E, that is

$$\mathcal{S} = \{\psi \in E : \|\psi\| = 1\}.$$

The corresponding energy functional of problem (1.1) is defined on E by

$$\mathcal{E}(\psi) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta\psi|^2 + |\nabla\psi|^2 + P(x)\psi^2\right) dx - \int_{\mathbb{R}^N} G(x,\psi)dx, \tag{2.2}$$

where $G(x,\psi) = \int_0^\psi g(x,t)dt$. Under the assumptions $(g_1)-(g_4)$ and (P), we can easily check that $\mathcal{E}(\psi) \in C^1(E,\mathbb{R})$ and

$$\langle\mathcal{E}'(\psi), v\rangle = \int_{\mathbb{R}^N} (\Delta\psi\Delta v + \nabla\psi\nabla v + P(x)\psi v)\, dx - \int_{\mathbb{R}^N} g(x,\psi)v dx, \tag{2.3}$$

for all $\psi, v \in E$. Thus, solutions to problem (1.1) can be obtained as the critical points of the functional $\mathcal{E}(\psi)$. We consider the Nehari manifold

$$\mathcal{M} = \{\psi \in E \setminus \{0\} : \langle\mathcal{E}'(\psi),\psi\rangle = 0\},$$

and let

$$c^* = \inf_{\psi \in \mathcal{M}} \mathcal{E}(\psi).$$

Note that $\mathcal{M}$ contains every nonzero solution of problem (1.1). For $t > 0$, we consider the fibering maps $\phi_\psi : t \to \mathcal{E}(t\psi)$ defined by

$$\phi_\psi(t) = \mathcal{E}(t\psi) = \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\Delta\psi|^2 + |\nabla\psi|^2 + P(x)\psi^2 \right) dx - \int_{\mathbb{R}^N} G(x, t\psi) dx.$$

Now we have the following lemma. Hereafter, we suppose that (P) and $(g_1) - (g_4)$ are satisfied.

Lemma 2.1. *(i) For each $\psi \in E\backslash\{0\}$, there is a unique $t_\psi > 0$ such that $\phi'_\psi(t) > 0$ for $0 < t < t_\psi$ and $\phi'_\psi(t) < 0$ for $t > t_\psi$. Moreover, $t\psi \in \mathcal{M}$ if and only if $t = t_\psi$.*
(ii) There exists $\rho > 0$ such that $c^* \geq \inf_{\psi \in S_\rho} \mathcal{E}(\psi) > 0$, where $S_\rho = \{\psi \in E : \|\psi\| = \rho\}$.
(iii) For all $\psi \in \mathcal{M}$, there holds $\|\psi\| \geq \sqrt{2c^*}$.
(iv) For all $\psi \in \mathcal{M}$, there holds $\mathcal{E}(\psi) \to \infty$ as $\|\psi\| \to \infty$.

Proof. (i) First, we claim that $\phi_\psi(t) > 0$ for $t > 0$ small. Indeed, the conditions (g_1) and (g_2) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $(x, \psi) \in \mathbb{R}^N \times \mathbb{R}$, there hold

$$|g(x, \psi)| \leq \varepsilon|\psi| + C_\varepsilon|\psi|^{q-1}, \quad |G(x, \psi)| \leq \varepsilon|\psi|^2 + C_\varepsilon|\psi|^q. \tag{2.4}$$

Then, by (2.4) and the Sobolev embedding theorem, for $\varepsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}
\phi_\psi(t) &\geq \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\Delta\psi|^2 + |\nabla\psi|^2 + P(x)\psi^2 \right) dx - \frac{\varepsilon t^2}{2} \int_{\mathbb{R}^N} |\psi|^2 dx - \frac{C_\varepsilon t^q}{q} \int_{\mathbb{R}^N} |\psi|^q dx \\
&\geq \frac{t^2}{2}\|\psi\|^2 - \frac{t^2}{2}\varepsilon C_1\|\psi\|^2 - \frac{t^q}{q} C_\varepsilon C_2 \|\psi\|^q,
\end{aligned}$$

since $q > 2$, $\phi_\psi(t) > 0$ whenever $t > 0$ is small enough.

On the other hand, we have

$$\begin{aligned}
\phi_\psi(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\Delta\psi|^2 + |\nabla\psi|^2 + P(x)\psi^2 \right) dx - \int_{\mathbb{R}^N} G(x, t\psi) dx \\
&= t^2 \left(\frac{1}{2}\|\psi\|^2 - \int_{\mathbb{R}^N} \frac{G(x, t\psi)}{(t\psi)^2} \cdot \psi^2 dx \right).
\end{aligned}$$

By (g_3) and Fatou's lemma, one has

$$\int_{\mathbb{R}^N} \frac{G(x, t\psi)}{(t\psi)^2} \cdot \psi^2 dx \to +\infty \ (t \to +\infty).$$

Hence $\phi_\psi(t) \to -\infty$ as $t \to +\infty$ and $\max_{t>0} \phi_\psi(t)$ is achieved at a $t = t_\psi > 0$. In addition, the condition $\phi'_\psi(t) = 0$ is equivalent to

$$\int_{\mathbb{R}^N} \left(|\Delta\psi|^2 + |\nabla\psi|^2 + P(x)\psi^2 \right) dx = \int_{\mathbb{R}^N} \frac{g(x, t\psi)}{t\psi} \cdot \psi^2 dx.$$

By (g_4), the function $\frac{g(x,t)}{t}$ is strictly increasing for $t > 0$, so there exists a unique $t_\psi > 0$ such that $\phi'_\psi(t_\psi) = 0$. On the other hand, we note that

$$\phi'_\psi(t) = t^{-1}\langle \mathcal{E}'(t\psi), t\psi\rangle.$$

Therefore, $t\psi \in \mathcal{M}$ if and only if $t = t_\psi$.

(ii) For $\psi \in E$, we have

$$\mathcal{E}(\psi) = \frac{1}{2}\|\psi\|^2 - \int_{\mathbb{R}^N} G(x,\psi)dx,$$

and thus by (2.3) there holds

$$\int_{\mathbb{R}^N} G(x,\psi)dx = o(\|\psi\|^2) \text{ as } \psi \to 0,$$

hence $\inf_{\psi\in S_\rho} \mathcal{E}(\psi) > 0$ if $\rho > 0$ is sufficiently small. The inequality $\inf_{\psi\in\mathcal{M}} \mathcal{E}(\psi) \geq \inf_{\psi\in S_\rho} \mathcal{E}(\psi)$ is a consequence of (i), since for every $\psi \in \mathcal{M}$ there exists $t > 0$ such that $t\psi \in S_\rho$, and $\mathcal{E}(t_\psi\psi) \geq \mathcal{E}(t\psi)$.

(iii) Note that by using (g_2) and (g_4), we can get

$$G(x,\psi) \geq 0, \quad g(x,\psi)\psi \geq 2G(x,\psi), \ \ \forall\, \psi \neq 0. \tag{2.5}$$

Then by the definition of c^* and (2.5), for $\psi \in \mathcal{M}$ one has

$$c^* \leq \frac{1}{2}\|\psi\|^2 - \int_{\mathbb{R}^N} G(x,\psi)dx \leq \frac{1}{2}\|\psi\|^2,$$

hence $\|\psi\| \geq \sqrt{2c^*}$.

(iv) Arguing by contradiction, suppose there exists a sequence $\{\psi_m\} \subset \mathcal{M}$ such that $\|\psi_m\| \to \infty$ and $\mathcal{E}(\psi_m) \leq d$ for some $d > 0$. Let $v_m = \frac{\psi_m}{\|\psi_m\|}$. Then $\{v_m\}$ is bounded ($\|v_m\| = 1$) in E, after passing to a subsequence, if necessary, we may assume that $v_m \rightharpoonup v$ in E and $v_m(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Choose $y_m \in \mathbb{R}^N$ to satisfy

$$\int_{B_1(y_m)} v_m^2 dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} v_m^2 dx.$$

Since $\mathcal{E}$ and $\mathcal{M}$ are invariant with respect to the action of $\mathbb{Z}^N$ given by (1.3), we may assume translating v_m, if necessary, that $\{y_m\}$ is bounded in $\mathbb{R}^N$. If

$$\lim_{m\to\infty} \int_{B_1(y_m)} v_m^2 dx = 0, \tag{2.6}$$

according to P. L. Lions' vanishing lemma (see [20], Lemma 1.21), we get $v_m \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_*$. By (2.4), fixing an $s > \sqrt{2c^*}$ and using the Lebesgue dominated convergence theorem, we have

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} G(x, sv_m)dx = \int_{\mathbb{R}^N} \lim_{m\to\infty} G(x, sv_m)dx = 0.$$

Note that $\{\psi_m\} \subset \mathcal{M}$, and then by Lemma 2.1, we obtain that

$$d \geq \mathcal{E}(\psi_m) \geq \mathcal{E}(sv_m) = \frac{s^2}{2}\|v_m\|^2 - \int_{\mathbb{R}^N} G(x, sv_m)dx \to \frac{s^2}{2},$$

which is a contradiction for $s > \sqrt{2d}$. Hence (2.6) cannot hold, and then $v_m \to v \neq 0$ in $L^2_{loc}(\mathbb{R}^N)$. Since $|\psi_m(x)| \to \infty$ if $v(x) \neq 0$, then by (g_3) and Fatou's lemma, we have

$$\int_{\mathbb{R}^N} \frac{G(x,\psi_m)}{\psi_m^2} v_m^2 dx \to +\infty \ (m \to \infty),$$

and therefore

$$0 \leq \frac{\mathcal{E}(\psi_m)}{\|\psi_m\|^2} = \frac{1}{2}\|v_m\|^2 - \int_{\mathbb{R}^N} \frac{G(x,\psi_m)}{\psi_m^2} v_m^2 dx \to -\infty,$$

as $m \to \infty$, which is a contradiction. This completes the proof. □

Lemma 2.2. *Let $\mathcal{V}$ be a compact subset of $E\backslash\{0\}$, then there exists $R > 0$ such that $\mathcal{E}(\psi) < 0$ on $(\mathbb{R}^+\mathcal{V})\backslash B_R(0)$ for all $\psi \in \mathcal{V}$, where $\mathbb{R}^+\mathcal{V} = \{t\psi : t \in \mathbb{R}^+, \psi \in \mathcal{V}\}$.*

Proof. Without loss of generality, we may assume that $\mathcal{V} \subset \mathcal{S}$, i.e., $\|\psi\| = 1$ for every $\psi \in \mathcal{V}$. Arguing by contradiction, suppose there exists $\psi_m \in \mathcal{V}$ and $w_m = t_m\psi_m$ such that $\mathcal{E}(w_m) \geq 0$ and $t_m \to +\infty$ as $m \to \infty$. Passing to a subsequence, we may assume that $\psi_m \to \psi \in \mathcal{S}$. Note that $|w_m(x)| \to \infty$ if $\psi(x) \neq 0$, then by (g_3) and Fatou's lemma we have

$$\int_{\mathbb{R}^N} \frac{G(x,w_m)}{t_m^2} dx = \int_{\mathbb{R}^N} \frac{G(x,w_m)}{w_m^2} \psi_m^2 dx \to +\infty \ (m \to \infty),$$

and therefore

$$0 \leq \frac{\mathcal{E}(w_m)}{t_m^2} = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G(x,w_m)}{t_m^2} dx \to -\infty,$$

which is a contradiction. This completes the proof. □

Recall that $\mathcal{S}$ is the unit sphere in E, and define the mapping $\varphi : \mathcal{S} \to \mathcal{M}$ by setting

$$\varphi(w) = t_w w,$$

where t_w is the same as in Lemma 2.1 (i). Note that $\|\varphi(w)\| = t_w$.

Lemma 2.3. *(i) The mapping φ is a homeomorphism between $\mathcal{S}$ and $\mathcal{M}$, and the inverse of φ is given by $\varphi^{-1}(\psi) = \frac{\psi}{\|\psi\|}$.*

(ii) The mapping φ^{-1} is Lipschitz continuous.

Proof. (i) See [19], Proposition 8.

(ii) For $\psi, u \in \mathcal{M}$, by Lemma 2.1(iii), we have

$$\begin{aligned}\|\varphi^{-1}(\psi) - \varphi^{-1}(u)\| &= \left\|\frac{\psi}{\|\psi\|} - \frac{u}{\|u\|}\right\| = \left\|\frac{\psi - u}{\|\psi\|} + \frac{(\|u\| - \|\psi\|)u}{\|\psi\|\|u\|}\right\| \\ &\leq \frac{2}{\|\psi\|}\|\psi - u\| \leq \sqrt{\frac{2}{c^*}}\|\psi - u\|,\end{aligned}$$

this implies that the mapping φ^{-1} is Lipschitz continuous. □

Now we consider the functional $\mathcal{I} : \mathcal{S} \to \mathbb{R}$ given by $\mathcal{I}(w) = \mathcal{E}(\varphi(w))$. Then we have

Lemma 2.4. *(i)* $\mathcal{I} \in C^1(\mathcal{S}, \mathbb{R})$ *and*

$$\langle \mathcal{I}'(w), z\rangle = \|\varphi(w)\|\langle \mathcal{E}'(\varphi(w)), z\rangle \text{ for all } z \in T_w(\mathcal{S}) = \{u \in E : \langle w, u\rangle = 0\}.$$

(ii) If $\{w_m\}$ *is a (PS) sequence for* $\mathcal{I}$*, then* $\{\varphi(w_m)\}$ *is a (PS) sequence for* $\mathcal{E}$*. If* $\{\psi_m\} \subset \mathcal{M}$ *is a bounded (PS) sequence for* $\mathcal{E}$*, then* $\{\varphi^{-1}(\psi_m)\}$ *is a (PS) sequence for* $\mathcal{I}$*.*

(iii)

$$\inf_{\psi\in\mathcal{S}} \mathcal{I}(\psi) = \inf_{\psi\in\mathcal{M}} \mathcal{E}(\psi) = c^*.$$

Moreover, w is a critical point of $\mathcal{I}$ *if and only if* $\varphi(w)$ *is a nontrivial critical point of* $\mathcal{E}(\psi)$*, and the corresponding critical values coincide.*

(iv) If $\mathcal{E}(\psi)$ *is even, then* $\mathcal{I}(\psi)$ *is also even.*

Proof. The proof is entirely analogous to that of Corollary 10 in [19]. By Lemmas 2.1 and 2.3, it can be concluded that the hypotheses in [19] are satisfied. Indeed, if $\phi_w(t) = \mathcal{E}(tw)$ and $w \in \mathcal{S}$, then $\phi_w'(t) > 0$ for $0 < t < t_w$ and $\phi_w'(t) < 0$ for $t > t_w$ by Lemma 2.1(i), $t_w \geq \delta > 0$ by Lemma 2.1 (ii) and $t_w \leq R$ for $w \in \mathcal{V} \subset \mathcal{S}$ by Lemma 2.2. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. From the conclusion (ii) of Lemma 2.1, we know that $c^* > 0$. Moreover, if $\psi_0 \in \mathcal{M}$ satisfies $\mathcal{E}(\psi_0) = c^*$, then $\varphi^{-1}(\psi_0) \in \mathcal{S}$ is a minimizer of $\mathcal{I}$, and therefore a critical point of $\mathcal{I}$, so that ψ_0 is a critical point of $\mathcal{E}$ by Lemma 2.4. It remains to show that there exists a minimizer ψ of $\mathcal{E}|_{\mathcal{M}}$. By Ekeland's variational principle [20], there exists a sequence $\{w_m\} \subset \mathcal{S}$ such that

$$\mathcal{I}(w_m) \to c^* \text{ and } \mathcal{I}'(w_m) \to 0 \text{ as } m \to \infty.$$

Set $\psi_m = \varphi(w_m) \in \mathcal{M}$ for all $m \in \mathbb{N}$. Then $\mathcal{E}(\psi_m) \to c^*$ and $\mathcal{E}'(\psi_m) \to 0$ as $m \to \infty$. By Lemma 2.1(iv), we know that $\{\psi_m\}$ is bounded and hence $\psi_m \rightharpoonup \psi$ after passing to a subsequence. Choose $y_m \in \mathbb{R}^N$ to satisfy

$$\int_{B_1(y_m)} \psi_m^2 dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} \psi_m^2 dx. \tag{2.7}$$

Since $\mathcal{E}$ and $\mathcal{M}$ are invariant with respect to the action of $\mathbb{Z}^N$ given by (1.3), we may assume that $\{y_m\}$ is bounded in $\mathbb{R}^N$. If

$$\lim_{m\to\infty} \int_{B_1(y_m)} \psi_m^2 dx = 0, \tag{2.8}$$

then by P. L. Lions' vanishing lemma, we have $\psi_m \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_*$. From (2.4) and the Sobolev embedding theorem, we infer that

$$\int_{\mathbb{R}^N} g(x, \psi_m)\psi_m dx = o(\|\psi_m\|) \text{ as } m \to \infty.$$

Hence

$$o(\|\psi_m\|) = \langle \mathcal{E}'(\psi_m), \psi_m\rangle = \|\psi_m\|^2 - \int_{\mathbb{R}^N} g(x, \psi_m)\psi_m dx = \|\psi_m\|^2 - o(\|\psi_m\|),$$

and therefore $\|\psi_m\| \to 0$, contrary to Lemma 2.1(iii). It follows that (2.8) cannot hold, and thus $\psi_m \rightharpoonup \psi \neq 0, \mathcal{E}'(\psi) = 0$.

In the following we claim that $\mathcal{E}(\psi) = c^*$. Notice that $\{\psi_m\}$ is bounded, by (2.5) and Fatou's lemma we get that

$$
\begin{aligned}
c^* &= \liminf_{m\to\infty}\left(\mathcal{E}(\psi_m) - \frac{1}{2}\langle \mathcal{E}'(\psi_m), \psi_m\rangle\right) \\
&= \liminf_{m\to\infty}\left(\int_{\mathbb{R}^N}\left(\frac{1}{2}g(x,\psi_m)\psi_m - G(x,\psi_m)\right)dx\right) \\
&\geq \int_{\mathbb{R}^N}\left(\frac{1}{2}g(x,\psi)\psi - G(x,\psi)\right)dx \\
&= \mathcal{E}(\psi) - \frac{1}{2}\langle \mathcal{E}'(\psi), \psi\rangle = \mathcal{E}(\psi).
\end{aligned}
$$

Hence $\mathcal{E}(\psi) \leq c^*$. On the other hand, by the definition of c^* and note that $\psi \in \mathcal{M}$, we have $c^* \leq \mathcal{E}(\psi)$, so we obtain that $\mathcal{E}(\psi) = c^*$. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We begin with the following lemma (see Lemma 2.13 in [18]).

Lemma 3.1. *Let $K = \{\psi \in \mathcal{S} : \mathcal{I}'(\psi) = 0\}$, then $\alpha := \inf\{\|\psi - w\| : \psi, w \in K, \psi \neq w\} > 0$.*

As a consequence of Lemma 2.4, we see as Remark 2.12 of [18] that since φ, φ^{-1} are equivariant and $\mathcal{E}, \mathcal{I}$ are invariant with respect to the action of $\mathbb{Z}^N$ given by (1.3), there is a one-to-one correspondence between the critical orbits of $\mathcal{E}|_{\mathcal{M}}$ and $\mathcal{I}$. Hence, the proof of Theorem 1.2 will be completed upon showing that $\mathcal{I}$ has infinitely many critical orbits. We shall proceed by contradiction. Namely let us suppose (to the contrary) that the set K only contains finitely many orbits.

Note that by Theorem 1.1 and Lemma 2.3, the set K is nonempty. Choose a subset $\mathcal{J}$ of K such that $\mathcal{J} = -\mathcal{J}$ and each orbit $O(\psi) \subset K$ has a unique representative in $\mathcal{J}$. So we assume by contradiction that

$$\mathcal{J} \text{ is a finite set.} \tag{3.1}$$

From now on, we assume that the nonlinearity $g(x,t)$ is odd in t. For a functional $\mathcal{F}$ we put

$$\mathcal{F}^d = \{\psi : \mathcal{F}(\psi) \leq d\}, \quad \mathcal{F}_c = \{\psi : \mathcal{F}(\psi) \geq c\}, \quad \mathcal{F}_c^d = \{\psi : c \leq \mathcal{F}(\psi) \leq d\}.$$

Lemma 3.2. *Let $d \geq c^*$. If $\{v_m^1\}$, $\{v_m^2\} \subset \mathcal{I}^d$ are two Palais-Smale sequences for $\mathcal{I}$, then*

$$\text{either } \|v_m^1 - v_m^2\| \to 0 \text{ as } m \to \infty \text{ or } \limsup_{m\to\infty} \|v_m^1 - v_m^2\| \geq \rho(d) > 0,$$

where $\rho(d)$ depends on d but not on the particular choice of Palais-Smale sequences.

Proof. We put $\psi_m^1 := \varphi(v_m^1)$ and $\psi_m^2 := \varphi(v_m^2)$. By Lemma 2.4(ii), both sequences $\{\psi_m^1\}$, $\{\psi_m^2\}$ are Palais-Smale sequences for $\mathcal{E}$ and since $\{\psi_m^1\}$, $\{\psi_m^2\} \subset \mathcal{E}^d$, $\{\psi_m^1\}$, $\{\psi_m^2\}$ are bounded. We consider two cases. **Case 1**. For $2 < q < 2_*$, $\|\psi_m^1 - \psi_m^2\|_q \to 0$ as $m \to \infty$. From (g_1) and (g_2), it follows that for each $\varepsilon > 0$

and m large enough, we have that

$$\|\psi_m^1 - \psi_m^2\|^2 = \langle \mathcal{E}'(\psi_m^1), \psi_m^1 - \psi_m^2\rangle - \langle \mathcal{E}'(\psi_m^2), \psi_m^1 - \psi_m^2\rangle + \int_{\mathbb{R}^N} [g(x,\psi_m^1) - g(x,\psi_m^2)](\psi_m^1 - \psi_m^2)dx$$
$$\leq \varepsilon\|\psi_m^1 - \psi_m^2\| + \int_{\mathbb{R}^N} [\varepsilon(|\psi_m^1| + |\psi_m^2|) + C_\varepsilon(|\psi_m^1|^{q-1} + |\psi_m^2|^{q-1})](\psi_m^1 - \psi_m^2)dx$$
$$\leq (1 + C_1)\varepsilon\|\psi_m^1 - \psi_m^2\| + D_\varepsilon\|\psi_m^1 - \psi_m^2\|_q,$$

where $\varepsilon > 0$ is arbitrary, and C_1 does not depend on the choice of ε. Notice that $\|\psi_m^1 - \psi_m^2\|_q \to 0$, therefore $\|\psi_m^1 - \psi_m^2\| \to 0$ and Lemma 3.1 implies

$$\|v_m^1 - v_m^2\| = \|\varphi^{-1}(\psi_m^1) - \varphi^{-1}(\psi_m^1)\| \to 0 \text{ as } m \to \infty.$$

Case 2. For $2 < q < 2_*$, $\|\psi_m^1 - \psi_m^2\|_q \nrightarrow 0$ as $m \to \infty$. It can be concluded from Lemma 1.21 in [20] that there exists $\varepsilon > 0$ and $y_m \in \mathbb{R}^N$ such that after passing to a subsequence,

$$\int_{B_1(y_m)} (\psi_m^1 - \psi_m^2)^2 dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} (\psi_m^1 - \psi_m^2)^2 dx \geq \varepsilon \quad \text{for all } m. \tag{3.2}$$

Since φ, φ^{-1} and $\mathcal{E}', \mathcal{I}'$ are equivariant with respect to the action of $\mathbb{Z}^N$ given by (1.3), we may assume that the sequence $\{y_m\}$ is bounded in $\mathbb{R}^N$. Passing to a subsequence once more, there exist ψ^1, ψ^2 and α^1, α^2 such that

$$\psi_m^1 \rightharpoonup \psi^1, \quad \psi_m^2 \rightharpoonup \psi^2, \quad \|\psi_m^1\| \to \alpha^1, \quad \|\psi_m^2\| \to \alpha^2,$$

and $\mathcal{E}'(\psi^1) = \mathcal{E}'(\psi^2) = 0$. According to (3.2), $\psi^1 \neq \psi^2$ and by Lemma 2.1(iii),

$$\sqrt{2c^*} \leq \alpha^i \leq \nu(d) < +\infty, (i = 1, 2), \quad \text{where } \nu(d) = \sup\{\|\psi\| : \psi \in \mathcal{E}^d \cap \mathcal{M}\}.$$

Suppose $\psi^1, \psi^2 \neq 0$. Then $\psi^1, \psi^2 \in \mathcal{M}$ and $v^1 := \varphi^{-1}(\psi^1) \in K$, $v^2 := \varphi^{-1}(\psi^2) \in K$, $v^1 \neq v^2$. Hence

$$\liminf_{m\to\infty} \|v_m^1 - v_m^2\| = \liminf_{m\to\infty} \left\| \frac{\psi_m^1}{\|\psi_m^1\|} - \frac{\psi_m^2}{\|\psi_m^2\|} \right\| \geq \left\| \frac{\psi_m^1}{\alpha^1} - \frac{\psi_m^2}{\alpha^2} \right\| = \|\beta_1 v^1 - \beta_2 v^2\|,$$

where

$$\beta_1 = \frac{\|\psi^1\|}{\alpha^1} \geq \frac{\sqrt{2c^*}}{\nu(d)}, \quad \beta_2 = \frac{\|\psi^2\|}{\alpha^2} \geq \frac{\sqrt{2c^*}}{\nu(d)}.$$

Since $\|v^1\| = \|v^2\| = 1$, it is easy to see from the inequalities above that

$$\liminf_{m\to\infty} \|v_m^1 - v_m^2\| \geq \|\beta_1 v^1 - \beta_2 v^2\| \geq \min\{\beta_1, \beta_2\}\|v^1 - v^2\| \geq \frac{\alpha\sqrt{2c^*}}{\nu(d)} > 0, \tag{3.3}$$

where α is the constant in Lemma 3.1. Hence, (3.3) implies that

$$\liminf_{m\to\infty} \|v_m^1 - v_m^2\| \geq \rho(d) > 0.$$

Now the case where either $\psi^1 = 0$ or $\psi^2 = 0$ remains to be considered. If $\psi^2 = 0$, then $\psi^1 \neq 0$ and

$$\liminf_{m\to\infty} \|v_m^1 - v_m^2\| = \liminf_{m\to\infty} \left\| \frac{\psi_m^1}{\|\psi_m^1\|} - \frac{\psi_m^2}{\|\psi_m^2\|} \right\| \geq \frac{\|\psi_m^1\|}{\alpha^1} \geq \frac{\sqrt{2c^*}}{\nu(d)} > 0.$$

The case $\psi^1 = 0$ is similar. □

It is well known that $\mathcal{I}$ admits a pseudo-gradient vector field, i.e., there exists a Lipschitz continuous map $H : \mathcal{S} \setminus K \to T\mathcal{S}$ (see [17], p.86) such that

$$\|H(w)\| < 2\|\nabla \mathcal{I}(w)\|, \ \ \langle H(w), \nabla \mathcal{I}(w)\rangle > \frac{1}{2}\|\nabla \mathcal{I}(w)\|^2,$$

where $T\mathcal{S}$ denotes the tangent bundle of $\mathcal{S}$. Moreover, seeing that $\mathcal{I}$ is even, we may assume H is odd. Let $\eta : \mathcal{G} \to \mathcal{S} \setminus K$ be the flow defined by the following Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}\eta(t,w) = -H(\eta(t,w)), \\ \eta(0,w) = w, \end{cases} \tag{3.4}$$

where

$$\mathcal{G} = \{(t,w) : w \in \mathcal{S} \setminus K, T^-(w) < t < T^+(w)\},$$

and $(T^-(w), T^+(w))$ is the maximal existence time for the trajectory $t \to \eta(t,w)$. Note that η is odd in w because H is odd, and $t \to \mathcal{I}(\eta(t,w))$ is strictly decreasing by the properties of a pseudo-gradient.

Remark 3.1. *We note that by the same argument as Lemma 2.15 of [18], we can get: For $w \in \mathcal{S}$, the limit $\lim_{t\to T^+(w)} \eta(t,w)$ exists and is a critical point of $\mathcal{I}$.*

Let $A \subset \mathcal{S}$, $\delta > 0$ and define $U_\delta(A) = \{w \in \mathcal{S} : \mathrm{dist}(w, A) < \delta\}$. Then we have

Lemma 3.3. *Let $K_d = \{\psi \in K : \mathcal{I}(\psi) = d$ and $\mathcal{I}'(\psi) = 0\}$ and $d \geq c^*$. Then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that there hold*

(i) $\mathcal{I}_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$.

(ii) $\lim\limits_{t\to T^+(w)} \mathcal{I}(\eta(t,w)) < d - \varepsilon$ for all $w \in \mathcal{I}^{d+\varepsilon} \setminus U_\delta(K_d)$.

Proof. The proof is virtually identical to Lemma 2.16 in [18], and the details are omitted. □

Now, we will prove the Theorem 1.2. For this purpose, we should first introduce the definition of genus.

Definition 3.1. *For a closed symmetric set A that does not contain the origin, we define the Krasnoselskii genus of A, denoted $\gamma(A)$, as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \backslash \{0\}$. If there is no such mapping for any k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$.*

Proof of Theorem 1.2. Define

$$d_k = \inf\{d \in \mathbb{R} : \gamma(\mathcal{I}^d) \geq k\}, \ (k \in \mathbb{N}).$$

Then the d_k are the numbers at which the sets $\mathcal{I}^d$ change genus. It is noticeable that $d_k \leq d_{k+1}$. Let $k \geq 1$ and set $d = d_k$. By Lemma 3.1, K_d is either empty set or discrete set, hence $\gamma(K_d) = 0$ or 1. By the continuity property of the genus, there exists $\delta > 0$ such that $\gamma(\overline{U_\delta(K_d)}) = \gamma(K_d)$, where $\delta < \frac{\alpha}{2}$. For the δ, we can choose $\varepsilon > 0$ such that the conclusions of Lemma 3.3 hold. Then for each $w \in \mathcal{I}^{d+\varepsilon} \setminus U_\delta(K_d)$ there exists $t \in [0, T^+(w))$ such that $\mathcal{I}(\eta(t,w)) < d - \varepsilon$. Let $\varrho = \varrho(w)$ be the infimum of the time for which $\mathcal{I}(\eta(t,w)) \leq d - \varepsilon$, that is

$$\varrho(w) = \inf\{t \in [0, T^+(w)) : \mathcal{I}(\eta(t,w)) \leq d - \varepsilon\}.$$

Since $d - \varepsilon$ is not a critical value of $\mathcal{I}$ by Lemma 3.3, it is apparent that by the implicit function theorem, $\varrho(w)$ is a continuous mapping and since $\mathcal{I}$ is even, $\varrho(-w) = \varrho(w)$. Define a mapping $h : \mathcal{I}^{d+\varepsilon} \setminus U_\delta(K_d) \to \mathcal{I}^{d-\varepsilon}$ by setting $h(w) := \eta(\varrho(w), w)$. Then h is odd and continuous, so it can be derived from the properties of the genus and the definition of d_k that

$$\gamma(\mathcal{I}^{d+\varepsilon}) \leq \gamma(\overline{U_\delta(K_d)}) + \gamma(\mathcal{I}^{d-\varepsilon}) \leq \gamma(\overline{U_\delta(K_d)}) + k - 1 = \gamma(K_d) + k - 1.$$

If $\gamma(K_d) = 0$, then $\gamma(\mathcal{I}^{d+\varepsilon}) \leq k - 1$, contrary to the definition of d_k, so $\gamma(K_d) = 1$ and $K_d \neq \emptyset$. If $d_{k+1} = d_k = d$, then $\gamma(K_d) > 1$ (see [16], Proposition 8.5). However, this is impossible, so we have $d_{k+1} > d_k$ and $K_{d_k} \neq \emptyset$ for all $k \geq 1$, hence there is an infinite sequence $\{\pm w_k\}$ of pairs of geometrically distinct critical points of $\mathcal{I}$ with $\mathcal{I}(w_k) = d_k$, which is a contradiction to (3.1), and Theorem 1.2 is proved. □

4. Conclusions

In this paper, we are interested in studying a class of biharmonic elliptic equations with potential functions. Our problem is more complicated by the fact that the classical (AR)-type condition is not assumed. We establish the existence results of ground state solutions for the biharmonic elliptic equation (1.1) by using the Nehari manifold method and critical point theories. Moreover, the existence of infinitely many geometrically distinct solutions for this equation is also investigated. We believe that the proposed approach in the present paper can also be applied to study other related equations and systems. An interesting question is whether similar results still hold for a class of biharmonic elliptic systems under the same conditions.

Author contributions

Dengfeng Lu: Investigation, Writing-original draft; Shuwei Dai: Writing-review & editing. The authors contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare that they have no competing interests.

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