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Research article

Invariant measures for stochastic FitzHugh-Nagumo delay lattice systems with long-range interactions in weighted space

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Abstract: The focus of this paper lies in exploring the limiting dynamics of stochastic FitzHugh-Nagumo delay lattice systems with long-range interactions and nonlinear noise in weighted space. To begin, we established the well-posedness of solutions to these stochastic delay lattice systems and subsequently proved the existence and uniqueness of invariant measures.

Keywords: FitzHugh-Nagumo lattice systems; nonlinear noise; weighted space; invariant measures; delay
Mathematics Subject Classification: 35B40, 35B41, 37L30

1. Introduction

The objective of this paper is to investigate the existence and uniqueness of invariant measures for the stochastic FitzHugh-Nagumo delay lattice system with long-range interactions on the integer set $\mathbb{Z}$:

$$\begin{cases} du_n(t) = (\sum\limits_{m\in\mathbb{Z}} J(n-m)u_m(t) - \alpha v_n(t) + f_n(u_n(t)) + a_n)dt \\ \qquad\qquad + \sum\limits_{j=1}^{\infty} (g_{j,n}(u_n(t), u_n(t-\rho)) + b_{j,n})dW_j(t), \\ dv_n(t) = (\beta u_n(t) - \lambda v_n(t) + c_n)dt + \sum\limits_{j=1}^{\infty} (h_{j,n}(v_n(t), v_n(t-\rho)) + l_{j,n})dW_j(t), \\ u_n(s) = \phi_n(s), v_n(s) = \varphi_n(s), s \in [-\rho, 0], \end{cases} \tag{1.1}$$

where $u_n, v_n \in \mathbb{R}$, $t > 0$, $\alpha, \rho, \beta, \lambda > 0$, the coupling parameters $J(m)$ are real numbers satisfying $J(m) = J(-m)$ for all positive integer m, $a = (a_n)_{n\in\mathbb{Z}}$, $c = (c_n)_{n\in\mathbb{Z}}$, $b = (b_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$, and $l = (l_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$ are given deterministic sequences in ℓ^2_η, $f_n, g_{j,n}, h_{j,n}$ are Lipschitz continuous functions for all $j \in \mathbb{N}, n \in \mathbb{Z}$, and $(W_j(t))_{j\in\mathbb{N}}$ is a sequence of independent two-sided real-valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in\mathbb{R}}, P)$.

The subsequent changes should be observed while considering the transformation of $J(m)$,

$$J(m) = \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j \delta_{m,j-k},$$

where k is any positive integer and $\delta_{m,n}$ is the Kronecker's delta. Then, lattice system (1.1) can be changed into

$$\begin{cases} du_n(t) = (\triangle^k u_n(t) - \alpha v_n(t) + f_n(u_n(t)) + a_n)dt + \sum\limits_{j=1}^{\infty} (g_{j,n}(u_n(t), u_n(t-\rho)) + b_{j,n})dW_j(t), \\ dv_n(t) = (\beta u_n(t) - \lambda v_n(t) + c_n)dt + \sum\limits_{j=1}^{\infty} (h_{j,n}(v_n(t), v_n(t-\rho)) + l_{j,n})dW_j(t), \\ u_n(s) = \phi_n(s), v_n(s) = \varphi_n(s), s \in [-\rho, 0], \end{cases}$$

where $t > 0, n \in \mathbb{Z}$, $\triangle^k = \triangle \circ \cdots \circ \triangle, k$ times, and $\triangle$ is defined by $\triangle u_n = u_{n+1} + u_{n-1} - 2u_n$.

The emergence of lattice equations from spatial discretization of partial differential equations is widely acknowledged. Lattice systems exhibiting long-range interactions have garnered significant attention in the literature. Of those, the dynamics of the DNA molecule were described by Schrödinger lattice systems in [1]. Subsequently, Pereira investigated the asymptotic behavior of Schrödinger lattice systems in [2] and delay lattice systems in [3], respectively. Recently, Chen et al. considered the long-term dynamics of stochastic complex Ginzburg-Landau systems in their study [4], and Wong-Zakai approximations of stochastic lattice systems in another study [5].

The FitzHugh-Nagumo systems were used to describe the transmission of signals across axons in neurobiology in [6]. The asymptotic behavior of FitzHugh-Nagumo systems were studied in both deterministic [7] and stochastic scenarios [8–12]. The FitzHugh-Nagumo lattice systems were employed to stimulate the propagation of action potentials in myelinated nerve axons in [13]. The attractors of FitzHugh-Nagumo lattice systems were investigated in the deterministic case by [7, 14], and in the stochastic case by [11, 12, 15–18]. Among these studies, Wang et al. [11] derived the existence and upper semi-continuity of random attractors for FitzHugh-Nagumo lattice systems in $\ell^2 \times \ell^2$, while Chen et al. [15] obtained the existence and uniqueness of weak pullback mean random attractors for FitzHugh-Nagumo lattice systems with nonlinear noises in weighted spaces $\ell^2_\sigma \times \ell^2_\sigma$.

Furthermore, time delays are a common occurrence in various systems, and can lead to instability, oscillation, and other changes in dynamical systems. Due to their practical and theoretical significance, there has been an increasing emphasis on the study of time-delay systems. Recent studies have delved into the exploration of random attractors for stochastic lattice systems featuring fixed delays in [18–22]. Additionally, investigations have also been carried out concerning systems with varying delays over time as documented in [3, 12, 23–25].

Currently, there has been a significant amount of research conducted on the dynamical behavior of differential equations driven by linear noise. In order to effectively handle stochastic systems with nonlinear noise, Kloeden [26] and Wang [27, 28] introduced the concept of weak pullback mean random attractors. The work described above has subsequently been widely applied in numerous studies on stochastic systems by a multitude of scholars in [15–17, 19–21, 25, 27–38]. Among them, Wang et al [25] studied the stochastic delay modified Swift-Hohenberg lattice systems, as well as Chen et al. [19] and Li et al. [20] considered the stochastic delay lattice systems. However, to the best of our knowledge, the current state of literature on the invariant measures for stochastic FitzHugh-Nagumo delay lattice systems with long-range interactions driven by nonlinear noise in weighted space is regrettably scarce.

The lattice system (1.1) is defined on $\mathbb{Z}$, which represents a spatially discrete analogue to stochastic partial differential equations (PDEs) defined on $\mathbb{R}$. Proving the existence of invariant measures for PDEs on unbounded domains poses a major challenge, primarily due to establishing the tightness of distribution laws of solutions caused by non-compactness in usual Sobolev embeddings on unbounded domains. Various approaches have been developed in literature to address the tightness of solution distributions for PDEs on unbounded domains, such as using weighted spaces in [39, 40], weak Feller property of solutions in [41, 42], and cut-off techniques in [43, 44]. In this paper, the cut-off method will be employed to establish the existence of invariant measures for the stochastic lattice system (1.1) in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. Specifically, we will demonstrate that when time is sufficiently large, the mean square of solution tails in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$ becomes uniformly small; based on this result, we can establish tightness in distribution laws for solutions in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. The tail-estimates method has previously been used to prove existence of global attractors for deterministic PDEs [45, 46] and stochastic PDEs with additive or linear multiplicative noise in [47, 48]. In this paper, we will apply the tail-estimates approach to handle nonlinear noise involved in (1.1) in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. For further information regarding existence of invariant measures for stochastic PDEs defined within bounded domains, please refer to [49] and its references.

The structure of this paper is organized as follows: Section 2 introduces the notations and discusses the well-posedness of lattice system (1.1). The subsequent section establishes necessary uniform estimates of solutions, which play a crucial role in demonstrating the main results in the following section. Sections 4 and 5 focus on establishing the existence and uniqueness of invariant measures for lattice system (1.1). Finally, we provide a summary and closing remarks in the last section.

2. Preliminaries

In this section, we will investigate the well-posedness of the stochastic Fitzhugh-Nagumo delay lattice system (1.1) in weighted space $\ell^2_\eta \times \ell^2_\eta$, where ℓ^2_η is defined by

$$\ell^2_\eta = \Big\{u = (u_n)_{n\in\mathbb{Z}} | u_n \in \mathbb{R}, \sum_{n\in\mathbb{Z}} \eta_n |u_n|^2 < \infty\Big\}.$$

ℓ^2_η is a Hilbert space with the inner product and norm given by

$$(u, v)_\eta = \sum_{n\in\mathbb{Z}} \eta_n u_n v_n, \ \|u\|^2_\eta = (u, u)_\eta, \ u, v \in \ell^2_\eta.$$

We further assume that weights $\eta = (\eta_n)_{n\in\mathbb{Z}}$ satisfy the conditions

$$\eta_n > 0, \ \forall n \in \mathbb{Z}, \ \sum_{n\in\mathbb{Z}} \eta_n < \infty, \tag{2.1}$$

and

$$\alpha_m := \sup_{n\in\mathbb{Z}} \frac{\eta_{n+m} + \eta_n}{\eta^{1/2}_{n+m}\eta^{1/2}_n} < \infty, \ \forall m \in \mathbb{N}. \tag{2.2}$$

To get the existence of invariant measures for lattice system (1.1) in ℓ^2_η, the interaction $J(m)$ should decrease at a sufficiently rapid rate such that

$$\tilde{\alpha} := \sum_{m=0}^{\infty} \alpha_m |J(m)| < \infty. \tag{2.3}$$

For sequences $a = (a_n)_{n\in\mathbb{Z}}$, $c = (c_n)_{n\in\mathbb{Z}}$, $b = (b_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$, and $l = (l_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$ in lattice system (1.1), we assume

$$\begin{aligned} \|a\|^2_\eta &= \sum_{n\in\mathbb{Z}} \eta_n |a_n|^2 < \infty, \ \|b\|^2_\eta = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \eta_n |b_{j,n}|^2 < \infty, \\ \|c\|^2_\eta &= \sum_{n\in\mathbb{Z}} \eta_n |c_n|^2 < \infty, \ \|l\|^2_\eta = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \eta_n |l_{j,n}|^2 < \infty. \end{aligned} \tag{2.4}$$

For the nonlinear term f_n in lattice system (1.1), we assume that f_n is a smooth function satisfying that there exists $\kappa \in \mathbb{R}$ such that for all $z \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$f_n(0) = 0, \ f'_n(z) \le \kappa. \tag{2.5}$$

Moreover, for each $n \in \mathbb{Z}$ and $z \in \mathbb{R}$, we assume that there are positive constants ι_n and δ such that

$$f_n(z)z \le -\delta|z|^2 + \iota_n, \tag{2.6}$$

where $\iota = (\iota_n)_{n\in\mathbb{Z}}$ belongs to ℓ^1_η and its norm is denoted by $\|\iota\|_{1,\eta}$.

For every $j \in \mathbb{N}$ and $n \in \mathbb{Z}$, we assume that $g_{j,n}, h_{j,n} : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous; that is, there is a constant $L > 0$ such that for all $z_1, z_2, z_1^*, z_2^* \in \mathbb{R}$,

$$|g_{j,n}(z_1, z_2) - g_{j,n}(z_1^*, z_2^*)| \bigvee |h_{j,n}(z_1, z_2) - h_{j,n}(z_1^*, z_2^*)| \le L(|z_1 - z_1^*| + |z_2 - z_2^*|). \tag{2.7}$$

We further assume that for each $z, z^* \in \mathbb{R}$, $j \in \mathbb{N}$, and $n \in \mathbb{Z}$,

$$|g_{j,n}(z, z^*)| \bigvee |h_{j,n}(z, z^*)| \le \gamma_{j,n}(1 + |z| + |z^*|), \tag{2.8}$$

where $\gamma_{j,n} > 0$, $\|\gamma\|^2 = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} |\gamma_{j,n}|^2 < \infty$, and $\|\gamma\|^2_\eta = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \eta_n |\gamma_{j,n}|^2 < \infty$.

For any $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2_\eta$ and $v = (v_n)_{n\in\mathbb{Z}} \in \ell^2_\eta$, denote by $f(u) = (f_n(u_n))_{n\in\mathbb{Z}}$ and $f(v) = (f_n(v_n))_{n\in\mathbb{Z}}$. By (2.5), we get

$$(f(u) - f(v), u - v)_\eta = \sum_{n\in\mathbb{Z}} \eta_n (f_n(u_n) - f_n(v_n))(u_n - v_n) = \sum_{n\in\mathbb{Z}} \eta_n f'_n(\xi_n)|u_n - v_n|^2 \le \kappa \|u - v\|^2_\eta, \tag{2.9}$$

where $\xi_n = \theta_n u_n + (1 - \theta_n)v_n$ for some $\theta_n \in (0, 1)$. Moreover, we can obtain that f is locally Lipschitz continuous from ℓ^2_η to ℓ^2_η; that is, there exists $L_C > 0$ such that for any $u, v \in \ell^2_\eta$ with $\|u\|^2_\eta \le C$ and $\|v\|^2_\eta \le C$,

$$\|f(u) - f(v)\|^2_\eta \le L^2_C \|u - v\|^2_\eta. \tag{2.10}$$

For each $u^1 = (u_n^1)_{n\in\mathbb{Z}}, u^2 = (u_n^2)_{n\in\mathbb{Z}}, v^1 = (v_n^1)_{n\in\mathbb{Z}}, v^2 = (v_n^2)_{n\in\mathbb{Z}} \in \ell_\eta^2$, and $j \in \mathbb{N}$, denote by $g_j(u^1, v^1) = (g_{j,n}(u_n^1, v_n^1))_{n\in\mathbb{Z}}$ and $h_j(u^1, v^1) = (h_{j,n}(u_n^1, v_n^1))_{n\in\mathbb{Z}}$. It follows from (2.7) and (2.8) that

$$\sum_{j\in\mathbb{N}} \|g_j(u^1, v^1)\|_\eta^2 \bigvee \sum_{j\in\mathbb{N}} \|h_j(u^1, v^1)\|_\eta^2 \le 2\|\gamma\|_\eta^2 + 4\|\gamma\|^2(\|u^1\|_\eta^2 + \|v^1\|_\eta^2) \tag{2.11}$$

and

$$\begin{aligned}&\sum_{j\in\mathbb{N}} \|g_j(u^1, v^1) - g_j(u^2, v^2)\|_\eta^2 \bigvee \sum_{j\in\mathbb{N}} \|h_j(u^1, v^1) - g_j(u^2, v^2)\|_\eta^2 \\ &\le 2L^2(\|u^1 - u^2\|_\eta^2 + \|v^1 - v^2\|_\eta^2).\end{aligned} \tag{2.12}$$

The system (1.1) can be reformulated as an abstract system in ℓ^2, for $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2$, and we set

$$(Au)_n = \sum_{m\in\mathbb{Z}} J(n-m)u_m. \tag{2.13}$$

By Lemma 3.1 of [4], we have

$$\|Au\|^2 \le 2|J(0)|^2\|u\|^2 + 8\Big(\sum_{m=1}^{\infty} |J(m)|\Big)^2\|u\|^2. \tag{2.14}$$

By the above notation, system (1.1) can be rewritten as follows: For all $t > 0$,

$$\begin{cases} du(t) = (Au(t) - \alpha v(t) + f(u(t)) + a)dt + \sum\limits_{j=1}^{\infty} (g_j(u(t), u(t-\rho)) + b_j)dW_j(t), \\ dv(t) = (\beta u(t) - \lambda v(t) + c)dt + \sum\limits_{j=1}^{\infty} (h_j(v(t), v(t-\rho)) + l_j)dW_j(t), \\ u(s) = \phi(s), v(s) = \varphi(s), s \in [-\rho, 0]. \end{cases} \tag{2.15}$$

Let $(\phi, \varphi) \in L^2(\Omega, C([-\rho, 0], \ell_\eta^2 \times \ell_\eta^2))$ be $\mathcal{F}_0$-measurable. Then, a continuous $\ell_\eta^2 \times \ell_\eta^2$-valued $\mathcal{F}_t$-adapted stochastic process $(u(t), v(t))$ is called a solution of stochastic lattice system (2.15) if $(u_0, v_0) = (\phi, \varphi)$,$(u(t), v(t)) \in L^2(\Omega, C([-\rho, T], \ell_\eta^2 \times \ell_\eta^2))$ for all $T > -\rho$, $t \ge 0$ and for almost all $\omega \in \Omega$,

$$\begin{cases} u(t) = \phi(0) + \int_0^t (Au(r) - \alpha v(r) + f(u(r)) + a)dr + \sum\limits_{j=1}^{\infty} \int_0^t (g_j(u(r), u(r-\rho)) + b_j)dW_j(r), \\ v(t) = \varphi(0) + \int_0^t (\beta u(r) - \lambda v(r) + c)dr + \sum\limits_{j=1}^{\infty} \int_0^t (h_j(v(r), v(r-\rho)) + l_j)dW_j(r). \end{cases}$$

By (2.1)–(2.8) and the theory of the functional differential equation, we can get that for any $(\phi, \varphi) \in L^2(\Omega, C([-\rho, 0], \ell_\eta^2 \times \ell_\eta^2))$, stochastic lattice system (2.15) has a solution $(u(t), v(t)) \in L^2(\Omega, C([-\rho, T], \ell_\eta^2 \times \ell_\eta^2))$ for every $T \ge -\rho$. Moreover, this solution is unique if $(u^*(t), v^*(t))$ is any other solution of system (2.15), then

$$P(\{(u(t), v(t)) = (u^*(t), v^*(t)) \text{ for all } t \ge -\rho\}) = 1.$$

Actually, the stochastic lattice system (2.15) has a unique solution defined for $t \in [t_0 - \rho, \infty)$, regardless of any initial time $t_0 \ge 0$ and any $\mathcal{F}_{t_0}$-measurable $(\phi, \varphi) \in L^2(\Omega, C([-\rho, 0], \ell_\eta^2 \times \ell_\eta^2))$.

Hereafter, for $t \in \mathbb{R}$, (u_t, v_t) is defined by

$$(u_t, v_t)(s) = (u_{n,t}(s), v_{n,t}(s))_{n\in\mathbb{Z}} = (u_n(t+s), v_n(t+s))_{n\in\mathbb{Z}} = (u(t+s), v(t+s)), \quad s \in [-\rho, 0],$$

and let $C_{\rho,\eta} = C([-\rho, 0], \ell^2_\eta)$ with the norm $\|\chi\|_{\rho,\eta} = \sup_{-\rho\le s\le 0} \|\chi(s)\|_\eta, \chi \in C_{\rho,\eta}$.

The establishment of Lipschitz continuity for solutions to stochastic lattice system (2.15) in relation to initial data will now be undertaken, which shall subsequently be employed.

Lemma 2.1. *Suppose (2.1)–(2.8) hold and* $(\phi_1, \varphi_1), (\phi_2, \varphi_2) \in L^2(\Omega, C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta))$. *If* $(u(t, \phi_1), v(t, \varphi_1))$ *and* $(u(t, \phi_2), v(t, \varphi_2))$ *are the solutions of stochastic lattice system (2.15) with initial data* (ϕ_1, φ_1) *and* (ϕ_2, φ_2)*, respectively, then for any* $t \ge 0$*,*

$$\mathbb{E}\Big[\sup_{-\rho\le r\le t} \|u(r, \phi_1) - u(r, \phi_2)\|^2_\eta + \sup_{-\rho\le r\le t} \|v(r, \varphi_1) - v(r, \varphi_2)\|^2_\eta\Big]$$
$$\le M_1\Big(1 + e^{M_1 t}\Big)\mathbb{E}\Big[\|\phi_1 - \phi_2\|^2_{C_{\rho,\eta}} + \|\varphi_1 - \varphi_2\|^2_{C_{\rho,\eta}}\Big],$$

where M_1 *is a positive constant independent of* (ϕ_1, φ_1)*,* (ϕ_2, φ_2)*, and* t*.*

Proof. By (2.15), we get that for all $t \ge 0$,

$$\begin{aligned} d(u(t,\phi_1) - u(t,\phi_2)) =& A(u(t,\phi_1) - u(t,\phi_2))dt - \alpha(v(t,\varphi_1) - v(t,\varphi_2))dt \\ &+ (f(u(t,\phi_1)) - f(u(t,\phi_2)))dt \\ &+ \sum_{j=1}^{\infty}(g_j(u(t,\phi_1), u(t-\rho,\phi_1)) - g_j(u(t,\phi_2), u(t-\rho,\phi_2)))dW_j(t), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} d(v(t,\varphi_1) - v(t,\varphi_2)) =& \beta(u(t,\phi_1) - u(t,\phi_2))dt - \lambda(v(t,\varphi_1) - v(t,\varphi_2))dt \\ &+ \sum_{j=1}^{\infty}(h_j(v(t,\varphi_1), v(t-\rho,\varphi_1)) - h_j(v(t,\varphi_2), v(t-\rho,\varphi_2)))dW_j(t), \end{aligned}$$

which along with (2.16) and Itô's formula shows that for all $t \ge 0$,

$$\begin{aligned} &\frac{1}{2}(\beta\|u(t,\phi_1) - u(t,\phi_2)\|^2_\eta + \alpha\|v(t,\varphi_1) - v(t,\varphi_2)\|^2_\eta) \\ &= \frac{1}{2}(\beta\|\phi_1(0) - \phi_2(0)\|^2_\eta + \alpha\|\varphi_1(0) - \varphi_2(0)\|^2_\eta) - \lambda\alpha\int_0^t \|v(s,\varphi_1) - v(s,\varphi_2)\|^2_\eta ds \\ &\quad + \beta\int_0^t \Big(A(u(s,\phi_1) - u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta ds \\ &\quad + \beta\int_0^t \Big(f(u(s,\phi_1)) - f(u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta ds \\ &\quad + \frac{\beta}{2}\sum_{j=1}^{\infty}\int_0^t \|g_j(u(s,\phi_1), u(s-\rho,\phi_1)) - g_j(u(s,\phi_2), u(s-\rho,\phi_2))\|^2_\eta ds \\ &\quad + \frac{\alpha}{2}\sum_{j=1}^{\infty}\int_0^t \|h_j(v(s,\varphi_1), v(s-\rho,\varphi_1)) - h_j(v(s,\varphi_2), v(s-\rho,\varphi_2))\|^2_\eta ds \\ &\quad + \beta\sum_{j=1}^{\infty}\int_0^t \Big(\mathbf{g}_j, u(s,\phi_1) - u(s,\phi_2)\Big)_\eta dW_j(s) + \alpha\sum_{j=1}^{\infty}\int_0^t \Big(\mathbf{h}_j, v(s,\varphi_1) - v(s,\varphi_2)\Big)_\eta dW_j(s), \end{aligned} \tag{2.17}$$

where

$$\mathbf{g}_j = g_j(u(s,\phi_1), u(s-\rho,\phi_1)) - g_j(u(s,\phi_2), u(s-\rho,\phi_2))$$

and

$$\mathbf{h}_j = h_j(v(s,\varphi_1), v(s-\rho,\varphi_1)) - h_j(v(s,\varphi_2), v(s-\rho,\varphi_2)).$$

By (2.13) and the fact of $J(m) = J(-m)$, we have

$$\begin{aligned}
&\Big(A(u(s,\phi_1) - u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta \\
&= J(0)\|u(s,\phi_1) - u(s,\phi_2)\|_\eta^2 + \sum_{n\in\mathbb{Z}} \eta_n \sum_{m=1}^{\infty} J(m)(u_n(s,\phi_1) - u_n(s,\phi_2)) \\
&\quad \times (u_{n-m}(s,\phi_1) - u_{n-m}(s,\phi_2) + u_{n+m}(s,\phi_1) - u_{n+m}(s,\phi_2)) \\
&= J(0)\|u(s,\phi_1) - u(s,\phi_2)\|_\eta^2 \\
&\quad + \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} J(m)\eta_{n+m}\big(u_{n+m}(s,\phi_1) - u_{n+m}(s,\phi_2)\big)\big(u_n(s,\phi_1) - u_n(s,\phi_2)\big) \\
&\quad + \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} J(m)\eta_n\big(u_n(s,\phi_1) - u_n(s,\phi_2)\big)\big(u_{n+m}(s,\phi_1) - u_{n+m}(s,\phi_2)\big) \\
&= J(0)\|u(s,\phi_1) - u(s,\phi_2)\|_\eta^2 \\
&\quad + \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} J(m)(\eta_n + \eta_{n+m})\big(u_n(s,\phi_1) - u_n(s,\phi_2)\big)\big(u_{n+m}(s,\phi_1) - u_{n+m}(s,\phi_2)\big),
\end{aligned} \tag{2.18}$$

which along with (2.2) and (2.3) implies that

$$\begin{aligned}
&\beta \int_0^t \Big(A(u(s,\phi_1) - u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta ds \\
&\le \beta J(0) \int_0^t \|u(s,\phi_1) - u(s,\phi_2)\|_\eta^2 ds \\
&\quad + \beta \int_0^t \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} |J(m)|\alpha_m \eta_n^{\frac{1}{2}} \eta_{n+m}^{\frac{1}{2}} |u_n(s,\phi_1) - u_n(s,\phi_2)||u_{n+m}(s,\phi_1) - u_{n+m}(s,\phi_2)| ds \\
&\le \beta\tilde{\alpha} \int_0^t \|u(s,\phi_1) - u(s,\phi_2)\|_\eta^2 ds.
\end{aligned} \tag{2.19}$$

By (2.9), we obtain

$$\beta \int_0^t \Big(f(u(s,\phi_1)) - f(u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta ds \le \beta\kappa \int_0^t \|u(s,\phi_1) - u(s,\phi_2)\|_\eta^2 ds. \tag{2.20}$$

By (2.12), we get

$$
\begin{aligned}
&\frac{\beta}{2}\sum_{j=1}^{\infty}\int_0^t \|g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2))\|_\eta^2 ds\\
&\quad+\frac{\alpha}{2}\sum_{j=1}^{\infty}\int_0^t \|h_j(v(s,\varphi_1),v(s-\rho,\varphi_1))-h_j(v(s,\varphi_2),v(s-\rho,\varphi_2))\|_\eta^2 ds\\
&\le 2\beta L^2\int_0^t \|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2 ds+\beta L^2\int_{-\rho}^0 \|\phi_1(s)-\phi_2(s)\|_\eta^2 ds\\
&\quad+2\alpha L^2\int_0^t \|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2 ds+\alpha L^2\int_{-\rho}^0 \|\varphi_1(s)-\varphi_2(s)\|_\eta^2 ds.
\end{aligned}
\tag{2.21}
$$

It follows from (2.17)–(2.21) that for all $t \ge 0$,

$$
\begin{aligned}
&\beta\|u(t,\phi_1)-u(t,\phi_2)\|_\eta^2+\alpha\|v(t,\varphi_1)-v(t,\varphi_2)\|_\eta^2\\
&\le \beta\|\phi_1(0)-\phi_2(0)\|_\eta^2+\alpha\|\varphi_1(0)-\varphi_2(0)\|_\eta^2+2\beta L^2\int_{-\rho}^0 \|\phi_1(s)-\phi_2(s)\|_\eta^2 ds\\
&\quad+2\alpha L^2\int_{-\rho}^0 \|\varphi_1(s)-\varphi_2(s)\|_\eta^2 ds+4\alpha L^2\int_0^t \|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2 ds\\
&\quad+2\beta(\tilde{\alpha}+|\kappa|+2L^2)\int_0^t \|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2 ds\\
&\quad+2\beta\Big|\sum_{j=1}^{\infty}\int_0^t \Big(\mathbf{g}_j,u(s,\phi_1)-u(s,\phi_2)\Big)_\eta dW_j(s)\Big|\\
&\quad+2\alpha\Big|\sum_{j=1}^{\infty}\int_0^t \Big(\mathbf{h}_j,v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta dW_j(s)\Big|,
\end{aligned}
$$

which implies that for all $t \ge 0$,

$$
\begin{aligned}
&\mathbb{E}\Big[\beta\sup_{0\le r\le t}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\sup_{0\le r\le t}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]\\
&\le (1+2\rho L^2)\Big(\mathbb{E}\Big[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2\Big]\Big)\\
&\quad+2\beta(\tilde{\alpha}+|\kappa|+2L^2)\int_0^t \mathbb{E}\Big[\sup_{0\le r\le s}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2\Big]ds\\
&\quad+4\alpha L^2\int_0^t \mathbb{E}\Big[\sup_{0\le r\le s}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]ds\\
&\quad+2\beta\mathbb{E}\Big[\sup_{0\le r\le t}\Big|\sum_{j=1}^{\infty}\int_0^r \Big(\mathbf{g}_j,u(s,\phi_1)-u(s,\phi_2)\Big)_\eta dW_j(s)\Big|\Big]\\
&\quad+2\alpha\mathbb{E}\Big[\sup_{0\le r\le t}\Big|\sum_{j=1}^{\infty}\int_0^r \Big(\mathbf{h}_j,v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta dW_j(s)\Big|\Big].
\end{aligned}
\tag{2.22}
$$

For the last two terms of (2.22), by (2.12), the Burkholder-Davis-Gundy (BDG) inequality, and the Minkowski inequality, we have

$$\begin{aligned}
&2\beta\mathbb{E}\Big[\sup_{0\le r\le t}\Big|\sum_{j=1}^{\infty}\int_0^r\Big(\mathbf{g}_j,u(s,\phi_1)-u(s,\phi_2)\Big)_\eta dW_j(s)\Big|\Big]\\
&\le\frac{\beta C_1}{\sqrt{2}}\mathbb{E}\Big[\Big(\int_0^t\sum_{j=1}^{\infty}\|\mathbf{g}_j\|_\eta^2\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2ds\Big)^{\frac{1}{2}}\Big]\\
&\le\frac{\beta C_1}{\sqrt{2}}\mathbb{E}\Big[\sup_{0\le s\le t}\|u(s,\phi_1)-u(s,\phi_2)\|_\eta\\
&\quad\times\Big(\int_0^t\sum_{j=1}^{\infty}\|g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2))\|_\eta^2ds\Big)^{\frac{1}{2}}\Big]\\
&\le\beta C_1L\mathbb{E}\Big[\sup_{0\le s\le t}\|u(s,\phi_1)-u(s,\phi_2)\|_\eta\Big(\int_0^t\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2ds\Big)^{\frac{1}{2}}\Big]\\
&\quad+\beta C_1L\mathbb{E}\Big[\sup_{0\le s\le t}\|u(s,\phi_1)-u(s,\phi_2)\|_\eta\Big(\int_0^t\|u(s-\rho,\phi_1)-u(s-\rho,\phi_2)\|_\eta^2ds\Big)^{\frac{1}{2}}\Big]\\
&\le\frac{\beta}{2}\mathbb{E}\Big[\sup_{0\le r\le t}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2\Big]+2\beta C_1^2L^2\int_0^t\mathbb{E}\Big[\sup_{0\le r\le s}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2\Big]ds\\
&\quad+\rho\beta C_1^2L^2\mathbb{E}\Big[\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2\Big],
\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}
&2\alpha\mathbb{E}\Big[\sup_{0\le r\le t}\Big|\sum_{j=1}^{\infty}\int_0^r\Big(\mathbf{h}_j,v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta dW_j(s)\Big|\Big]\\
&\le\frac{\alpha}{2}\mathbb{E}\Big[\sup_{0\le r\le t}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]+2\alpha C_1^2L^2\int_0^t\mathbb{E}\Big[\sup_{0\le r\le s}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]ds\\
&\quad+\rho\alpha C_1^2L^2\mathbb{E}\Big[\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2\Big],
\end{aligned}$$

which along with (2.22) and (2.23) shows that

$$\begin{aligned}
&\mathbb{E}\Big[\beta\sup_{0\le r\le t}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\sup_{0\le r\le t}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]\\
&\le C_2\mathbb{E}\Big[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2\Big]\\
&\quad+C_3\int_0^t\mathbb{E}\Big[\beta\sup_{0\le r\le s}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\sup_{0\le r\le s}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]ds,
\end{aligned}\tag{2.24}$$

where $C_2=2(1+2\rho L^2+\rho C_1^2L^2)$, $C_3=4(\tilde{\alpha}+2L^2+|\kappa|+C_1^2L^2)$. It follows from (2.24) and the Gronwall inequality that for all $t\ge0$,

$$\begin{aligned}
&\mathbb{E}\Big[\beta\sup_{0\le r\le t}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\sup_{0\le r\le t}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2\Big]\\
&\le C_2e^{C_3t}\mathbb{E}\Big[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2\Big].
\end{aligned}$$

This completes the proof. □

The existence of invariant measures of the stochastic lattice system (2.15) in the subsequent analysis necessitates the fulfillment of the following inequality.

$$\|\gamma\|^2 < \frac{1}{96e^{\rho\nu}} \max\Big\{2\delta - 4\tilde{\alpha} - \frac{27\beta^4}{\lambda^3} - 8, 2\lambda - \frac{27\alpha^4}{\delta^3} - 6\Big\}, \tag{2.25}$$

where $\nu > 0, 2\delta - 4\tilde{\alpha} - \frac{27\beta^4}{\lambda^3} - 8 > 0, 2\lambda - \frac{27\alpha^4}{\delta^3} - 6 > 0$.

3. Uniform estimates

In this section, we obtain uniform estimates of the solutions to stochastic lattice system (2.15), which play a pivotal role in proving the existence of invariant measures. More specifically, we will showcase the compactness of a family of probability distributions pertaining to (u_t, v_t) in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. Initially, our focus lies on discussing uniform estimates of solutions to stochastic lattice system (2.15) in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$ for all $t \geq 0$.

Lemma 3.1. *Suppose (2.1)–(2.8) and (2.25) hold. Let $(\phi, \varphi) \in L^2(\Omega, C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta))$ be the initial data of stochastic lattice system (2.15), then the solution (u, v) of the system (2.15) satisfies*

$$\sup_{t\geq-\rho} \mathbb{E}\Big[\|u(t)\|^2_\eta + \|v(t)\|^2_\eta\Big] \leq M_2\Big(1 + \mathbb{E}\Big[\|\phi\|^2_{C_{\rho,\eta}} + \|\varphi\|^2_{C_{\rho,\eta}}\Big]\Big),$$

where M_2 is a positive constant independent of (ϕ, φ).

Proof. By (2.15) and Itô's formula, we get that for all $t \geq 0$,

$$\begin{cases} d\|u(t)\|^2_\eta = 2\Big(\Big(Au(t), u(t)\Big)_\eta - \alpha\Big(v(t), u(t)\Big)_\eta + \Big(f(u(t)), u(t)\Big)_\eta + \Big(a, u(t)\Big)_\eta\Big)dt \\ \qquad + \sum\limits_{j=1}^{\infty} \|g_j(u(t), u(t-\rho)) + b_j\|^2_\eta dt + 2\sum\limits_{j=1}^{\infty} \Big(g_j(u(t), u(t-\rho)) + b_j, u(t)\Big)_\eta dW_j(t), \\ d\|v(t)\|^2_\eta = 2\Big(\beta\Big(u(t), v(t)\Big)_\eta - \lambda\|v(t)\|^2_\eta + \Big(c, v(t)\Big)_\eta\Big)dt \\ \qquad + \sum\limits_{j=1}^{\infty} \|h_j(v(t), v(t-\rho)) + l_j\|^2_\eta dt + 2\sum\limits_{j=1}^{\infty} \Big(h_j(v(t), v(t-\rho)) + l_j, v(t)\Big)_\eta dW_j(t). \end{cases} \tag{3.1}$$

Let ν be a positive constant which will be specified later. We get from (3.1) that for all $t \geq 0$,

$$\begin{aligned} & e^{\nu t}(\beta\|u(t)\|^2_\eta + \alpha\|v(t)\|^2_\eta) - \nu\beta\int_0^t e^{\nu s}\|u(s)\|^2_\eta ds - (\nu - 2\lambda)\alpha\int_0^t e^{\nu s}\|v(s)\|^2_\eta ds \\ & = \beta\|\phi(0)\|^2_\eta + \alpha\|\varphi(0)\|^2_\eta + 2\beta\int_0^t e^{\nu s}\Big(Au(s), u(s)\Big)_\eta ds + 2\beta\int_0^t e^{\nu s}\Big(a, u(s)\Big)_\eta ds \\ & \quad + 2\beta\int_0^t e^{\nu s}\Big(f(u(s)), u(s)\Big)_\eta ds + \beta\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\|g_j(u(s), u(s-\rho)) + b_j\|^2_\eta ds \\ & \quad + 2\alpha\int_0^t e^{\nu s}\Big(c, v(s)\Big)_\eta ds + \alpha\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\|h_j(v(s), v(s-\rho)) + l_j\|^2_\eta ds \\ & \quad + 2\beta\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\Big(g_j(u(s), u(s-\rho)) + b_j, u(s)\Big)_\eta dW_j(s) \\ & \quad + 2\alpha\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\Big(h_j(v(s), v(s-\rho)) + l_j, v(s)\Big)_\eta dW_j(s). \end{aligned} \tag{3.2}$$

Taking the expectation, we obtain that for $t \geq 0$,

$$
\begin{aligned}
& e^{\nu t}\mathbb{E}\Big[\beta\|u(t)\|_\eta^2 + \alpha\|v(t)\|_\eta^2\Big] - \nu\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds - (\nu - 2\lambda)\alpha\int_0^t e^{\nu s}\mathbb{E}\Big[\|v(s)\|_\eta^2\Big]ds \\
&= \mathbb{E}\Big[\beta\|\phi(0)\|_\eta^2 + \alpha\|\varphi(0)\|_\eta^2\Big] + 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(Au(s), u(s)\Big)_\eta\Big]ds \\
&\quad + 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(a, u(s)\Big)_\eta\Big]ds + 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(f(u(s)), u(s)\Big)_\eta\Big]ds \\
&\quad + 2\alpha\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(c, v(s)\Big)_\eta\Big]ds + \beta\sum_{j=1}^\infty\int_0^t e^{\nu s}\mathbb{E}\Big[\|g_j(u(s), u(s-\rho)) + b_j\|_\eta^2\Big]ds \\
&\quad + \alpha\sum_{j=1}^\infty\int_0^t e^{\nu s}\mathbb{E}\Big[\|h_j(v(s), v(s-\rho)) + l_j\|_\eta^2\Big]ds.
\end{aligned}
\tag{3.3}
$$

Similar to (2.18) and (2.19), we get

$$
2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(Au(s), u(s)\Big)_\eta\Big]ds \leq 2\beta\tilde{\alpha}\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds. \tag{3.4}
$$

By (2.6), we have

$$
2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(f(u(s)), u(s)\Big)_\eta\Big]ds \leq -2\beta\delta\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds + \frac{2\beta\|\iota\|_{1,\eta}}{\nu}e^{\nu t}. \tag{3.5}
$$

Note that

$$
\begin{aligned}
& 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(a, u(s)\Big)_\eta\Big]ds + 2\alpha\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(c, v(s)\Big)_\eta\Big]ds \\
& \leq \beta\delta\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds + \lambda\alpha\int_0^t e^{\nu s}\mathbb{E}\Big[\|v(s)\|_\eta^2\Big]ds + \frac{\beta}{\delta\nu}\|a\|_\eta^2 e^{\nu t} + \frac{\alpha}{\lambda\nu}\|c\|_\eta^2 e^{\nu t}.
\end{aligned}
\tag{3.6}
$$

By (2.11), we obtain

$$
\begin{aligned}
& \beta\sum_{j=1}^\infty\int_0^t e^{\nu s}\mathbb{E}\Big[\|g_j(u(s), u(s-\rho)) + b_j\|_\eta^2\Big]ds \\
& \leq 2\beta\sum_{j=1}^\infty\int_0^t e^{\nu s}\mathbb{E}\Big[\|g_j(u(s), u(s-\rho))\|_\eta^2\Big]ds + 2\beta\sum_{j=1}^\infty\int_0^t e^{\nu s}\mathbb{E}\Big[\|b_j\|_\eta^2\Big]ds \\
& \leq 8\beta\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2 + \|u(s-\rho)\|_\eta^2\Big]ds + \frac{4\beta}{\nu}\|\gamma\|_\eta^2 e^{\nu t} + \frac{2\beta}{\nu}\|b\|_\eta^2 e^{\nu t} \\
& \leq 8\beta\rho e^{\rho\nu}\|\gamma\|^2\mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^2\Big] + 16\beta e^{\rho\nu}\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds + \frac{4\beta}{\nu}\|\gamma\|_\eta^2 e^{\nu t} + \frac{2\beta}{\nu}\|b\|_\eta^2 e^{\nu t},
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
& \alpha\sum_{j=1}^\infty\int_0^t e^{\nu s}\mathbb{E}\Big[\|h_j(v(s), v(s-\rho)) + l_j\|_\eta^2\Big]ds \\
& \leq 8\alpha\rho e^{\rho\nu}\|\gamma\|^2\mathbb{E}\Big[\|\varphi\|_{C_{\rho,\eta}}^2\Big] + 16\alpha e^{\rho\nu}\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|v(s)\|_\eta^2\Big]ds + \frac{4\alpha}{\nu}\|\gamma\|_\eta^2 e^{\nu t} + \frac{2\alpha}{\nu}\|l\|_\eta^2 e^{\nu t}.
\end{aligned}
\tag{3.8}
$$

For $t \geq 0$, it follows from (3.3)–(3.8) that

$$\begin{aligned} e^{\nu t}\mathbb{E}\Big[\beta\|u(t)\|_\eta^2 + \alpha\|v(t)\|_\eta^2\Big] \leq{}& (1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\beta\|\phi\|_{C_{\rho,\eta}}^2 + \alpha\|\varphi\|_{C_{\rho,\eta}}^2\Big] \\ &+ \frac{e^{\nu t}}{\nu}\Big(4(\beta+\alpha)\|\gamma\|_\eta^2 + \frac{\beta}{\delta}\|a\|_\eta^2 + \frac{\alpha}{\lambda}\|c\|_\eta^2 + 2\beta\|b\|_\eta^2 + 2\alpha\|l\|_\eta^2 + 2\beta\|\iota\|_{1,\eta}\Big) \\ &+ \beta\Big(\nu - \delta + 2\tilde{\alpha} + 16e^{\rho\nu}\|\gamma\|^2\Big)\int_0^t e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds \\ &+ \alpha\Big(\nu - \lambda + 16e^{\rho\nu}\|\gamma\|^2\Big)\int_0^t e^{\nu s}\mathbb{E}\Big[\|v(s)\|_\eta^2\Big]ds. \end{aligned} \tag{3.9}$$

For $t \geq 0$, by (2.25) and (3.9), we get that there exists $\nu_1 > 0$ such that for all $\nu \in (0, \nu_1)$,

$$\begin{aligned} \mathbb{E}\Big[\beta\|u(t)\|_\eta^2 + \alpha\|v(t)\|_\eta^2\Big] \leq{}&(1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\beta\|\phi\|_{C_{\rho,\eta}}^2 + \alpha\|\varphi\|_{C_{\rho,\eta}}^2\Big]e^{-\nu t} \\ &+ \frac{1}{\nu}\Big(4(\beta+\alpha)\|\gamma\|_\eta^2 + \frac{\beta}{\delta}\|a\|_\eta^2 + \frac{\alpha}{\lambda}\|c\|_\eta^2 + 2\beta\|b\|_\eta^2 + 2\alpha\|l\|_\eta^2 + 2\beta\|\iota\|_{1,\eta}\Big). \end{aligned} \tag{3.10}$$

Note that

$$\sup_{-\rho\leq t\leq 0}\mathbb{E}\Big[\beta\|u(t)\|_\eta^2 + \alpha\|v(t)\|_\eta^2\Big] \leq \mathbb{E}\Big[\beta\|\phi\|_{C_{\rho,\eta}}^2 + \alpha\|\varphi\|_{C_{\rho,\eta}}^2\Big],$$

which along with (3.10) implies the desired result. □

Lemma 3.2. *Suppose (2.1)–(2.8) and (2.25) hold. Let $(\phi, \varphi) \in L^4(\Omega, C([-\rho, 0], \ell_\eta^2 \times \ell_\eta^2))$ be the initial data of stochastic lattice system (2.15), then the solution (u, v) of the system (2.15) satisfies*

$$\sup_{t\geq-\rho}\mathbb{E}\Big[\|u(t)\|_\eta^4 + \|v(t)\|_\eta^4\Big] \leq M_3\Big(1 + \mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^4 + \|\varphi\|_{C_{\rho,\eta}}^4\Big]\Big),$$

where M_3 is a positive constant independent of (ϕ, φ).

Proof. Given $n \in \mathbb{N}$, define τ_n by

$$\tau_n = \inf\{t \geq 0 : \|u(t)\|_\eta + \|v(t)\|_\eta > n\},$$

and $\tau_n = \infty$ if the set $\{t \geq 0 : \|u(t)\|_\eta + \|v(t)\|_\eta > n\} = \emptyset$. By (3.1) and Itô's formula, we get for all $t \geq 0$,

$$\begin{aligned} &d(\|u(t)\|_\eta^4 + \|v(t)\|_\eta^4) + 4\lambda\|v(t)\|_\eta^4 dt - 4(\beta\|v(t)\|_\eta^2 - \alpha\|u(t)\|_\eta^2)\Big(u(t), v(t)\Big)_\eta dt \\ &= 4\|u(t)\|_\eta^2\Big(Au(t), u(t)\Big)_\eta dt + 4\|u(t)\|_\eta^2\Big(f(u(t)), u(t)\Big)_\eta dt + 4\|u(t)\|_\eta^2\Big(a, u(t)\Big)_\eta dt \\ &\quad + 2\|u(t)\|_\eta^2\sum_{j=1}^\infty\|g_j(u(t), u(t-\rho)) + b_j\|_\eta^2 dt + 4\sum_{j=1}^\infty\Big|\Big(g_j(u(t), u(t-\rho)) + b_j, u(t)\Big)_\eta\Big|^2 dt \\ &\quad + 4\|u(t)\|_\eta^2\sum_{j=1}^\infty\Big(g_j(u(t), u(t-\rho)) + b_j, u(t)\Big)_\eta dW_j(t) + 4\|v(t)\|_\eta^2\Big(c, v(t)\Big)_\eta dt \\ &\quad + 2\|v(t)\|_\eta^2\sum_{j=1}^\infty\|h_j(v(t), v(t-\rho)) + l_j\|_\eta^2 dt + 4\sum_{j=1}^\infty\Big|\Big(h_j(v(t), v(t-\rho)) + l_j, v(t)\Big)_\eta\Big|^2 dt \\ &\quad + 4\|v(t)\|_\eta^2\sum_{j=1}^\infty\Big(h_j(v(t), v(t-\rho)) + l_j, v(t)\Big)_\eta dW_j(t). \end{aligned} \tag{3.11}$$

Let ν be a positive constant which will be specified later, and we get from (3.11) that for all $t \geq 0$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\nu(t\wedge\tau_n)}(\|u(t\wedge\tau_n)\|_\eta^4 + \|v(t\wedge\tau_n)\|_\eta^4)\Big] + 4\lambda\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^4 ds\Big] \\
&= \mathbb{E}\Big[\|\phi(0)\|_\eta^4 + \|\varphi(0)\|_\eta^4\Big] + \nu\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}(\|u(s)\|_\eta^4 + \|v(s)\|_\eta^4) ds\Big] \\
&\quad + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}(\beta\|v(s)\|_\eta^2 - \alpha\|u(s)\|_\eta^2)\Big(u(s), v(s)\Big)_\eta ds\Big] \\
&\quad + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\Big(A(u(s)), u(s)\Big)_\eta ds\Big] + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\Big(a, u(s)\Big)_\eta ds\Big] \\
&\quad + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\Big(f(u(s)), u(s)\Big)_\eta ds\Big] + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^2\Big(c, v(s)\Big)_\eta ds\Big] \\
&\quad + 2\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2 \sum_{j=1}^\infty \|g_j(u(s), u(s-\rho)) + b_j\|_\eta^2 ds\Big] \\
&\quad + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s} \sum_{j=1}^\infty \Big|\Big(g_j(u(s), u(s-\rho)) + b_j, u(s)\Big)_\eta\Big|^2 ds\Big] \\
&\quad + 2\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^2 \sum_{j=1}^\infty \|h_j(v(s), v(s-\rho)) + l_j\|_\eta^2 ds\Big] \\
&\quad + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s} \sum_{j=1}^\infty \Big|\Big(h_j(v(s), v(s-\rho)) + l_j, v(s)\Big)_\eta\Big|^2 ds\Big].
\end{aligned}
\tag{3.12}
$$

Similar to (2.18) and (2.19), we get

$$
4\int_0^{t\wedge\tau_n} e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big(Au(s), u(s)\Big)_\eta\Big] ds \leq 4\tilde{\alpha}\int_0^{t\wedge\tau_n} e^{\nu s}\mathbb{E}\Big[\|u(s)\|_\eta^4\Big] ds. \tag{3.13}
$$

By (2.6) and Young's inequality, we have

$$
\begin{aligned}
4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\Big(f(u(s)), u(s)\Big)_\eta ds\Big] &\leq 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2(-\delta\|u(s)\|_\eta^2 + \|\iota\|_{1,\eta}) ds\Big] \\
&\leq 2(1-2\delta)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^4 ds\Big] + \frac{2\|\iota\|_{1,\eta}^2}{\nu} e^{\nu t}.
\end{aligned}
\tag{3.14}
$$

Note that

$$
\begin{aligned}
&4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\Big(a, u(s)\Big)_\eta ds\Big] + 4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^2\Big(c, v(s)\Big)_\eta ds\Big] \\
&\leq \delta\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^4 ds\Big] + \lambda\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^4 ds\Big] + \frac{27}{\delta^3\nu}\|a\|_\eta^4 e^{\nu t} + \frac{27}{\lambda^3\nu}\|c\|_\eta^4 e^{\nu t},
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&4\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}(\beta\|v(s)\|_\eta^2-\alpha\|u(s)\|_\eta^2)\Big(u(s),v(s)\Big)_\eta ds\Big]\\
&\le 4\beta\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^3\|u(s)\|_\eta ds\Big]+4\alpha\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^3\|v(s)\|_\eta ds\Big]\\
&\le \Big(\lambda+\frac{27\alpha^4}{\delta^3}\Big)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^4 ds\Big]+\Big(\delta+\frac{27\beta^4}{\lambda^3}\Big)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^4 ds\Big].
\end{aligned}
\tag{3.16}
$$

By (2.8), we get

$$
\begin{aligned}
&2\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\|g_j(u(s),u(s-\rho))+b_j\|_\eta^2 ds\Big]\\
&\quad+4\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_0^{t\wedge\tau_n} e^{\nu s}\Big|\Big(g_j(u(s),u(s-\rho))+b_j,u(s)\Big)_\eta\Big|^2 ds\Big]\\
&\le 6\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\|g_j(u(s),u(s-\rho))+b_j\|_\eta^2 ds\Big]\\
&\le 12\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^2\Big(\sum_{j=1}^{\infty}\|b_j\|_\eta^2+2\|\gamma\|_\eta^2\Big)ds\Big]\\
&\quad+72\|\gamma\|^2\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^4 ds\Big]+24\|\gamma\|^2\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s-\rho)\|_\eta^4 ds\Big]\\
&\le (96e^{\rho\nu}\|\gamma\|^2+6)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^4 ds\Big]+6\Big(\|b\|_\eta^2+2\|\gamma\|_\eta^2\Big)^2\frac{e^{\nu t}}{\nu}+24\rho e^{\rho\nu}\|\gamma\|^2\mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^4\Big],
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
&2\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^2\|h_j(v(s),v(s-\rho))+l_j\|_\eta^2 ds\Big]\\
&\quad+4\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_0^{t\wedge\tau_n} e^{\nu s}\Big|\Big(h_j(v(s),v(s-\rho))+l_j,v(s)\Big)_\eta\Big|^2 ds\Big]\\
&\le (96e^{\rho\nu}\|\gamma\|^2+6)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^4 ds\Big]+6\Big(\|l\|_\eta^2+2\|\gamma\|_\eta^2\Big)^2\frac{e^{\nu t}}{\nu}+24\rho e^{\rho\nu}\|\gamma\|^2\mathbb{E}\Big[\|\varphi\|_{C_{\rho,\eta}}^4\Big].
\end{aligned}
\tag{3.18}
$$

For $t\ge 0$, it follows from (3.12)–(3.18) that

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\nu(t\wedge\tau_n)}(\|u(t\wedge\tau_n)\|_\eta^4+\|v(t\wedge\tau_n)\|_\eta^4)\Big]\\
&\le (1+24\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^4+\|\varphi\|_{C_{\rho,\eta}}^4\Big]\\
&\quad+\Big(\nu-2\delta+4\tilde{\alpha}+96e^{\rho\nu}\|\gamma\|^2+\frac{27\beta^4}{\lambda^3}+8\Big)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|u(s)\|_\eta^4 ds\Big]\\
&\quad+\Big(\nu-2\lambda+96e^{\rho\nu}\|\gamma\|^2+\frac{27\alpha^4}{\delta^3}+6\Big)\mathbb{E}\Big[\int_0^{t\wedge\tau_n} e^{\nu s}\|v(s)\|_\eta^4 ds\Big]\\
&\quad+\Big(2\|\iota\|_{1,\eta}^2+6\Big(\|b\|_\eta^2+2\|\gamma\|_\eta^2\Big)^2+6\Big(\|l\|_\eta^2+2\|\gamma\|_\eta^2\Big)^2+\frac{27}{\delta^3}\|a\|_\eta^4+\frac{27}{\lambda^3}\|c\|_\eta^4\Big)\frac{e^{\nu t}}{\nu},
\end{aligned}
$$

which along with (2.25) implies that there exists $\nu_2 > 0$ such that for all $\nu \in (0, \nu_2)$,

$$\begin{aligned}&\mathbb{E}\Big[e^{\nu(t\wedge\tau_n)}(\|u(t\wedge\tau_n)\|_\eta^4 + \|v(t\wedge\tau_n)\|_\eta^4)\Big]\\&\le (1+24\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^4 + \|\varphi\|_{C_{\rho,\eta}}^4\Big]\\&\quad + \Big(2\|\iota\|_{1,\eta}^2 + 6\Big(\|b\|_\eta^2 + 2\|\gamma\|_\eta^2\Big)^2 + 6\Big(\|l\|_\eta^2 + 2\|\gamma\|_\eta^2\Big)^2 + \frac{27}{\delta^3}\|a\|_\eta^4 + \frac{27}{\lambda^3}\|c\|_\eta^4\Big)\frac{e^{\nu t}}{\nu}.\end{aligned}$$

Letting $n \to \infty$, we obtain from the above inequality that for all $t \ge 0$,

$$\begin{aligned}\mathbb{E}\Big[\|u(t)\|_\eta^4 + \|v(t)\|_\eta^4\Big] &\le (1+24\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^4 + \|\varphi\|_{C_{\rho,\eta}}^4\Big]e^{-\nu t}\\&\quad + \Big(2\|\iota\|_{1,\eta}^2 + 6\Big(\|b\|_\eta^2 + 2\|\gamma\|_\eta^2\Big)^2 + 6\Big(\|l\|_\eta^2 + 2\|\gamma\|_\eta^2\Big)^2 + \frac{27}{\delta^3}\|a\|_\eta^4 + \frac{27}{\lambda^3}\|c\|_\eta^4\Big)\frac{1}{\nu}.\end{aligned} \tag{3.19}$$

Note that for all $t \in [-\rho, 0]$,

$$\mathbb{E}\Big[\|u(t)\|_\eta^4 + \|v(t)\|_\eta^4\Big] \le \mathbb{E}\Big[\|\phi\|_{C_{\rho,\eta}}^4 + \|\varphi\|_{C_{\rho,\eta}}^4\Big],$$

which along with (3.19) concludes the proof. □

Lemma 3.3. *Suppose (2.1)–(2.8) and (2.25) hold. Let $(\phi, \varphi) \in L^4(\Omega, C([-\rho, 0], \ell_\eta^2 \times \ell_\eta^2))$ be the initial data of stochastic lattice system (2.15), then the solution (u, v) of the system (2.15) satisfies, for any $t > r \ge 0$,*

$$\mathbb{E}\Big[\|u(t) - u(r)\|_\eta^4 + \|v(t) - v(r)\|_\eta^4\Big] \le M_4(|t-r|^2 + |t-r|^4),$$

where M_4 is a positive constant depending on (ϕ, φ), but is independent of t and r.

Proof. For $t > r \ge 0$, by (2.15), we get

$$\begin{cases} u(t) - u(r) = \int_r^t (Au(s) - \alpha v(s) + f(u(s)) + a)ds + \sum\limits_{j=1}^{\infty} \int_r^t (g_j(u(s), u(s-\rho)) + b_j)dW_j(s),\\ v(t) - v(r) = \int_r^t (\beta u(s) - \lambda v(s) + c)dt + \sum\limits_{j=1}^{\infty} \int_r^t (h_j(v(s), v(t-\rho)) + l_j)dW_j(s),\end{cases}$$

which together with (2.5), (2.10), and (2.14) implies that, for $t > r \ge 0$,

$$\begin{cases} \|u(t) - u(r)\|_\eta \le \int_r^t (C_4\|u(s)\|_\eta + \alpha\|v(s)\|_\eta)ds + \|a\|_\eta|t-r|\\ \qquad\qquad + \Big\|\sum\limits_{j=1}^{\infty} \int_r^t (g_j(u(s), u(s-\rho)) + b_j)dW_j(s)\Big\|_\eta,\\ \|v(t) - v(r)\|_\eta \le \int_r^t (\beta\|u(s)\|_\eta + \lambda\|v(s)\|_\eta)ds + \|c\|_\eta\||t-r|\\ \qquad\qquad + \Big\|\sum\limits_{j=1}^{\infty} \int_r^t (h_j(v(s), v(t-\rho)) + l_j)dW_j(s)\Big\|_\eta,\end{cases} \tag{3.20}$$

where $C_4 = \sqrt{2|J(0)|^2 + 8\Big(\sum\limits_{m=1}^{\infty}|J(m)|\Big)^2 + L_C}$. By (3.20), we get

$$
\begin{aligned}
&\mathbb{E}\Big[\|u(t)-u(r)\|_\eta^4 + \|v(t)-v(r)\|_\eta^4\Big] \\
&\le 64(C_4^4+\beta^4)\mathbb{E}\Big[\Big(\int_r^t \|u(s)\|_\eta ds\Big)^4\Big] + 64(\alpha^4+\lambda^4)\mathbb{E}\Big[\Big(\int_r^t \|v(s)\|_\eta ds\Big)^4\Big] \\
&\quad + 64(\|a\|_\eta^4 + \|c\|_\eta^4)|t-r|^4 + 64\mathbb{E}\Big[\Big\|\sum_{j=1}^{\infty}\int_r^t (g_j(u(s),u(s-\rho))+b_j)dW_j(s)\Big\|_\eta^4\Big] \\
&\quad + 64\mathbb{E}\Big[\Big\|\sum_{j=1}^{\infty}\int_r^t (h_j(v(s),v(s-\rho))+l_j)dW_j(s)\Big\|_\eta^4\Big].
\end{aligned}
\tag{3.21}
$$

By Schwarz's inequality and Lemma 3.2, we have

$$
\begin{aligned}
&64(C_4^4+\beta^4)\mathbb{E}\Big[\Big(\int_r^t \|u(s)\|_\eta ds\Big)^4\Big] + 64(\alpha^4+\lambda^4)\mathbb{E}\Big[\Big(\int_r^t \|v(s)\|_\eta ds\Big)^4\Big] \\
&\le 64(C_4^4+\beta^4+\alpha^4+\lambda^4)|t-r|^3\int_r^t \mathbb{E}\Big[\|u(s)\|_\eta^4 + \|v(s)\|_\eta^4\Big]ds \\
&\le C_5|t-r|^4.
\end{aligned}
\tag{3.22}
$$

For the last two terms of (3.21), by (2.11), Lemma 3.2, and the BDG inequality, we get

$$
\begin{aligned}
&64\mathbb{E}\Big[\Big\|\sum_{j=1}^{\infty}\int_r^t (g_j(u(s),u(s-\rho))+b_j)dW_j(s)\Big\|_\eta^4\Big] \\
&\le C_6\mathbb{E}\Big[\Big(\int_r^t\sum_{j=1}^{\infty}\|g_j(u(s),u(s-\rho))+b_j\|_\eta^2 ds\Big)^2\Big] \\
&\le 8C_6\Big(2\|\gamma\|_\eta^2+\|b\|_\eta^2\Big)^2|t-r|^2 + 128C_6\|\gamma\|^4\mathbb{E}\Big[\Big(\int_r^t (\|u(s)\|_\eta^2 + \|u(s-\rho)\|_\eta^2)ds\Big)^2\Big] \\
&\le C_7|t-r|^2,
\end{aligned}
\tag{3.23}
$$

and

$$
64\mathbb{E}\Big[\Big\|\sum_{j=1}^{\infty}\int_r^t (h_j(v(s),v(s-\rho))+l_j)dW_j(s)\Big\|_\eta^4\Big] \le C_7|t-r|^2,
$$

which along with (3.21)–(3.23) implies the desired result. □

The subsequent step entails acquiring uniform estimates on the tails of solutions to stochastic lattice system (2.15), which play a pivotal role in proving the tightness of a family of solution distributions.

Lemma 3.4. *Suppose (2.1)–(2.8) and (2.25) hold. For any compact subset $E \subset L^2(\Omega, C([-\rho,0],\ell_\eta^2\times\ell_\eta^2))$, the solution (u,v) of stochastic lattice system (2.15) satisfies*

$$
\limsup_{k\to\infty}\sup_{(\phi,\varphi)\in E}\sup_{t\ge-\rho}\sum_{|n|\ge k}\mathbb{E}[\eta_n(|u_n(t,\phi)|^2 + |v_n(t,\varphi)|^2)] = 0.
$$

Proof. Let ϑ be a smooth function which is defined on $\mathbb{R}^+$ such that $0 \le \vartheta(z) \le 1$ for all $z \in \mathbb{R}^+$, and

$$\vartheta(z) = \begin{cases} 0, & 0 \le z \le 1; \\ 1, & z \ge 2. \end{cases}$$

For $k \in N$, set $\vartheta_k = (\vartheta(\frac{|n|}{k}))_{n\in\mathbb{Z}}$, $\vartheta_k u = (\vartheta(\frac{|n|}{k})u_n)_{n\in\mathbb{Z}}$, and $\vartheta_k v = (\vartheta(\frac{|n|}{k})v_n)_{n\in\mathbb{Z}}$. By (2.15), we have

$$\begin{cases} d(\vartheta_k u(t)) = (\vartheta_k Au(t) - \alpha\vartheta_k v(t) + \vartheta_k f(u(t)) + \vartheta_k a)dt + \sum_{j=1}^{\infty} (\vartheta_k g_j(u(t), u(t-\rho)) + \vartheta_k b_j)dW_j(t), \\ d(\vartheta_k v(t)) = (\beta\vartheta_k u(t) - \lambda\vartheta_k v(t) + \vartheta_k c)dt + \sum_{j=1}^{\infty} (\vartheta_k h_j(v(t), v(t-\rho)) + \vartheta_k l_j)dW_j(t), \end{cases}$$

which along with Itô's formula implies that

$$\begin{aligned} & d(\beta\|\vartheta_k u(t)\|_\eta^2 + \alpha\|\vartheta_k v(t)\|_\eta^2) \\ &= 2\beta\Big(\vartheta_k Au(t), \vartheta_k u(t)\Big)_\eta dt + 2\beta\Big(\vartheta_k f(u(t)), \vartheta_k u(t)\Big)_\eta dt + 2\beta\Big(\vartheta_k a, \vartheta_k u(t)\Big)_\eta dt \\ &\quad + \beta\sum_{j=1}^{\infty} \|\vartheta_k g_j(u(t), u(t-\rho)) + \vartheta_k b_j\|_\eta^2 dt - 2\lambda\alpha\|\vartheta_k v(t)\|_\eta^2 dt \\ &\quad + 2\alpha\Big(\vartheta_k c, \vartheta_k v(t)\Big)_\eta dt + \alpha\sum_{j=1}^{\infty} \|\vartheta_k h_j(v(t), v(t-\rho)) + \vartheta_k l_j\|_\eta^2 dt \\ &\quad + 2\beta\sum_{j=1}^{\infty} \Big(\vartheta_k g_j(u(t), u(t-\rho)) + \vartheta_k b_j, \vartheta_k u(t)\Big)_\eta dW_j(t) \\ &\quad + 2\alpha\sum_{j=1}^{\infty} \Big(\vartheta_k h_j(v(t), v(t-\rho)) + \vartheta_k l_j, \vartheta_k v(t)\Big)_\eta dW_j(t). \end{aligned} \tag{3.24}$$

Then, we get that for all $t \ge 0$,

$$\begin{aligned} & e^{\nu t}\mathbb{E}\Big[\beta\|\vartheta_k u(t)\|_\eta^2 + \alpha\|\vartheta_k v(t)\|_\eta^2\Big] + (2\lambda - \nu)\alpha\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k v(s)\|_\eta^2\Big]ds \\ &= \mathbb{E}\Big[\beta\|\vartheta_k\phi(0)\|_\eta^2 + \alpha\|\vartheta_k\varphi(0)\|_\eta^2\Big] + \nu\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2\Big]ds \\ &\quad + 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(\vartheta_k Au(s), \vartheta_k u(s)\Big)_\eta\Big]ds + 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(\vartheta_k f(u(s)), \vartheta_k u(s)\Big)_\eta\Big]ds \\ &\quad + 2\beta\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(\vartheta_k a, \vartheta_k u(s)\Big)_\eta\Big]ds + 2\alpha\int_0^t e^{\nu s}\mathbb{E}\Big[\Big(\vartheta_k c, \vartheta_k v(s)\Big)_\eta\Big]ds \\ &\quad + \beta\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j\|_\eta^2\Big]ds \\ &\quad + \alpha\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j\|_\eta^2\Big]ds, \end{aligned} \tag{3.25}$$

where ν is a positive constant which will be specified later. Furthermore, we find that

$$
\begin{aligned}
\Big(\vartheta_k Au, \vartheta_k u\Big)_\eta &= \sum_{n\in\mathbb{Z}} \eta_n \sum_{m\in\mathbb{Z}} J(m)\vartheta^2\Big(\frac{|n|}{k}\Big)u_{n-m}u_n \\
&= J(0)\sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big)\eta_n|u_n|^2 + \sum_{n\in\mathbb{Z}}\sum_{m=1}^{\infty} J(m)\vartheta^2\Big(\frac{|n|}{k}\Big)\eta_n u_{n+m}u_n \\
&\quad + \sum_{n\in\mathbb{Z}}\sum_{m=1}^{\infty} J(m)\vartheta^2\Big(\frac{|n+m|}{k}\Big)\eta_{n+m}u_n u_{n+m} \\
&= J(0)\sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big)\eta_n|u_n|^2 + \sum_{n\in\mathbb{Z}}\sum_{m=1}^{\infty} J(m)\Big(\vartheta^2\Big(\frac{|n+m|}{k}\Big)\eta_{n+m} + \vartheta^2\Big(\frac{|n|}{k}\Big)\eta_n\Big)u_n u_{n+m} \\
&= J(0)\sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big)\eta_n|u_n|^2 + J_1 + J_2,
\end{aligned}
\tag{3.26}
$$

where

$$
J_1 = \sum_{n\in\mathbb{Z}}\sum_{m=1}^{\infty} J(m)\Big(\vartheta^2\Big(\frac{|n+m|}{k}\Big) - \vartheta^2\Big(\frac{|n|}{k}\Big)\Big)\eta_{n+m}u_n u_{n+m},
$$

and

$$
J_2 = \sum_{n\in\mathbb{Z}}\sum_{m=1}^{\infty} J(m)\vartheta^2\Big(\frac{|n|}{k}\Big)(\eta_{n+m} + \eta_n)u_n u_{n+m}.
$$

For any $n \in \mathbb{Z}$ and $m \in \mathbb{N}^+$, by the definition of $\vartheta(z)$ we can get that there exists a constant $C_8 > 0$ such that

$$
\Big|\vartheta\Big(\frac{|n+m|}{k}\Big) - \vartheta\Big(\frac{|n|}{k}\Big)\Big| \leq \frac{m}{k}C_8. \tag{3.27}
$$

By (2.2), we have

$$
\eta_{n+m}^{1/2} \leq \alpha_m \eta_n^{1/2},\ \forall n \in \mathbb{Z}, m \geq 1,
$$

which together with (3.27) implies that for any $p > 1$,

$$
\begin{aligned}
|J_1| &\leq \sum_{n\in\mathbb{Z}}\sum_{m=1}^{\infty} |J(m)|\Big|\vartheta^2\Big(\frac{|n+m|}{k}\Big) - \vartheta^2\Big(\frac{|n|}{k}\Big)\Big|\eta_{n+m}|u_{n+m}||u_n| \\
&\leq \frac{2C_8}{k}\sum_{m=1}^{p} m\alpha_m|J(m)|\sum_{n\in\mathbb{Z}}\eta_{n+m}^{1/2}\eta_n^{1/2}|u_{n+m}||u_n| + \sum_{m=p+1}^{\infty}\alpha_m|J(m)|\sum_{n\in\mathbb{Z}}\eta_{n+m}^{1/2}\eta_n^{1/2}|u_{n+m}||u_n| \\
&\leq \frac{2C_8}{k}\sum_{m=1}^{p} m\alpha_m|J(m)|\|u\|_\eta^2 + \sum_{m=p+1}^{\infty}\alpha_m|J(m)|\|u\|_\eta^2.
\end{aligned}
\tag{3.28}
$$

By (2.2), we obtain

$$
\begin{aligned}
|J_2| &\le \sum_{m=1}^{\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big) \eta_{n+m}^{1/2} \eta_n^{1/2} |u_{n+m}||u_n| \\
&\le \frac{1}{2} \sum_{m=1}^{\infty} \alpha_m |J(m)| \Big(\sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big) \eta_{n+m} |u_{n+m}|^2 + \sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big) \eta_n |u_n|^2 \Big),
\end{aligned}
$$

which together with (3.27) and (2.3) implies that for any $p > 1$,

$$
\begin{aligned}
|J_2| \le{}& \sum_{m=1}^{\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big) \eta_n |u_n|^2 + \frac{1}{2} \sum_{m=1}^{p} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \eta_{n+m} \Big| \vartheta^2\Big(\frac{|n+m|}{k}\Big) - \vartheta^2\Big(\frac{|n|}{k}\Big) \Big| |u_{n+m}|^2 \\
&+ \frac{1}{2} \sum_{m=p+1}^{\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \eta_{n+m} \Big| \vartheta^2\Big(\frac{|n+m|}{k}\Big) - \vartheta^2\Big(\frac{|n|}{k}\Big) \Big| |u_{n+m}|^2 \\
\le{}& \sum_{m=1}^{\infty} \alpha_m |J(m)| \sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big) \eta_n |u_n|^2 + \frac{C_8}{k} \sum_{m=1}^{p} m \alpha_m |J(m)| \|u\|_\eta^2 + \sum_{m=p+1}^{\infty} \alpha_m |J(m)| \|u\|_\eta^2 .
\end{aligned}
\tag{3.29}
$$

For any $p > 1$, it follows from (3.26), (3.28), and (3.29) that

$$
2\beta \Big| \Big(\vartheta_k A u, \vartheta_k u\Big)_\eta \Big| \le 2\beta\tilde{\alpha} \sum_{n\in\mathbb{Z}} \vartheta^2\Big(\frac{|n|}{k}\Big) \eta_n |u_n|^2 + \frac{6\beta C_8}{k} \sum_{m=1}^{p} m\alpha_m |J(m)| \|u\|_\eta^2 + 4\beta \sum_{m=p+1}^{\infty} \alpha_m |J(m)| \|u\|_\eta^2 . \tag{3.30}
$$

By (2.6) and Young's inequality, we have

$$
2\beta \int_0^t e^{\nu s} \mathbb{E}\Big[\Big(\vartheta_k f(u(s)), \vartheta_k u(s)\Big)_\eta \Big] ds \le -2\beta\delta \int_0^t e^{\nu s} \mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2 \Big] ds + \frac{2\beta e^{\nu t}}{\nu} \sum_{|n|\ge k} \eta_n |\iota_n| . \tag{3.31}
$$

Note that

$$
\begin{aligned}
&2\beta \int_0^t e^{\nu s} \mathbb{E}\Big[\Big(\vartheta_k a, \vartheta_k u(s)\Big)_\eta \Big] ds + 2\alpha \int_0^t e^{\nu s} \mathbb{E}\Big[\Big(\vartheta_k c, \vartheta_k v(s)\Big)_\eta \Big] ds \\
&\le \beta\delta \int_0^t e^{\nu s} \mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2 \Big] ds + \frac{\beta e^{\nu t}}{\delta\nu} \sum_{|n|\ge k} \eta_n |a_n|^2 + \lambda\alpha \int_0^t e^{\nu s} \mathbb{E}\Big[\|\vartheta_k v(s)\|_\eta^2 \Big] ds + \frac{\alpha e^{\nu t}}{\lambda\nu} \sum_{|n|\ge k} \eta_n |c_n|^2 .
\end{aligned}
\tag{3.32}
$$

For the last two terms of (3.25), by (2.8), we get

$$
\begin{aligned}
&\beta\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k g_j(u(s),u(s-\rho))+\vartheta_k b_j\|_\eta^2\Big]ds\\
&\quad+\alpha\sum_{j=1}^{\infty}\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k h_j(v(s),v(s-\rho))+\vartheta_k l_j\|_\eta^2\Big]ds\\
&\le \frac{2\beta}{\nu}e^{\nu t}\sum_{|n|\ge k}\sum_{j=1}^{\infty}\eta_n\Big(b_{j,n}^2+2\gamma_{j,n}^2\Big)+8\beta\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2+\|\vartheta_k u(s-\rho)\|_\eta^2\Big]ds\\
&\quad+\frac{2\alpha}{\nu}e^{\nu t}\sum_{|n|\ge k}\sum_{j=1}^{\infty}\eta_n\Big(l_{j,n}^2+2\gamma_{j,n}^2\Big)+8\alpha\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k v(s)\|_\eta^2+\|\vartheta_k v(s-\rho)\|_\eta^2\Big]ds \qquad (3.33)\\
&\le \frac{2\beta}{\nu}e^{\nu t}\sum_{|n|\ge k}\sum_{j=1}^{\infty}\eta_n\Big(b_{j,n}^2+2\gamma_{j,n}^2\Big)+16\beta e^{\rho\nu}\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2\Big]ds\\
&\quad+\frac{2\alpha}{\nu}e^{\nu t}\sum_{|n|\ge k}\sum_{j=1}^{\infty}\eta_n\Big(l_{j,n}^2+2\gamma_{j,n}^2\Big)+16\alpha e^{\rho\nu}\|\gamma\|^2\int_0^t e^{\nu s}\mathbb{E}\Big[\|\vartheta_k v(s)\|_\eta^2\Big]ds\\
&\quad+8\beta e^{\rho\nu}\|\gamma\|^2\int_{-\rho}^0 e^{\nu s}\mathbb{E}\Big[\|\vartheta_k\phi(s)\|_\eta^2\Big]ds+8\alpha e^{\rho\nu}\|\gamma\|^2\int_{-\rho}^0 e^{\nu s}\mathbb{E}\Big[\|\vartheta_k\varphi(s)\|_\eta^2\Big]ds.
\end{aligned}
$$

Then, it follows from (3.25) and (3.30)–(3.33) that for $p>1$,

$$
\begin{aligned}
&\mathbb{E}\Big[\beta\|\vartheta_k u(t)\|_\eta^2+\alpha\|\vartheta_k v(t)\|_\eta^2\Big]\\
&\le(1+8\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\beta\|\vartheta_k\phi(0)\|_\eta^2+\alpha\|\vartheta_k\varphi(0)\|_\eta^2\Big]e^{-\nu t}\\
&\quad+\beta\Big(\nu-\delta+16e^{\rho\nu}\|\gamma\|^2+2\tilde{\alpha}+\frac{6C_8}{k}\sum_{m=1}^{p}m\alpha_m|J(m)|+4\sum_{m=p+1}^{+\infty}\alpha_m|J(m)|\Big)\int_0^t e^{\nu(s-t)}\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds\\
&\quad+\alpha\Big(\nu-\lambda+16e^{\rho\nu}\|\gamma\|^2\Big)\int_0^t e^{\nu(s-t)}\mathbb{E}\Big[\|v(s)\|_\eta^2\Big]ds+\frac{\beta}{\delta\nu}\sum_{|n|\ge k}\eta_n|a_n|^2+\frac{\alpha}{\lambda\nu}\sum_{|n|\ge k}\eta_n|c_n|^2 \qquad (3.34)\\
&\quad+\frac{2\beta}{\nu}\sum_{|n|\ge k}\sum_{j=1}^{\infty}\eta_n(b_{j,n}^2+2\gamma_{j,n}^2)+\frac{2\alpha}{\nu}\sum_{|n|\ge k}\sum_{j=1}^{\infty}\eta_n(l_{j,n}^2+2\gamma_{j,n}^2)+\frac{2\beta}{\nu}\sum_{|n|\ge k}\eta_n|\iota_n|.
\end{aligned}
$$

Furthermore, it follows from (2.3) that there is a $K_1=K_1(\nu)>0$ such that for all $k\ge K_1$,

$$\frac{6C_8}{k}\sum_{m=1}^{p}m\alpha_m|J(m)|\le\frac{\nu}{2}. \qquad (3.35)$$

By (2.3) again, we can choose $p=p(\nu)$ large enough such that

$$4\sum_{m=p+1}^{\infty}\alpha_m|J(m)|\le\frac{\nu}{2},$$

which along with (3.35) and (2.25) implies that there exists $\nu_3>0$ such that for all $\nu\in(0,\nu_3)$,

$$\nu-\delta+16e^{\rho\nu}\|\gamma\|^2+2\tilde{\alpha}+\frac{6C_8}{k}\sum_{m=1}^{p}m\alpha_m|J(m)|+4\sum_{m=p+1}^{+\infty}\alpha_m|J(m)|\le 2\nu-\delta+16e^{\rho\nu}\|\gamma\|^2\le 0,$$

which together with (3.34) and (2.25) shows that for all $t \geq 0$ and $k \geq K_1$,

$$\begin{aligned}\mathbb{E}\Big[\beta\|\vartheta_k u(t)\|_\eta^2 + \alpha\|\vartheta_k v(t)\|_\eta^2\Big] \leq &(1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\beta\|\vartheta_k\phi(0)\|_\eta^2 + \alpha\|\vartheta_k\varphi(0)\|_\eta^2\Big]e^{-\nu t}\\ &+ \frac{\beta}{\delta\nu}\sum_{|n|\geq k}\eta_n|a_n|^2 + \frac{\alpha}{\lambda\nu}\sum_{|n|\geq k}\eta_n|c_n|^2 + \frac{2\beta}{\nu}\sum_{|n|\geq k}\sum_{j=1}^{\infty}\eta_n(b_{j,n}^2 + 2\gamma_{j,n}^2)\\ &+ \frac{2\alpha}{\nu}\sum_{|n|\geq k}\sum_{j=1}^{\infty}\eta_n(l_{j,n}^2 + 2\gamma_{j,n}^2) + \frac{2\beta}{\nu}\sum_{|n|\geq k}\eta_n|\iota_n|.\end{aligned} \tag{3.36}$$

Note that $(\phi, \varphi) \in E$ and E is a compact subset in $L^2(\Omega, C([-\rho, 0], \ell_\eta^2 \times \ell_\eta^2))$. Then, for each $\varepsilon > 0$, there exists $K_2 = K_2(\varepsilon, \phi, \varphi) \geq 1$ such that for all $k \geq K_2$,

$$\sum_{|n|\geq k}\mathbb{E}[\eta_n(\beta|\phi_n(0)|^2 + \alpha|\varphi_n(0)|^2)] \leq \varepsilon. \tag{3.37}$$

It follows from (3.37) that for all $k \geq K_2$,

$$\begin{aligned}&(1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\mathbb{E}\Big[\beta\|\vartheta_k\phi(0)\|_\eta^2 + \alpha\|\vartheta_k\varphi(0)\|_\eta^2\Big]\\ &= (1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\sum_{n\in\mathbb{Z}}\mathbb{E}\Big[\eta_n\Big(\beta\Big|\vartheta\Big(\frac{|n|}{k}\Big)\phi_n(0)\Big|^2 + \alpha\Big|\vartheta\Big(\frac{|n|}{k}\Big)\varphi_n(0)\Big|^2\Big)\Big]\\ &\leq (1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\sum_{|n|\geq k}\mathbb{E}\Big[\eta_n\Big(\beta\Big|\phi_n(0)\Big|^2 + \alpha\Big|\varphi_n(0)\Big|^2\Big)\Big]\\ &\leq (1 + 8\rho e^{\rho\nu}\|\gamma\|^2)\varepsilon.\end{aligned} \tag{3.38}$$

Since $a = (a_n)_{n\in\mathbb{Z}}$, $c = (c_n)_{n\in\mathbb{Z}}$, $b = (b_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$, $l = (l_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$, $\gamma = (\gamma_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}} \in \ell_\eta^2$ and $\iota = (\iota_n)_{n\in\mathbb{Z}} \in \ell_\eta^1$, we get that there exists $K_3 = K_3(\varepsilon) \geq 1$ such that for all $t \geq 0$ and $k \geq K_3$,

$$\begin{aligned}&\frac{\beta}{\delta\nu}\sum_{|n|\geq k}\eta_n|a_n|^2 + \frac{\alpha}{\lambda\nu}\sum_{|n|\geq k}\eta_n|c_n|^2 + \frac{2\beta}{\nu}\sum_{|n|\geq k}\sum_{j=1}^{\infty}\eta_n(b_{j,n}^2 + 2\gamma_{j,n}^2)\\ &\quad + \frac{2\alpha}{\nu}\sum_{|n|\geq k}\sum_{j=1}^{\infty}\eta_n(l_{j,n}^2 + 2\gamma_{j,n}^2) + \frac{2\beta}{\nu}\sum_{|n|\geq k}\eta_n|\iota_n| \leq \varepsilon,\end{aligned}$$

which along with (3.36) and (3.38) implies that for all $t \geq 0$, $k \geq \max\{K_1, K_2, K_3\}$, and $(\phi, \varphi) \in E$,

$$\sum_{|n|\geq 2k}\mathbb{E}\Big[\eta_n(\beta|u_n(t)|_\eta^2 + \alpha|v_n(t)|_\eta^2)\Big] \leq \mathbb{E}\Big[\beta\|\vartheta_k u(t)\|_\eta^2 + \alpha\|\vartheta_k v(t)\|_\eta^2\Big] \leq (2 + 8\rho e^{\rho\nu}\|\gamma\|^2)\varepsilon. \tag{3.39}$$

Observe that $\{(\phi(s), \varphi(s)) \in L^2(\Omega, \ell_\eta^2 \times \ell_\eta^2) : s \in [-\rho, 0]\}$ is a compact subset in $L^2(\Omega, \ell_\eta^2 \times \ell_\eta^2)$. Then, for each $\varepsilon > 0$, there are $s_1, s_2, \cdots s_m \in [-\rho, 0]$ such that

$$\{(\phi(s), \varphi(s)) \in L^2(\Omega, \ell_\eta^2 \times \ell_\eta^2) : s \in [-\rho, 0]\} \subseteq \bigcup_{j=1}^{m} B\Big((\phi(s_j), \varphi(s_j)), \frac{1}{2}\sqrt{\varepsilon}\Big), \tag{3.40}$$

where $B((\phi(s_j), \varphi(s_j)), \frac{1}{2}\sqrt{\varepsilon})$ is an open ball in $L^2(\Omega, \ell^2_\eta \times \ell^2_\eta)$ centered at $(\phi(s_j), \varphi(s_j))$ with radius $\frac{1}{2}\sqrt{\varepsilon}$. Since $(\phi(s_j), \varphi(s_j)) \in L^2(\Omega, \ell^2_\eta \times \ell^2_\eta)$, for $j = 1, \cdots, m$, there exists $K_4 = K_4(\varepsilon, \phi, \varphi) \geq 1$, such that for all $k \geq K_4$,

$$\sum_{|n|\geq k} \mathbb{E}[\eta_n(|\phi_n(s_j)|^2 + |\varphi_n(s_j)|^2)] \leq \frac{1}{4}\varepsilon, \quad j = 1, 2, \cdots, m. \tag{3.41}$$

It follows from (3.40) and (3.41) that for all $k \geq K_4$ and $s \in [-\rho, 0]$,

$$\sum_{|n|\geq k} \mathbb{E}[\eta_n(|\phi_n(s)|^2 + |\varphi_n(s)|^2)] \leq \varepsilon,$$

which along with (3.39) implies the desired result. □

The tail estimates given by Lemma 3.4 have been enhanced to obtain uniform estimates on the tails of solutions, which are crucial for achieving tightness in the probability distributions of solution segments in the space $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$.

Lemma 3.5. *Suppose (2.1)–(2.8) and (2.25) hold. For any compact subset $E \subset L^2(\Omega, C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta))$, the solution (u, v) of stochastic lattice system (2.15) satisfies*

$$\limsup_{n\to\infty} \sup_{(\phi,\varphi)\in E} \sup_{t\geq\rho} \mathbb{E}\Big[\sup_{t-\rho\leq r\leq t} \sum_{|n|\geq k} \eta_n(|u_n(r, \phi)|^2 + |v_n(r, \varphi)|^2)\Big] = 0.$$

Proof. Let ϑ be the function defined in Lemma 3.4. For all $t \geq \rho$ and $t-\rho \leq r \leq t$, it follows from (3.24) that

$$\begin{aligned}
&\beta\|\vartheta_k u(r)\|^2_\eta + \alpha\|\vartheta_k v(r)\|^2_\eta + 2\lambda\alpha \int_{t-\rho}^r \|\vartheta_k v(s)\|^2_\eta ds \\
&= \beta\|\vartheta_k u(t-\rho)\|^2_\eta + \alpha\|\vartheta_k v(t-\rho)\|^2_\eta + 2\beta \int_{t-\rho}^r \Big(\vartheta_k Au(s), \vartheta_k u(s)\Big)_\eta ds \\
&\quad + 2\beta \int_{t-\rho}^r \Big(\vartheta_k f(u(s)), \vartheta_k u(s)\Big)_\eta ds + 2\beta \int_{t-\rho}^r \Big(\vartheta_k a, \vartheta_k u(s)\Big)_\eta ds \\
&\quad + 2\alpha \int_{t-\rho}^r \Big(\vartheta_k c, \vartheta_k v(s)\Big)_\eta ds + \beta \sum_{j=1}^\infty \int_{t-\rho}^r \|\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j\|^2_\eta ds \\
&\quad + 2\beta \sum_{j=1}^\infty \int_{t-\rho}^r \Big(\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j, \vartheta_k u(s)\Big)_\eta dW_j(s) \\
&\quad + \alpha \sum_{j=1}^\infty \int_{t-\rho}^r \|\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j\|^2_\eta ds \\
&\quad + 2\alpha \sum_{j=1}^\infty \int_{t-\rho}^r \Big(\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j, \vartheta_k v(s)\Big)_\eta dW_j(s),
\end{aligned}$$

which shows that for all $t \geq \rho$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}(\beta\|\vartheta_k u(r)\|_\eta^2+\alpha\|\vartheta_k v(r)\|_\eta^2)\Big]+2\lambda\alpha\mathbb{E}\Big[\int_{t-\rho}^t\|\vartheta_k v(s)\|_\eta^2 ds\Big]\\
&\leq \mathbb{E}\Big[\beta\|\vartheta_k u(t-\rho)\|_\eta^2+\alpha\|\vartheta_k v(t-\rho)\|_\eta^2\Big]+2\beta\mathbb{E}\Big[\int_{t-\rho}^t\Big|\Big(\vartheta_k Au(s),\vartheta_k u(s)\Big)_\eta\Big|ds\Big]\\
&\quad+2\beta\mathbb{E}\Big[\int_{t-\rho}^t\Big|\Big(\vartheta_k f(u(s)),\vartheta_k u(s)\Big)_\eta\Big|ds\Big]+2\beta\mathbb{E}\Big[\int_{t-\rho}^t\|\vartheta_k a\|_\eta\|\vartheta_k u(s)\|_\eta ds\Big]\\
&\quad+2\alpha\mathbb{E}\Big[\int_{t-\rho}^t\|\vartheta_k c\|\|\vartheta_k v(s)\|_\eta ds\Big]+\beta\mathbb{E}\Big[\sum_{j=1}^\infty\int_{t-\rho}^t\|\vartheta_k g_j(u(s),u(s-\rho))+\vartheta_k b_j\|_\eta^2 ds\Big]\\
&\quad+\alpha\mathbb{E}\Big[\sum_{j=1}^\infty\int_{t-\rho}^t\|\vartheta_k h_j(v(s),v(s-\rho))+\vartheta_k l_j\|_\eta^2 ds\Big]\\
&\quad+2\beta\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\Big|\sum_{j=1}^\infty\int_{t-\rho}^r\Big(\vartheta_k g_j(u(s),u(s-\rho))+\vartheta_k b_j,\vartheta_k u(s)\Big)_\eta dW_j(s)\Big|\Big]\\
&\quad+2\alpha\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\Big|\sum_{j=1}^\infty\int_{t-\rho}^r\Big(\vartheta_k h_j(v(s),v(s-\rho))+\vartheta_k l_j,\vartheta_k v(s)\Big)_\eta dW_j(s)\Big|\Big].
\end{aligned}
\tag{3.42}
$$

For any $\varepsilon > 0$, by Lemma 3.4, we get that there is a $K_5 = K_5(\varepsilon, E) \geq 1$ such that for all $k \geq K_5$ and $t \geq -\rho$,

$$
\sum_{|n|\geq k}\mathbb{E}\Big[\eta_n(\beta|u_n(t)|^2+\alpha|v_n(t)|^2)\Big]\leq\varepsilon,
$$

which shows that for all $k \geq K_5$ and $t \geq -\rho$,

$$
\begin{aligned}
\mathbb{E}\Big[\beta\|\vartheta_k u(t)\|_\eta^2+\alpha\|\vartheta_k v(t)\|_\eta^2\Big]&=\sum_{|n|\geq k}\mathbb{E}\Big[\eta_n(\beta|\vartheta_k u_n(t)|^2+\alpha|\vartheta_k v_n(t)|^2)\Big]\\
&\leq\sum_{|n|\geq k}\mathbb{E}\Big[\eta_n(\beta|u_n(t)|^2+\alpha|v_n(t)|^2)\Big]\leq\varepsilon.
\end{aligned}
\tag{3.43}
$$

Then, for all $k \geq K_5$ and $t \geq \rho$,

$$
\mathbb{E}\Big[\beta\|\vartheta_k u(t-\rho)\|_\eta^2+\alpha\|\vartheta_k v(t-\rho)\|_\eta^2\Big]\leq\varepsilon.
\tag{3.44}
$$

Proceeding as in (3.30), we have

$$
\begin{aligned}
&2\beta\mathbb{E}\Big[\int_{t-\rho}^t\Big|\Big(\vartheta_k Au(s),\vartheta_k u(s)\Big)_\eta\Big|ds\Big]\\
&\leq 2\beta\tilde{\alpha}\int_{t-\rho}^t\mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2\Big]ds+\frac{6\beta C_8}{k}\sum_{m=1}^p m\alpha_m|J(m)|\int_{t-\rho}^t\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds\\
&\quad+4\beta\sum_{m=p+1}^\infty\alpha_m|J(m)|\int_{t-\rho}^t\mathbb{E}\Big[\|u(s)\|_\eta^2\Big]ds.
\end{aligned}
\tag{3.45}
$$

Then, by (3.43), we get that for all $k \geq K_5$ and $t \geq -\rho$,

$$2\beta\tilde{\alpha} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2\Big] ds \leq 2\tilde{\alpha}\rho\varepsilon. \tag{3.46}$$

Furthermore, it follows from (2.3) and Lemma 3.1 that there is a $K_6 = K_6(\varepsilon, E) \geq K_5$, such that for all $k \geq K_6$ and $t \geq \rho$,

$$\frac{6\beta C_8}{k} \sum_{m=1}^{p} m\alpha_m |J(m)| \int_{t-\rho}^{t} \mathbb{E}\Big[\|u(s)\|_\eta^2\Big] ds \leq \rho\varepsilon. \tag{3.47}$$

By (2.3) and Lemma 3.1 again, we can choose $p = p(\varepsilon)$ large enough such that for all $t \geq \rho$,

$$4\beta \sum_{m=p+1}^{\infty} \alpha_m |J(m)| \int_{t-\rho}^{t} \mathbb{E}\Big[\|u(s)\|_\eta^2\Big] ds \leq \rho\varepsilon. \tag{3.48}$$

Since $\iota = (\iota_n)_{n\in\mathbb{Z}} \in \ell_\eta^1$, we get that there exists $K_7 = K_7(\varepsilon, E) \geq K_6$ such that for all $k \geq K_7$,

$$2\beta \int_{t-\rho}^{t} \mathbb{E}\Big[\Big|\Big(\vartheta_k f(u(s)), \vartheta_k u(s)\Big)_\eta\Big|\Big] ds \leq -2\beta\delta \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k u(s)\|^2\Big] ds + 2\beta\rho \sum_{|n|>k} \eta_n |\iota_n| \leq \rho\varepsilon. \tag{3.49}$$

By (3.43), we get for all $k \geq K_5$ and $t \geq \rho$,

$$\begin{aligned}
&2\beta\mathbb{E}\Big[\int_{t-\rho}^{t} \|\vartheta_k a\| \|\vartheta_k u(s)\|_\eta ds\Big] + 2\alpha\mathbb{E}\Big[\int_{t-\rho}^{t} \|\vartheta_k c\| \|\vartheta_k v(s)\|_\eta ds\Big] \\
&\leq \int_{t-\rho}^{t} \mathbb{E}\Big[\beta\|\vartheta_k u(s)\|_\eta^2 + \alpha\|\vartheta_k v(s)\|_\eta^2\Big] ds + \int_{t-\rho}^{t} \mathbb{E}\Big[\beta\|\vartheta_k a\|_\eta^2 + \alpha\|\vartheta_k c\|_\eta^2\Big] ds \\
&\leq \rho\varepsilon + \rho \sum_{|n|\geq k} \eta_n (\beta|a_n|^2 + \alpha|c_n|^2).
\end{aligned} \tag{3.50}$$

Since $a = (a_n)_{n\in\mathbb{Z}}, c = (c_n)_{n\in\mathbb{Z}} \in \ell_\eta^2$, it follows from (3.50) that there exists $K_8 = K_8(\varepsilon, E) \geq K_7$ such that for all $k \geq K_8$ and $t \geq \rho$,

$$2\beta\mathbb{E}\Big[\int_{t-\rho}^{t} \Big(\vartheta_k a, \vartheta_k u(s)\Big)_\eta ds\Big] + 2\alpha\mathbb{E}\Big[\int_{t-\rho}^{t} \Big(\vartheta_k c, \vartheta_k v(s)\Big)_\eta ds\Big] \leq 2\rho\varepsilon. \tag{3.51}$$

By (2.8), (3.43), and Lemma 3.4, we get for all $t \geq \rho$ and $k \geq K_5$,

$$\begin{aligned}
&\beta \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j\|_\eta^2\Big] ds \\
&\leq 2\beta \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k g_j(u(s), u(s-\rho))\|_\eta^2\Big] ds + 2\beta \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k b_j\|_\eta^2\Big] ds \\
&\leq 2\rho\beta \sum_{j=1}^{\infty} \sum_{|n|\geq k} \eta_n (b_{j,n}^2 + 2\gamma_{j,n}^2) + 8\beta\|\gamma\|^2 \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2 + \|\vartheta_k u(s-\rho)\|_\eta^2\Big] ds \\
&\leq 2\rho\beta \sum_{j=1}^{\infty} \sum_{|n|\geq k} \eta_n (b_{j,n}^2 + 2\gamma_{j,n}^2) + 8\beta\|\gamma\|^2 \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2\Big] ds + 8\beta\|\gamma\|^2 \int_{t-2\rho}^{t-\rho} \mathbb{E}\Big[\|\vartheta_k u(s)\|_\eta^2\Big] ds \\
&\leq 2\rho\beta \sum_{j=1}^{\infty} \sum_{|n|\geq k} \eta_n (b_{j,n}^2 + 2\gamma_{j,n}^2) + 16\beta\|\gamma\|^2\rho\varepsilon,
\end{aligned} \tag{3.52}$$

and

$$\alpha \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j\|_{\eta}^2\Big] ds \leq 2\rho\alpha \sum_{j=1}^{\infty} \sum_{|n|\geq k} \eta_n (l_{j,n}^2 + 2\gamma_{j,n}^2) + 16\alpha\|\gamma\|^2 \rho\varepsilon. \tag{3.53}$$

Since $b = (b_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$, $l = (l_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$, and $\gamma = (\gamma_{j,n})_{j\in\mathbb{N},n\in\mathbb{Z}}$ belong to ℓ_η^2, we infer from (3.52) and (3.53) that there exists $K_9 = K_9(\varepsilon, E) \geq K_8$ such that for all $k \geq K_9$ and $t \geq \rho$,

$$\begin{aligned} &\beta \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j\|_{\eta}^2\Big] ds \\ &\quad + \alpha \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}\Big[\|\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j\|_{\eta}^2\Big] ds \\ &\leq \rho(\beta + \alpha)(2 + 16\|\gamma\|^2)\varepsilon. \end{aligned} \tag{3.54}$$

For the last two terms of (3.42), by the BDG inequality, (2.8), and (3.54), we have for all $k \geq K_9$ and $t \geq \rho$,

$$\begin{aligned} &2\beta\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\Big|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\Big(\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j, \vartheta_k u(s)\Big)_{\eta} dW_j(s)\Big|\Big] \\ &\quad + 2\alpha\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\Big|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\Big(\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j, \vartheta_k v(s)\Big)_{\eta} dW_j(s)\Big|\Big] \\ &\leq 2\beta C_9\mathbb{E}\Big[\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\Big|\Big(\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j, \vartheta_k u(s)\Big)_{\eta}\Big|^2 ds\Big)^{\frac{1}{2}}\Big] \\ &\quad + 2\alpha C_9\mathbb{E}\Big[\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\Big|\Big(\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k c_j, \vartheta_k v(s)\Big)_{\eta}\Big|^2 ds\Big)^{\frac{1}{2}}\Big] \\ &\leq \frac{\beta}{2}\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|_{\eta}^2\Big] + 2\beta C_9^2\mathbb{E}\Big[\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\|\vartheta_k g_j(u(s), u(s-\rho)) + \vartheta_k b_j\|_{\eta}^2 ds\Big] \\ &\quad + \frac{\alpha}{2}\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\|\vartheta_k v(r)\|_{\eta}^2\Big] + 2\alpha C_9^2\mathbb{E}\Big[\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\|\vartheta_k h_j(v(s), v(s-\rho)) + \vartheta_k l_j\|_{\eta}^2 ds\Big] \\ &\leq \frac{\beta}{2}\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|_{\eta}^2\Big] + \frac{\alpha}{2}\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\|\vartheta_k v(r)\|_{\eta}^2\Big] + 2C_9^2\rho(\beta + \alpha)(2 + 16\|\gamma\|^2)\varepsilon. \end{aligned} \tag{3.55}$$

By (3.42)–(3.55), we get that for all $t \geq \rho$ and $k \geq K_9$,

$$\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\sum_{|n|\geq 2k}\eta_n(\beta|u_n(r)|^2 + \alpha|v_n(r)|^2)\Big] \leq \mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}(\beta\|\vartheta_k u(r)\|^2 + \alpha\|\vartheta_k v(r)\|^2)\Big] \leq C_{10}\varepsilon,$$

where $C_{10} = 2\Big(1 + 2\tilde{\alpha}\rho + 5\rho + \rho(\beta + \alpha)(2 + 16\|\gamma\|^2)(1 + 2C_9^2)\Big)\varepsilon$. This completes the proof. □

4. Existence of invariant measures

The focus of this section is to establish the existence of invariant measures for lattice system (2.15) in $C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$. Initially, we introduce the transition operators of the lattice system and subsequently demonstrate the tightness of a family of probability distributions for solutions of the lattice system.

Given every $t_0\geq 0$ and $\mathcal{F}_{t_0}$-measurable $(\phi,\varphi)\in L^2(\Omega,C([-\rho,0],\ell^2_\eta\times\ell^2_\eta))$, lattice system (2.15) possesses a distinct solution that is valid for all $t\geq t_0-\rho$. Given $t\geq t_0$ and $(\phi,\varphi)\in L^2(\Omega,C([-\rho,0],\ell^2_\eta\times\ell^2_\eta))$, we use $(u_t(t_0,\phi),v_t(t_0,\phi))$ to represent the segment of the solution $(u(t,t_0,\phi),v(t,t_0,\varphi))$ which is given by

$$(u_t(t_0,\phi),v_t(t_0,\varphi))(s)=(u(s+t,t_0,\phi),v(s+t,t_0,\varphi)),\quad \forall s\in[-\rho,0].$$

Then, we have $(u_t(t_0,\phi),v_t(t_0,\varphi))\in L^2(\Omega,C([-\rho,0],\ell^2_\eta\times\ell^2_\eta))$ for all $t\geq t_0$.

Suppose $\psi: C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)\to\mathbb{R}$ is a bounded Borel function. For $0\leq r\leq t$, we set

$$(p_{r,t}\psi)(\phi,\varphi)=\mathbb{E}[\psi((u_t(r,\phi),v_t(r,\varphi)))],\quad \forall(\phi,\varphi)\in C([-\rho,0],\ell^2_\eta\times\ell^2_\eta).$$

In addition, for $G\in\mathcal{B}(C([-\rho,0],\ell^2_\eta\times\ell^2_\eta))$, $0\leq r\leq t$, and $(\phi,\varphi)\in C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$, we set

$$p(r,\phi,\varphi;t,G)=(p_{r,t}1_G)(\phi,\varphi)=P\{\omega\in\Omega:(u_t(r,\phi),v_t(r,\varphi))\in G\},$$

where 1_G is the characteristic function of G. Then, we can get that $p(r,\phi,\varphi;t,\cdot)$ is the probability distribution of $(u_t(r,\phi),v_t(r,\varphi))$ in $C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$. Furthermore, the transition operator $p_{0,t}$ is denoted as p_t for the sake of convenience.

Definition 4.1. *A probability measure μ on $C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$ is called an invariant measure of lattice system (2.15) if*

$$\int_{C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)}(p_t\psi)(\phi,\varphi)d\mu(\phi,\varphi)=\int_{C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)}\psi(\phi,\varphi)d\mu(\phi,\varphi),\quad \forall t\geq 0.$$

Now, we show the properties of transition operators $\{p_{r,t}\}_{0\leq r\leq t}$ as follows.

Lemma 4.1. *Suppose (2.1)–(2.8) and (2.25) hold. Then, we have*

(i) The family $\{p_{r,t}\}_{0\leq r\leq t}$ is Feller; that is, if $\psi: C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)\to\mathbb{R}$ is bounded and continuous, then $p_{r,t}\psi: C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)\to\mathbb{R}$ is bounded and continuous.

(ii) The family $\{p_{r,t}\}_{0\leq r\leq t}$ is homogeneous; that is,

$$p(r,\phi,\varphi;t,\cdot)=p(0,\phi,\varphi;t-r,\cdot),\forall r\in[0,t],(\phi,\varphi)\in C([-\rho,0],\ell^2_\eta\times\ell^2_\eta).$$

(iii) Given $r\geq 0$ and $(\phi,\varphi)\in C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$, the process $\{(u_t(r,\phi),v_t(r,\varphi))\}_{t\geq r}$ is a $C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$-value Markov process. Consequently, if $\psi: C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)\to\mathbb{R}$ is a bounded Borel function, then for any $0\leq s\leq r\leq t$, P-a.s.

$$(p_{s,t}\psi)(\phi,\varphi)=(p_{s,t}(p_{r,t}\psi))(\phi,\varphi),\forall(\phi,\varphi)\in C([-\rho,0],\ell^2_\eta\times\ell^2_\eta),$$

for all $(\phi,\varphi)\in C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)$ and $G\in\mathcal{B}(C([-\rho,0],\ell^2_\eta\times\ell^2_\eta))$, the Chapman-Kolmogorov equation is valid:

$$p(s,\phi,\varphi;t,G)=\int_{C([-\rho,0],\ell^2_\eta\times\ell^2_\eta)}p(s,\phi,\varphi;r,dy)p(r,y;t,G).$$

Proof. By Lemma 2.1 and the standard arguments as in [49], we can get the Feller property (i)–(iii). □

Lemma 4.2. *Suppose (2.1)–(2.8) and (2.25) hold. Then, the distribution laws of the process* $\{(u_t(0,0), v_t(0,0))\}_{t\geq 0}$ *is tight on* $C([-\rho,0], \ell^2_\eta \times \ell^2_\eta)$.

Proof. For all $t \geq 0$, by Lemma 3.1 and Chebyshev's inequality, we have

$$P\{\|u_t(0)\|_\eta + \|v_t(0)\|_\eta \geq R\} \leq \frac{2}{R^2}\mathbb{E}\Big[\|u_t(0)\|^2_\eta + \|v_t(0)\|^2_\eta\Big] \leq \frac{C_{11}}{R^2} \to 0, \text{ as } R \to \infty.$$

Then, for each $\varepsilon > 0$, there exists a constant $R_1 = R_1(\varepsilon) > 0$ such that

$$P\{\|u_t(0)\|_\eta + \|v_t(0)\|_\eta \geq R_1\} \leq \frac{\varepsilon}{3}, \quad \forall t \geq 0. \tag{4.1}$$

By Lemma 3.3, we get that for all $r, s \in [-\rho, 0]$ and $t \geq \rho$,

$$\begin{aligned} &\mathbb{E}\Big[\|u(t+r) - u(t+s)\|^4_\eta + \|v(t+r) - v(t+s)\|^4_\eta\Big] \\ &\leq C_{12}(1 + |t-s|^2)|r-s|^2 \leq C_{12}(1+\rho^2)|r-s|^2 \end{aligned} \tag{4.2}$$

for some $C_{12} > 0$. Given $\varepsilon > 0$, it follows from the usual technique of diadic division and (4.2) that there exists a constant $R_2 = R_2(\varepsilon) > 0$ such that for all $t \geq 0$,

$$P\Big(\Big\{\sup_{-\rho\leq s<r\leq 0} \frac{\|u_t(r) - u_t(s)\|_\eta + \|v_t(r) - v_t(s)\|_\eta}{|r-s|^{\frac{1}{8}}} \leq R_2\Big\}\Big) > 1 - \frac{1}{3}\varepsilon. \tag{4.3}$$

By Lemma 3.5, we obtain that for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an integer $k_m = k_m(\varepsilon, m) \geq 1$ such that for all $t \geq 0$,

$$\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t} \sum_{|n|\geq k_m} \eta_n(|u_n(r)|^2 + |v_n(r)|^2)\Big] \leq \frac{\varepsilon}{2^{2m+2}}. \tag{4.4}$$

Then, for all $t \geq 0$ and $m \in \mathbb{N}$,

$$P\Big(\bigcup_{m=1}^{\infty}\Big\{\sup_{t-\rho\leq r\leq t} \sum_{|n|\geq k_m} \eta_n(|u_n(r)|^2 + |v_n(r)|^2) \geq \frac{1}{2^m}\Big\}\Big) \leq \sum_{m=1}^{\infty} 2^m \mathbb{E}\Big[\sup_{t-\rho\leq r\leq t} \sum_{|n|\geq k_m} \eta_n(|u_n(r)|^2 + |v_n(r)|^2)\Big] \leq \frac{\varepsilon}{4},$$

which shows that for all $t \geq 0$,

$$P\Big(\Big\{\sup_{t-\rho\leq r\leq t} \sum_{|n|\geq k_m} \eta_n(|u_n(r)|^2 + |v_n(r)|^2) \leq \frac{1}{2^m}, \forall m \in \mathbb{N}\Big\}\Big) > 1 - \frac{1}{3}\varepsilon. \tag{4.5}$$

For $\varepsilon > 0$, set $\mathcal{Z}_\varepsilon = \mathcal{Z}_{1,\varepsilon} \bigcap \mathcal{Z}_{2,\varepsilon} \bigcap \mathcal{Z}_{3,\varepsilon}$, where

$$\mathcal{Z}_{1,\varepsilon} = \{(u, v) \in C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta) : \|u(0)\|_\eta + \|v(0)\|_\eta \leq R_1(\varepsilon)\}, \tag{4.6}$$

$$\mathcal{Z}_{2,\varepsilon} = \Big\{(u, v) \in C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta) : \sup_{-\rho\leq s<r\leq 0} \frac{\|u(r) - u(s)\|_\eta + \|v(r) - v(s)\|_\eta}{|r-s|^{\frac{1}{8}}} \leq R_2(\varepsilon)\Big\}, \tag{4.7}$$

$$\mathcal{Z}_{3,\varepsilon} = \Big\{(u, v) \in C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta) : \sup_{-\rho\leq r\leq 0} \sum_{|n|\geq k_m} \eta_n(|u_n(r)|^2 + |v_n(r)|^2) \leq \frac{1}{2^m}, \forall m \in \mathbb{N}\Big\}. \tag{4.8}$$

It follows from (4.1), (4.3), and (4.5)–(4.8) that for all $t \geq 0$,

$$P(\{(u_t, v_t) \in \mathcal{Z}_\varepsilon\}) > 1 - \varepsilon. \tag{4.9}$$

By Arzela-Ascoli theorem, we can establish the pre-compactness of $\mathcal{Z}_\varepsilon$ in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. Specifically, by (4.7), we get that $\mathcal{Z}_\varepsilon$ is equi-continuous in $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. On the other hand, by (4.6) and (4.7), we have for every $r \in [-\rho, 0]$,

$$\begin{aligned}\|u(r)\|_\eta + \|v(r)\|_\eta &\leq \|u(r) - u(0)\|_\eta + \|u(0)\|_\eta + \|v(r) - v(0)\|_\eta + \|v(0)\|_\eta \\ &\leq R_2(\varepsilon)|r|^{\frac{1}{8}} + R_1(\varepsilon) \leq \rho^{\frac{1}{8}} R_2(\varepsilon) + R_1(\varepsilon),\end{aligned}$$

which along with (4.8) shows that $\{(u(r), v(r)), (u, v) \in \mathcal{Z}_\varepsilon\}$ is pre-compact in $\ell^2_\eta \times \ell^2_\eta$. This completes the proof. □

Now, the main outcome of this paper has been showed: The existence of invariant measures for lattice system (2.15) on $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$.

Theorem 4.1. *Suppose (2.1)–(2.8) and (2.25) hold. Then, lattice system (2.15) has an invariant measure on* $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$.

Proof. By using Krylov-Bogolyubov's method, for each $n \in \mathbb{N}$, the probability measure μ_n is defined by

$$\mu_n = \frac{1}{n} \int_0^n p(0, 0; t, \cdot) dt. \tag{4.10}$$

It follows from Lemma 4.2 that the sequence $(\mu_n)_{n=1}^\infty$ is tight on $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$. Consequently, there exists a probability measure μ on $C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta)$ and a subsequence (still denoted by $(\mu_n)_{n=1}^\infty$) such that

$$\mu_n \to \mu, \ as\ n \to \infty. \tag{4.11}$$

By (4.10)–(4.11) and Lemma 4.1, we can get for every $t \geq 0$ and every bounded and continuous function $\psi : C([-\rho, 0], \ell^2_\eta \times \ell^2_\eta) \to \mathbb{R}$,

$$\begin{aligned}&\int_{C([-\rho,0],\ell^2_\eta \times \ell^2_\eta)} \psi(y) d\mu(y) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_0^n \Big(\int_{C([-\rho,0],\ell^2_\eta \times \ell^2_\eta)} \psi(y) p(0, 0; s, dy) \Big) ds \\ &= \lim_{n\to\infty} \frac{1}{n} \int_{-t}^{n-t} \Big(\int_{C([-\rho,0],\ell^2_\eta \times \ell^2_\eta)} \psi(y) p(0, 0; s + t, dy) \Big) ds \\ &= \lim_{n\to\infty} \frac{1}{n} \int_0^n \Big(\int_{C([-\rho,0],\ell^2_\eta \times \ell^2_\eta)} \psi(y) p(0, 0; s + t, dy) \Big) ds \\ &= \lim_{n\to\infty} \frac{1}{n} \int_0^n \Big(\int_{C([-\rho,0],\ell^2_\eta \times \ell^2_\eta)} \Big(\int_{C([-\rho,0],\ell^2_\eta \times \ell^2_\eta)} \psi(y) p(s, \phi, \varphi; s + t, dy) \Big) p(0, 0; s, d(\phi, \varphi)) \Big) ds\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{n} \int_0^n \Big(\int_{C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)} \Big(\int_{C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)} \psi(y) p(0,\phi,\varphi;t,dy) \Big) p(0,0;s,d(\phi,\varphi)) \Big) ds$$
$$= \int_{C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)} \Big(\int_{C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)} \psi(y) p(0,\phi,\varphi;t,dy) \Big) d\mu(\phi,\varphi)$$
$$= \int_{C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)} (p_{0,t}\psi)(\phi,\varphi) d\mu(\phi,\varphi),$$

which implies that μ is an invariant measure of lattice system (2.15). This completes the proof. □

5. Uniqueness of invariant measures

In this section, we examine the uniqueness of invariant measures for system (2.15) under additional constraints on the diffusion and drift terms. Specifically, we impose the following assumption:

$$2L^2 < \max\{\lambda, -\tilde{\alpha} - \kappa\}, \tag{5.1}$$

which implies that there exists a small number $\varsigma > 0$ such that

$$\max\{4L^2 + 2\tilde{\alpha} + 2\kappa + \varsigma, 4L^2 - 2\lambda + \varsigma\} \le 0. \tag{5.2}$$

From now on, we fix such a $\varsigma > 0$ satisfying (5.2). We will demonstrate that, subject to condition (5.2), any two solutions of Eq (2.15) converge toward each other at an exponential rate, which immediately implies the uniqueness of invariant measures. To begin with, we establish uniform estimates in $C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)$.

Lemma 5.1. *Suppose (2.1)–(2.8) and (5.1) hold, and* $(\phi_1,\varphi_1),(\phi_2,\varphi_2) \in L^2(\Omega, C([-\rho,0],\ell_\eta^2\times\ell_\eta^2))$. *If* $(u(t,\phi_1),v(t,\varphi_1))$ *and* $(u(t,\phi_2),v(t,\varphi_2))$ *are the solutions of system (2.15) with initial data* (ϕ_1,φ_1) *and* (ϕ_2,φ_2)*, respectively, then for any* $t \ge -\rho$,

$$\mathbb{E}[\|u(t,\phi_1) - u(t,\phi_2)\|_\eta^2 + \|v(t,\varphi_1) - v(t,\varphi_2)\|_\eta^2] \le M_5 \mathbb{E}[\|\phi_1 - \phi_2\|_{C_{\rho,\eta}}^2 + \|\varphi_1 - \varphi_2\|_{C_{\rho,\eta}}^2] e^{-\varsigma t},$$

where M_5 *is a positive constant depending on* (ϕ,φ).

Proof. By (2.17), we get that for $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}[\beta\|u(t,\phi_1) - u(t,\phi_2)\|_\eta^2 + \alpha\|v(t,\varphi_1) - v(t,\varphi_2)\|_\eta^2] \\
&\le \mathbb{E}[\beta\|\phi_1(0) - \phi_2(0)\|_\eta^2 + \alpha\|\varphi_1(0) - \varphi_2(0)\|_\eta^2] - 2\lambda\alpha \int_0^t \mathbb{E}\Big[\|v(s,\varphi_1) - v(s,\varphi_2)\|_\eta^2\Big] ds \\
&\quad + 2\beta \int_0^t \mathbb{E}\Big[\Big(A(u(s,\phi_1) - u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta\Big] ds \\
&\quad + 2\beta \int_0^t \mathbb{E}\Big[\Big(f(u(s,\phi_1)) - f(u(s,\phi_2)), u(s,\phi_1) - u(s,\phi_2)\Big)_\eta\Big] ds \\
&\quad + \beta \sum_{j=1}^\infty \int_0^t \mathbb{E}[\|g_j(u(s,\phi_1), u(s-\rho,\phi_1)) - g_j(u(s,\phi_2), u(s-\rho,\phi_2))\|_\eta^2] ds \\
&\quad + \alpha \sum_{j=1}^\infty \int_0^t \mathbb{E}[\|h_j(v(s,\varphi_1), v(s-\rho,\varphi_1)) - h_j(v(s,\varphi_2), v(s-\rho,\varphi_2))\|_\eta^2] ds.
\end{aligned} \tag{5.3}$$

Similar to (2.18) and (2.19), we obtain

$$2\beta\int_0^t \mathbb{E}\Big[\Big(A(u(s,\phi_1)-u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta\Big]ds \le 2\beta\tilde{\alpha}\int_0^t \mathbb{E}[\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2]ds. \quad (5.4)$$

By (2.9), we have

$$2\beta\int_0^t \mathbb{E}\Big[\Big(f(u(s,\phi_1))-f(u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta\Big]ds \le 2\beta\kappa\int_0^t \mathbb{E}[\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2]ds. \quad (5.5)$$

By (2.12), we get

$$\begin{aligned}
&\beta\sum_{j=1}^{\infty}\int_0^t [\|g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2))\|_\eta^2]ds\\
&\quad+\alpha\sum_{j=1}^{\infty}\int_0^t [\|h_j(v(s,\varphi_1),v(s-\rho,\varphi_1))-h_j(v(s,\varphi_2),v(s-\rho,\varphi_2))\|_\eta^2]ds\\
&\le 4\beta L^2\int_0^t [\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2]ds+2\beta L^2\int_{-\rho}^0 [\|\phi_1(s)-\phi_2(s)\|_\eta^2]ds\\
&\quad+4\alpha L^2\int_0^t [\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2]ds+2\alpha L^2\int_{-\rho}^0 [\|\varphi_1(s)-\varphi_2(s)\|_\eta^2]ds.
\end{aligned} \quad (5.6)$$

It follows from (5.2)–(5.6) that for all $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}[\beta\|u(t,\phi_1)-u(t,\phi_2)\|_\eta^2+\alpha\|v(t,\varphi_1)-v(t,\varphi_2)\|_\eta^2]\\
&\le (1+2\rho L^2)\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]\\
&\quad-\varsigma\int_0^t \mathbb{E}[\beta\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2+\alpha\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2]ds,
\end{aligned}$$

which implies that for all $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}[\beta\|u(t,\phi_1)-u(t,\phi_2)\|_\eta^2+\alpha\|v(t,\varphi_1)-v(t,\varphi_2)\|_\eta^2]\\
&\le (1+2\rho L^2)\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma t}.
\end{aligned} \quad (5.7)$$

On the other hand, for $t \in [-\rho, 0]$, we have

$$\begin{aligned}
&\mathbb{E}[\beta\|u(t,\phi_1)-u(t,\phi_2)\|_\eta^2+\alpha\|v(t,\varphi_1)-v(t,\varphi_2)\|_\eta^2]\\
&=\mathbb{E}[\beta\|\phi_1(t)-\phi_2(t)\|_\eta^2+\alpha\|\varphi_1(t)-\varphi_2(t)\|_\eta^2]\\
&\le \mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma t},
\end{aligned}$$

which along with (5.7) concludes the proof. □

Lemma 5.2. *Suppose (2.1)–(2.8) and (5.1) hold, and $(\phi_1,\varphi_1),(\phi_2,\varphi_2) \in L^2(\Omega, C([-\rho,0],\ell_\eta^2\times\ell_\eta^2))$. If $(u(t,\phi_1),v(t,\varphi_1))$ and $(u(t,\phi_2),v(t,\varphi_2))$ are the solutions of system (2.15) with initial data (ϕ_1,φ_1) and (ϕ_2,φ_2), respectively, then for any $t \ge \rho$,*

$$\mathbb{E}\Big[\sup_{t-\rho\le r\le t}(\|u(r,\phi_1)-u(r,\phi_2)\|^2+\|v(r,\varphi_1)-u(r,\varphi_2)\|^2)\Big] \le M_6\mathbb{E}[\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma t},$$

where M_6 is a positive constant independent of (ϕ_1,φ_1) and (ϕ_2,φ_2).

Proof. By (2.17), we get that for $t \ge \rho$ and $r \ge t-\rho$,

$$\begin{aligned}
&\beta\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2+2\lambda\alpha\int_{t-\rho}^r\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2ds\\
&=\beta\|u(t-\rho,\phi_1)-u(t-\rho,\phi_2)\|_\eta^2+\alpha\|v(t-\rho,\varphi_1)-v(t-\rho,\varphi_2)\|_\eta^2\\
&\quad+2\beta\int_{t-\rho}^r\Big(A(u(s,\phi_1)-u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta ds\\
&\quad+2\beta\int_{t-\rho}^r\Big(f(u(s,\phi_1))-f(u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta ds\\
&\quad+\beta\sum_{j=1}^\infty\int_{t-\rho}^r\|g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2))\|_\eta^2ds\\
&\quad+\alpha\sum_{j=1}^\infty\int_{t-\rho}^r\|h_j(v(s,\varphi_1),v(s-\rho,\varphi_1))-h_j(v(s,\varphi_2),v(s-\rho,\varphi_2))\|_\eta^2ds\\
&\quad+2\beta\sum_{j=1}^\infty\int_{t-\rho}^r\Big(\mathbf{g}_j,u(s,\phi_1)-u(s,\phi_2)\Big)_\eta dW_j(s)\\
&\quad+2\alpha\sum_{j=1}^\infty\int_{t-\rho}^r\Big(\mathbf{h}_j,v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta dW_j(s),
\end{aligned}\tag{5.8}$$

where

$$\mathbf{g}_j=g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2)),$$

and

$$\mathbf{h}_j=h_j(v(s,\varphi_1),v(s-\rho,\varphi_1))-h_j(v(s,\varphi_2),v(s-\rho,\varphi_2)).$$

By (5.8), we get that for all $t\ge\rho$,

$$\begin{aligned}
&\mathbb{E}[\beta\sup_{t-\rho\le r\le t}\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\sup_{t-\rho\le r\le t}\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2]\\
&\le\mathbb{E}[\beta\|u(t-\rho,\phi_1)-u(t-\rho,\phi_2)\|_\eta^2+\alpha\|v(t-\rho,\varphi_1)-v(t-\rho,\varphi_2)\|_\eta^2]\\
&\quad+2\beta\mathbb{E}\Big[\int_{t-\rho}^t\Big|\Big(A(u(s,\phi_1)-u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta\Big|ds\Big]\\
&\quad+2\beta\mathbb{E}\Big[\sup_{t-\rho\le r\le t}\int_{t-\rho}^r\Big(f(u(s,\phi_1))-f(u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta ds\Big]\\
&\quad+\beta\mathbb{E}\Big[\sum_{j=1}^\infty\int_{t-\rho}^t\|g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2))\|_\eta^2ds\Big]\\
&\quad+\alpha\mathbb{E}\Big[\sum_{j=1}^\infty\int_{t-\rho}^t\|h_j(v(s,\varphi_1),v(s-\rho,\varphi_1))-h_j(v(s,\varphi_2),v(s-\rho,\varphi_2))\|_\eta^2ds\Big]\\
&\quad+2\beta\mathbb{E}\Big[\sup_{t-\rho\le r\le t}\Big|\sum_{j=1}^\infty\int_{t-\rho}^r\Big(\mathbf{g}_j,u(s,\phi_1)-u(s,\phi_2)\Big)_\eta dW_j(s)\Big|\Big]\\
&\quad+2\alpha\mathbb{E}\Big[\sup_{t-\rho\le r\le t}\Big|\sum_{j=1}^\infty\int_{t-\rho}^r\Big(\mathbf{h}_j,v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta dW_j(s)\Big|\Big].
\end{aligned}\tag{5.9}$$

By Lemma 5.1, we see that for all $t \geq \rho$,

$$\begin{aligned}&\mathbb{E}[\beta\|u(t-\rho,\phi_1)-u(t-\rho,\phi_2)\|_\eta^2+\alpha\|v(t-\rho,\varphi_1)-v(t-\rho,\varphi_2)\|_\eta^2]\\&\leq(1+2\rho L^2)\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma(t-\rho)}.\end{aligned}\tag{5.10}$$

Similar to (2.18) and (2.19), we obtain

$$2\beta\mathbb{E}\Big[\int_{t-\rho}^{t}\Big|\Big(A(u(s,\phi_1)-u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta\Big|ds\Big]\leq 2\beta\tilde{\alpha}\int_{t-\rho}^{t}\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2ds,$$

which along with Lemma 5.1 implies that

$$\begin{aligned}&2\beta\mathbb{E}\Big[\int_{t-\rho}^{t}\Big|\Big(A(u(s,\phi_1)-u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta\Big|ds\Big]\\&\leq\frac{2\tilde{\alpha}(1+2\rho L^2)}{\varsigma}\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma(t-\rho)}.\end{aligned}\tag{5.11}$$

By (2.9) and (5.1), we have

$$2\beta\mathbb{E}\Big[\sup_{t-\rho\leq r\leq t}\int_{t-\rho}^{r}\Big(f(u(s,\phi_1))-f(u(s,\phi_2)),u(s,\phi_1)-u(s,\phi_2)\Big)_\eta ds\Big]\leq 0.\tag{5.12}$$

By (2.12) and Lemma 5.1 we get

$$\begin{aligned}&\beta\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\|g_j(u(s,\phi_1),u(s-\rho,\phi_1))-g_j(u(s,\phi_2),u(s-\rho,\phi_2))\|_\eta^2\Big]ds\\&\quad+\alpha\mathbb{E}\Big[\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\|h_j(v(s,\varphi_1),v(s-\rho,\varphi_1))-h_j(v(s,\varphi_2),v(s-\rho,\varphi_2))\|_\eta^2\Big]ds\\&\leq 4\beta L^2\int_{t-\rho}^{t}\mathbb{E}\Big[\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2\Big]ds+2\beta L^2\int_{t-2\rho}^{t-\rho}\mathbb{E}\Big[\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2\Big]ds\\&\quad+4\alpha L^2\int_{t-\rho}^{t}\mathbb{E}\Big[\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2\Big]ds+2\alpha L^2\int_{t-2\rho}^{t-\rho}\mathbb{E}\Big[\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2\Big]ds\\&\leq\frac{4L^2(1+2\rho L^2)}{\varsigma}\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma(t-\rho)}\\&\quad+\frac{2L^2(1+2\rho L^2)}{\varsigma}\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma(t-2\rho)}\\&\leq\frac{6L^2(1+2\rho L^2)}{\varsigma}\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma(t-\rho)}.\end{aligned}\tag{5.13}$$

For the last two terms of (5.9), by the BDG inequality and (5.13), we get

$$
\begin{aligned}
&2\beta\mathbb{E}\Big[\sup_{t-\rho\le r\le t}\Big|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\Big(\mathbf{g}_j, u(s,\phi_1)-u(s,\phi_2)\Big)_\eta dW_j(s)\Big|\Big]\\
&\quad+2\alpha\mathbb{E}\Big[\sup_{t-\rho\le r\le t}\Big|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\Big(\mathbf{h}_j, v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta dW_j(s)\Big|\Big]\\
&\le C_{13}\beta\mathbb{E}\Big[\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\Big|\Big(\mathbf{g}_j, u(s,\phi_1)-u(s,\phi_2)\Big)_\eta\Big|^2 ds\Big)^{\frac12}\Big]\\
&\quad+C_{13}\alpha\mathbb{E}\Big[\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\Big|\Big(\mathbf{h}_j, v(s,\varphi_1)-v(s,\varphi_2)\Big)_\eta\Big|^2 ds\Big)^{\frac12}\Big]\\
&\le C_{13}\beta\mathbb{E}\Big[\sup_{t-\rho\le s\le t}\|u(s,\phi_1)-u(s,\phi_2)\|_\eta\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\|\mathbf{g}_j\|_\eta^2 ds\Big)^{\frac12}\Big]\\
&\quad+C_{13}\alpha\mathbb{E}\Big[\sup_{t-\rho\le s\le t}\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\|\mathbf{h}_j\|_\eta^2 ds\Big)^{\frac12}\Big]\\
&\le \frac{\beta}{2}\mathbb{E}\Big[\sup_{t-\rho\le s\le t}\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2\Big]+\frac{\beta}{2}C_{13}^2\mathbb{E}\Big[\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\|\mathbf{g}_j\|_\eta^2 ds\Big]\\
&\quad+\frac{\alpha}{2}\mathbb{E}\Big[\sup_{t-\rho\le s\le t}\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2+\frac{\alpha}{2}C_{13}^2\mathbb{E}\Big[\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\|\mathbf{h}_j\|_\eta^2 ds\Big]\\
&\le \frac{1}{2}\mathbb{E}\Big[\sup_{t-\rho\le s\le t}(\beta\|u(s,\phi_1)-u(s,\phi_2)\|_\eta^2+\alpha\|v(s,\varphi_1)-v(s,\varphi_2)\|_\eta^2)\Big]\\
&\quad+C_{14}\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma(t-\rho)},
\end{aligned}
\tag{5.14}
$$

where $C_{14}=\frac{3C_{13}^2L^2(1+2\rho L^2)}{\varsigma}$. It follows from (5.9)–(5.14) that for all $t\ge 0$,

$$
\mathbb{E}\Big[\sup_{t-\rho\le r\le t}(\beta\|u(r,\phi_1)-u(r,\phi_2)\|_\eta^2+\alpha\|v(r,\varphi_1)-v(r,\varphi_2)\|_\eta^2)\Big]\le C_{15}\mathbb{E}[\beta\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\alpha\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma t},
$$

where $C_{15}=2((1+2\rho L^2)(1+\frac{2\tilde{\alpha}+6L^2}{\varsigma})+C_{14})$. This completes the proof. □

Theorem 5.1. *Suppose (2.1)–(2.8) and (5.1) hold. Then, stochastic lattice system (2.15) has a unique invariant measure in* $C([-\rho,0],\ell_\eta^2\times\ell_\eta^2)$.

Proof. For any $(\phi_1,\varphi_1),(\phi_2,\varphi_2)\in L^2(\Omega, C([-\rho,0],\ell_\eta^2\times\ell_\eta^2))$, by Lemma 5.2, we see that the segments of the solutions $(u_t(\phi_1),v_t(\varphi_1))$ and $(u_t(\phi_2),v_t(\varphi_2))$ of (2.15) satisfy, for all $t\ge\rho$,

$$
\mathbb{E}[\|u_t(\phi_1)-u_t(\phi_2)\|_{C_{\rho,\eta}}^2+\|v_t(\varphi_1)-v_t(\varphi_2)\|_{C_{\rho,\eta}}^2]\le M_7\mathbb{E}[\|\phi_1-\phi_2\|_{C_{\rho,\eta}}^2+\|\varphi_1-\varphi_2\|_{C_{\rho,\eta}}^2]e^{-\varsigma t},
$$

which along with the standard arguments (see, e.g., [50]) implies the uniqueness of invariant measures for the lattice system (2.15). This completes the proof. □

6. Conclusions

The current focus lies in the theoretical proof of the well-posedness of solutions and the existence and uniqueness of invariant measures for these stochastic delay lattice systems. In the future, our research group will investigate the convergence and approximation of invariant measures for the systems under noise perturbation, as well as explore large deviation principles for the systems. Additionally, we will employ finite-dimensional numerical approximation methods to address both the existence of numerical solutions and numerical invariant measures.

Author contributions

Xintao Li and Lianbing She: Conceptualization, Writing original draft and writing-review and editing; Rongrui Lin: Writing original draft and writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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