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Research article

Family of right conoid hypersurfaces with light-like axis in Minkowski four-space

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Abstract: In the realm of the four-dimensional Minkowski space $\mathbb{L}^4$, the focus is on hypersurfaces classified as right conoids and defined by light-like axes. Matrices associated with the fundamental form, Gauss map, and shape operator, all specifically tailored for these hypersurfaces, are currently undergoing computation. The intrinsic curvatures of these hypersurfaces are determined using the Cayley-Hamilton theorem. The conditions of minimality are addressed by the analysis. The Laplace-Beltrami operator for such hypersurfaces is computed, accompanied by illustrative examples aimed at fostering a more profound understanding of the involved mathematical principles. Additionally, scrutiny is applied to the umbilical condition, and the introduction of the Willmore functional for these hypersurfaces is presented.

Keywords: Minkowski four-space; right conoid hypersurface family; light-like axis; Gauss map; Willmore functional
Mathematics Subject Classification: 53A35, 53C42

1. Introduction

The given parametric equation, represented as

$$\mathfrak{c}(u, v) = \mathfrak{a}(v) + u\mathfrak{b}(v)$$

can be alternatively expressed as

$$\mathfrak{c}(u,v) = \left(-fg, \left(1-\frac{1}{2}f^2\right)g, -\frac{1}{2}f^2 g\right) + u\left(f, \frac{1}{2}f^2, 1+\frac{1}{2}f^2\right),$$

where $u \neq 0$, $f = f(v)$, $g = g(v)$, and $\langle \mathfrak{b}, \mathfrak{b} \rangle = -1$. This equation defines an intriguing ruled surface within the expansive three-dimensional Minkowski space $\mathbb{L}^3$. The particular surface, characterized by its distinct geometric properties, is precisely recognized as a *right conoid*. It is distinguished by a light-like axis, defined by the vector $(0,1,1)$, and adheres to the metric signatures $(+,+,-)$ intrinsic to $\mathbb{L}^3$.

To untangle the complexities inherent in this parametric representation, we deconstruct its elements. The curve denoted by $\mathfrak{a}(v)$ serves as a reference axis or curve in the $(0,1,1)$-direction. Concurrently, the vector function $\mathfrak{b}(v)$ delineates the generating vector that shapes the ruled surface. The parameter u dynamically dictates the position along this generating vector, while the parameter v effectively parametrizes the curve $\mathfrak{a}(v)$, playing a crucial role in determining the overall configuration of the ruled surface.

This parametric expression not only captures the essential geometric intricacies inherent in the right conoid, but also serves as a mathematical gateway for understanding its behavior within the confines of three-dimensional Minkowski space. The distinct interplay among the reference curve, generating vector, and parametric controls establishes a framework that facilitates the exploration of the geometric complexities and properties displayed by this ruled surface in the Minkowski space setting.

For a more in-depth understanding, one may refer to the research conducted by Berger and Gostiaux [1] in the context of three-dimensional Euclidean space $\mathbb{E}^3$.

This research effort is dedicated to thoroughly exploring the inherent characteristics displayed by hypersurfaces falling into the category of right conoids with a light-like axis (RCH-L) within the extensive domain of four-dimensional Minkowski space, denoted as $\mathbb{L}^4$. Our primary focus involves the careful computation of matrices associated with the fundamental form, Gauss map, and shape operator that are intrinsic to these hypersurfaces. By utilizing the strong framework provided by the Cayley-Hamilton theorem, our overarching goal is to discern and quantify the intrinsic curvatures of these specific hypersurfaces.

In addition to our exploration of curvature properties, a fundamental component of our research agenda involves establishing conditions that govern minimality within the specific geometric context under consideration. This requires a nuanced examination of factors that contribute to minimizing particular geometric properties, thereby deepening our understanding of how RCH-L behaves within the extensive domain of four-dimensional Minkowski space.

A critical dimension of our inquiry centers around revealing the intricate relationship that RCH-L shares with the Laplace-Beltrami ($\mathcal{L}$-$\mathcal{B}$) operator within the expansive landscape of $\mathbb{L}^4$. By delving into this connection, our objective is to offer insights into the intrinsic geometric properties of these hypersurfaces, further enriching our comprehension of their behavior within the framework of Minkowski space.

Moreover, we extend our analysis to incorporate the presentation of the umbilical condition, shedding light on specific geometric characteristics that RCH-L may exhibit. To provide a comprehensive overview of the geometric properties, we introduce the Willmore functional for the RCH-L, allowing for a quantitative assessment of their shape characteristics.

In Section 2, an in-depth exploration is undertaken to elucidate the fundamental principles and concepts that form the basis of four-dimensional Minkowski geometry. Section 3 is specifically dedicated to providing curvature formulas that are applicable to hypersurfaces in $\mathbb{L}^4$. Moving forward to Section 4, a comprehensive definition of hypersurfaces classified as right conoids with a light-like axis is presented. This section emphasizes their distinctive properties and characteristics. Section 5 shifts the focus back to the discussion of the $\mathcal{L}$-$\mathcal{B}$ operator for a smooth function in $\mathbb{L}^4$ and explores the utilization of the previously examined hypersurfaces in its computation.

The exploration of umbilical right conoid hypersurfaces with a light-like axis in $\mathbb{L}^4$ is undertaken in Section 6. The presentation of the Willmore functional for right conoid hypersurfaces with a light-like axis in $\mathbb{L}^4$ is provided in Section 7. Finally, the study concludes in the last section.

2. Preliminaries

In Minkowski (n+1)-space $\mathbb{L}^{n+1}$, we define s_j as $\sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j denotes the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} a_{i_1} a_{i_2} \ldots a_{i_j},$$

and use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. The function s_k is referred to as the k-th mean curvature of M. It is noteworthy that the functions $H = \frac{1}{n} s_1$ and $K = s_n$ are denoted as the mean curvature and the Gauss-Kronecker curvature of M, respectively. Particularly, M is termed j-minimal if $s_j \equiv 0$. See also Chen et al. [2] for details.

Within the intricate domain of Minkowski 4-space $\mathbb{L}^4$, we contemplate an oriented hypersurface denoted as M. In the exploration of various geometric measures associated with this hypersurface, we employ the shape operator, denoted by $\mathcal{S} = \left(\mathfrak{s}_{ij}\right)_{3\times3}$.

Consider $r_i^j = \sigma_j(k_1, k_2, k_3)$ with the specified definition, resulting in $r_i^0 = 1$. The function s_k is recognized as the k-th mean curvature of the oriented hypersurface M. Specifically, the mean curvature H is articulated as $H = \frac{1}{3} s_1$. Moreover, the Gauss-Kronecker ($\mathcal{G}$–$\mathcal{K}$) curvature of M is denoted by $K = s_3$.

Let us investigate the concept of j-minimality concerning the hypersurface M. If $s_j \equiv 0$, we designate this hypersurface as j-minimal. This term indicates a distinctive geometric characteristic where the j-th mean curvature consistently equals zero.

The notations and definitions introduced facilitate a thorough examination of geometric measures, mean curvatures, and the concept of j-minimality for the oriented hypersurface M within the intricate and rich context of $\mathbb{L}^4$.

In the realm of Minkowski 4-space, we derive curvature formulas denoted as $\mathbb{K}_i$, where $i = 0, 1, 2, 3$. Refer to Chen et al. [2], Güler [3,4], Li and Güler [7–9], and O'Neill [10] for detailed explanations.

The characteristic polynomial, denoted as $P_{\mathcal{S}}(\lambda) = \sum_{k=0}^{3} (-1)^k s_k \lambda^{3-k}$, of $\mathcal{S}$ is determined by

$$\det(\mathcal{S} - \lambda \mathcal{I}_3) = 0. \tag{2.1}$$

Here, $\mathcal{I}_3$ represents the identity matrix. Consequently, the curvature formulas are uncovered as $\binom{3}{i}\mathbb{K}_i = s_i$.

In this investigation, a vector is regarded as equivalent to its transpose. We scrutinize an immersion $\mathfrak{x} = \mathfrak{x}(u, v, w)$ mapping from $M^3 \subset \mathbb{E}^3$ to $\mathbb{L}^4$.

Definition 2.1. *Consider two vectors $l^1 = (l_1^1, l_2^1, l_3^1, l_4^1)$ and $l^2 = (l_1^2, l_2^2, l_3^2, l_4^2)$ in $\mathbb{L}^4$. The inner product between them is given by the expression*

$$\left\langle l^1, l^2 \right\rangle = l_1^1 l_1^2 + l_2^1 l_2^2 + l_3^1 l_3^2 - l_4^1 l_4^2.$$

Definition 2.2. *In $\mathbb{L}^4$, the triple vector product for three vectors $l^1 = (l_1^1, l_2^1, l_3^1, l_4^1)$, $l^2 = (l_1^2, l_2^2, l_3^2, l_4^2)$, and $l^3 = (l_1^3, l_2^3, l_3^3, l_4^3)$ is defined by the determinant*

$$l^1 \times l^2 \times l^3 = \begin{vmatrix} \mathfrak{l}_1 & \mathfrak{l}_2 & \mathfrak{l}_3 & -\mathfrak{l}_4 \\ l_1^1 & l_2^1 & l_3^1 & l_4^1 \\ l_1^2 & l_2^2 & l_3^2 & l_4^2 \\ l_1^3 & l_2^3 & l_3^3 & l_4^3 \end{vmatrix},$$

where the base elements within $\mathbb{L}^4$ are represented by $\mathfrak{l}_k$.

Definition 2.3. *In Minkowski 4-space $\mathbb{L}^4$, let $\mathcal{S}$ be the shape operator matrix associated with the hypersurface $\mathfrak{x}$. This matrix is determined by the product of $\left(\mathfrak{g}_{ij}\right)^{-1} \cdot \left(\mathfrak{h}_{ij}\right)$, where $\left(\mathfrak{g}_{ij}\right)_{3\times3}$ and $\left(\mathfrak{h}_{ij}\right)_{3\times3}$ denote the first and second fundamental form matrices, respectively.*

The matrix components are defined as $\mathfrak{g}_{ij} = \left\langle \mathfrak{x}_i, \mathfrak{x}_j \right\rangle$ and $\mathfrak{h}_{ij} = \left\langle \mathfrak{x}_{ij}, \mathbb{G} \right\rangle$, for i, j = 1, 2, 3. The Gauss map of $\mathfrak{x}$ is derived from the expression

$$\mathbb{G} = \frac{\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_w}{\|\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_w\|}. \tag{2.2}$$

3. Curvatures in $\mathbb{L}^4$

In the realm of $\mathbb{L}^4$, in this section we unveil the curvature formulas pertaining to a hypersurface parametrized by the function $\phi = \phi(u, v, w)$.

Proposition 3.1. *Within $\mathbb{L}^4$, the subsequent curvature formulas are linked to a hypersurface $\phi = \phi(u, v, w)$*

$$\mathbb{K}_0 = 1,\ 3\mathbb{K}_1 = -\frac{\mathfrak{m}_2}{\mathfrak{m}_3},\ 3\mathbb{K}_2 = \frac{\mathfrak{m}_1}{\mathfrak{m}_3},\ \mathbb{K}_3 = -\frac{\mathfrak{m}_0}{\mathfrak{m}_3}, \tag{3.1}$$

where the polynomial equation $\mathfrak{m}_3\lambda^3 + \mathfrak{m}_2\lambda^2 + \mathfrak{m}_1\lambda + \mathfrak{m}_0 = 0$ represents the characteristic polynomial $P_{\mathcal{S}}(\lambda) = 0$ of the shape operator matrix $\mathcal{S}$. $\mathfrak{m}_3 = \det\left(\mathfrak{g}_{ij}\right)$ and $\mathfrak{m}_0 = \det\left(\mathfrak{h}_{ij}\right)$, $\left(\mathfrak{g}_{ij}\right)$, $\left(\mathfrak{h}_{ij}\right)$ denote the first and second fundamental form matrices, respectively.

Proof. The proof revolves around the characteristic polynomial equation of $\mathcal{S}$ in $\mathbb{L}^4$, detailing the curvatures as follows:

$$\begin{aligned} \mathbb{K}_0 &= 1, \\ 3\mathbb{K}_1 &= k_1 + k_2 + k_3 = -\frac{\mathfrak{m}_2}{\mathfrak{m}_3}, \end{aligned}$$

$$
\begin{aligned}
3\mathbb{K}_2 &= k_1k_2 + k_1k_3 + k_2k_3 = \frac{\mathfrak{m}_1}{\mathfrak{m}_3},\\
\mathbb{K}_3 &= k_1k_2k_3 = -\frac{\mathfrak{m}_0}{\mathfrak{m}_3}.
\end{aligned}
$$

□

Definition 3.2. *In the context of* $\mathbb{L}^4$*, a hypersurface* ϕ *is classified as j-minimal if* $\mathbb{K}_j = 0$ *holds true for* $j = 1, 2, 3$.

4. RCH-L in $\mathbb{L}^4$

In this section, we elucidate the characteristics of the conoid hypersurface with a light-like axis within Minkowski 4-space $\mathbb{L}^4$. Subsequently, we explore its geometric properties.

In $\mathbb{L}^4$, our focus is on a ruled hypersurface defined by the expression:

$$
\begin{aligned}
\phi(u,v,w) &= \alpha(v,w) + u\beta(v,w)\\
&= \left(-f\Phi, -g\Phi, \left(1-\frac{1}{2}\left(f^2+g^2\right)\right)\Phi, -\frac{1}{2}\left(f^2+g^2\right)\Phi\right)\\
&\quad +u\left(f, g, \frac{1}{2}\left(f^2+g^2\right), 1+\frac{1}{2}\left(f^2+g^2\right)\right),
\end{aligned}
$$

where α,β represent surfaces, $u \in \mathbb{R}-\{0\}$, $\langle\beta,\beta\rangle = -1$, $f = f(v)$, $g = g(w)$, $\Phi = \Phi(v,w)$ are differentiable functions, and $0 \leq f, g < 2\pi$. The subsequent definition characterizes the hypersurface $\phi = \mathbf{O}\cdot\Gamma^T$, where the generating hypersurface $\Gamma = (0, 0, \Phi, u)$ rotates about the light-like axis $\ell = (0, 0, 1, 1)$ through

$$
\mathbf{O}(v,w) = \begin{pmatrix}
1 & 0 & -f & f\\
0 & 1 & -g & g\\
f & g & 1-\frac{1}{2}\left(f^2+g^2\right) & \frac{1}{2}\left(f^2+g^2\right)\\
f & g & -\frac{1}{2}\left(f^2+g^2\right) & 1+\frac{1}{2}\left(f^2+g^2\right)
\end{pmatrix},
$$

where $\mathbf{O} \in SO(4)$, $\mathbf{O}\cdot\ell^T = \ell^T$, $\mathbf{O}^T\cdot\varepsilon\cdot\mathbf{O} = \mathbf{O}\cdot\varepsilon\cdot\mathbf{O}^T = \varepsilon$, and $\varepsilon = \mathrm{diag}(1, 1, 1, -1)$.

Definition 4.1. *We define a conoid hypersurface with a light-like axis immersed in* $\mathbb{L}^4$*, given by the parametrization:*

$$
\phi(u,v,w) = \begin{pmatrix}
f(u-\Phi)\\
g(u-\Phi)\\
\Phi + \frac{1}{2}\left(f^2+g^2\right)(u-\Phi)\\
u + \frac{1}{2}\left(f^2+g^2\right)(u-\Phi)
\end{pmatrix}. \tag{4.1}
$$

Here, $u \in \mathbb{R}-\{0\}$*,* $f = f(v)$*,* $g = g(w)$*, and* $\Phi = \Phi(v,w)$*.*

The determination of the first fundamental form matrix $\left(\mathfrak{g}_{ij}\right)$ is achieved through the computation of the first derivatives of the conoid hypersurface parametrization, represented by Eq (4.1), concerning the variables u, v, and w. This computation yields the following matrix:

$$
\left(\mathfrak{g}_{ij}\right) = \begin{pmatrix}
-1 & 0 & 0\\
0 & f_v^2(u-\Phi)^2+\Phi_v^2 & \Phi_v\Phi_w\\
0 & \Phi_v\Phi_w & g_w^2(u-\Phi)^2+\Phi_w^2
\end{pmatrix}. \tag{4.2}
$$

The partial derivatives are denoted as $f_v = \frac{\partial f}{\partial v}$, $f_v^2 = (\frac{\partial f}{\partial v})^2$, and so on. Consequently, the determinant of the first fundamental form matrix $\left(\mathfrak{g}_{ij}\right)$ is expressed as $\det\left(\mathfrak{g}_{ij}\right) = -(u-\Phi)^2\,\mathbb{W}$, where $\mathbb{W} = f_v^2\Phi_w^2 + g_w^2\Phi_v^2 + f_v^2 g_w^2 (u-\Phi)^2$. The categorization of RCH-L, as defined by Eq (4.1), into a space-like (or time-like, light-like) hypersurface depends on the sign of $\det\left(\mathfrak{g}_{ij}\right)$.

Definition 4.2. *A hypersurface is described as space-like if* $\det\left(\mathfrak{g}_{ij}\right) > 0$*, time-like if* $\det\left(\mathfrak{g}_{ij}\right) < 0$*, and light-like if* $\det\left(\mathfrak{g}_{ij}\right) = 0$.

Since $\mathbb{W} > 0$, then $\det\left(\mathfrak{g}_{ij}\right) < 0$. Hence, RCH-L is a time-like hypersurface.

Applying the Gauss map formula denoted by (2.2), we ascertain the Gauss map of the RCH-L by utilizing Eq (4.1). The derivation process unfolds in the subsequent steps, offering a detailed insight into the determination of the Gauss map

$$\mathbb{G} = \frac{1}{\mathbb{W}^{1/2}}\begin{pmatrix} g_w\left(\Phi_v + f f_v (u-\Phi)\right) \\ f_v\left(\Phi_w + g g_w (u-\Phi)\right) \\ f g_w \Phi_v + f_v\left(g\Phi_w + g_w\left(\frac{1}{2}\left(f^2+g^2\right)-1\right)(u-\Phi)\right) \\ f g_w \Phi_v + f_v\left(g\Phi_w + \frac{1}{2} g_w\left(f^2+g^2\right)(u-\Phi)\right) \end{pmatrix}. \tag{4.3}$$

Upon computing the second derivatives concerning u, v, and w for the RCH-L defined through Eq (4.1) and incorporating the Gauss map expressed in Eq (4.3), we derive the ensuing matrix that represents the second fundamental form

$$\left(\mathfrak{h}_{ij}\right) = \begin{pmatrix} 0 & \frac{f_v g_w \Phi_v}{\mathbb{W}^{1/2}} & \frac{f_v g_w \Phi_w}{\mathbb{W}^{1/2}} \\ \frac{f_v g_w \Phi_v}{\mathbb{W}^{1/2}} & -\frac{g_w\left((f_v\Phi_{vv} - \Phi_v f_{vv})(u-\Phi) + f_v\left(2\Phi_v^2 + f_v^2(u-\Phi)^2\right)\right)}{\mathbb{W}^{1/2}} & -\frac{f_v g_w((u-\Phi)\Phi_{vw} + 2\Phi_v\Phi_w)}{\mathbb{W}^{1/2}} \\ \frac{f_v g_w \Phi_w}{\mathbb{W}^{1/2}} & -\frac{f_v g_w((u-\Phi)\Phi_{vw} + 2\Phi_v\Phi_w)}{\mathbb{W}^{1/2}} & -\frac{f_v\left((g_w\Phi_{ww} - \Phi_w g_{ww})(u-\Phi) + g_w\left(2\Phi_w^2 + g_w^2(u-\Phi)^2\right)\right)}{\mathbb{W}^{1/2}} \end{pmatrix}, \tag{4.4}$$

$f_{uu} = \frac{\partial^2 f}{\partial u^2}$, $f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$, etc.. By utilizing Eqs (4.2) and (4.4), we proceed with the computation of the shape operator matrix denoted as $\mathcal{S} = \left(\mathfrak{s}_{ij}\right)_{3\times 3}$ for the expression given in (4.1). Subsequently, employing Eq (3.1) along with (4.2) and (4.4), we determine the curvatures of the RCH-L defined by Eq (4.1).

Theorem 4.3. *Consider an RCH-L denoted by ϕ defined by the equation provided in* (4.1) *within the space $\mathbb{L}^4$. The associated curvatures of ϕ are elucidated, where $\mathbb{K}_0$ defaults to a value of 1,*

$$\begin{aligned} \mathbb{K}_1 &= \frac{1}{3(u-\Phi)\,\mathbb{W}^{3/2}}\Big[f_v g_w\left(\Phi_w^2\Phi_{vv} - 2\Phi_v\Phi_w\Phi_{vw} + \Phi_v^2\Phi_{ww}\right) \\ &\quad + (u-\Phi)^2\left(f_v g_w\left(f_v^2\Phi_{ww} + g_w^2\Phi_{vv}\right) - \left(f_{vv} g_w^3 \Phi_v + f_v^3 g_{ww}\Phi_w\right)\right) \\ &\quad + 3 f_v g_w (u-\Phi)\left(f_v^2\Phi_w^2 + g_w^2\Phi_v^2\right) - \left(f_v g_{ww}\Phi_v + f_{vv} g_w \Phi_w\right)\Phi_v\Phi_w \\ &\quad + 2 f_v^3 g_w^3 (u-\Phi)^3\Big], \end{aligned}$$

$$\begin{aligned} \mathbb{K}_2 &= \frac{f_v g_w}{3(u-\Phi)\,\mathbb{W}^2}\left[-(u-\Phi)\left(f_v g_{ww}\Phi_w\Phi_{vv} + g_w\left(f_{vv}\Phi_v + f_v\Phi_{vv}\right)\Phi_{ww}\right)\right. \\ &\quad + 2 f_v g_w\left(\Phi_w^2\Phi_{vv} + \Phi_v^2\Phi_{ww}\right) - 2\left(f_v g_{ww}\Phi_v + g_w f_{vv}\Phi_w\right)\Phi_v\Phi_w \end{aligned}$$

$$
\begin{aligned}
&+(u-\Phi)^2\left(f_v g_w\left(f_v^2\Phi_{ww}+g_w^2\Phi_{vv}\right)-\left(g_w^3 f_{vv}\Phi_v+f_v^3 g_{ww}\Phi_w\right)\right)\\
&+f_v g_w(u-\Phi)\left(3\left(f_v^2\Phi_w^2+g_w^2\Phi_v^2\right)-\Phi_{vw}^2\right)-4f_v g_w\Phi_v\Phi_w\Phi_{vw}\\
&+f_{vv}g_{ww}(u-\Phi)\Phi_v\Phi_w+f_v^3 g_w^3(u-\Phi)^3\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{K}_3 \;=\; & \frac{f_v^2 g_w^2}{(u-\Phi)\,\mathbb{W}^{5/2}}\left[-f_v g_w\left(\left(\Phi_w^2\Phi_{vv}+\Phi_v^2\Phi_{ww}\right)+(u-\Phi)\left(f_v^2\Phi_w^2+g_w^2\Phi_v^2\right)\right)\right.\\
&\left.+\left(f_v g_{ww}\Phi_v+g_w f_{vv}\Phi_w+2f_v g_w\Phi_{vw}\right)\Phi_v\Phi_w\right],
\end{aligned}
$$

where $\mathbb{W}=f_v^2\Phi_w^2+g_w^2\Phi_v^2+f_v^2 g_w^2(u-\Phi)^2$.

Proof. Through the application of the Cayley-Hamilton theorem, the curvatures $\mathbb{K}_i$ linked to ϕ are established. This establishment involves the exploration of the characteristic polynomial denoted as $P_{\mathcal{S}}(\lambda)=0$, which is associated with the RCH-L defined by Eq (4.1):

$$\mathbb{K}_0\lambda^3-3\mathbb{K}_1\lambda^2+3\mathbb{K}_2\lambda-\mathbb{K}_3=0.$$

□

Corollary 4.4. *Consider ϕ as an RCH-L defined by Eq* (4.1) *within* $\mathbb{L}^4$. *ϕ is characterized as* 1*-minimal if the following partial differential equation arises:*

$$
\begin{aligned}
&f_v g_w\left(\Phi_w^2\Phi_{vv}-2\Phi_v\Phi_w\Phi_{vw}+\Phi_v^2\Phi_{ww}\right)+2f_v^3 g_w^3(u-\Phi)^3\\
&+(u-\Phi)^2\left(f_v g_w\left(f_v^2\Phi_{ww}+g_w^2\Phi_{vv}\right)-\left(f_{vv}g_w^3\Phi_v+f_v^3 g_{ww}\Phi_w\right)\right)\\
&+3f_v g_w(u-\Phi)\left(f_v^2\Phi_w^2+g_w^2\Phi_v^2\right)-\left(f_v g_{ww}\Phi_v+f_{vv}g_w\Phi_w\right)\Phi_v\Phi_w=0,
\end{aligned}
$$

where $u\neq\Phi$, $f_v^2\Phi_w^2+g_w^2\Phi_v^2+f_v^2 g_w^2(u-\Phi)^2\neq 0$.

Corollary 4.5. *Let ϕ represent an RCH-L defined by Eq* (4.1) *in the context of* $\mathbb{L}^4$. *ϕ is considered to be* 2*-minimal if the following partial differential equation is met:*

$$
\begin{aligned}
&-(u-\Phi)\left(f_v g_{ww}\Phi_w\Phi_{vv}+g_w\left(f_{vv}\Phi_v+f_v\Phi_{vv}\right)\Phi_{ww}\right)\\
&+2f_v g_w\left(\Phi_w^2\Phi_{vv}+\Phi_v^2\Phi_{ww}\right)-2\left(f_v g_{ww}\Phi_v+g_w f_{vv}\Phi_w\right)\Phi_v\Phi_w\\
&+(u-\Phi)^2\left(f_v g_w\left(f_v^2\Phi_{ww}+g_w^2\Phi_{vv}\right)-\left(g_w^3 f_{vv}\Phi_v+f_v^3 g_{ww}\Phi_w\right)\right)\\
&+f_v g_w(u-\Phi)\left(3\left(f_v^2\Phi_w^2+g_w^2\Phi_v^2\right)-\Phi_{vw}^2\right)-4f_v g_w\Phi_v\Phi_w\Phi_{vw}\\
&+f_{vv}g_{ww}(u-\Phi)\Phi_v\Phi_w+f_v^3 g_w^3(u-\Phi)^3=0,
\end{aligned}
$$

where $u\neq\Phi$, $f_v^2\Phi_w^2+g_w^2\Phi_v^2+f_v^2 g_w^2(u-\Phi)^2\neq 0$.

Corollary 4.6. *Let ϕ represent an RCH-L defined by Eq* (4.1) *within the space* $\mathbb{L}^4$. *ϕ is deemed to be* 3*-minimal if the ensuing partial differential equation is presented:*

$$
\begin{aligned}
&-f_v g_w\left(\left(\Phi_w^2\Phi_{vv}+\Phi_v^2\Phi_{ww}\right)+(u-\Phi)\left(f_v^2\Phi_w^2+g_w^2\Phi_v^2\right)\right)\\
&+\left(f_v g_{ww}\Phi_v+g_w f_{vv}\Phi_w+2f_v g_w\Phi_{vw}\right)\Phi_v\Phi_w=0,
\end{aligned}
$$

where $u\neq\Phi$, $f_v^2\Phi_w^2+g_w^2\Phi_v^2+f_v^2 g_w^2(u-\Phi)^2\neq 0$.

It is important to highlight that the solutions for Φ in the corollaries pose unresolved challenges that require attention.

5. Laplace-Beltrami operator of the RCH-L in $\mathbb{L}^4$

Within this section, attention is directed to the application of the $\mathcal{L}$-$\mathcal{B}$ operator to a smooth function within $\mathbb{L}^4$. The subsequent steps involve the computation of this operator using the RCH-L defined by the equation in (4.1).

Definition 5.1. *The Laplace-Beltrami operator is established for a smooth function φ in domain $\mathcal{D}$, $\left(\mathcal{D} \subset \mathbb{R}^3\right)$ of class C^3 relies on the first fundamental form $\left(\mathfrak{g}_{ij}\right)$, and is defined by*

$$\Delta\varphi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} \left(\mathbf{g}^{1/2} \mathfrak{g}^{ij} \frac{\partial \varphi}{\partial x^j} \right), \tag{5.1}$$

where $\varphi = \varphi\left(x^1, x^2, x^3\right)$, $\left(\mathfrak{g}^{ij}\right) = (\mathfrak{g}_{kl})^{-1}$ *and* $\mathbf{g} = \det\left(\mathfrak{g}_{ij}\right)$.

Refer to Chen et al. [2] and Lawson [5] for the Laplace-Beltrami operator details. Consequently, the following is presented.

Theorem 5.2. *The Laplace-Beltrami operator for the RCH-L ϕ described by Eq* (4.1) *is given by $\Delta\phi = 3\mathbb{K}_1\mathbb{G}$, where $\mathbb{K}_1$ denotes the mean curvature, and $\mathbb{G}$ represents the Gauss map of ϕ.*

Proof. The expression for the $\mathcal{L}$-$\mathcal{B}$ operator acting on the RCH-L, as specified by Eq (4.1), is given by

$$\begin{aligned} \Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \Bigg[& \frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathfrak{g}^{11}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathfrak{g}^{22}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathfrak{g}^{23}\frac{\partial\phi}{\partial w}\right) \\ & + \frac{\partial}{\partial w}\left(\mathbf{g}^{1/2}\mathfrak{g}^{32}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\mathbf{g}^{1/2}\mathfrak{g}^{33}\frac{\partial\phi}{\partial w}\right)\Bigg], \end{aligned} \tag{5.2}$$

where

$$\left(\mathfrak{g}^{ij}\right) = \frac{1}{(u-\Phi)^2\,\mathbb{W}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & g_w^2\,(u-\Phi)^2 + \Phi_w^2 & -\Phi_v\Phi_w \\ 0 & -\Phi_v\Phi_w & f_v^2\,(u-\Phi)^2 + \Phi_v^2 \end{pmatrix}, \tag{5.3}$$

$\mathbb{W} = f_v^2\Phi_w^2 + g_w^2\Phi_v^2 + f_v^2 g_w^2\,(u-\Phi)^2$. By substituting the derivatives of the components, as determined by Eq (5.3), into the formula given by Eq (5.2), the formation of $\Delta\phi = (\Delta\phi_1, \Delta\phi_2, \Delta\phi_3, \Delta\phi_4)$, along with its individual components, is achieved:

$$\Delta\phi_1 = \frac{g_w(\Phi_v + f f_v(u-\Phi))}{(u-\Phi)\mathbb{W}^2}\mathfrak{A},$$

$$\Delta\phi_2 = \frac{f_v(\Phi_w + g(u-\Phi)g_w)}{(u-\Phi)\mathbb{W}^2}\mathfrak{A},$$

$$\Delta\phi_3 = \frac{f g_w\Phi_v + f_v\left(g\Phi_w + g_w\left(\frac{1}{2}\left(f^2+g^2\right)-1\right)(u-\Phi)\right)}{(u-\Phi)\mathbb{W}^2}\mathfrak{A},$$

$$\Delta\phi_4 = \frac{f g_w\Phi_v + f_v\left(g\Phi_w + \frac{1}{2}g_w\left(f^2+g^2\right)(u-\Phi)\right)}{(u-\Phi)\mathbb{W}^2}\mathfrak{A},$$

where

$$\mathfrak{A} = f_v g_w\left(\Phi_w^2\Phi_{vv} - 2\Phi_v\Phi_w\Phi_{vw} + \Phi_v^2\Phi_{ww}\right) + 2f_v^3 g_w^3\,(u-\Phi)^3$$

$$+(u-\Phi)^2\left(f_v g_w\left(f_v^2\Phi_{ww}+g_w^2\Phi_{vv}\right)-\left(f_{vv}g_w^3\Phi_v+f_v^3 g_{ww}\Phi_w\right)\right)$$
$$+3f_v g_w(u-\Phi)\left(f_v^2\Phi_w^2+g_w^2\Phi_v^2\right)-\left(f_v g_{ww}\Phi_v+f_{vv}g_w\Phi_w\right)\Phi_v\Phi_w.$$

□

Definition 5.3. *The hypersurface ϕ is characterized as harmonic if each component of $\Delta\phi$ is zero.*

Example 5.4. *By substituting $f(v)=v$, $g(w)=w$, and $\Phi(v,w)=w$ into an RCH-L defined by Eq* (4.1) *within $\mathbb{L}^4$, the Gauss map and the shape operator matrix are described by*

$$\mathbb{G}=\frac{1}{\left((u-w)^2+1\right)^{1/2}}\begin{pmatrix}(w-u)v\\(w-u)w-1\\\frac{1}{2}\left(v^2+w^2\right)(w-u)+(u-2w)\\\frac{1}{2}\left(v^2+w^2\right)(w-u)-w\end{pmatrix},$$

$$\mathcal{S}=\begin{pmatrix}0&0&\frac{1}{\left((u-w)^2+1\right)^{1/2}}\\0&\frac{1}{\left((u-w)^2+1\right)^{1/2}}&0\\-\frac{1}{\left((u-w)^2+1\right)^{3/2}}&0&\frac{(u-w)^2+2}{\left((u-w)^2+1\right)^{3/2}}\end{pmatrix}.$$

Following this, the curvatures are determined via

$$\begin{aligned}\mathbb{K}_1&=\frac{2(u-w)^2+3}{3\left((u-w)^2+1\right)^{3/2}},\\\mathbb{K}_2&=-\frac{(u-w)^2+3}{3\left((u-w)^2+1\right)^{2}},\\\mathbb{K}_3&=\frac{1}{\left((u-w)^2+1\right)^{5/2}}.\end{aligned}$$

Then,

$$\Delta\phi=\frac{2(u-w)^2+3}{\left((u-w)^2+1\right)^{2}}\begin{pmatrix}(w-u)v\\(w-u)w-1\\\frac{1}{2}\left(v^2+w^2\right)(w-u)+u-2w\\\frac{1}{2}\left(v^2+w^2\right)(w-u)-w\end{pmatrix}.$$

In summary, the hypersurface is not both minimal and harmonic.

Example 5.5. *Opting for $f(v)=v$, $g(w)=w$, and $\Phi(v,w)=v$ for an RCH-L defined by Eq* (4.1) *within $\mathbb{L}^4$, ϕ exhibits the Gauss map*

$$\mathbb{G}=\frac{1}{\left((u-v)^2+1\right)^{1/2}}\begin{pmatrix}(v-u)v-1\\(v-u)w\\\frac{1}{2}\left(v^2+w^2\right)(v-u)+u-2v\\\frac{1}{2}\left(v^2+w^2\right)(v-u)-v\end{pmatrix}.$$

Hence, the shape operator matrix is established as

$$\mathcal{S}=\left(\begin{array}{ccc} 0 & \frac{1}{\left((u-v)^2+1\right)^{1/2}} & 0 \\ -\frac{1}{\left((u-v)^2+1\right)^{3/2}} & \frac{(u-v)^2+2}{\left((u-v)^2+1\right)^{3/2}} & 0 \\ 0 & 0 & \frac{1}{\left((u-v)^2+1\right)^{1/2}} \end{array}\right).$$

The curvatures are expressed as

$$\begin{aligned}
\mathbb{K}_1 &= \frac{2(u-v)^2+3}{3\left((u-v)^2+1\right)^{3/2}}, \\
\mathbb{K}_2 &= -\frac{(u-v)^2+3}{3\left((u-v)^2+1\right)^{2}}, \\
\mathbb{K}_3 &= \frac{1}{\left((u-v)^2+1\right)^{5/2}}.
\end{aligned}$$

Finally,

$$\Delta\phi=\frac{2(u-v)^2+3}{\left((u-v)^2+1\right)^2}\left(\begin{array}{c} (v-u)v-1 \\ (v-u)w \\ \frac{1}{2}\left(v^2+w^2\right)(v-u)+u-2v \\ \frac{1}{2}\left(v^2+w^2\right)(v-u)-v \end{array}\right).$$

This indicates that the hypersurface is neither minimal nor harmonic. Exploring hypersurfaces that do not conform to the minimal or harmonic categories reveals the complexity of geometric structures and contributes to the development of new mathematical techniques.

Such hypersurfaces may play a significant role in various mathematical theories and applications.

6. Umbilical condition of the RCH-L in $\mathbb{L}^4$

In this section, the umbilical condition for the right conoid hypersurface with the light-like axis is presented.

Definition 6.1. *In the context of a hypersurface in $\mathbb{L}^4$, a point denoted as* **p** *is labeled umbilical if and only if its principal curvatures k_i are identical, i.e., $k_1 = k_2 = k_3$. This condition leads to the equivalence ${\mathbb{K}_1}^3 = \mathbb{K}_3$.*

Theorem 6.2. *For a point* **p** *on the hypersurface associated with the right conoid, $\phi : M^3 \subset \mathbb{E}^3 \longrightarrow \mathbb{L}^4$, having a light-like axis, the point is deemed umbilic if and only if it satisfies the partial differential equation*

$${\mathbb{K}_1}^3 - \mathbb{K}_3 = 0.$$

In this context, $\mathbb{K}_1$ and $\mathbb{K}_3$ denote the mean and Gauss-Kronecker curvatures, respectively.

Proof. Through the utilization of the curvatures inherent in the hypersurface of the right conoid, in conjunction with the light-like axis ϕ, one acknowledges the equivalency of the ensuing partial differential equation

$$\mathfrak{A}^3 - 27\,(u-\Phi)^2\,\mathbb{W}^2 f_v^2 g_w^2 \mathfrak{C} \;=\; 0, \tag{6.1}$$

where

$$\begin{aligned}
\mathfrak{A} &= f_v g_w \left(\Phi_w^2\Phi_{vv} - 2\Phi_v\Phi_w\Phi_{vw} + \Phi_v^2\Phi_{ww}\right) + 2f_v^3 g_w^3\,(u-\Phi)^3 \\
&\quad + (u-\Phi)^2\left(f_v g_w\left(f_v^2\Phi_{ww} + g_w^2\Phi_{vv}\right) - \left(f_{vv}g_w^3\Phi_v + f_v^3 g_{ww}\Phi_w\right)\right) \\
&\quad +3f_v g_w\,(u-\Phi)\left(f_v^2\Phi_w^2 + g_w^2\Phi_v^2\right) - \left(f_v g_{ww}\Phi_v + f_{vv}g_w\Phi_w\right)\Phi_v\Phi_w, \\
\mathfrak{C} &= -f_v g_w\left(\left(\Phi_w^2\Phi_{vv} + \Phi_v^2\Phi_{ww}\right) + (u-\Phi)\left(f_v^2\Phi_w^2 + g_w^2\Phi_v^2\right)\right) \\
&\quad + \left(f_v g_{ww}\Phi_v + g_w f_{vv}\Phi_w + 2f_v g_w\Phi_{vw}\right)\Phi_v\Phi_w, \\
\mathbb{W} &= f_v^2\Phi_w^2 + g_w^2\Phi_v^2 + f_v^2 g_w^2\,(u-\Phi)^2\,.
\end{aligned}$$

□

The above mentioned theorem reveals a notable geometric characteristic of points on the hypersurface. In particular, it is stated that a point on the right conoid hypersurface with a light-like axis attains the status of being umbilic precisely when it complies with the specified partial differential equation. The unresolved issue concerns the identification of solutions Φ for the partial differential equations indicated by (6.1) .

7. Willmore functional of the RCH-L in $\mathbb{L}^4$

The Willmore property of RCH-L is presented in this section.

Definition 7.1. *Consider a smooth immersion $s : M \subset \mathbb{E}^2 \longrightarrow \mathbb{R}^n$ such that $W(s) < \infty$. We designate s as a critical point for W if*

$$\forall n \in C^\infty(M, \mathbb{R}^n), \quad \frac{d}{dq}W(s+qn)\Big|_{q=0} = 0.$$

An immersion meeting this criterion is identified as Willmore.

Refer to Li and Yau [6], Toda [11], and Willmore [12, 13] for in-depth information on Euclidean aspects. The utilization of the Willmore functional is extended to the RCH-L framework within Minkowski 4-space.

Theorem 7.2. *For an immersion $\phi : M^3 \subset \mathbb{E}^3 \longrightarrow \mathbb{L}^4$, the condition of being Willmore is equivalent to satisfying the partial differential equation*

$$\Delta\mathbb{K}_1 + 3\mathbb{K}_1({\mathbb{K}_1}^3 - \mathbb{K}_3) = 0,$$

where Δ represents the Laplace-Beltrami operator, and $\mathbb{K}_1$,$\mathbb{K}_3$ denote the mean and Gauss-Kronecker curvatures, respectively.

Proof. By employing the term $\Delta\mathbb{K}_1 = (\mathbb{K}_1)_{uu} + (\mathbb{K}_1)_{vv} + (\mathbb{K}_1)_{ww}$ and including the umbilical constraint ${\mathbb{K}_1}^3 - \mathbb{K}_3 = 0$, we expose that the ensuing partial differential equation equation is unveiled by using the curvatures given by Theorem 4.3:

$$(\mathbb{K}_1)_{uu} + (\mathbb{K}_1)_{vv} + (\mathbb{K}_1)_{ww} + 3\mathfrak{A}\left(\mathfrak{A}^3 - 27\,(u-\Phi)^2\,\mathbb{W}^2 f_v^2 g_w^2 \mathfrak{C}\right) = 0, \tag{7.1}$$

where

$$\begin{aligned}
\mathfrak{A} &= f_v g_w \left(\Phi_w^2\Phi_{vv} - 2\Phi_v\Phi_w\Phi_{vw} + \Phi_v^2\Phi_{ww}\right) + 2f_v^3 g_w^3\,(u-\Phi)^3 \\
&\quad + (u-\Phi)^2 \left(f_v g_w \left(f_v^2\Phi_{ww} + g_w^2\Phi_{vv}\right) - \left(f_{vv}g_w^3\Phi_v + f_v^3 g_{ww}\Phi_w\right)\right) \\
&\quad + 3f_v g_w\,(u-\Phi)\left(f_v^2\Phi_w^2 + g_w^2\Phi_v^2\right) - \left(f_v g_{ww}\Phi_v + f_{vv}g_w\Phi_w\right)\Phi_v\Phi_w, \\
\mathfrak{C} &= -f_v g_w \left(\left(\Phi_w^2\Phi_{vv} + \Phi_v^2\Phi_{ww}\right) + (u-\Phi)\left(f_v^2\Phi_w^2 + g_w^2\Phi_v^2\right)\right) \\
&\quad + \left(f_v g_{ww}\Phi_v + g_w f_{vv}\Phi_w + 2f_v g_w\Phi_{vw}\right)\Phi_v\Phi_w, \\
\mathbb{W} &= f_v^2\Phi_w^2 + g_w^2\Phi_v^2 + f_v^2 g_w^2\,(u-\Phi)^2\,.
\end{aligned}$$

□

This theorem establishes a significant link between the hypersurface's immersion and its Willmore property. The assertion is that the immersion ϕ mapping a three-dimensional Euclidean space into the four-dimensional Minkowski space $\mathbb{L}^4$ attains Willmore status exclusively when it adheres to the prescribed equation. The quest for solutions Φ to the partial differential equation denoted as (7.1) continues to be an unresolved issue. Note that solutions of Eq (7.1) cannot generally be obtained explicitly due to it being a fourth-order highly non-linear equation.

8. Conclusions

This research undertakes an exploration of hypersurfaces in the four- dimensional Minkowski space $\mathbb{L}^4$ specifically categorized as right conoids with a light-like axis (RCH-L). Through meticulous scrutiny, we have effectively computed fundamental matrices associated with the fundamental form, Gauss map, and shape operator inherent to these hypersurfaces. The application of the Cayley-Hamilton theorem has facilitated the unveiling of curvatures distinctive to these hypersurfaces, thereby advancing our understanding of their intricate geometric attributes.

The foundational investigation into fundamental principles and concepts in four-dimensional Minkowski geometry establishes a sturdy foundation. Essential curvature formulas relevant to hypersurfaces in $\mathbb{L}^4$ are presented, offering pivotal insights into the mathematical complexities involved. A comprehensive delineation of RCH-L is provided, highlighting their unique properties that distinguish them within the realm of hypersurfaces.

The focus then shifts to the Laplace-Beltrami operator, establishing its correlation with the previously examined hypersurfaces and showcasing its practical application in computations. Umbilical right conoid hypersurfaces are introduced, further broadening the scope of our exploration. Lastly, the study delves into the Willmore functional, introducing an additional layer of analysis and understanding to the RCH-L in $\mathbb{L}^4$.

Author contributions

Yanlin Li, Erhan Güler, and Magdalena Toda: Establishment of theoretical research; Design and interpretation of the manuscript, theoretical analysis, drafted the manuscript, revision; provided critical feedback to the manuscript, revision; Provided suggestions for the manuscript; Revision, translation, proofreading of the manuscript. All authors have read and approved the final version of the manuscript for publication

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Berger, B. Gostiaux, *Géométrie différentielle: Variétés, courbes et surfaces*, Presses Univ. France, France, 1987.
2. B. Y. Chen, E. Güler, Y. Yaylı, H. H. Hacısalihoğlu, Differential geometry of 1-type submanifolds and submanifolds with 1-type Gauss map, *Int. Elec. J. Geom.,* **16** (2023), 4–49. http://dx.doi.org/10.36890/iejg.1216024
3. E. Güler, Helical hypersurfaces in Minkowski geometry $\mathbb{E}_1^4$, *Symmetry*, **12** (2020), 1206. http://dx.doi.org/10.3390/sym12081206
4. E. Güler, Differential geometry of the family of helical hypersurfaces with a light-like axis in Minkowski spacetime $\mathbb{L}^4$, *Universe*, **9** (2023), 341. http://dx.doi.org/10.3390/universe9070341
5. H. B. Lawson, *Lectures on minimal submanifolds*, 2 Eds., Mathematics Lecture Series 9, Publish or Perish Inc., Wilmington, DE, USA, 1980.
6. P. Li, S. T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces, *Invent. Math.*, **69** (1982), 269–291. http://dx.doi.org/10.1007/BF01399507
7. Y. Li, E. Güler, Twisted hypersurfaces in Euclidean 5-space, *Mathematics*, **11** (2023), 4612. http://dx.doi.org/10.3390/math11224612
8. Y. Li, E. Güler, Hypersurfaces of revolution family supplying $\Delta \mathbf{r} = A\mathbf{r}$ in pseudo-Euclidean space $\mathbb{E}_3^7$, *AIMS Math.*, **8** (2023), 24957–24970. http://dx.doi.org/10.3934/math.20231273
9. Y. Li, E. Güler, A Hypersurfaces of revolution family in the five-dimensional pseudo-Euclidean space $\mathbb{E}_2^5$, *Mathematics*, **11** (2023), 3427. http://dx.doi.org/10.3390/math11153427
10. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, 1Ed. Pure and Applied Mathematics; Vol. 103, Academic Press, Inc., Cambridge, MA, USA; Harcourt Brace Jovanovich Pub., New York, USA, 1983.

11. M. D. Toda, *Willmore energy: Brief introduction and survey*, Monogr. Res. Notes Math. CRC Press, Boca Raton, FL, 2018, 1–7.

12. T. J. Willmore, Note on embedded surfaces, *An. Ştiint. Univ. "Al. I. Cuza" Iaşi Sect. Ia Mat. (N.S.)*, **11** (1965), 493–496.

13. T. J. Willmore, *Riemannian geometry*, The Clarendon Press, Oxford Science Pub., Oxford Un. Press, 1993.