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Research article

Uncertainty quantification based on residual Tsallis entropy of order statistics

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Abstract: In this study, we focused on investigating the properties of residual Tsallis entropy for order statistics. The reliability of engineering systems is highly influenced by order statistics, for example, when modeling the lifetime of a series system and the lifetime of a parallel system. The residual Tsallis entropy of the ith order statistic from a continuous distribution function and its deviation from the residual Tsallis entropy of the ith order statistics from a uniform distribution were investigated. In the mathematical framework, a method was provided to represent the residual Tsallis entropy of the ith order statistic in the continuous case with respect to the case where the distribution was uniform. This approach can provide insight into the behavior and properties of the residual Tsallis entropy for order statistics. We also investigated the monotonicity of the new uncertainty measure under different conditions. An investigation of these properties leads to a deeper understanding of the relationship between the position of the order statistics and the resulting Tsallis entropy. Finally, we presented the computational results and proposed estimators for estimating the residual Tsallis entropy of an exponential distribution. For this purpose, we derived a maximum likelihood estimator.

Keywords: order statistics; residual Tsallis entropy; Shannon entropy; residual lifetime
Mathematics Subject Classification: 62N05, 94A17

1. Introduction

Information theory is a rich and wide-ranging field that has laid the foundation for new mathematical questions and advances in mathematical techniques. Various information measures seem useful for deriving insightful results in many areas of mathematics. In the literature, the quantification of uncertainty that exists in random phenomena is largely enabled by the information theory. The extensive range of its applications is expounded upon in Shannon's seminal work [23]. When a non-negative random variable (rv) X, representing the lifetime of a system, or a unit or living organism, is given for a continuous cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$, as stated in [28], the Tsallis entropy of order α becomes a pertinent metric which is defined as follows:

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1-\alpha}\left[\int_0^\infty f^\alpha(x)dx - 1\right], \\ &= \frac{1}{1-\alpha}[E(f^{\alpha-1}(F^{-1}(U))) - 1], \end{aligned} \tag{1.1}$$

in which $\alpha > 0,\ \alpha \neq 1$, $E(\cdot)$ signifies the expectation, $F^{-1}(u) = \inf\{x; F(x) \geq u\}$, for $u \in [0,1]$, amounts to the quantile function, and U is uniformly distributed on $[0,1]$. It is worth pointing out that the Tsallis entropy is a nondecreasing function of the Renyi one given by: $R_\alpha(X) = 1/(1-\alpha)\log(1+(1-\alpha)H_\alpha(X))$, where $R_\alpha(X)$ is the Renyi entropy of order α, (see [16]). The Tsallis entropy can generally take negative values, but by selecting the right values for α, it can be nonnegative. One crucial finding is that $H(X) = \lim_{\alpha\to 1} H_\alpha(X)$, demonstrating how Tsallis entropy and Shannon differential entropy converge. The Tsallis entropy shows nonadditivity in contrast to the additivity of the Shannon entropy. Specifically, under the Shannon framework, $H(X,Y) = H(X) + H(Y)$ holds for two independent random variables X and Y. By contrast, the Tsallis framework yields $H_\alpha(X,Y) = H_\alpha(X) + H_\alpha(Y) + (1-\alpha)H_\alpha(X)H_\alpha(Y)$. Because it is not additive like the Shannon entropy, the Tsallis entropy is more versatile and can be applied to a wide range of subjects such as information theory, physics, chemistry, and technology; see, e.g., [18]. It is worth noting that the Tsallis entropy finds extensive applications in parameter estimation problems, including areas like seismic imaging and natural information. This demonstrates the significant utility of Tsallis entropy as an effective tool for addressing complexity and nonadditivity challenges encountered in parameter estimation across diverse fields.

Assuming that the lifetime of a recently introduced system is denoted by the rv X, the Tsallis entropy $H_\alpha(X)$ function is a measure of the system's intrinsic uncertainty. However, there exist scenarios in which the actors are aware of the system's age. For example, let us assume that one knows that the system is operational at time t and wishes to assess future uncertainty, that is, the uncertainty in the rv $X_t = X - t \mid X > t$ ([9]). In these circumstances, the traditional Tsallis entropy $H_\alpha(X)$ is no longer able to offer the desired insight. Consequently, a new measure known as the residual Tsallis entropy (RTE) is implemented, defined as follows:

$$\begin{aligned} H_\alpha(X;t) &= \frac{1}{1-\alpha}\left[\int_0^\infty f_t^\alpha(x)dx - 1\right] \\ &= \frac{1}{1-\alpha}\left[\int_t^\infty \left(\frac{f(x)}{S(t)}\right)^\alpha dx - 1\right], \end{aligned} \tag{1.2}$$

where

$$f_t(x) = \frac{f(x+t)}{S(t)},\ x,t > 0,$$

is the pdf of X_t and $S(t) = P(X > t)$ is the survival function (sf) of X. Numerous studies have been conducted in the literature to investigate different facets of Tsallis entropy, which can be seen in [2, 5, 6, 15, 17, 31].

Engineers widely acknowledge that highly uncertain components or systems possess inherent unreliability. However, they often face challenges in quantifying this uncertainty. For instance, during the system design phase, engineers typically rely on available information about deterioration, part wear, and other relevant factors to generate hazard rate functions or mean residual lifetime functions. These functions offer insights into the expected behavior of the system based on the provided information (see [11] for more details). So, the purpose of this paper is to investigate RTE of order statistics which measures the concentration of conditional probabilities. The results include expressions, bounds, and monotonicity properties of RTE of order statistics.

We take a random sample of size n from a distribution F represented by the sample $X_1, X_2, \ldots, X_n$. The ordered values of the sample are called order statistics of the sample, which are denoted by: $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Since they can be used to explain probability distributions, evaluate how well a dataset fits a particular model, regulate the quality of a process or product, assess the reliability of a system or component, and for a variety of other purposes, these statistics are crucial for many different fields.

Numerous applications of order statistics include robust estimation, outlier detection, and probability distribution characterization [4]. The information qualities of order statistics have been studied by a number of writers. It was demonstrated by [29] that there is a constant difference between the parent random variable and the average entropy of order statistics. For order statistics, [19] derived a few recurrence relations. Using order statistics, [12] investigated Kullback-Leibler information measure and Shannon entropy. Renyi entropy is known to not uniquely determine the underlying distribution function. [7] used order statistics to study Renyi entropy and demonstrated that, for certain values of n, this measure characterizes the distribution function. Using the Renyi entropy of ith order statistics, Abbasnejad and [1] examined a few stochastic comparisons and talked about some bounds for this measure. For ith order statistics, [30] proposed the residual Renyi entropy. They have derived certain bounds for the residual Renyi entropy of order statistics and record values, and they have simplified the definition of the residual Renyi entropy of the ith order statistics. [25] studied the residual extropy as a measure of uncertainty of order statistics. [26] considered some aspects of past Tsallis entropy of order statistics.

Furthermore, they play a crucial role in reliability theory, particularly in the analysis of coherent systems and lifespan testing of data acquired through different censorship techniques. Several scholars have made good use of the information properties of ordered variables; these findings are reported in [13, 19, 29], as well as their references. Properties of Renyi entropy of excess amounts of ordered variables and record values were examined by [30]; also see [8]. By examining elements of residual Tsallis entropy in terms of ordered variables, our study seeks to advance the field. [3] has investigated the characteristics of a coherent and mixed system's Tsallis entropy in more recent times.

The outline of the rest of the paper is as follows: The RTE form for order statistics, $X_{i:n}$, is shown in Section 2. It is based on a sample taken from any arbitrary continuous distribution function F. We translate these RTE into terms of RTE for order statistics derived from a unit-distributed sample. We derive upper and lower bounds to approximate the RTE, since closed-form equations for the RTE of order statistics are frequently unavailable for many statistical models. To illustrate these bounds' applicability and practicality, we offer multiple illustrative instances. Furthermore, we examine the monotonicity characteristics of RTE for a sample's extremum in mild circumstances. As the sample size grows, we observe that the RTEs at the extremum of a random sample follow a monotonic pattern. We refute this observation, however, with a counterexample showing that RTE for other order statistics $X_{i:n}$ is non-monotonic concerning sample size. We investigate the RTE of order statistics

$X_{i:n}$ concerning the index of order statistics to further explore the monotonic behavior i. Our findings demonstrate that over the whole support of F, the RTE of $X_{i:n}$ is not a monotonic function of i.

In Section 4, we conclude by presenting some computational results that validate some of the conclusions drawn from this paper. We also offer estimators for calculating the exponential distribution's RTE. For this reason, the maximum likelihood estimator (MLE) is derived. Section 4 concludes the article with some open problems for the future.

Let us assume random lifetimes X and Y with probability density functions (pdfs) f and g, and survival functions S_X and S_Y, respectively. We recall that X is less than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if $S_X(x) \leq S_Y(x)$ for all $x > 0$, and X is less than Y in the likelihood ratio order, denoted by $X \leq_{lr} Y$ if $g(x)/f(x)$ is increasing in $x > 0$.

2. Residual Tsallis entropy of order statistics

The RTE of order statistics dependent on the RTE of ordered uniformly distributed variables is thus expressed here. Considering $i = 1, \ldots, n$, the pdf and sf of $X_{i:n}$ are denoted by $f_{i:n}(x)$ and $S_{i:n}(x)$, respectively. It can be written that

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} (F(x))^{i-1} (S(x))^{n-i} f(x), \ x > 0, \tag{2.1}$$

$$S_{i:n}(x) = \sum_{k=0}^{i-1} \binom{n}{k} (1 - S(x))^k (S(x))^{n-k}, \ x > 0, \tag{2.2}$$

where

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx, \ a > 0, b > 0,$$

is known as the complete beta function; see, e.g., [10]. Furthermore, we can express the survival function $S_{i:n}(x)$ as follows:

$$S_{i:n}(x) = \frac{\bar{B}_{F(x)}(i, n-i+1)}{B(i, n-i+1)}, \tag{2.3}$$

where

$$\bar{B}_x(a, b) = \int_x^1 u^{a-1}(1-u)^{b-1} du, \ 0 < x < 1,$$

is known as the incomplete beta functions. In this section, we shortly write $Y \sim \bar{B}_t(a, b)$ to denote that the random variable Y follows the pdf:

$$f_Y(y) = \frac{1}{\bar{B}_t(a, b)} y^{a-1}(1-y)^{b-1}, \ t \leq y \leq 1. \tag{2.4}$$

The study of $X_{i:n}$'s residual Tsallis entropy, which is based on the conditional rv $[X_{i:n} - t | X_{i:n} > t]$, is the focus of this paper. It measures the degree of uncertainty regarding the predictability of the system's residual lifespan. When $i = 1, 2, \ldots, n$, the $(n-i+1)$-out-of-n systems are important structures in the field of reliability engineering. An $(n-i+1)$-out-of-n system functions in this scenario if, and only if, at least $(n-i+1)$ components are active. We consider a system consisting of identical components that are dispersed independently; these components' lifetimes are represented by the notation $X_1, X_2, \ldots, X_n$.

The random lifetime of the system is equal to $X_{i:n}$, where i denotes the ordered variable's position. When $i = 1$, a parallel system is shown, and a serial system is indicated by $i = n$. The RTE of $X_{i:n}$ functions as a measure of entropy related to the system's residual lifetime in the context of $(n-i+1)$-out-of-n systems running at time t. System designers can learn important information about the entropy of $(n-i+1)$-out-of-n structures connected to systems that are operating at a specific time t from this dynamic entropy metric.

The following lemma establishes a connection between the incomplete beta function and the RTE of ordered variables from a uniform distribution, thus improving computational efficiency. From a practical perspective, this link is essential since it makes the computation of RTE easier. Since it only requires a few simple calculations, the proof of this lemma—which is obtained by the definition of the RTE—is not included here.

Lemma 2.1. *Let $U_{i:n}$ be the ith ordered value of a random sample with uniformly distributed units on (0,1). Then,*

$$H_\alpha(U_{i:n};t) = \frac{1}{1-\alpha}\left[\frac{\bar{B}_t(\alpha(i-1)+1, \alpha(n-i)+1)}{\bar{B}_t^\alpha(i,n-i+1)} - 1\right],\ 0 < t < 1,$$

for all $\alpha > 0,\ \alpha \neq 1$.

Proof. The pdf and survival function of $U_{i:n}$ are represented by

$$g_{i:n}(u) = \frac{1}{B(i,n-i+1)}u^{i-1}(1-u)^{n-i}, \tag{2.5}$$

$$S_{U_{i:n}}(x) = \frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}, \tag{2.6}$$

for all $0 < u < 1$. From (1.2), (2.5), and (2.6), one gets

$$\begin{aligned} H_\alpha(U_{i:n};t) &= \frac{1}{1-\alpha}\left[\int_t^1 \left(\frac{g_{i:n}(u)}{S_{U_{i:n}}(t)}\right)^\alpha du - 1\right] \\ &= \frac{1}{1-\alpha}\left[\int_t^1 \left(\frac{u^{i-1}(1-u)^{n-i}(x)}{\bar{B}_t(i,n-i+1)}\right)^\alpha du - 1\right] \\ &= \frac{1}{1-\alpha}\left[\frac{1}{\bar{B}_t^\alpha(i,n-i+1)}\int_t^1 u^{\alpha(i-1)}(1-u)^{\alpha(n-i)}du - 1\right] \\ &= \frac{1}{1-\alpha}\left[\frac{\bar{B}_t(\alpha(i-1)+1, \alpha(n-i)+1)}{\bar{B}_t^\alpha(i,n-i+1)} - 1\right]. \end{aligned}$$

Hence, the theorem is proved. □

This lemma makes use of the well-known incomplete beta function to make it simple for scholars and practitioners to compute the RTE of order statistics from a uniform distribution. The application and usability of RTE are enhanced in a variety of scenarios by this computational simplification. In Figure 1, the plot of $H_\alpha(U_{i:n};t)$ is depicted for different values of α and $i = 1, 2, \cdots, 5$ when the total number of observations is $n = 5$. It is expected from the figure that there is no inherent monotonicity between the order statistics. However, in the following Lemma 2.3, we establish conditions under which a monotonic relationship can be established between the index i and the number of components. This lemma will provide valuable insight into the arrangement of the system components and the resulting effect on the reliability of the system.

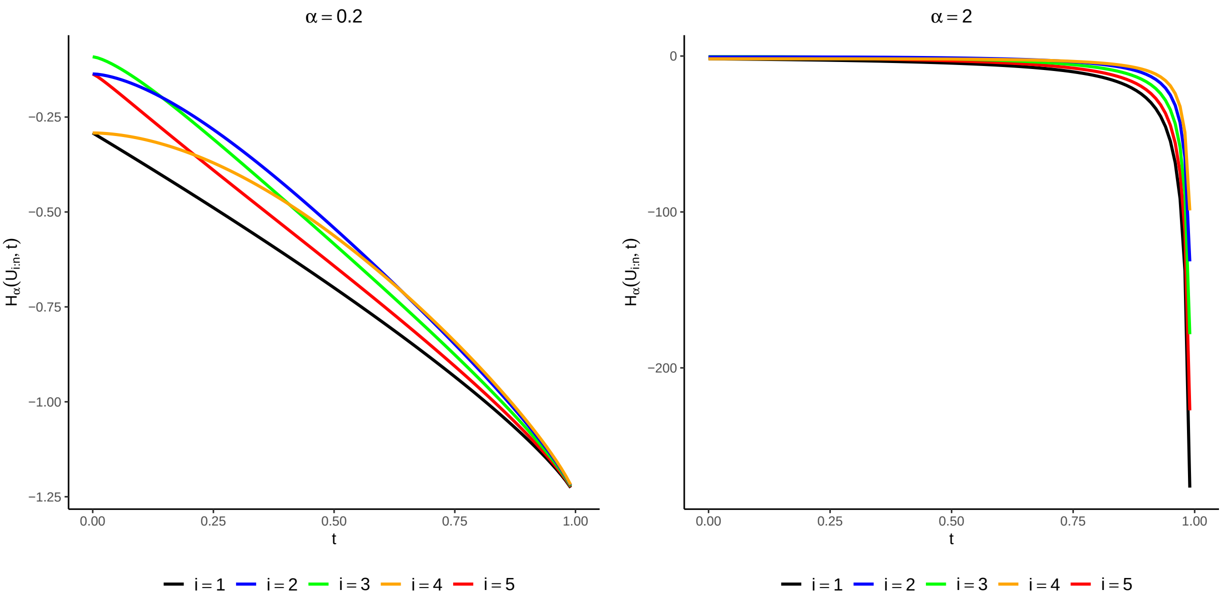

Figure 1. The amounts of $H_\alpha(U_{i:n}; t)$ for $\alpha = 0.2$ (left panel) and $\alpha = 2$ (right panel) when $0 < t < 1$.

The utilization of the probability integral transformation $U_{i:n} \overset{d}{=} F(X_{i:n}),\ i = 1, 2, \cdots, n$, where $\overset{d}{=}$ means equality in distribution and F is a continuous distribution function, is widely recognized in the literature. It is a well-established fact that this transformation yields a beta distribution with parameters i and $n - i + 1$. This fundamental property plays a pivotal role in the achievement of our results. Using this, the upcoming theorem establishes a relationship between the RTE of order statistics $X_{i:n}$ and the RTE of order statistics from a uniform distribution.

Theorem 2.1. *For all $\alpha > 0,\ \alpha \neq 1$, we have:*

$$H_\alpha(X_{i:n}; t) = \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)E[f^{\alpha-1}(F^{-1}(Y_i))] - 1\right],\ t > 0, \tag{2.7}$$

where $Y_i \sim \bar{B}_{F(t)}(\alpha(i-1)+1, \alpha(n-i)+1)$.

Proof. By making the change of variable as $u = F(x)$, from (1.2), (2.1), and (2.3), one gets

$$\begin{aligned} H_\alpha(X_{i:n}; t) &= \frac{1}{1-\alpha}\left[\int_t^\infty \left(\frac{f_{i:n}(x)}{S_{i:n}(t)}\right)^\alpha dx - 1\right] \\ &= \frac{1}{1-\alpha}\left[\int_t^\infty \left(\frac{F^{i-1}(x)S^{n-i}(x)f(x)}{\bar{B}_{F(t)}(i, n-i+1)}\right)^\alpha dx - 1\right] \end{aligned}$$

$$= \frac{1}{1-\alpha}\left[\frac{\bar{B}_{F(t)}(\alpha(i-1)+1,\alpha(n-i)+1)}{\bar{B}^{\alpha}_{F(t)}(i,n-i+1)}\int_t^{\infty}\frac{F^{\alpha(i-1)}(x)S^{\alpha(n-i)}(x)f^{\alpha}(x)}{\bar{B}_{F(t)}(\alpha(i-1)+1,\alpha(n-i)+1)}dx-1\right]$$

$$= \frac{1}{1-\alpha}\left[\frac{\bar{B}_{F(t)}(\alpha(i-1)+1,\alpha(n-i)+1)}{\bar{B}^{\alpha}_{F(t)}(i,n-i+1)}\int_{F(t)}^{1}\frac{u^{\alpha(i-1)}(1-u)^{\alpha(n-i)}f^{\alpha-1}(F^{-1}(u))}{\bar{B}_{F(t)}(\alpha(i-1)+1,\alpha(n-i)+1)}du-1\right]$$

$$= \frac{1}{1-\alpha}\left[((1-\alpha)H_{\alpha}(U_{i:n};F(t))+1)E[f^{\alpha-1}(F^{-1}(Y_i))]-1\right],\ t>0. \tag{2.8}$$

The last equality is obtained from Lemma 2.1, and this completes the proof. □

It is worth pointing out that Eq (2.7) demonstrates how the RTE of $[X_{i:n} - t|X_{i:n} > t]$ can be expressed as the product of two distinct terms, both of which are dependent on time t. However, the first term is influenced by the RTE of order statistics from a uniform distribution, while the second term is dependent on the distribution of the component lifetimes. By explicitly acknowledging this decomposition, we provide a deeper understanding of the factors influencing entropy and shed light on the role of the RTE and component lifetimes in the analysis. After some calculation, it can be seen that when in (2.7) the order α goes to unity, the Shannon entropy of the ith order statistic from a sample of F can be written as follows:

$$H(X_{i:n};t) = H(U_{i:n};F(t)) - E[\log f(F^{-1}(Y_i))], \tag{2.9}$$

where $Y_i \sim \bar{B}_{F(t)}(i, n-i+1)$. The particular case where $t = 0$, has already been derived by [13]. It is clear that Eq (2.9) demonstrates how the residual entropy of $[X_{i:n} - t|X_{i:n} > t]$ can be expressed as the differences of two distinct terms, both of which are dependent on time t. The first term is the residual entropy of order statistics consisting of (independent and identically distributed) i.i.d. random variables of uniform distribution on $[0, 1]$ while the second term depends on the truncated beta distribution.

Even if we have been able to construct a closed equation for the RTE of the initially ordered variable in the exponential case, considering the higher order statistics in some other distributions makes the process much more challenging. Regretfully, there are typically no closed-form equations available for the RTE of higher-order statistics in these distributions or in many other distributions. We are encouraged to investigate different methods for characterizing the RTE of order statistics in light of this constraint. In light of this, we offer the following theorem as a convincing demonstration that sheds light on the characteristics of these constraints and their use in real-world situations.

Theorem 2.2. *Let X and $X_{i:n}$ have RTEs $H_\alpha(X;t)$ and $H_\alpha(X_{i:n};t)$, respectively.*

(a) *Let $M_i = f_{Y_i}(m_i)$ where $m_i = \max\{F(t), \frac{i-1}{n-1}\}$ is the mode of the distribution of Y_i, then for $\alpha > 1$, we have*

$$H_\alpha(X_{i:n};t) \geq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n};F(t))+1)((1-\alpha)H_\alpha(X;t)+1)M_iS^{\alpha}(t)-1\right],$$

and for $0 < \alpha < 1$, we have

$$H_\alpha(X_{i:n};t) \leq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n};F(t))+1)((1-\alpha)H_\alpha(X;t)+1)M_iS^{\alpha}(t)-1\right].$$

(b) *Suppose that we have $M = f(m) < \infty$, thus m is the mode of f and $f(x) \leq M$. Then, for any $\alpha > 0$, we obtain*

$$H_\alpha(X_{i:n}; t) \geq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)M^{\alpha-1} - 1\right].$$

Proof. **(a)** By applying Theorem 2.3, we only need to establish a bound for $E[f^{\alpha-1}(F^{-1}(Y_i))]$. To do this, for $\alpha > 1$ one has

$$\begin{aligned}
E[f^{\alpha-1}(F^{-1}(Y_i))] &= \int_{F(t)}^{1} \frac{u^{\alpha(i-1)}(1-u)^{\alpha(n-i)}}{\bar{B}_{F(t)}(\alpha(i-1)+1, \alpha(n-i)+1)} f^{\alpha-1}(F^{-1}(u))du \\
&\geq M_i \int_{F(t)}^{1} f^{\alpha-1}(F^{-1}(u))du \\
&= M_i \int_{t}^{\infty} f^{\alpha}(x)dx \\
&= M_i[(1-\alpha)H_\alpha(X; t) + 1]S^{\alpha}(t).
\end{aligned}$$

The result now is easily obtained by recalling (2.7). The proof for $0 < \alpha < 1$ is easily obtained by reversing the inequality.

(b) Since for $\alpha > 1$ it holds that

$$f^{\alpha-1}(F^{-1}(u)) \leq M^{\alpha-1},$$

one can write

$$E[f^{\alpha-1}(F^{-1}(Y_i))]M^{\alpha-1}.$$

The result now is easily obtained from relation (2.7) and this completes the proof. By reversing the inequality, it is easy to obtain the proof for $0 < \alpha < 1$. □

The preceding theorem splits into two parts. The RTE associated with $X_{i:n}$, $H_\alpha(X_{i:n}; t)$ is lower bound in the first subdivision, represented by (a). Note that under some facts, the specified lower bound can be changed to an upper bound. This bound is generated by combining the RTE in the original situation with the incomplete beta function. On the other hand, we proposed a lower bound on the RTE of $X_{i:n}$ in part (b) of the theory, which we refer to as $H_\alpha(X_{i:n}; t)$. This lower bound is expressed via the RTE of ordered uniformly distributed variables and the mode, represented by m, of the underlying distribution. This finding provides interesting insights into the information properties of $X_{i:n}$ and provides a quantifiable measure of the lower bound of RTE with respect to the mode of the distribution. In Table 1, we list the bounds of the RTE of the order statistics based on Theorem 2.2 for some well-known distributions.

Table 1. Bounds on $H_\alpha(X_{i:n}; t)$ derived from Theorem 2.2 (parts (i) and (ii)).

Probability Density Function	Bounds
Standard half-Cauchy distribution	
$f(x) = \frac{2}{\pi(1+x^2)},\ x > 0,$	$\geq (\leq) \frac{1}{1-\alpha}\left[\frac{M_i 2^{\alpha-1}}{\pi^\alpha}((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)\bar{B}_{\frac{t^2}{1+t^2}}(\alpha - \frac{1}{2}, \frac{1}{2}) - 1\right]$
	$\geq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)\left(\frac{2}{\pi}\right)^{\alpha-1} - 1\right]$
Standard half-normal distribution	
$f(x) = \frac{2}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2},\ x > \mu > 0,$	$\geq (\leq) \frac{1}{1-\alpha}\left[\frac{M_i\sqrt{2^{\alpha+1}}}{\sigma^{\alpha-1}\pi^\alpha}((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)\overline{\Phi}(\sqrt{\frac{\alpha}{2}}(\frac{t-\mu}{\sigma})) - 1\right]$
	$\geq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)\left(\frac{2}{\sigma\sqrt{2\pi}}\right)^{\alpha-1} - 1\right]$
Generalized exponential distribution	
$f(x) = \frac{\lambda}{\beta}e^{-\frac{(x-\mu)}{\beta}}(1 - e^{-\frac{(x-\mu)}{\beta}})^{\lambda-1},\ x > \mu > 0,$	$\geq (\leq) \frac{1}{1-\alpha}\left[\frac{M_i\lambda^\alpha}{\beta^{\alpha-1}}((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)\bar{B}_{1-e^{-\frac{(x-\mu)}{\beta}}}(\alpha(\lambda-1) + 1, \alpha) - 1\right]$
	$\geq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)(\beta(1 - \frac{1}{\lambda})^{1-\lambda})^{1-\alpha} - 1\right]$
Generalized gamma distribution	
$f(x) = \frac{b^c}{\Gamma(c)}x^{c-1}e^{-bx},\ x > 0,$	$\geq (\leq) \frac{1}{1-\alpha}\left[\frac{M_i b^{\alpha-1}}{(\Gamma(c))^\alpha \alpha^{\alpha(c-1)+1}}((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)\Gamma(\alpha(c-1) + 1, \alpha bt) - 1\right]$
	$\geq \frac{1}{1-\alpha}\left[((1-\alpha)H_\alpha(U_{i:n}; F(t)) + 1)(\frac{b(c-1)^{c-1}e^{1-c}}{\Gamma(c)})^{\alpha-1} - 1\right]$

We address the monotonic behavior of the RTE of order statistics in the ensuing lemma. We first offer a core lemma which is fundamental to our research and serves as a foundation for our later discoveries.

Lemma 2.2. *Consider two nonnegative functions, $q(x)$ and $s_\beta(x)$, where $q(x)$ increases in x. Let t and c be real numbers such that $0 \leq t < c < \infty$. Additionally, let the random variable Z_β follow pdf $f_\beta(z)$, where $\beta > 0$, as*

$$f_\beta(z) = \frac{q^{r\beta}(z)s_\beta(z)}{\int_t^c q^{r\beta}(x)s_\beta(x)dx},\ z \in (t, c). \tag{2.10}$$

Suppose r is real-valued and let K_α be defined as:

$$K_\alpha(r) = \frac{1}{1-\alpha}\left[\frac{\int_t^c q^{r\alpha}(x)s_\alpha(x)dx}{\left(\int_t^c q^r(x)s_1(x)dx\right)^\alpha} - 1\right],\ \alpha > 0,\ \alpha \neq 1. \tag{2.11}$$

(i) *If for $\alpha > 1$ $(0 < \alpha < 1)$, $Z_\alpha \leq_{st} (\geq_{st})Z_1$, then $K_\alpha(r)$ increases in r.*

(ii) *If for $\alpha > 1$ $(0 < \alpha < 1)$, $Z_\alpha \geq_{st} (\leq_{st})Z_1$, then $K_\alpha(r)$ decreases in r.*

Proof. Proof of part (i) is similar to part (ii). Assuming that $K_\alpha(r)$ is differentiable in r, one obtains

$$\frac{\partial K_\alpha(r)}{\partial r} = \frac{1}{1-\alpha}\frac{\partial g_\alpha(r)}{\partial r},$$

where

$$g_\alpha(r) = \frac{\int_t^c q^{r\alpha}(x)s_\alpha(x)dx}{\left(\int_t^c q^r(x)s_1(x)dx\right)^\alpha}.$$

It is evident that

$$\begin{aligned}\frac{\partial g_\alpha(r)}{\partial r} &= \frac{\alpha}{\left(\int_t^c q^r(x)s_1(x)dx\right)^{\alpha+1}}\\ &\times \left[\int_t^c \log q(x)q^{r\alpha}(x)s_\alpha(x)dx\int_t^c q^r(x)s_1(x)dx - \int_t^c \log q(x)q^r(x)s_1(x)dx\int_t^c q^{r\alpha}(x)s_\alpha(x)dx\right]\\ &= \frac{\alpha\int_t^c q^r(x)s_1(x)dx\int_t^c q^{r\alpha}(x)s_\alpha(x)dx}{\left(\int_t^c q^r(x)s_1(x)dx\right)^{\alpha+1}}\left[E[\log q(Z_\alpha)] - E[\log q(Z_1)]\right] \le (\ge) 0. \qquad (2.12)\end{aligned}$$

Since $Z_\alpha \le_{st} (\ge_{st}) Z_1$ and, further, since $\log(\cdot)$ is increasing, thus one can show that $E[\log q(Z_\alpha)] \le (\ge) E[\log q(Z_1)]$. This implies that (2.12) is nonpositive (nonnegative). Therefore, $K_\alpha(r)$ increases in r. □

Corollary 2.1. In the setting of Lemma 2.2, if $q(x)$ decreases in x, then

(i) For $\alpha > 1$ $(0 < \alpha < 1)$, $Z_\alpha \le_{st} (\ge_{st}) Z_1$, then $K_\alpha(r)$ decreases when r increases.

(ii) If for $\alpha > 1$ $(0 < \alpha < 1)$, $Z_\alpha \ge_{st} (\le_{st}) Z_1$, then $K_\alpha(r)$ increases when r increases.

Due to Lemma 2.2, the next corollary can be regarded in the context of $(n-i+1)$-out-of-n structures where components have uniformly distributed random lifetimes.

Lemma 2.3. (i) *Take into account a parallel (series) system with n components with uniformly distributed lifetime on (0,1). The RTE of the system's lifetime decreases as the number of components rises.*

(ii) *If $i_1 \le i_2 \le n$, are integers, then $H_\alpha(U_{i_1:n}; t) \le H_\alpha(U_{i_2:n}; t)$ for $t \ge \frac{i_2-1}{n-1}$.*

Proof. (i) The presumption is that the system operates in parallel. Analogous reasoning can be applied to a series system to authenticate the outcome via Remark 2.9. From Lemma 2.1, we get

$$H_\alpha(U_{n:n}; t) = \frac{1}{1-\alpha}\left[\frac{\int_t^1 x^{\alpha(n-1)}dx}{\int_t^1 x^{n-1}dx} - 1\right],\ 0 < t < 1.$$

Therefore, Lemma 2.2 readily reveals that $H_\alpha(U_{i:n}; t)$ can be depicted as (2.11) where $q(x) = x$ and $s_\alpha(x) = x^\alpha$. We adopt the assumption, devoid of any generality loss, that $n \ge 1$ is continuous. Considering $\alpha > 1$ $(0 < \alpha < 1)$,

$$\frac{\int_t^1 x^{\alpha(n-1)}dx}{\int_t^1 x^{n-1}dx},$$

increases (decreases) in t. Hence, we can establish the inequality

$$Z_\alpha \ge_{st} (\le_{st}) Z_1,$$

where the pdf of Z_β, $\beta > 0$, is defined in Eq (2.10). By applying Lemma 2.2, we can deduce that the RTE for the parallel structure decreases as more components are included in the system.

(ii) To start, we observe that

$$\begin{aligned} H_\alpha(U_{i:n};t) &= \frac{1}{1-\alpha}\left[\frac{\int_t^1 x^{\alpha(i-1)}(1-x)^{\alpha(n-i)}dx}{\left(\int_t^1 x^{i-1}(1-x)^{n-i}dx\right)^\alpha}-1\right] \\ &= \frac{1}{1-\alpha}\left[\frac{\int_t^1 \left(\frac{x}{1-x}\right)^{\alpha i}\frac{(1-x)^{n\alpha}}{x^\alpha}dx}{\left(\int_t^1 \left(\frac{x}{1-x}\right)^{i}\frac{(1-x)^{n}}{x}dx\right)^\alpha}-1\right]. \end{aligned}$$

Furthermore, the pdf of Z_α as stated in (2.10) is

$$f_\alpha(z) = \frac{\left(\frac{z}{1-z}\right)^{\alpha i}\frac{(1-z)^{n\alpha}}{z^\alpha}}{\int_t^1 \left(\frac{x}{1-x}\right)^{\alpha i}\frac{(1-x)^{n\alpha}}{x^\alpha}dx},\ z \in (t,1),$$

where $q(x) = \frac{x}{1-x}$ and $s_\alpha(x) = \frac{(1-x)^{n\alpha}}{x^\alpha}$. Hence, for $1 \geq z \geq t \geq \frac{i_2-1}{n-1}$ and $\alpha > 1$ (or $0 < \alpha < 1$), we can write

$$Z_\alpha \leq_{st} (\geq_{st}) Z_1.$$

In conclusion, it can be inferred for $i_1 \leq i_2 \leq n$ that

$$H_\alpha(U_{i_1:n};t) \leq H_\alpha(U_{i_2:n};t), \quad t \geq \frac{i_2-1}{n-1},$$

and this completes the proof. □

Theorem 2.3. *Let f (pdf of lifetime components of a parallel (series) system) be increasing (decreasing). Then, the associated RTE of the systems's lifetime decreases as n increases.*

Proof. Assuming that $Y_n \sim \bar{B}_{F(t)}(\alpha(n-1)+1,1)$, $f_{Y_n}(y)$ indicates the pdf of Y_n. It is clear that

$$\frac{f_{Y_{n+1}}(y)}{f_{Y_n}(y)} = \frac{\bar{B}_{F(t)}(\alpha(n-1)+1,1)}{\bar{B}_{F(t)}(\alpha n+1,1)}y^\alpha,\ F(t) < y < 1,$$

increases in y. Consequently, $Y_n \leq_{lr} Y_{n+1}$ and, therefore, $Y_n \leq_{st} Y_{n+1}$. Moreover, for $\alpha > 1$ $(0 < \alpha < 1)$, it is shown that $f^{\alpha-1}(F^{-1}(x))$ increases (decreases) in x. Therefore,

$$E[f^{\alpha-1}(F^{-1}(Y_n))] \leq (\geq) E[f^{\alpha-1}(F^{-1}(Y_{n+1})]. \tag{2.13}$$

From Theorem 2.1, for $\alpha > 1$ $(0 < \alpha < 1)$, we have

$$\begin{aligned} (1-\alpha)H_\alpha(X_{n:n};t)+1 &= [(1-\alpha)H_\alpha(U_{n:n};F(t))+1]E[f^{\alpha-1}(F^{-1}(Y_n))] \\ &\leq (\geq) [(1-\alpha)H_\alpha(U_{n:n};F(t))+1]E[f^{\alpha-1}(F^{-1}(Y_{n+1}))] \\ &\leq (\geq) [(1-\alpha)H_\alpha(U_{n+1:n+1};F(t))+1]E[f^{\alpha-1}(F^{-1}(Y_{n+1}))] \\ &= (1-\alpha)H_\alpha(X_{n+1:n+1};t)+1. \end{aligned}$$

The first inequality is obtained by noting that $(1-\alpha)H_\alpha(U_{n:n};F(t))+1$ is nonnegative. The last inequality is obtained from Part (i) of Lemma 2.3. Thus, one can conclude that $H_\alpha(X_{n:n};t) \geq H_\alpha(X_{n+1:n+1};t)$ for all $t > 0$. □

According to the reliability theory, if a series system has more components, we can envision a situation where the pdf decreases and the RTE of the system decreases as well. This occurs when we have a lifespan model with a time-dependent failure rate ($h(t) = f(t)/S(t)$). The density function of the data distribution must therefore likewise decrease. There are certain lifetime distributions in the dependability domain, where RTE becomes down as the scale parameter is up. This is the case, for instance, when the shape parameter for the Weibull distribution is $a \leq 1$, while the shape parameter for the Gamma distribution is $b \leq 1$. Consequently, as the number of components increases, the RTE, which is related to the random lifespan brought about by the series structure and where the component lifetimes follow the Gamma (or Weibull) distribution, gets lower.

Now, we want to observe how the RTE of order statistics $X_{i:n}$ changes with i. We use Part (ii) of Lemma 2.3, which gives us a formula for the RTE of $X_{i:n}$ in terms of i.

Theorem 2.4. *Let f be a decreasing function. Let i_1 and i_2 be two integers such that $i_1 \leq i_2 \leq n$. Then, the RTE of the i_1-th smallest value of X among n samples, $X_{i_1:n}$, is less than or equal to the RTE of the i_2-th smallest value, $X_{i_2:n}$, for all values of X that are greater than or equal to the $F^{-1}(\frac{i_2-1}{n-1})$th percentile of F.*

Proof. For $i_1 \leq i_2 \leq n$, one can verify that $Y_{i_1} \leq_{lr} Y_{i_2}$. Thus, $Y_{i_1} \leq_{st} Y_{i_2}$. Now, we have

$$\begin{aligned}
(1-\alpha)H_\alpha(X_{i_1:n};t)+1 &= [(1-\alpha)H_\alpha(U_{i_1:n};t)+1]E[f^{\alpha-1}(F^{-1}(Y_{i_1}))] \\
&\geq (\leq) [(1-\alpha)H_\alpha(U_{i_1:n};t)+1]E[f^{\alpha-1}(F^{-1}(Y_{i_2}))] \\
&\geq (\leq) [(1-\alpha)H_\alpha(U_{i_2:n};t)+1]E[f^{\alpha-1}(F^{-1}(Y_{i_2}))] \\
&= (1-\alpha)H_\alpha(X_{i_2:n};t)+1.
\end{aligned}$$

Using Part (ii) of Lemma 2.3 and by a similar discussion as in the proof of Theorem 2.3, we can obtain the result. □

We can get a useful result from Theorem 2.4.

Corollary 2.2. *Let f be a decreasing function. Let i be a whole number that is less than or equal to half of $n+1$. Then, the RTE of $X_{i:n}$ increases in i as t exceeds the distribution median.*

Proof. Suppose $i_1 \leq i_2 \leq \frac{n+1}{2}$. This means that

$$m \geq F^{-1}(\frac{i_2-1}{n-1}),$$

in which $m = F^{-1}(\frac{1}{2})$ is the middle value of F. By Theorem 2.4, we get for $t \geq m$ that $H_\alpha(X_{i_1:n};t) \leq H_\alpha(X_{i_2:n};t)$. □

3. Numerical results

In this section, we provide some numerical examples in order to showcase the applicability of the obtained results in the previous section and estimate the value of $H_\alpha(X_{i:n};t)$, $i = 1, 2, \cdots, n$, for an exponential distribution with mean $1/\lambda$. Below, we provide an example for illustration of Theorem 2.1.

Example 3.1. Let X be a standard exponential distribution with mean one. Then, $f(F^{-1}(u)) = 1 - u,\ 0 < u < 1$, and

$$E[f^{\alpha-1}(F^{-1}(Y_i))] = \frac{\bar{B}_{1-e^{-t}}(\alpha(i-1)+1, \alpha(n-i+1))}{\bar{B}_{1-e^{-t}}(\alpha(i-1)+1, \alpha(n-i)+1)}. \tag{3.1}$$

Thus, from (2.7), we obtain

$$H_\alpha(X_{i:n};t) = \frac{1}{1-\alpha}\left[\frac{\bar{B}_{1-e^{-t}}(\alpha(i-1)+1, \alpha(n-i+1))}{\bar{B}^{\alpha}_{1-e^{-t}}(i, n-i+1)} - 1\right],\ i = 1, 2, \cdots, n. \tag{3.2}$$

In Figure 2, we plotted $H_\alpha(X_{i:n};t)$ for some values of α and $i = 1, 2, \cdots, 5$ when $n = 5$. When $i = 1$, we can use (3.2) to get

$$H_\alpha(X_{1:n};t) = \frac{n^{\alpha-1} - \alpha}{(1-\alpha)\alpha},\ t > 0.$$

Also, it is known that

$$H_\alpha(X;t) = \frac{1-\alpha}{(1-\alpha)\alpha},\ t > 0.$$

Therefore, we have

$$H_\alpha(X_{1:n};t) - H_\alpha(X;t) = \frac{n^{\alpha-1} - 1}{\alpha(1-\alpha)},\ t > 0.$$

This result exposes an interesting property: Time has no effect on the disparity between the RTE of the lifetime of the series system and the RTE of its constituent parts. Rather, it depends only on two variables: The total number of components in the system and, in the exponential case, the parameter α.

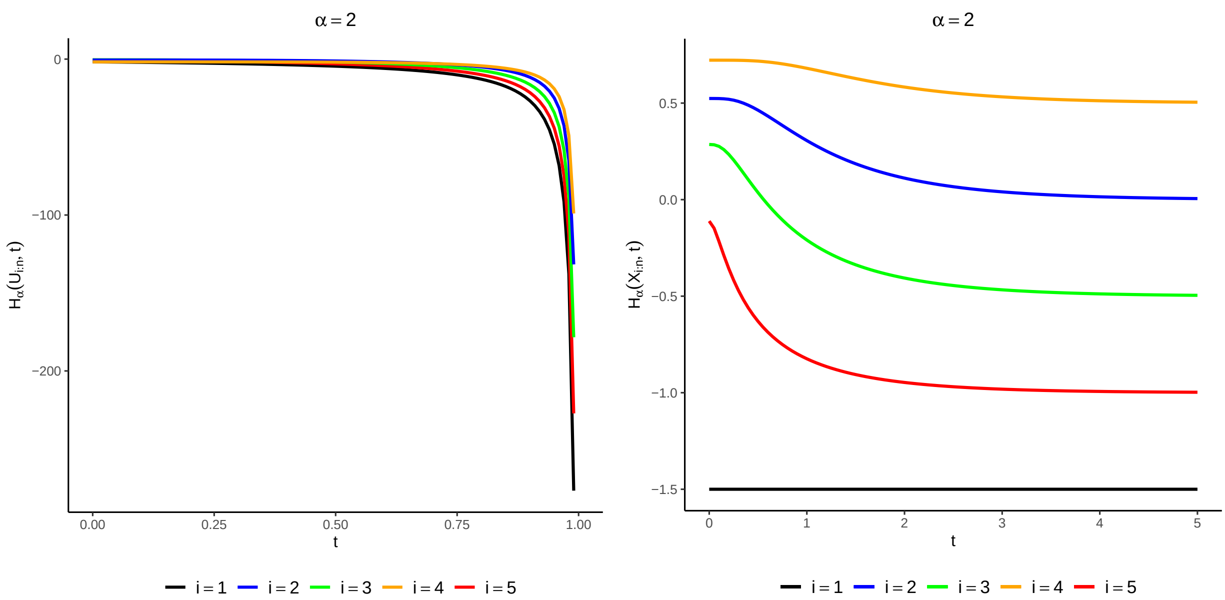

Figure 2. The exact values of $H_\alpha(X_{i:n};t)$ for $\alpha = 0.2$ (left panel) and $\alpha = 2$ (right panel) with respect to t.

Decreasing pdfs are found in many distributions, including mixtures of Pareto and exponential distributions. Conversely, some distributions such as the power distribution and associated density function have increasing pdfs. For distributions whose pdfs increase or decrease, we can establish a theorem using part (i) of the Corollary 2.1. However, as the following example demonstrates, this theory is not applicable to all types of $(n-i+1)$-out-of-n systems, thus caution is advised.

Example 3.2. Let us consider the system operates if, at least, $(n-1)$ of its n components are in action. Then, the system's lifetime is the second smallest component lifetime, $X_{2:n}$. The components have the same distribution, uniform on $(0,1)$. In Figure 3, we see how the RTE of $X_{2:n}$ changes with n when $a=2$ and $t=0.02$. The graph shows that the RTE of the system does not always decrease as n increases. For example, it reveals that $H_\alpha(X_{2:2};0.02)$ is less than that of $H_\alpha(X_{2:3};0.02)$.

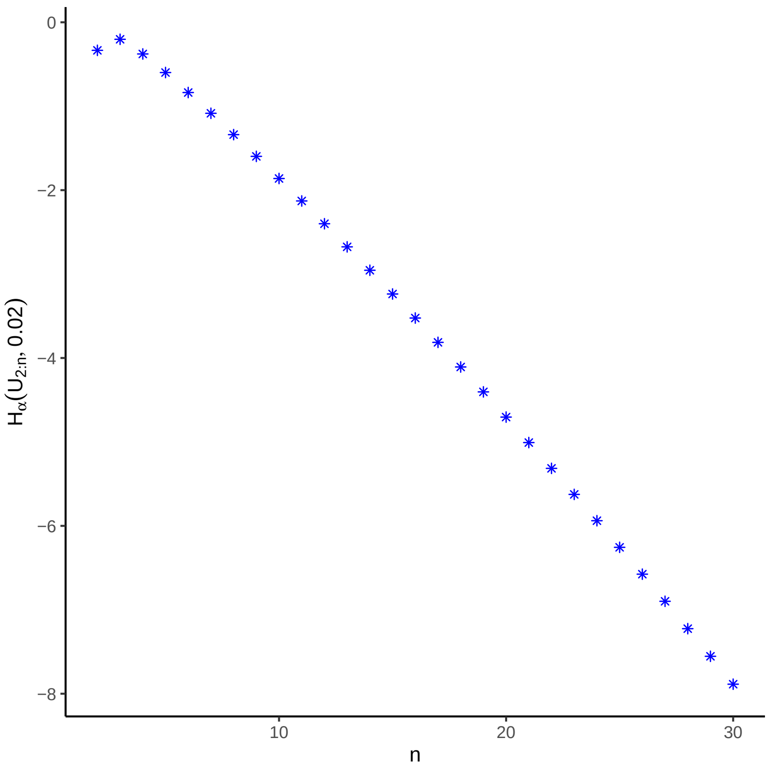

Figure 3. The RTE values for different n in a $(n-1)$-out-of-n system with a uniform parent distribution and $\alpha=2$ when $t=0.02$.

Here, we demonstrate that the condition $t \geq F^{-1}(\frac{i_2-1}{n-1})$ is a necessary condition in Theorem 2.4.

Example 3.3. Assume the sf of X is as

$$S(x)=\frac{1}{(1+x)^2},\ x>0.$$

In Figure 4, we see how the RTE of order statistics $X_{i:5}$, for $i=3,4$ and $\alpha=2$, changes with t, $t\in(0,5)$. The plots show that the RTE of the ordered variables does not always increase or decrease as i goes up for all values of t. For example, for $t<F^{-1}(\frac{3}{4})$, the RTE is not monotonic in i.

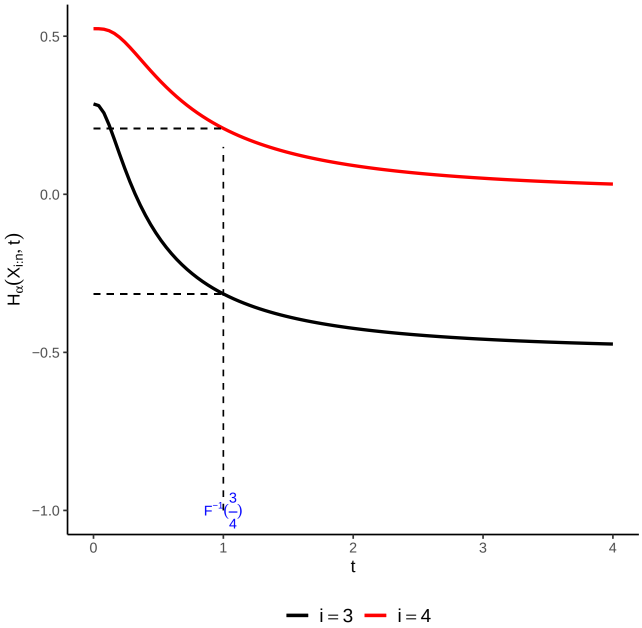

Figure 4. The plot of RTE of $X_{i:5}$, for $i = 3, 4$ and $\alpha = 2$ based on the survival function given in Example 3.3.

Hereafter, we carry out a simulation study for illustrating the estimation procedures developed in previous sections. To begin, we obtain the MLE of RTE. Using (2.7), we obtain the residual Tsallis entropy of order statistics based on an exponential distribution with mean $1/\lambda$ as follows:

$$H_\alpha(X_{i:n};t) = \frac{1}{1-\alpha}\left[\frac{\lambda^{\alpha-1}\bar{B}_{1-e^{-\lambda t}}(\alpha(i-1)+1, \alpha(n-i+1))}{\bar{B}^{\alpha}_{1-e^{-\lambda t}}(i, n-i+1)} - 1\right], \; i = 1, 2, \cdots, n.$$

Particularly for the case of $\alpha = 1$, and based on Eq (2.9), the Shannon entropy of the ith order statistic can be expressed as follows:

$$H(X_{i:n};t) = H(U_{i:n}; 1-e^{-\lambda t}) - E[\log(1-Y_i))],$$

where $Y_i \sim \bar{B}_{1-e^{-\lambda t}}(i, n-i+1)$.

To estimate $H_\alpha(X_{i:n};t)$ for simulated exponential data, we use the MLE of λ and we analyze its average bias and root mean square error (RMSE). We compute the bias and RMSE for various number of components ($n = 5, 10, 15$), different parameter values of $\lambda = 1, 2$, and $t = 0.5, 1, 1.5, 2$. The estimates are based on 5000 repetitions, and the results are shown in Tables 2 and 3. Suppose we have a random sample $X_1, X_2, \cdots, X_m$ drawn from an exponential distribution with mean $1/\lambda$. Then, the MLE of λ is given by $\widehat{\lambda} = \frac{m}{\sum_{i=1}^{m} X_i} = \frac{1}{\bar{X}}$. It is worth noting that the statistical data is generated based on the Monte-Carlo simulation. We estimate the values based on 5000 samples with different sample sizes $m = n = 5, 10$ and different values of the parameters λ, α, and t. Since MLE is an invariant estimator, we can estimate $H_\alpha(X_{i:n};t)$ for an exponential distribution via the MLE given by

$$\widehat{H}_\alpha(X_{i:n};t) = \frac{1}{1-\alpha}\left[\frac{\widehat{\lambda}^{\alpha-1}\bar{B}_{1-e^{-\widehat{\lambda} t}}(\alpha(i-1)+1, \alpha(n-i+1))}{\bar{B}^{\alpha}_{1-e^{-\widehat{\lambda} t}}(i, n-i+1)} - 1\right]$$

$$= \frac{1}{1-\alpha}\left[\frac{\bar{B}_{1-e^{-\frac{t}{\overline{X}}}}(\alpha(i-1)+1, \alpha(n-i+1))}{\overline{X}^{\alpha-1}\bar{B}^{\alpha}_{1-e^{-\frac{t}{\overline{X}}}}(i, n-i+1)} - 1\right],$$

for $i = 1, 2, \cdots, n$.

Furthermore, an estimation of the residual entropy of the ith order statistic can be formulated as follows:

$$H(X_{i:n}; t) = H(U_{i:n}; 1 - e^{-\widehat{\lambda}t}) - E[\log(1 - Y_i))] = H(U_{i:n}; 1 - e^{-\frac{t}{\overline{X}}}) - E[\log(1 - Y_i))],$$

where $Y_i \sim \bar{B}_{1-e^{-\frac{t}{\overline{X}}}}(i, n-i+1)$.

For simplicity, we only present the results for series and parallel systems for values of $\alpha = 0.2, 1, 1.2, 2$. However, similar trends have been observed for other values of the parameters and sample sizes. The results are displayed in Tables 2 and 3.

Table 2. The bias and RMSE of the estimate of $H_\alpha(X_{i:n}; t)$ for $\alpha = 0.2, 1$ with $i = 1$ and $i = n$.

			$H_{0.2}(X_{1:n}; t)$		$H_{0.2}(X_{n:n}; t)$		$H(X_{1:n}; t)$		$H(X_{n:n}; t)$	
n	λ	t	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
20	1	0.0	-0.004111	0.101963	0.023669	1.400607	-0.024673	0.232202	-0.023956	0.224504
		0.5	-0.002940	0.100067	-0.015000	1.364974	-0.025537	0.226118	-0.026153	0.229583
		1.0	-0.001218	0.100482	-0.022491	1.452236	-0.029445	0.227282	-0.028627	0.237049
		1.5	-0.005024	0.101233	-0.048425	1.515576	-0.024896	0.231191	-0.044507	0.279241
		2.0	-0.001714	0.102707	-0.013366	1.513197	-0.019139	0.224888	-0.057983	0.336823
30	1	0.0	-0.001575	0.060328	-0.021352	1.138932	-0.020142	0.184149	-0.011940	0.188139
		0.5	-0.000900	0.060146	-0.021600	1.151689	-0.015972	0.181127	-0.020274	0.183855
		1.0	0.000489	0.060145	-0.049570	1.154575	-0.014018	0.185117	-0.018327	0.185383
		1.5	-0.002124	0.060362	-0.079052	1.233653	-0.015212	0.183027	-0.022842	0.205462
		2.0	-0.001410	0.059558	-0.051625	1.276884	-0.014601	0.184373	-0.045161	0.252842
40	1	0.0	-0.000932	0.041486	-0.006984	0.973690	-0.013025	0.158373	-0.014442	0.157876
		0.5	-0.000291	0.041090	0.010585	0.988650	-0.010680	0.160016	-0.012075	0.160450
		1.0	-0.000965	0.041978	-0.028449	1.012318	-0.010349	0.158212	-0.012004	0.162294
		1.5	-0.001168	0.041333	-0.034202	1.053832	-0.010614	0.161088	-0.016247	0.163795
		2.0	0.000397	0.041325	-0.027568	1.110563	-0.011821	0.157864	-0.025932	0.197784
20	2	0.0	-0.002446	0.058226	-0.007118	0.824543	-0.026646	0.228696	-0.020955	0.227106
		0.5	-0.001087	0.058753	-0.038594	0.833928	-0.025026	0.226999	-0.030133	0.233960
		1.0	-0.001123	0.057952	-0.012696	0.869793	-0.031726	0.224602	-0.055647	0.330609
		1.5	-0.001031	0.059087	0.027672	0.804698	-0.022875	0.227325	-0.025042	0.332517
		2.0	-0.002530	0.058846	0.020016	0.720627	-0.016961	0.230532	-0.014888	0.297939
30	2	0.0	-0.000432	0.034163	-0.012610	0.655555	-0.013714	0.183895	-0.016170	0.189696
		0.5	-0.000803	0.034582	-0.029846	0.676429	-0.012091	0.182673	-0.017238	0.183837
		1.0	-0.000402	0.035017	-0.015309	0.736819	-0.018067	0.183517	-0.043436	0.253968
		1.5	-0.000598	0.034962	-0.005882	0.714880	-0.018234	0.185157	-0.017685	0.301256
		2.0	-0.000930	0.034343	0.027765	0.641635	-0.016395	0.182450	-0.006860	0.268161
40	2	0.0	-0.000400	0.023527	-0.009999	0.574911	-0.008702	0.158877	-0.013224	0.162924
		0.5	-0.001212	0.023618	-0.004161	0.584604	-0.011959	0.160160	-0.012536	0.157172
		1.0	-0.001119	0.023521	-0.015661	0.646895	-0.013997	0.160547	-0.031483	0.197244
		1.5	-0.000656	0.023497	-0.004489	0.632596	-0.014761	0.156546	-0.028053	0.265230
		2.0	0.000276	0.024035	0.019073	0.588827	-0.013667	0.155949	-0.005080	0.251386

Table 3. The bias and RMSE of the estimate of $H_\alpha(X_{i:n};t)$ for $\alpha = 1.2, 2$ with $i = 1$ and $i = n$.

			$H_{1.2}(X_{1:n};t)$		$H_{1.2}(X_{n:n};t)$		$H_2(X_{1:n};t)$		$H_2(X_{n:n};t)$	
n	λ	t	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
20	1	0.0	-0.036685	0.348336	-0.023340	0.173762	-0.452599	2.466906	-0.013824	0.064536
		0.5	-0.042676	0.353032	-0.022528	0.170276	-0.496878	2.514799	-0.012206	0.065292
		1.0	-0.051450	0.351932	-0.028482	0.178117	-0.484141	2.532462	-0.014892	0.067977
		1.5	-0.042779	0.349915	-0.043827	0.216008	-0.505927	2.552919	-0.025610	0.092888
		2.0	-0.040903	0.356226	-0.054122	0.260015	-0.526016	2.547214	-0.041800	0.131094
30	1	0.0	-0.040302	0.307586	-0.013597	0.136502	-0.550497	2.972020	-0.008951	0.049803
		0.5	-0.027936	0.308898	-0.015890	0.137044	-0.521227	3.066964	-0.009581	0.050169
		1.0	-0.031743	0.307648	-0.013536	0.137030	-0.578099	3.001601	-0.009409	0.051671
		1.5	-0.035009	0.308558	-0.020906	0.151498	-0.566610	2.933522	-0.010304	0.054673
		2.0	-0.042133	0.309328	-0.035845	0.198142	-0.560626	2.969982	-0.021695	0.082308
40	1	0.0	-0.026488	0.280460	-0.011294	0.119258	-0.493912	3.465678	-0.007291	0.042477
		0.5	-0.029448	0.281887	-0.012519	0.115980	-0.475527	3.386782	-0.005798	0.042768
		1.0	-0.025177	0.271100	-0.009937	0.118043	-0.484233	3.316020	-0.006592	0.043252
		1.5	-0.024838	0.286013	-0.014468	0.123224	-0.537798	3.378410	-0.006441	0.044033
		2.0	-0.023824	0.280455	-0.021237	0.146304	-0.479358	3.363165	-0.011877	0.054860
20	2	0.0	-0.056318	0.411699	-0.022944	0.198262	-1.072021	4.977434	-0.028913	0.128285
		0.5	-0.050415	0.398105	-0.027264	0.202833	-0.993129	5.071529	-0.029730	0.134172
		1.0	-0.056414	0.397686	-0.062243	0.301835	-1.138403	4.961945	-0.082598	0.266717
		1.5	-0.032251	0.401509	-0.043805	0.321194	-1.028292	5.080591	-0.074257	0.323894
		2.0	-0.053717	0.398513	-0.021448	0.283873	-0.955008	5.100835	-0.045678	0.303842
30	2	0.0	-0.038296	0.352777	-0.018590	0.160516	-1.072724	5.855002	-0.018836	0.098837
		0.5	-0.043430	0.351647	-0.016782	0.160062	-0.966672	5.946254	-0.017346	0.102989
		1.0	-0.033139	0.349205	-0.039773	0.223367	-1.060875	5.911871	-0.039096	0.168524
		1.5	-0.039873	0.352490	-0.030791	0.279365	-0.999007	5.851596	-0.063707	0.255772
		2.0	-0.040241	0.358401	-0.014475	0.250281	-1.062045	6.059834	-0.039921	0.256821
40	2	0.0	-0.038086	0.322851	-0.011819	0.137116	-1.005885	6.886099	-0.013194	0.083738
		0.5	-0.029491	0.319172	-0.012025	0.135845	-1.052631	6.825047	-0.010727	0.084993
		1.0	-0.032562	0.318250	-0.023726	0.172899	-1.043127	6.809147	-0.021793	0.109989
		1.5	-0.036946	0.326066	-0.032002	0.243537	-1.036542	6.796656	-0.053233	0.214256
		2.0	-0.030189	0.321165	-0.006727	0.231400	-0.995804	6.914287	-0.033001	0.230448

In order to evaluate the performance of the suggested estimators, we present a practical application using real-world data, allowing us to analyze and validate their performance in a real-life context.

Example 3.4. We utilize a dataset obtained from [27], which comprises measurements of the strength of 1.5 cm glass fibers conducted at the National Physical Laboratory in England. This dataset serves as the basis for our empirical analysis, which is given as follows:

Data Set: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89

In a study conducted by [22], it was verified that an exponential distribution provides a good fit for the available data. Using this dataset, we computed the MLE of the parameter λ as $\widehat{\lambda} = 0.663647$. Consequently, for $n = 5$ and $t = 2$, we obtained the following estimates: $\widehat{H}_{0.2}(X_{1:5};2) = 1.144167$, $\widehat{H}_{0.2}(X_{5:5};2) = 8.037384$, $\widehat{H}_{1.2}(X_{1:5};2) = -0.2962707$, $\widehat{H}_{1.2}(X_{5:5};2) = 1.367607$, $\widehat{H}_2(X_{1:5};2) = -0.6591172$, and $\widehat{H}_2(X_{5:5};2) = 0.7645877$.

4. Conclusions

The concept of RTE for order statistics was discussed in this article. In the context of the RTE of the order statistics of a random sample with uniformly distributed units, we introduce a new method to express the RTE of the order statistics of a continuous random variable. This relationship sheds light on the properties and behavior of the RTE for various distributions. In addition, we have constructed boundary conditions that allow a better understanding of the properties and provide a realistic approximation since it is difficult to find closed-form equations for the RTE of ordered variables. These boundary conditions are useful tools to study and compare the RTE values in different contexts. In addition, we examined how the total number of observations n and the index of order statistics i affect the RTE. We were able to better understand the relationship between the entropy of the general distribution and the position of the order statistic by looking at the variations of the RTE with respect to i and n. We provide illustrative examples to support our results and demonstrate the application of our method. These examples emphasize the usefulness of the RTE for order statistics and show how adaptable our method is for other distributions. In summary, this study improves our understanding of RTE for order statistics by establishing relationships, generating bounds, and examining the effects of index and sample size. The results of this study can be applied to other information measures commonly addressed in the literature, such as historical cumulative Tsallis entropy and dynamic cumulative residual Tsallis entropy.

Author contributions

Mansour Shrahili and Mohamed Kayid: Writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest declared by the authors.

References

1. M. Abbasnejad, N. R. Arghami, Renyi entropy properties of order statistics, *Commun. Stat.-Theory M.,* **40** (2010), 40–52. https://doi.org/10.1080/03610920903353683

2. S. Abe, Axioms and uniqueness theorem for Tsallis entropy, *Phys. Lett. A,* **271** (2000), 74–79. https://doi.org/10.1016/S0375-9601(00)00337-6
3. G. Alomani, M. Kayid, Further properties of Tsallis entropy and its application, *Entropy,* **25** (2023), 199. https://doi.org/10.3390/e25020199
4. B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, *A first course in order statistics*, SIAM, 2008. https://doi.org/10.1137/1.9780898719062
5. M. Asadi, N. Ebrahimi, E. S. Soofi, Dynamic generalized information measures, *Stat. Probab. Lett.,* **71** (2005), 85–98. https://doi.org/10.1016/j.spl.2004.10.033
6. M. Asadi, N. Ebrahimi, E. S. Soofi, Connections of Gini, Fisher, and Shannon by Bayes risk under proportional hazards, *J. Appl. Probab.,* **54** (2017), 1027–1050. https://doi.org/10.1017/jpr.2017.51
7. S. Baratpour, J. Ahmadi, N. R. Arghami, Entropy properties of record statistics, *Stat. Pap.,* **48** (2007), 197–213. https://doi.org/10.1007/s00362-006-0326-7
8. S. Baratpour, J. Ahmadi, N. R. Arghami, Characterizations based on Rényi entropy of order statistics and record values, *J. Stat. Plan. Infer.,* **138** (2008), 2544–2551. https://doi.org/10.1016/j.jspi.2007.10.024
9. J. E. Contreras-Reyes, D. I. Gallardo, O. Kharazmi, Time-dependent residual Fisher information and distance for some special continuous distributions, *Commun. Stat. Simul. Comput.,* 2022, 1–21. https://doi.org/10.1080/03610918.2022.2146136
10. H. A. David, H. N. Nagaraja, *Order statistics*, John Wiley and Sons, 2004. https://doi.org/10.1002/0471722162
11. N. Ebrahimi, How to measure uncertainty in the residual life time distribution, *Sankhyā: Indian J. Stat., Ser. A,* **50** (1996), 48–56. Available from: `http://www.jstor.org/stable/25051082.`
12. N. Ebrahimi, E. S. Soofi, H. Zahedi, Information properties of order statistics and spacings, *IEEE T. Inform. Theory,* **50** (2004), 177–183. https://doi.org/10.1109/TIT.2003.821973
13. N. Ebrahimi, E. S. Soofi, R. Soyer, Information measures in perspective, *Int. Stat. Rev.,* **78** (2010), 383–412. https://doi.org/10.1111/j.1751-5823.2010.00105.x
14. N. Ebrahimi, E. S. Soofi, R. Soyer, Fractional cumulative residual inaccuracy information measure and its extensions with application to chaotic maps, *Int. J. Bifurcat. Chaos,* **34** (2024), 2450006. https://doi.org/10.1142/S0218127424500068
15. E. Maasoumi, The measurement and decomposition of multi-dimensional inequality, *Econometrica: J. Economet. Soc.,* **54** (1986), 991–997. https://doi.org/10.2307/1912849
16. A. M. Mariz, On the irreversible nature of the Tsallis and Renyi entropies, *Phys. Lett. A,* **78** (1992), 409–411. https://doi.org/10.1016/0375-9601(92)90339-N
17. A. K. Nanda, P. Paul, Some results on generalized residual entropy, *Inf. Sci.,* **176** (2006), 27–47. https://doi.org/10.1016/j.ins.2004.10.008
18. O. Nicolis, J. Mateu, J. E. Contreras-Reyes, Wavelet-based entropy measures to characterize two-dimensional fractional Brownian fields, *Entropy,* **22** (2020), 196. https://doi.org/10.3390/e22020196

19. S. Park, The entropy of consecutive order statistics, *IEEE T. Inform. Theory,* **41** (1995), 2003–2007. https://doi.org/10.1109/18.476325

20. S. Park, A goodness-of-fit test for normality based on the sample entropy of order statistics, *Stat. Prob. Lett.,* **44** (1999), 359–363. https://doi.org/10.1016/S0167-7152(99)00027-9

21. M. Shaked, J. G. Shanthikumar, *Stochastic orders*, Springer Science and Business Media, 2007. https://doi.org/10.1007/978-0-387-34675-5

22. R. Shanker, F. Hagos, S. Sujatha, On modeling of lifetimes data using exponential and Lindley distributions, *Biom. Biostat. Int. J.,* **2** (2015), 1–9. https://doi.org/10.15406/bbij.2015.02.00042

23. C. E. Shannon, A mathematical theory of communication, *Bell Syst. Tech. J.,* **27** (1948), 379–423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x

24. M. Shrahili, M. Kayid, Uncertainty quantification based on residual Tsallis entropy of order statistics, *Preprints,* 2023. https://doi.org/10.20944/preprints202309.1495.v1

25. M. Shrahili, M. Kayid, Excess lifetime extropy of order statistics, *Axioms,* **12** (2023), 1024. https://doi.org/10.3390/axioms12111024

26. M. Shrahili, M. Kayid, Some new results involving past Tsallis entropy of order statistics, *Entropy,* **25** (2023), 1581. https://doi.org/10.3390/e25121581

27. R. L. Smith, J. C. Naylor, A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *J. Roy. Stat. Soc. C-Appl.,* **36** (1987), 358–369. https://doi.org/10.2307/2347795

28. C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *J. Stat. Phys.,* **52** (1988), 479–487. https://doi.org/10.1007/BF01016429

29. K. M. Wong, S. Chen, The entropy of ordered sequences and order statistics, *IEEE T. Inform. Theory,* **36** (1990), 276–284. https://doi.org/10.1109/18.52473

30. S. Zarezadeh, M. Asadi, Results on residual Rényi entropy of order statistics and record values, *Inf. Sci.,* **180** (2010), 4195–4206. https://doi.org/10.1016/j.ins.2010.06.019

31. Z. Zhang, Uniform estimates on the Tsallis entropies, *Lett. Math. Phys.,* **80** (2007), 171–181. https://doi.org/10.1007/s11005-007-0155-1