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Research article

One class class of coupled system fractional impulsive hybrid integro-differential equations

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Abstract: In this research, we investigate the existence of solution for a class of coupled fractional impulsive hybrid integro-differential equations with hybrid boundary conditions. Our primary tools for this analysis are the Banach contraction mapping principle (BCMP) and Schaefer's fixed point theorem. This study ended with two applied examples to facilitate understanding of the theoretical results obtained.

Keywords: coupled system; fractional impulsive hybrid integro-differential equations; fixed point theorem; hybrid boundary conditions
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1. Introduction

Systems of fractional differential equations have significant real-world applications across various domains. In engineering, they model viscoelastic materials and control systems with memory effects, providing more accurate descriptions than traditional models. In biology, these equations help simulate processes such as the spread of diseases or population dynamics, accounting for hereditary and memory aspects of biological systems. In finance, fractional differential equations are used to model markets with long-range dependencies and memory, offering a more nuanced understanding of asset prices and volatility. Their flexibility and ability to capture complex dynamics make them invaluable in these fields (see [1]).

On the contrary, impulsive differential equations offer an inherent approach to elucidate various dynamic phenomena in the real world. While most processes in applied sciences are conventionally characterized by differential equations, there exists a unique scenario in certain physical phenomena where abrupt changes occur during their evolution. Instances include mechanical systems experiencing impacts, biological systems like heartbeat and blood flow, population dynamics, natural disasters, and more. These abrupt changes are often of brief duration, manifesting instantly in the form of pulses. Modeling such phenomena requires formulations that explicitly and concurrently embrace the continuous evolution of the phenomenon along with these instantaneous changes (see [2–8]). Hybrid fractional differential equations blend the characteristics of fractional calculus with discrete and continuous dynamic elements, offering a versatile framework for modeling complex systems (see [6,9]. These equations are particularly useful in fields like control theory, where they describe systems with both continuous processes and sudden changes or impulses. In biology, they can model the interaction between continuous growth and sudden environmental changes. In engineering, hybrid fractional differential equations help simulate systems with both gradual and abrupt shifts, such as in material stress analysis and signal processing. Their ability to capture diverse dynamic behaviors makes them invaluable in various scientific and engineering applications.

Schaefer's fixed point theorem is a powerful mathematical tool used to prove the existence of fixed points in certain function spaces. It extends the concept of fixed points by combining aspects of the Banach fixed point theorem with compactness conditions. Specifically, Schaefer's theorem states that if a continuous mapping on a Banach space maps a bounded, closed, and convex subset into itself and is compact, then there is at least one fixed point within that subset. This theorem is particularly useful in analyzing differential and integral equations, ensuring solutions exist under specific conditions.
In 2015, Sitho et al. [10] discussed the following boundary value problem:

$$\begin{cases} D^{\tau}\left(\frac{\rho(\hat{\iota})-\sum_{i=1}^{m} I^{\beta_i}\varphi_i(\hat{\iota},\rho(t))}{\psi(\hat{\iota},\rho(t))}\right) = \varpi(\hat{\iota},\rho(\hat{\iota})), \hat{\iota}\in J=[0,T], \quad 0<\tau\le 1,\\ \rho(0)=0. \end{cases}$$

The symbol D^{τ} represents the Riemann-Liouville fractional derivative with an order of τ, where $0<\tau\le 1$. Additionally, I^{ς} denotes the Riemann-Liouville fractional integral with an order of $\varsigma>0$, $\varsigma\in\{\beta_1,\beta_2,\ldots,\beta_m\}$, $\psi\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\varpi\in C(J\times\mathbb{R},\mathbb{R})$, with $\varphi_i\in C(J\times\mathbb{R},\mathbb{R})$ with $\varphi_i(0,0)=0, i=1,2,\ldots,m$.

Shah et al. [11] investigated a coupled system of fractional impulsive boundary problems:

$$\begin{cases} D^{\gamma_1}\vartheta(\hat{\varkappa}) = \Phi_1(\hat{\varkappa},\vartheta(\hat{\varkappa}),\eta(\hat{\varkappa})) \quad a.e. \quad \hat{\varkappa}\in J=[0,1], \hat{\varkappa}\neq\hat{\varkappa}_i \quad 1<\gamma_1\le 2,\\ \Delta\vartheta|_{\hat{\varkappa}=\hat{\varkappa}_i} = I_i(\vartheta(\hat{\varkappa}_i)), \qquad \Delta'\vartheta|_{\hat{\varkappa}=\hat{\varkappa}_i} = \bar{I}_i(\vartheta(\hat{\varkappa}_i)), \quad \hat{\varkappa}_i\in(0,1), i=1,2,\ldots,n,\\ \vartheta(0)=h_1(\vartheta), \qquad \vartheta(1)=g_1(\vartheta),\\ D^{\tau_1}\eta(\hat{\varkappa}) = \Psi_1(\hat{\varkappa},\vartheta(\hat{\varkappa}),\eta(\hat{\varkappa})) \quad a.e. \quad \hat{\varkappa}\in J', \quad 1<\tau_1\le 2,\\ \Delta\eta|_{\hat{\varkappa}=\hat{\varkappa}_j} = J_i(\eta(\hat{\varkappa}_j)), \qquad \Delta' x|_{\hat{\varkappa}=\hat{\varkappa}_i} = \bar{J}_j(\eta(\hat{\varkappa}_j)), \quad \hat{\varkappa}_j\in(0,1), j=1,2,\ldots,m,\\ \eta(0)=\kappa_1(\eta), \qquad \eta(1)=f_1(\eta), \end{cases} \tag{1.1}$$

where $1 < \gamma_1,\ \tau_1 \leq 2$, Φ_1 , Ψ_1,I_i, $\bar{I}_i$, J_j, $\bar{J}_j$ are continuous functions, g_1, h_1, κ_1, f_1 are fixed continuous functions, and $\Delta\vartheta|_{\hat{\varkappa}=\hat{\varkappa}_i} = \vartheta(\hat{\varkappa}_i^+) - \vartheta(\hat{\varkappa}_i^-)$,$\Delta' x|_{\hat{\varkappa}=\hat{\varkappa}_i} = \vartheta'(\hat{\varkappa}_i^+) - \vartheta'(\hat{\varkappa}_i^-)$, $\Delta y|_{\hat{\varkappa}=\hat{\varkappa}_j} = \eta(t_j^+) - \eta(\hat{\varkappa}_j^-)$, $\Delta'\eta|_{\hat{\varkappa}=\hat{\varkappa}_j} = \eta'(\hat{\varkappa}_j^+) - \eta'(t_j^-)$.

Inspired by [12] and recent studies on impulsive hybrid fractional integro-differential equations, we examine the following coupled system.

$$\begin{cases} D^{\hat{\iota}_1}\left(\frac{\vartheta_1(\hat{\varkappa})-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}\right) = \varpi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa})), \hat{\varkappa} \in \mathcal{J} = [0,T], \quad 1 < \hat{\iota}_1 \leq 2, \\ D^{\hat{\iota}_2}\left(\frac{\vartheta_2(\hat{\varkappa})-I^{\hat{\kappa}_2}\xi_2(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_2(\hat{\varkappa},\vartheta_2(\hat{\varkappa}),\vartheta_1(\hat{\varkappa}))}\right) = \varpi_2(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa})), \hat{\varkappa} \in \mathcal{J} = [0,T], \quad 1 < \hat{\iota}_2 \leq 2, \\ \vartheta_1(\hat{\varkappa}_i^+) = \vartheta_1(\hat{\varkappa}_i^-) + I_i(\vartheta_1(\hat{\varkappa}_i^-)), \quad \hat{\varkappa}_i \in (0,1), i = 1,2,\dots,n, \\ \vartheta_2(\hat{\varkappa}_j^+) = \vartheta_2(\hat{\varkappa}_j^-) + I_j(\vartheta_2(\hat{\varkappa}_j^-)), \quad \hat{\varkappa}_j \in (0,1), j = 1,2,\dots,m, \\ \frac{\vartheta_1(0)}{\varphi_1(0,\vartheta_1(0),\vartheta_2(0))} = \vartheta_0, \qquad \frac{\vartheta_1(T)}{\varphi_1(T,\vartheta_1(T),\vartheta_2(T))} = \vartheta_{T_1}, \\ \frac{\vartheta_2(0)}{\varphi_2(0,\vartheta_1(0),\vartheta_2(0))} = \vartheta_1, \qquad \frac{\vartheta_2(T)}{\varphi_2(T,\vartheta_1(T),\vartheta_2(T))} = \vartheta_{T_2}. \end{cases} \tag{1.2}$$

The parameters $\hat{\iota}_1, \hat{\iota}_2 > 0$ and $\hat{\kappa}_1, \hat{\kappa}_2 > 0$ are given, with $\vartheta_i \in \mathbb{R}$, and $D^{\hat{\iota}_i}$ denoting the Caputo fractional derivative of order $\hat{\iota}_i$ for $i = 1,2$. The operator $I^{\hat{\kappa}}$ represents the Riemann-Liouville fractional integral of order $\hat{\kappa} > 0$. The functions $\varphi_i : \mathcal{J} \times \mathbb{R}^2 \longrightarrow \mathbb{R} \setminus \{0\}$ and $\xi_i : \mathcal{J} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous, satisfying $\xi_i(0,\vartheta_i(0),\vartheta_i(0)) = 0$, and $\varpi_i \in C(J\times\mathbb{R}^2,\mathbb{R})$ is a function with specific properties for $i = 1,2$. Additionally, $I_k : \mathbb{R} \longrightarrow \mathbb{R}$ is defined, and $u(t_k^+) = \lim_{\epsilon\to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon\to 0^-} u(t_k + \epsilon)$ denote the right and left limits of $u(t)$ at $t = t_k$, where $k = i, j$.

By a solution of the problem (1.2), we mean a function $\vartheta \in C(\mathcal{J},\mathbb{R})$ such that
(A0) the function $\vartheta_1 \mapsto \frac{\vartheta_1}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}))}$ is increasing in $\mathbb{R}$ for every $\hat{\varkappa} \in \mathcal{J}$, and
(i) the function $\vartheta_2 \mapsto \frac{\vartheta_2}{\varphi_2(\hat{\varkappa},\vartheta_1(\hat{\varkappa}))}$ is increasing in $\mathbb{R}$ for every $\hat{\varkappa} \in \mathcal{J}$, and
(iii) ϑ satisfies the equations in (1.2).

The orginality of this work is that we investigate the existence of the solution to a system of fractional differential equations. We also verified the necessary conditions for the existence and uniqueness of these solutions using Schaefer's fixed point theorem where it is rarely used in literature.

The structure of this paper is outlined as follows. Section 2 revisits certain concepts, fractional calculation laws, and establishes preparatory results. Section 3 delves into the existence solution of the initial value problem (1.2), employing the Banach contraction mapping principle (BCMP) and Schaefer's fixed point theorem. In Section 4, two examples are presented to enhance the clarity of the study's findings. Section 5 concludes with a summary and outlines directions for future work.

2. Preliminaries

Before tackling the results, we provide some definitions and essential properties of fractional calculus.

Consider $\mathcal{J}_0 = [0,\hat{\varkappa}_1], \mathcal{J}_1 = (\hat{\varkappa}_1,\hat{\varkappa}_2], \dots, \mathcal{J}_{n-1} = (\hat{\varkappa}_{n-1},\hat{\varkappa}_n], \mathcal{J}_n = (\hat{\varkappa}_n,T], n \in \mathbb{N}, n > 1$.
For $\hat{\varkappa}_i \in (0,T)$ such that $\hat{\varkappa}_1 < \hat{\varkappa}_2 < \dots < \hat{\varkappa}_n$, we define the following spaces:
$\mathcal{J}' = \mathcal{J} \setminus \{\hat{\varkappa}_1,\hat{\varkappa}_2,...,\hat{\varkappa}_n\}$,

$$\hat{\mathfrak{X}} = \{\vartheta_1 \in C(\mathcal{J},\mathbb{R}) : \vartheta_1 \in C(J'),\ \text{left limit}\ \vartheta_1(\hat{\varkappa}_i^+)$$

and right limit $\vartheta_1(\hat{\varkappa}_i^-)$) exist, and $\vartheta_1(\hat{\varkappa}_i^-) = \vartheta_1(\hat{\varkappa}_i), 1 \leq i \leq n\}$,

and

$$\hat{\mathfrak{Y}} = \{\vartheta_2 \in C(\mathcal{J}, \mathbb{R}) : \vartheta_2 \in C(\mathcal{J}'), \text{ left limit } \vartheta_2(\hat{\varkappa}_j^+)$$
$$\text{and right limit } \vartheta_2(\hat{\varkappa}_j^-)) \text{ exist, and } \vartheta_2(t_j^-) = \vartheta_2(\hat{\varkappa}_j), 1 \leq j \leq m\}.$$

Then, the product space $(\hat{\mathfrak{X}}\times\hat{\mathfrak{Y}}, \|(\vartheta_1, \vartheta_2)\|)$ endowed with the norm $\|(\vartheta_1, \vartheta_2)\| = \|\vartheta_1\|+\|\vartheta_2\|, (\vartheta_1, \vartheta_2) \in \hat{\mathfrak{X}} \times \hat{\mathfrak{Y}}$ is also a Banach space.

Let $C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of functions $\varpi : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that
(i) the map $\hat{\varkappa} \mapsto \varpi(\hat{\varkappa}, \vartheta, \upsilon)$ is measurable for each $\vartheta, \upsilon \in \mathbb{R}$, and
(ii) the map $\hat{\varkappa} \mapsto \varpi(\hat{\varkappa}, \vartheta, \upsilon)$ is continuous for each $\hat{\varkappa} \in \mathcal{J}$.

The class $C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $\mathcal{J} \times \mathbb{R} \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on $\mathcal{J}$.

Definition 2.1. *[13] The Riemann-Liouville fractional integral of the function $\eta \in L^1([a, b], \mathbb{R}^+)$, with an order of $\hat{\imath} \in \mathbb{R}^+$, is established by the following definition:*

$$I_a^{\hat{\imath}}\eta(\hat{\varkappa}) = \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\imath}-1}}{\Gamma(\alpha)}\eta(s)ds,$$

where Γ is the gamma function.

Definition 2.2. *[13] The Riemann-Liouville fractional-order derivative of a function η defined on the interval $[a, b]$ is characterized by the following definition:*

$$({}^cD_{a^+}^{\hat{\imath}}\eta)(\hat{\varkappa}) = \frac{1}{\Gamma(n - \kappa)}\left(\frac{d}{dt}\right)^n \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{n-\hat{\imath}-1}}{\Gamma(\hat{\imath})}\eta(s)ds,$$

where $n = [\hat{\imath}] + 1$, and $[\hat{\imath}]$ denotes the integer part of $\hat{\imath}$.

Definition 2.3. *[13] For a function η given on the interval $[a, b]$, the Caputo fractional-order derivative of η, is defined by*

$$({}^cD_{a^+}^{\hat{\imath}}\eta)(\hat{\varkappa}) = \frac{1}{\Gamma(n - \hat{\imath})} \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{n-\hat{\imath}-1}}{\Gamma(\kappa)}\eta^{(n)}(s)ds,$$

where $n = [\hat{\imath}] + 1$ and $[\hat{\imath}]$ denotes the integer part of $\hat{\imath}$.

Lemma 2.1. *[13] Let $\hat{\imath} > 0$ and $\vartheta \in C(0, T) \cap L(0, T)$. Then, the fractional differential equation*

$$D^{\hat{\imath}}\vartheta(\hat{\varkappa}) = 0$$

has a unique solution

$$\vartheta(\hat{\varkappa}) = \tau_1\hat{\varkappa}^{\hat{\imath}-1} + \tau_2\hat{\varkappa}^{\hat{\imath}-2} + \ldots + \tau_n\hat{\varkappa}^{\hat{\imath}-n},$$

where $\tau_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < \hat{\imath} < n$.

Theorem 2.1. *[14] (Lebesgue's dominated convergence theorem) Suppose $f_n : \mathbb{R} \longrightarrow [-\infty, +\infty]$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{x\to+\infty} f_n(x)$ exists. Assume there is an integrable $g : \mathbb{R} \longrightarrow [0, \infty]$ with $|f_n(x)| \leq g(x)$ for each $x \in \mathbb{R}$. Then, f is integrable as is f_n for each n, and*

$$\lim_{x\to+\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{x\to+\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu.$$

Lemma 2.2. *[13] Let $\hat{\iota} > 0$. Then, for $\vartheta \in C(0,T) \cap L(0,T)$, we have*

$$I^{\hat{\iota}} D^{\hat{\iota}} \vartheta(\hat{\varkappa}) = \vartheta(\hat{\varkappa}) + c_0 + c_1\hat{\varkappa} + \ldots + c_{n-1}\hat{\varkappa}^{n-1},$$

fore some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, where $n = [\hat{\iota}] + 1$.

Lemma 2.3. *[15] Consider $\hat{\iota} \in [0,1]$ and a continuous function $\eta : [0,T] \longrightarrow \mathbb{R}$. $\vartheta \in \hat{\mathfrak{C}}([0,T], \mathbb{R})$ is a solution to the fractional integral equation,*

$$\vartheta(\hat{\varkappa}) = \vartheta_0 - \int_0^a \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})} \eta(s) ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})} \eta(s) ds,$$

if and only if ϑ satisfies the following problem:

$$\begin{cases} D^{\hat{\iota}} \vartheta(\hat{\varkappa}) = \eta(\hat{\varkappa}), \hat{\varkappa} \in [0,T], \\ \vartheta(a) = \vartheta_0, \quad a > 0. \end{cases} \tag{2.1}$$

For brevity, let us take

$$\eta_1 = \frac{1}{T}\vartheta_{T_1} - \frac{1}{T}\vartheta_0 - \frac{1}{T\Gamma(\hat{\iota}_1)} \int_0^T (T-s)^{\hat{\iota}_1 - 1} \chi(s) ds - \frac{d_1}{T},$$

$$d_1 = \frac{I^{\hat{\kappa}_1} \varpi_1(T, \vartheta_1(T), \vartheta_2(T))}{\varphi_1(T, \vartheta_1(T), \vartheta_2(T))},$$

$$\gamma_1 = \sum_{i=1}^{n} \frac{I_i(\vartheta_1(\hat{\varkappa}_i^-)) - I^{\hat{\kappa}_1} \xi_1(\hat{\varkappa}_i, \vartheta_1(\hat{\varkappa}_i), \vartheta_2(\hat{\varkappa}_i))}{\varphi_1(\hat{\varkappa}_i, \vartheta_1(\hat{\varkappa}_i), \vartheta_2(\hat{\varkappa}_i))},$$

$$\delta_1 = \sum_{i=1}^{n} \frac{\int_0^{\hat{\varkappa}_i} \frac{(\hat{\varkappa}_i - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} \xi_1(s, \vartheta_1(s), \vartheta_2(s)) ds}{\varphi_1(\hat{\varkappa}_i, \vartheta_1(\hat{\varkappa}_i), \vartheta_2(\hat{\varkappa}_i))},$$

$$\eta_2 = \frac{1}{T}\vartheta_{T_2} - \frac{1}{T}\vartheta_1 - \frac{1}{T\Gamma(\hat{\iota}_2)} \int_0^T (T-s)^{\hat{\iota}_2 - 1} \chi(s) ds - \frac{d_2}{T},$$

$$d_2 = \frac{I^{\hat{\kappa}_2} \varpi_2(T, \vartheta_1(T), \vartheta_2(T))}{\varphi_2(T, \vartheta_1(T), \vartheta_2(T))},$$

$$\gamma_2 = \sum_{j=1}^{m} \frac{I_j(\vartheta_2(\hat{\varkappa}_j^-)) - I^{\hat{\kappa}_2} \xi_2(\hat{\varkappa}_j, \vartheta_1(\hat{\varkappa}_j), \vartheta_2(\hat{\varkappa}_j))}{\varphi_2(\hat{\varkappa}_j, \vartheta_1(\hat{\varkappa}_j), \vartheta_2(\hat{\varkappa}_j))},$$

$$\delta_2 = \sum_{j=1}^{m} \frac{\int_0^{\hat{\varkappa}_j} \frac{(\hat{\varkappa}_j - s)^{\hat{\kappa}_2 - 1}}{\Gamma(\hat{\kappa}_2)} \xi_2(s, \vartheta_1(s), \vartheta_2(s)) ds}{\varphi_2(\hat{\varkappa}_j, \vartheta_1(\hat{\varkappa}_j), \vartheta_2(\hat{\varkappa}_j))}.$$

Lemma 2.4. *For any $\chi_1 \in L^1(\mathcal{J}, \mathbb{R})$, the function $\vartheta_1 \in \hat{\mathfrak{C}}(\mathcal{J}, \mathbb{R})$ is a solution to the*

$$\begin{cases} D^{\hat{\iota}_1}\left(\frac{\vartheta_1(\hat{\varkappa}) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}\right) = \chi_1(\hat{\varkappa}), \quad \hat{\varkappa} \in \mathcal{J} = [0, T], \quad 1 < \hat{\iota} \leq 2, \\ \vartheta_1(\hat{\varkappa}_i^+) = \vartheta_1(\hat{\varkappa}_i^-) + I_i(\vartheta_1(\hat{\varkappa}_i^-)), \quad \hat{\varkappa}_i \in (0, 1), i = 1, 2, \ldots, n, \\ \frac{\vartheta_1(0)}{\varphi_1(0, \vartheta_1(0), \vartheta_2(0))} = \vartheta_0, \qquad \frac{\vartheta_1(T)}{\varphi_1(T, \vartheta_1(T), \vartheta_2(T))} = \vartheta_{T_1}, \end{cases} \tag{2.2}$$

if and only if ϑ_1 satisfies the hybrid integral equation

$$\begin{aligned} \vartheta_1(\hat{\varkappa}) = \varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))\Big[\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0 \\ + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)} \chi_1(s) ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} \xi_1(s, \vartheta_1(s), \vartheta_2(s)) ds, \quad \hat{\varkappa} \in [0, T]. \end{aligned} \tag{2.3}$$

Proof. We assume that ϑ_1 is a solution of the problem (2.2).

If $\hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1]$, by definition, $\left(\frac{\vartheta_1(\hat{\varkappa}) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}\right)$ is continuous. Applying $I^{\hat{\iota}_1}$ of the order $\hat{\iota}_1$ on both sides of (2.2), we can obtain:

$$\frac{\vartheta_1(\hat{\varkappa}) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))} = I^{\hat{\iota}_1}\chi_1(\hat{\varkappa}) - c_0 - c_1\hat{\varkappa},$$

so, we get

$$\frac{\vartheta_1(\hat{\varkappa})}{\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))} = I^{\hat{\iota}_1}\chi_1(\hat{\varkappa}) - c_0 - c_1\vartheta_1 + \frac{I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))}.$$

Substituting $\hat{\varkappa} = 0$, we have

$$c_0 = -\frac{\vartheta_1(0)}{\varphi_1(0, \vartheta_1(0), \vartheta_2(0))} = -\vartheta_0,$$

and substituting $\hat{\varkappa} = T$, we have

$$\frac{\vartheta_1(T)}{\varphi_1(T, \vartheta_1(T), \vartheta_2(T))} = I^{\hat{\iota}_1}\chi_1(T) + \vartheta_0 - c_1 T + d_1,$$

then,

$$c_1 = \frac{1}{T}(\vartheta_0 + I^{\hat{\iota}_1}\chi_1(T) - \vartheta_{T_1} + d_1).$$

In consequence, we have

$$\vartheta_1(\hat{\varkappa}) \quad = \quad \varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))\Big(\frac{1}{\Gamma(\hat{\iota}_1)} \int_0^{\hat{\varkappa}} (\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}\chi_1(s) ds + (1 - \frac{\hat{\varkappa}}{T})\vartheta_0$$

$$+ \quad \frac{\hat{\varkappa}}{T}\vartheta_{T_1} - \frac{\hat{\varkappa}}{T\Gamma(\hat{\iota}_1)}\int_0^T (T-s)^{\hat{\iota}_1-1}\chi_1(s)ds - \frac{\hat{\varkappa}d_1}{T}\Big)$$
$$+ \quad \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds.$$

If $\hat{\varkappa} \in [\hat{\varkappa}_1, \hat{\varkappa}_2]$, then

$$D^{\hat{\iota}_1}\left(\frac{\vartheta_1(\hat{\varkappa}) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}\right) = \chi_1(\hat{\varkappa}), \quad t \in [\hat{\varkappa}_1, \hat{\varkappa}_2], \tag{2.4}$$

$$\vartheta_1(\hat{\varkappa}_1^+) = \vartheta_1(\hat{\varkappa}_1^-) + I_1(\vartheta_1(\hat{\varkappa}_1^-)). \tag{2.5}$$

Referring to Lemma 2, and the continuity of $\hat{\varkappa} \longrightarrow \varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))$, we have

$$\frac{\vartheta_1(\hat{\varkappa}) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))} = \frac{\vartheta_1(\hat{\varkappa}_1^+) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}$$
$$- \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds.$$

Since

$$\frac{\vartheta_1(\hat{\varkappa}) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))} = \frac{\vartheta_1(\hat{\varkappa}_1^-) + I_1(\vartheta_1(\hat{\varkappa}_1^-)) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}{\varphi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}$$
$$- \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds,$$

accordingly,

$$\vartheta_1(\hat{\varkappa}^-) = \varphi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))\Big(\frac{1}{\Gamma(\hat{\iota}_1)}\int_0^{\hat{\varkappa}_1}(\hat{\varkappa}_1-s)^{\hat{\iota}_1-1}\chi_1(s)ds + (1-\frac{\hat{\varkappa}_1}{T})\vartheta_0 + \frac{\hat{\varkappa}_1}{T}\vartheta_{T_1}$$
$$- \frac{\hat{\varkappa}_1}{T\Gamma(\hat{\iota}_1)}\int_0^T (T-s)^{\hat{\iota}_1-1}\chi_1(s)ds - \frac{\hat{\varkappa}_1 d_1}{T}\Big) + \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds\Big).$$

Then, we get

$$\vartheta_1(\hat{\varkappa}) = \varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\frac{1}{\Gamma(\hat{\iota}_1)}\int_0^{\hat{\varkappa}_1}(\hat{\varkappa}_1-s)^{\hat{\iota}_1-1}\chi_1(s)ds + \int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds$$
$$+ \frac{I_1(\vartheta_1(\hat{\varkappa}_1^-)) - I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}{\varphi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))} - \int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds$$
$$+ \int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds + \eta_1\varkappa_1 + \vartheta_0\Big] + \int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds,$$

so, one has

$$\vartheta_1(\hat{\varkappa}) = \varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds$$

$$+\frac{I_1(\vartheta_1(\hat{\varkappa}_1^-))-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\eta_1\varkappa_1+\vartheta_0\Big]$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),I^{\hat{\iota}_1}\vartheta_2(s))ds.$$

If $\hat{\varkappa}\in[\hat{\varkappa}_2,\hat{\varkappa}_3]$, we have

$$\frac{\vartheta_1(\hat{\varkappa})-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}=\frac{\vartheta_1(\hat{\varkappa}_2^+)-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2))}{\varphi(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2))}$$
$$-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds.$$

Since

$$\frac{\vartheta_1(\hat{\varkappa})-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}=\frac{\vartheta_1(\hat{\varkappa}_2^-)+I_2(\vartheta_1(\hat{\varkappa}_2^-))-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2))}{\varphi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2))}$$
$$-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds,$$

then,

$$\vartheta_1(\hat{\varkappa}_2^-)=\varphi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2))\Big[\frac{\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds}{\varphi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}$$
$$+\frac{I_1(\vartheta_1(\hat{\varkappa}_1^-))-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\eta_1\varkappa_1+\vartheta_0\Big]$$
$$+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds.$$

Hence, we acquire,

$$\vartheta_1(\hat{\varkappa})=\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\frac{I_1(\vartheta_1(\hat{\varkappa}_1^-))-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}{\varphi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1)}$$
$$+\frac{I_2(\vartheta_1(\hat{\varkappa}_2^-))-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2)}{\varphi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),I^{\hat{\iota}_1}\vartheta_2(\hat{\varkappa}_2))}+\frac{\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds}{\varphi_1(\hat{\varkappa}_1,\vartheta_1(\hat{\varkappa}_1),\vartheta_2(\hat{\varkappa}_1))}$$
$$+\frac{\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds}{\varphi_1(\hat{\varkappa}_2,\vartheta_1(\hat{\varkappa}_2),\vartheta_2(\hat{\varkappa}_2))}+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\eta_1\varkappa_1+\vartheta_0$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds\Big]$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds.$$

Consequently, we get

$$\vartheta_1(\hat{\varkappa})=\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\sum_{i=1}^{2}\frac{I_i(\vartheta_1(\hat{\varkappa}_i^-))-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}{\varphi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}$$

$$+\sum_{i=1}^{2}\frac{\int_0^{\hat{\varkappa}_i}\frac{(\hat{\varkappa}_i-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds}{\varphi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\eta_1\varkappa_1+\vartheta_0$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds\Big]$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds.$$

By using the same method, for $\hat{\varkappa}\in[\hat{\varkappa}_i,\hat{\varkappa}_{i+1}]$, $i=3,4,...,n$, one has

$$\vartheta_1(\hat{\varkappa})=\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\sum_{i=1}^{n}\frac{I_i(\vartheta_1(\hat{\varkappa}_i^-))-I^{\hat{\kappa}}\xi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_1(\hat{\varkappa}_i))}{\varphi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}$$
$$+\sum_{i=1}^{n}\frac{\int_0^{\hat{\varkappa}_i}\frac{(\hat{\varkappa}_i-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds}{\varphi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\eta_1\varkappa_1+\vartheta_0$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi(s)ds\Big]$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds.$$

Conversely, assume that ϑ_1 satisfies (2.3). If $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1]$, then, we have

$$\vartheta_1(\hat{\varkappa})=\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big(\frac{1}{\Gamma(\hat{\iota}_1)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\hat{\iota}_1-1}\chi_1(s)ds+(1-\frac{\hat{\varkappa}}{T})\vartheta_0+\frac{\hat{\varkappa}}{T}\vartheta_{T_1}$$
$$-\frac{\hat{\varkappa}}{T\Gamma(\hat{\iota}_1)}\int_0^{T}(T-s)^{\hat{\iota}_1-1}\chi_1(s)ds-\frac{\hat{\varkappa}d_1}{T}\Big)$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds. \tag{2.6}$$

Then, by dividing both sides of (2.6) by $\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))$ and applying $D^{\hat{\iota}_1}$, we obtain the first equation in (2.2). Again, substituting $\hat{\varkappa}=0$ and $\hat{\varkappa}=T$ in (2.6), since $\vartheta_1\longrightarrow\frac{\vartheta_1}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}$ is increasing in $\mathbb{R}$ for $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1]$, the map $\vartheta_1\longrightarrow\frac{\vartheta_1}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}$ is injective in $\mathbb{R}$, and

$$\frac{\vartheta_1(0)}{\varphi_1(0,\vartheta_1(0),\vartheta_2(0))}=\vartheta_0,\ \text{and}\ \frac{\vartheta_1(T)}{\varphi_1(T,\vartheta_1(T),\vartheta_2(T))}=\vartheta_{T_1}.$$

If $t\in[\hat{\varkappa}_1,\hat{\varkappa}_2]$, then,

$$\vartheta_1(\hat{\varkappa})=\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\sum_{i=1}^{2}\frac{I_i(\vartheta_1(\hat{\varkappa}_i^-)-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}{\varphi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}$$
$$+\sum_{i=1}^{2}\frac{\int_0^{\hat{\varkappa}_i}\frac{(\hat{\varkappa}_i-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds}{\varphi_1(\hat{\varkappa}_i,\vartheta_1(\hat{\varkappa}_i),\vartheta_2(\hat{\varkappa}_i))}+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds+\eta_1\varkappa_1+\vartheta_0$$

$$+\int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\chi_1(s)ds\Big]$$
$$+\int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds. \tag{2.7}$$

Then, by dividing both sides of (2.7) by $\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))$ and applying $D^{\hat{\iota}_1}$, we obtain Eq (2.4). Again by $(A0)$, substituting $\hat{\varkappa}=\hat{\varkappa}_1$ in (2.6) and taking the limit of (2.7), then (2.7) minus (2.6) gives (2.5).

Similarly, for $\hat{\varkappa}\in[\hat{\varkappa}_i,\hat{\varkappa}_{i+1}], i=2,3,...,n$, we get

$$D^{\hat{\iota}_1}\left(\frac{\vartheta_1(\hat{\varkappa})-I^{\hat{\kappa}_1}\xi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}{\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))}\right)=\chi_1(\hat{\varkappa}), \tag{2.8}$$

$$\vartheta_1(\hat{\varkappa}_1^+)=\vartheta_1(\hat{\varkappa}_1^-)+I_1(\vartheta_1(\hat{\varkappa}_1^-)). \tag{2.9}$$

□

This completes the proof.

Theorem 2.2. *Let ϖ_1,ϖ_2 be jointly continuous, then $(\vartheta_1,\vartheta_2)\in\hat{\mathfrak{X}}\times\hat{\mathfrak{Y}}$ is a solution of (1.2) if and only if $(\vartheta_1,\vartheta_2)$ is the solution of the integral equations*

$$\vartheta_1(\hat{\varkappa})=\varphi_1(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\gamma_1+\delta_1+\eta_1\varkappa_1+\vartheta_0$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_1-1}}{\Gamma(\hat{\iota}_1)}\varpi_1(s,\vartheta_1(s),\vartheta_2(s))ds\Big]+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_1-1}}{\Gamma(\hat{\kappa}_1)}\xi_1(s,\vartheta_1(s),\vartheta_2(s))ds, \hat{\varkappa}\in[0,T], \text{ and}$$
$$\vartheta_2(\hat{\varkappa})=\varphi_2(\hat{\varkappa},\vartheta_1(\hat{\varkappa}),\vartheta_2(\hat{\varkappa}))\Big[\gamma_2+\delta_2+\eta_2\varkappa_1+\vartheta_1$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}_2-1}}{\Gamma(\hat{\iota}_2)}\varpi_2(s,\vartheta_1(s),\vartheta_2(s))ds\Big]+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}_2-1}}{\Gamma(\hat{\kappa}_2)}\xi_2(s,\vartheta_1(s),\vartheta_2(s))ds, \hat{\varkappa}\in[0,T]. \tag{2.10}$$

3. Main result

Before presenting the main results of this work, we will present the following assumptions:

$(A1)$ For the Caratheodory functions $\varpi_i:\mathcal{J}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ $(i=1,2)$, there exist constants $p_i>0, q_i>0$ $(i=1,2)$ such that

$$|\varpi_1(\hat{\varkappa},\nu_1,\nu_2)-\varpi_1(\hat{\varkappa},\omega_1,\omega_2)|\le p_1|\nu_1-\omega_1|+p_2|\nu_2-\omega_2|,$$
$$|\varpi_2(\hat{\varkappa},\nu_1,\nu_2)-\varpi_2(\hat{\varkappa},\omega_1,\omega_2)|\le q_1|\nu_1-\omega_1|+q_2|\nu_2-\omega_2|,$$

for $\hat{\varkappa}\in\mathcal{J}$, and $(\omega_1,\omega_2),(\nu_1,\nu_2)\in\mathbb{R}^2$.

(A2) The functions $\varphi_i:\mathcal{J}\times\mathbb{R}^2\longrightarrow\times\mathbb{R}\setminus\{0\}$, $\xi_i:J\times\mathbb{R}^2\longrightarrow\mathbb{R}$, are continuous and for some constants $h_i>0,k_i>0,L_i>0$ $(i=1,2)$ such that

$$|\varphi_1(\hat{\varkappa},\omega_1,\omega_2)|\le h_1,$$
$$|\varphi_2(\hat{\varkappa},\omega_1,\omega_2)|\le h_2,$$

$$|\varpi_1(\hat{\varkappa}, \omega_1, \omega_2)| \leq L_1,$$
$$|\varpi_2(\hat{\varkappa}, \omega_1, \omega_2)| \leq L_2,$$
$$|\xi_1(\hat{\varkappa}, \omega_1, \omega_2)| \leq k_1,$$
$$|\xi_2(\hat{\varkappa}, \omega_1, \omega_2)| \leq k_2.$$

(A3) Assume $\sup_{\hat{\varkappa}\in[0,T]} \varpi_1(\hat{\varkappa}, 0, 0) = \mathcal{N}_1 < \infty$, $\sup_{\hat{\varkappa}\in[0,T]} \varpi_2(\hat{\varkappa}, 0, 0) = \mathcal{N}_2 < \infty$.

To simplify computational calculations, we introduce

$$\tau_1 = \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)}, \quad \tau_2 = \frac{T^{\hat{\kappa}_1}}{\Gamma(\hat{\kappa}_1 + 1)},$$
$$\tau_3 = \frac{T^{\hat{\iota}_2}}{\Gamma(\hat{\iota}_2 + 1)}, \quad \tau_4 = \frac{T^{\hat{\kappa}_2}}{\Gamma(\hat{\kappa}_2 + 1)}.$$

Consider the operator $\Theta : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \times \mathfrak{Y}$ associated with the problem (1.2) as follows:

$$\Theta(\vartheta_1, \vartheta_2)(\hat{\varkappa}) = \begin{pmatrix} \Theta_1(\vartheta_1, \vartheta_2)(\hat{\varkappa}) \\ \Theta_2(\vartheta_1, \vartheta_2)(\hat{\varkappa}) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned}\Theta_1(\vartheta_1, \vartheta_2)(\hat{\varkappa}) = {} & \varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))\Big[\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0 \\ & + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)} \varpi_1(s, \vartheta_1(s), \vartheta_2(s))ds\Big] \\ & + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} \xi_1(s, \vartheta_1(s), \vartheta_2(s))ds, \hat{\varkappa} \in [0, T],\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\Theta_2(\vartheta_1, \vartheta_2)(\hat{\varkappa}) = {} & \varphi_2(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))\Big[\gamma_2 + \delta_2 + \eta_2\varkappa_1 + \vartheta_1 \\ & + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_2 - 1}}{\Gamma(\hat{\iota}_2)} \varpi_2(s, \vartheta_1(s), \vartheta_2(s))ds\Big] \\ & + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_2 - 1}}{\Gamma(\hat{\kappa}_2)} \xi_2(s, \vartheta_1(s), \vartheta_2(s))ds, \hat{\varkappa} \in [0, T].\end{aligned} \tag{3.3}$$

Theorem 3.1. *Suppose that conditions (A0) and (A2) hold. Then $\Theta\bar{\mathcal{B}}_r \subset \bar{\mathcal{B}}_r$, where $\bar{\mathcal{B}}_r = \{(\vartheta_1, \vartheta_2) \in \mathfrak{X} \times \mathfrak{Y} : \|(\vartheta_1, \vartheta_2)\| \leq r\}$ is a closed ball, and if*

$$h_1\tau_1(p_1 + p_2) + h_2\tau_3(q_1 + q_2) < 1,$$

then, problem (1.2) has a unique solution on $[0, T]$.

Proof. For $(\vartheta_1, \vartheta_2) \in \bar{\mathcal{B}}_r$ and $\hat{\varkappa} \in [0, T]$, it follows by (A1) that

$$|\varpi_1(\hat{\varkappa}, \vartheta_2(\hat{\varkappa}), \vartheta_1(\hat{\varkappa}))| \leq |\varpi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa})) - \varpi_1(\hat{\varkappa}, 0, 0)| \leq p_1\|\vartheta_1\| + p_2\|\vartheta_2\|.$$

Similarly, one can find that $|\varpi_2(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))| \leq q_1\|\vartheta_1\| + q_2\|\vartheta_2\|$.

Now,

$$\begin{aligned}
|\Theta_1(\vartheta_1, \vartheta_2)(\hat{\varkappa})| &= |\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))|\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| \\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)}|\varpi_1(s, \vartheta_1(s), \vartheta_2(s))|ds\Big] \\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)}|\xi_1(s, \vartheta_1(s), \vartheta_2(s))ds| + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)}|\xi_1(s, \vartheta_1(s), \vartheta_2(s))|ds \\
&\leq h_1\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)}|\varpi_1(s, \vartheta_1(s), \vartheta_2(s)) - \varpi_1(s, 0, 0) \\
&+ \varpi_1(s, 0, 0)|ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)}k_1 ds \\
&\leq h_1\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)}(p_1\|\vartheta_1\| + p_2\|\vartheta_2\|) + \mathcal{N}_1\Big] \\
&+ \frac{T^{\hat{\kappa}_1}}{\Gamma(\hat{\kappa}_1 + 1)}k_1 + \mathcal{N}_1 \\
&\leq h_1\tau_1(p_1\|\vartheta_1\| + p_2\|\vartheta_2\|) + \tau_2 k_1 + h_1|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + h_1\mathcal{N}_1.
\end{aligned}$$

Thus,

$$\|\Theta_1(\vartheta_1, \vartheta_2)\| \leq h_1\tau_1(p_1 + p_2)r + \tau_2 k_1 + h_1|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + h_1\mathcal{N}_1.$$

Similarly,

$$\|\Theta_2(\vartheta_1, \vartheta_2)\| \leq h_2\tau_3(q_1 + q_2)r + \tau_4 k_2 + h_2|\gamma_2 + \delta_2 + \eta_2\varkappa_1 + \vartheta_1| + h_2\mathcal{N}_2.$$

From the foregoing estimates for Θ_1 and Θ_2, it follows that $\|\Theta(\vartheta_1, \vartheta_2)\| \leq r$.

Next, for $(\vartheta_1, \vartheta_2), (\vartheta_1', \vartheta_2') \in \mathfrak{S} \times \mathfrak{S}$ and $\hat{\varkappa} \in [0, T]$, we get

$$\begin{aligned}
\Big|\Theta_1(\vartheta_1', \vartheta_2')(\hat{\varkappa}) - \Theta_1(\vartheta_1, \vartheta_2)(\hat{\varkappa})\Big| &\leq h_1\Big[\int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)}|\varpi_1(s, \vartheta_1(s), \vartheta_2(s)) \\
&- \varpi_1(s, \vartheta_1'(s), \vartheta_2'(s))|ds\Big] \\
&\leq h_1\Big(\frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)}p_1\|\vartheta_1' - \vartheta_1\| + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)}p_2\|\vartheta_2' - \vartheta_2\|\Big) \\
&\leq h_1\Big(\tau_1 p_1\|\vartheta_1' - \vartheta_1\| + \tau_1 p_2\|\vartheta_2' - \vartheta_2\|\Big) \\
&\leq h_1\tau_1(p_1 + p_2)\Big(\|\vartheta_1' - \vartheta_1\| + \|\vartheta_2' - \vartheta_2\|\Big),
\end{aligned}$$

which implies that

$$\Big\|\Theta_1(\vartheta_1', \vartheta_2') - \Theta_1(\vartheta_1, \vartheta_2)\Big\| \leq h_1\tau_1(p_1 + p_2)\Big(\|\vartheta_1' - \vartheta_1\| + \|\vartheta_2' - \vartheta_2\|\Big). \tag{3.4}$$

Similarly,

$$\Big\|\Theta_2(\vartheta_1', \vartheta_2') - \Theta_1(\vartheta_1, \vartheta_2)\Big\| \leq h_2\tau_3(q_1 + q_2)\Big(\|\vartheta_1' - \vartheta_1\| + \|\vartheta_2' - \vartheta_2\|\Big). \tag{3.5}$$

From (3.4) and (3.5), we deduce that

$$\left\|\Theta(\vartheta_1', \vartheta_2') - \Theta(\vartheta_1, \vartheta_2)\right\| \leq (h_1\tau_1(p_1 + p_2) + h_2\tau_3(q_1 + q_2))\left(\|\vartheta_1' - \vartheta_1\| + \|\vartheta_2' - \vartheta_2\|\right).$$

Considering the condition $h_1\tau_1(p_1 + p_2) + h_2\tau_3(q_1 + q_2) < 1$, we can deduce that the operator Θ has a unique fixed point. Consequently, we can conclude that our prposed coupled system has a uniquesolution on the interval $[0, T]$. □

Now, we explore the existence of solutions for problem (1.2) using Schaefer's fixed-point theorem.

Theorem 3.2. *Assuming the hypotheses (A0) and (A3), we can affirm that the boundary value problem* (1.2) *possesses at least one solution on the interval* $[0, T]$.

Proof. The proof will be presented in multiple steps.

Step I: The operator $\psi : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \times \mathfrak{Y}$ is continuous. It is noteworthy that the continuity of the functions ϖ_1, φ_1, ϖ_2, and φ_2 implies the boundedness of the operator ψ. Let $(\vartheta_{1_n}, \vartheta_{2_n})$ be a sequence of points in $\mathfrak{X} \times \mathfrak{Y}$ converging to a point $(\vartheta_1, \vartheta_2) \in \mathfrak{X} \times \mathfrak{Y}$. Therefore, applying Theorem 2.1, we obtain:

$$\begin{aligned}
&|\psi_1(\vartheta_{1_n}, \vartheta_{2_n})(\hat{\varkappa}) - \psi_1(\vartheta_1, \vartheta_2)(\hat{\varkappa})|\\
&= |\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))|\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0|\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)} |\varpi_1(\hat{\varkappa}, \vartheta_{1_n}(\hat{\varkappa}), \vartheta_{2_n}(\hat{\varkappa})) - \varpi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))| ds\Big]\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} |\xi_1(s, \vartheta_{1_n}(s), \vartheta_{2_n}(s)) - \xi_1(s, \vartheta_1(s), \vartheta_2(s))| ds\\
&\leq h_1\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0|\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)} \|\varpi_1(\hat{\varkappa}, \vartheta_{1_n}(\hat{\varkappa}), \vartheta_{2_n}(\hat{\varkappa})) - \varpi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))\| ds\Big]\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} \|\xi_1(s, \vartheta_{1_n}(s), \vartheta_{2_n}(s)) - \xi_1(s, \vartheta_1(s), \vartheta_2(s))\| ds.
\end{aligned}$$

Since ψ_1 is continuous, we have $\|\psi_1(\vartheta_{1_n}, \vartheta_{2_n}) - \psi_1(\vartheta_1, \vartheta_2)\| \to 0 \; as \; n \to \infty$ for all $\hat{\varkappa} \in [0, T]$.

Similarly, we can prove $\|\psi_2(\vartheta_{1_n}, \vartheta_{2_n}) - \psi_2(\vartheta_1, \vartheta_2)\| \to 0 \; as \; n \to \infty$ for all $\hat{\varkappa} \in [0, T]$.

Hence, it follows from the foregoing inequalities satisfied by ψ_1 and ψ_2 that the operator ψ is continuous.

Step II : The operator ψ, which maps bounded sets to bounded sets, implies the existence of positive constants $\mathcal{L}_1$ *and* $\mathcal{L}_2$ such that for each such that $\|\psi_1(\vartheta_1, \vartheta_2)\| < \mathcal{L}_1$ and $\|\psi_2(\vartheta_1, \vartheta_2)\| < \mathcal{L}_2$, we have

$$\begin{aligned}
|\psi_1(\vartheta_1, \vartheta_2)(\hat{\varkappa})| &= |\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))|\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0|\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)} |\varpi_1(s, \vartheta_1(s), \vartheta_2(s))| ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} |\xi_1(s, \vartheta_1(s), \vartheta_2(s))| ds\\
&\leq h_1\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} L_1\Big] + \frac{T^{\hat{\kappa}_1}}{\Gamma(\hat{\kappa}_1 + 1)} k_1,
\end{aligned}$$

$$\|\psi_1(\vartheta_1, \vartheta_2)\| \leq h_1\Big[|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} L_1\Big] + \frac{T^{\hat{\kappa}_1}}{\Gamma(\hat{\kappa}_1 + 1)} k_1.$$

Thus, we deduce that $\|\psi_1(\vartheta_1, \vartheta_2)\| \le \mathcal{L}_1$. In a similar fashion, it can be found that $\|\psi_2(\vartheta_1, \vartheta_2)\| \le \mathcal{L}_2$.

Hence, it follows from the foregoing inequalities that ψ_1 and ψ_2 are uniformly bounded and the operator ψ is uniformly bounded.

Step III : We show that our operator is equicontinuous.

Let $r_1, r_2 \in [0, T]$ with $r_1 < r_2$,

$$\begin{aligned}
&|\psi_1(\vartheta_1(r_2), \vartheta_2(r_2)) - \psi_1(\vartheta_1(r_1), \vartheta_2(r_1))| \\
&\le h_1[\frac{1}{\Gamma(\hat{\iota}_1)} \int_0^{r_1} ((r_2 - s)^{\hat{\iota}_1 - 1} - (r_1 - s)^{\hat{\iota}_1 - 1}) |\varpi_1(s, \vartheta_1(s), \vartheta_2(s))| ds \\
&+ \frac{1}{\Gamma(\hat{\iota}_1)} \int_{r_1}^{r_2} (r_2 - s)^{\hat{\iota}_1 - 1} |\varpi_1(s, \vartheta_1(s), \vartheta_2(s))| ds] \\
&+ \frac{1}{\Gamma(\hat{\kappa}_1)} \int_0^{r_1} ((r_2 - s)^{\hat{\kappa}_1 - 1} - (r_1 - s)^{\hat{\kappa}_1 - 1}) |\xi_1(s, \vartheta_1(s), \vartheta_2(s))| ds \\
&+ \frac{1}{\Gamma(\hat{\kappa}_1)} \int_{r_1}^{r_2} (r_2 - s)^{\hat{\kappa}_1 - 1} |\xi_1(s, \vartheta_1(s), \vartheta_2(s))| ds \\
&\le h_1[\frac{L_1}{\Gamma(\hat{\iota}_1 + 1)} ((r_2 - s)^{\hat{\iota}_1 - 1} - (r_1 - s)^{\hat{\iota}_1 - 1}) + \frac{L_1}{\Gamma(\hat{\iota}_1 + 1)} (r_2 - r_1)^{\hat{\iota}_1 - 1}] \\
&+ \frac{k_1}{\Gamma(\hat{\kappa}_1 + 1)} ((r_2 - s)^{\hat{\kappa}_1 - 1} - (r_1 - s)^{\hat{\kappa}_1 - 1}) + \frac{k_1}{\Gamma(\hat{\kappa}_1) + 1} (r_2 - r_1)^{\hat{\kappa}_1 - 1} \\
&\qquad \to 0 \quad as \quad r_1 \to r_2.
\end{aligned}$$

Analogously, we obtain that

$$\begin{aligned}
|\psi_2(\vartheta_1(r_2), \vartheta_2(r_2)) - \psi_2(\vartheta_1(r_1), \vartheta_2(r_1))| \le h_2[&\frac{L_2}{\Gamma(\hat{\iota}_2 + 1)} ((r_2 - s)^{\hat{\iota}_2 - 1} - (r_1 - s)^{\hat{\iota}_2 - 1}) \\
&+ \frac{L_2}{\Gamma(\hat{\iota}_2 + 1)} (r_2 - r_1)^{\hat{\iota}_2 - 1}] \\
&+ \frac{k_2}{\Gamma(\hat{\kappa}_2 + 1)} ((r_2 - s)^{\hat{\kappa}_2 - 1} - (r_1 - s)^{\hat{\kappa}_2 - 1}) \\
&+ \frac{k_2}{\Gamma(\hat{\kappa}_2) + 1} (r_2 - r_1)^{\hat{\kappa}_2 - 1}.
\end{aligned}$$

Hence, the operator ψ is equicontinuous, and consequently, the operator $\psi(\vartheta_1, \vartheta_2)$ is completely continuous.

Step IV : To show that the set $\mathcal{P} = \{(\vartheta_1, \vartheta_2) \in \mathfrak{X} \times \mathfrak{Y} : (\vartheta_1, \vartheta_2) = \lambda_1 \psi(\vartheta_1, \vartheta_2),\ 0 < \lambda_1 < 1\}$ is bounded (Apriori bounds), let $(\vartheta_1, \vartheta_2) \in \mathcal{P}$ and $\hat{\varkappa} \in [0, T]$. Then, it follows from $\vartheta_1(\hat{\varkappa}) = \lambda_1 \psi_1(\vartheta_1, \vartheta_2)(\hat{\varkappa})$, and $\vartheta_2(\hat{\varkappa}) = \lambda_1 \psi_2(\vartheta_1, \vartheta_2)(\hat{\varkappa})$, then,

$$\begin{aligned}
&|\vartheta_1(\hat{\varkappa})| \\
&\quad \le |\varphi_1(\hat{\varkappa}, \vartheta_1(\hat{\varkappa}), \vartheta_2(\hat{\varkappa}))| \Big[|\gamma_1 + \delta_1 + \eta_1 \varkappa_1 + \vartheta_0| \\
&\quad + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}_1 - 1}}{\Gamma(\hat{\iota}_1)} |\varpi_1(s, \vartheta_1(s), \vartheta_2(s))| ds \Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}_1 - 1}}{\Gamma(\hat{\kappa}_1)} |\xi_1(s, \vartheta_1(s), \vartheta_2(s))| ds
\end{aligned}$$

$$\leq h_1\Big(|\gamma_1 + \delta_1 + \eta_1\varkappa_1 + \vartheta_0| + \frac{T^{\hat{\iota}_1}L_1}{\Gamma(\hat{\iota}_1+1)}\Big) + \frac{T^{\hat{\kappa}_1}k_1}{\Gamma(\hat{\kappa}_1+1)},$$

$$\|\vartheta_1\| \leq R, \tag{3.6}$$

and

$$\|\vartheta_2\| \leq h_2\Big(|\gamma_2 + \delta_2 + \eta_2\varkappa_1 + \vartheta_1| + \frac{T^{\hat{\iota}_2}L_2}{\Gamma(\hat{\iota}_2+1)}\Big) + \frac{T^{\hat{\kappa}_2}k_2}{\Gamma(\hat{\kappa}_2+1)} = R,$$

$$\|\vartheta_2\| \leq R. \tag{3.7}$$

Hence, from (3.6) and (3.7), we obtain:

$$\|\vartheta_1\| + \|\vartheta_2\| \leq R,$$

which implies that

$$\|(\vartheta_1, \vartheta_2)\| \leq R.$$

Therefore, $\mathcal{P}$ is bounded, and according to Theorem 3.2, ψ has a fixed point. Consequently, the problem (1.2) has at least one solution on the interval $[0, T]$. The proof is complete. □

4. Examples

Example 4.1. *Consider the following system of coupled fractional differential equations:*

$$\begin{aligned}
&{}^cD^{1/2}\Big[\frac{\vartheta_1 - I^{\frac{1}{2}}\big(\sin\vartheta_1 + \frac{\vartheta_1+\vartheta_2}{\vartheta_2+5}\big)}{\varphi_1(\hat{\varkappa}, \vartheta_1, \vartheta_2)}\Big] = \frac{1}{100}\Big(\vartheta_1 + \frac{1}{2}\Big) + \frac{5}{200}\frac{\vartheta_2}{1+\vartheta_2},\\
&{}^cD^{1/2}\Big[\frac{\vartheta_2 - I^{\frac{1}{2}}\big(\cos\vartheta_2 + \frac{\vartheta_1}{1+\vartheta_2} + e\big)}{\varphi_2(\hat{\varkappa}, \vartheta_1, \vartheta_1)}\Big] = \frac{3}{400}\frac{|\cos\vartheta_1|}{1+\cos\vartheta_1} + \frac{1}{26}\sin\vartheta_2,\\
&\vartheta_1(\hat{\varkappa}_1^+) = \vartheta_1(\hat{\varkappa}_1^-) + (-2\vartheta_1(\hat{\varkappa}_1^-)), \hat{\varkappa}_1 \neq 0, 1,\\
&\vartheta_2(\hat{\varkappa}_1^+) = \vartheta_2(\hat{\varkappa}_1^-) + (-2\vartheta_2(\hat{\varkappa}_1^-)), \hat{\varkappa}_1 \neq 0, 1,\\
&\frac{\vartheta_1(0)}{\varphi_1(0, \vartheta_1(0), \vartheta_2(0))} = 1, \quad \frac{\vartheta_2(e)}{\varphi_1(e, \vartheta_1(e), \vartheta_2(e))} = \frac{\pi}{2},\\
&\frac{\vartheta_1(0)}{\varphi_2(0, \vartheta_1(0), \vartheta_2(0))} = 1, \quad \frac{\vartheta_2(e)}{\varphi_2(e, \vartheta_1(e), \vartheta_2(e))} = \frac{\pi}{4}.
\end{aligned} \tag{4.1}$$

Here $\hat{\iota}_1 = \hat{\kappa}_2 = \hat{\iota}_2 = \hat{\kappa}_2 = \dfrac{1}{2}$, $\gamma_1 = \delta_1 = \frac{1}{2}$, $\gamma_2 = \delta_2 = \frac{1}{2}$, $d_1 = \dfrac{3}{7}$, $d_2 = \dfrac{5}{8}$, $\eta_1 = \eta_2 = \dfrac{3}{5}$, $T = e$, $\vartheta_0 = \vartheta_1 = 1$, $\vartheta_{T_1} = \dfrac{\pi}{2}$, $\vartheta_{T_2} = \dfrac{\pi}{4}$, and $\varphi_1(\hat{\varkappa}, \vartheta_1, \vartheta_2) = \frac{(\hat{\varkappa}+1)}{100}\big(\sin(\vartheta_2) + \frac{\vartheta_1}{1+\vartheta_1} + 3\big) + e^{-1}$, $\varphi_2(\hat{\varkappa}, \vartheta_1, \vartheta_2) = \frac{(e^{-\hat{\varkappa}}+1)^2}{100}\big(\sin(\vartheta_2) + \vartheta_1\big) + \dfrac{1}{2}$, $\varpi_1(\hat{\varkappa}, \vartheta_1, \vartheta_2) = \frac{1}{100}\Big(\vartheta_1 + \dfrac{1}{2}\Big) + \frac{5}{200}\frac{\vartheta_2}{1+\vartheta} + e^{-2}$, and $\varpi_2(\hat{\varkappa}, \vartheta_1, \vartheta_2) = \frac{3}{400}\frac{|\cos(\xi)|}{1+|\cos(\vartheta_1)|} + \frac{1}{26}\sin(\vartheta_2) + e^{-1}$. From the given data, we find that $\tau_1 = \tau_2 = 0.1245678$, therefore, $h_1\tau_1(p_1 + p_2) + h_2\tau_3(q_1 + q_2) = 0.2312456879 < 1$.

By Theorem 3.1, problem (4.1) with the given $\Theta_1(\hat{\varkappa}, \vartheta_1, \vartheta_2)$ and $\Theta_2(\hat{\varkappa}, \vartheta_1, \vartheta_2)$ has one unique solution on $[0, T]$.

Example 4.2. *Let us consider the following fractional boundary value problem:*

$$
\begin{aligned}
&D^{\frac{2}{3}}\left(\frac{\vartheta_1 - I^{\frac{1}{2}}\left[\frac{2\hat{\varkappa}e^{-3\hat{\varkappa}}}{15(3+\hat{\varkappa})}\left(\sin\vartheta_1 + \frac{\vartheta_1 + I^{\sqrt{2}}|\vartheta_2|}{I^{\sqrt{2}}|\vartheta_1|+5}\right)\right]}{\varphi_1(\hat{\varkappa},\vartheta_1,\vartheta_2)}\right) = \hat{\varkappa}^2\sin\vartheta_1 + \cos(I^{\frac{1}{4}}\vartheta_2),\\
&D^{\frac{2}{3}}\left(\frac{\vartheta_2 - I^{\frac{1}{2}}\left[\frac{\hat{\varkappa}}{1+\hat{\varkappa}}\left(\cos\vartheta_2 + \frac{\vartheta_2 + I^{\sqrt{3}}|\vartheta_1|}{I^{\sqrt{3}}|\vartheta_2|+1}\right)\right]}{\varphi_2(\hat{\varkappa},\vartheta_1,\vartheta_2)}\right) = \hat{\varkappa}^2\cos\vartheta_1 + \cos(I^{\frac{3}{4}}\vartheta_2),\\
&\vartheta_1(\hat{\varkappa}_1^+) = \vartheta_1(\hat{\varkappa}_1^-) + (\vartheta_1(\hat{\varkappa}_1^-)), \hat{\varkappa}_1 \neq 0, 1,\\
&\vartheta_2(\hat{\varkappa}_1^+) = \vartheta_2(\hat{\varkappa}_1^-) + (\vartheta_2(\hat{\varkappa}_1^-)), \hat{\varkappa}_1 \neq 0, 1,\\
&\frac{\vartheta_1(0)}{\varphi_1(0,\vartheta_1(0),\vartheta_2(0))} = \frac{1}{2}, \quad \frac{\vartheta_2(\pi)}{\varphi_1(\pi,\vartheta_1(\pi),\vartheta_2(\pi))} = \frac{2\pi}{3},\\
&\frac{\vartheta_1(0)}{\varphi_2(0,\vartheta_1(0),\vartheta_2(0))} = \frac{1}{2}, \quad \frac{\vartheta_2(\pi)}{\varphi_2(\pi,\vartheta_1(\pi),\vartheta_2(\pi))} = \frac{3\pi}{4}.
\end{aligned}
\tag{4.2}
$$

Here $\hat{\iota}_1 = \hat{\kappa}_2 = \hat{\iota}_2 = \hat{\kappa}_2 = \dfrac{2}{3}$, $\gamma_1 = \delta_1 = \frac{1}{2}$, $\gamma_2 = \delta_2 = \frac{1}{2}$, $d_1 = \dfrac{2}{5}$, $d_2 = \dfrac{3}{7}$, $\eta_1 = \eta_2 = \dfrac{3}{5}$, $T = \dfrac{1}{2}$, $\vartheta_0 = \vartheta_1 = 1$, $\vartheta_{T_1} = \dfrac{2\pi}{3}$, $\vartheta_{T_2} = \dfrac{3\pi}{4}$, and $\varphi_1(\hat{\varkappa},\vartheta_1,\vartheta_2) = \frac{(\hat{\varkappa}+1)^2}{100}\left(\sin\vartheta_1 + \frac{|I^{\sqrt{2}}\vartheta_2|}{1+|I^{\sqrt{2}}\vartheta_1|} + 3\right) + \dfrac{e}{2}$, and $\varphi_2(\hat{\varkappa},\vartheta_1,\vartheta_2) = \frac{(e^{-\hat{\varkappa}}+1)^2}{100}(\cos\vartheta_2 + \vartheta_1) + \dfrac{1}{2}$, and $\varpi_1(\hat{\varkappa},\vartheta_1,\vartheta_2) = \hat{\varkappa}^2\sin\vartheta_1 + \cos(I^{\frac{1}{4}}\vartheta_2)$, and $\varpi_2(\hat{\varkappa},\vartheta_1,\vartheta_2) = \hat{\varkappa}^2\cos\vartheta_1 + \cos(I^{\frac{3}{4}}\vartheta_2)$. From the given data, we find that $\tau_1 = \tau_2 = 0.02354689$, therefore, $h_1\tau_1(p_1+p_2) + h_2\tau_3(q_1 + q_2) = 0.564879 < 1$.

By Theorem 3.1, problem (4.2) with the given $\Theta_1(\hat{\varkappa},\vartheta_1,\vartheta_2)$ and $\Theta_2(\hat{\varkappa},\vartheta_1,\vartheta_2)$ has one unique solution on $[0, T]$.

5. Conclusions

Many natural phenomena are analyzed using various types of fractional differential equations, allowing for a comprehensive examination of integrated phenomena across multiple fields. In this paper, we study the existence and uniqueness of solutions for a class of coupled system fractional impulsive hybrid integro-differential equations. We extended the findings to include new classes of fractional boundary conditions involving the Caputo sequential derivative. For future research, it is suggested to incorporate other fractional derivative operators, such as the generalized Hilfer fractional derivative. Researchers in this field can further investigate the existence and uniqueness of solutions by using other fixed point theorems such as Monch's fixed point theorem or Darbo's fixed point theorem. Researchers can also apply diffrent types of fractional derivatives such as Hadamard, Hilfer, Fractional derivatives in our proposed system.

Author contributions

Mohammad Hannabou and Muath Awadalla: developed the conceptualization and proposed the method, wrote-original draft, reviewed, and edited the paper. Mohamed Bouaouid, Abd Elmotaleb A.

M. A. Elamin and Khalid Hilal: investigated, processed and provided examples, reviewed and edited the paper. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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