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Research article

Sufficiency criteria for a class of convex functions connected with tangent function

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Abstract: The research here was motivated by a number of recent studies on Hankel inequalities and sharp bounds. In this article, we define a new subclass of holomorphic convex functions that are related to tangent functions. We then derive geometric properties like the necessary and sufficient conditions, radius of convexity, growth, and distortion estimates for our defined function class. Furthermore, the sharp coefficient bounds, sharp Fekete-Szegö inequality, sharp 2nd order Hankel determinant, and Krushkal inequalities are given. Moreover, we calculate the sharp coefficient bounds, sharp Fekete-Szegö inequality, and sharp second-order Hankel determinant for the functions whose coefficients are logarithmic.

Keywords: holomorphic convex functions; convolution; tangent function; Krushkal inequality; logarithmic coefficients
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1. Introduction

First, some fundamental ideas must be explained in order to fully comprehend the basic concepts utilized throughout the attainment of our major findings. For this, let $\mathcal{A}$ denote the family of all holomorphic (regular) functions f defined in the open unit disc $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, whose Taylor series representation is given as follows:

$$f(z) = z + \sum_{j=2}^{\infty} \xi_j z^j, \qquad z \in \mathbb{D}. \tag{1.1}$$

A subfamily containing all of the univalent functions of the family $\mathcal{A}$ in $\mathbb{D}$ is denoted by $\mathcal{S}$. A useful technique for examining different inclusion and radii concerns for families of holomorphic functions is known as subordination. A function f is subordinate to g in $\mathbb{D}$ written as $f \prec g$, if there exists a Schwarz function ω, which is regular in $\mathbb{D}$ and $\omega(0) = 0$ with $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. In addition, if the function g is univalent in $\mathbb{D}$ then we have

$$f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

The known subclasses of S are represented by the letters $\mathcal{S}^*$, C, $\mathcal{K}$ and $\mathcal{R}$. These subclasses include starlike, convex, close to convex, and functions with bounded turnings. Two regular functions, f and ς, are convolved in $\mathbb{D}$, the series representation of f is provided in (1.1) and $\varsigma = z + \sum_{j=2}^{\infty} b_j z^j$ is defined as follows:

$$(f * \varsigma)(z) = z + \sum_{j=2}^{\infty} \xi_j b_j z^j, \qquad z \in \mathbb{D}. \tag{1.2}$$

The integrated families of starlike and convex functions were developed in 1985 by Padmanabhan and Parvatham [1] who utilized the theory of convolution along with the function $\frac{z}{(1-z)^a}$, where $a \in \mathbb{R}$. By taking a regular function $\phi(z)$ with $\phi(0) = 1$, and $h(z) \in \mathcal{A}$,, Shanmugam [2] expanded on the concept presented in [1] and introduced the generic form of the function class $\mathcal{S}_h^*(\phi)$ as follows:

$$\mathcal{S}_h^*(\phi) = \left\{ f \in \mathcal{A} : \frac{z(f * h)'}{(f * h)} \prec \phi(z), \quad z \in \mathbb{D} \right\}. \tag{1.3}$$

By taking $h(z) = \frac{z}{1-z}$ or $\frac{z}{(1-z)^2}$, we derive the famous classes $\mathcal{S}^*(\phi)$ and $C(\phi)$ of Ma and Minda type starlike and convex functions defined in [3]. Further, by choosing $\phi(z) = \frac{1+z}{1-z}$ these classes can be reduced to $\mathcal{S}^*$ and C.

By limiting $\phi(z)$ in the generic form of $\mathcal{S}^*(\phi)$ and $C(\phi)$, numerous scholars have defined and investigated a variety of intriguing subclasses of analytic and univalent functions in the recent past. Here, we highlight few of them.

Let $\phi(z) = \frac{1+Fz}{1+Gz}$, $-1 \leq G < F \leq 1$. Then $\mathcal{S}^*[F, G] = \mathcal{S}^*\left(\frac{1+Fz}{1+Gz}\right)$ is the class of Janowski starlike functions; see [4]. For $\phi(z) = \cos z$, the class $\mathcal{S}^*_{\cos z}$ was studied by Bano and Raza [5], while for $\phi(z) = \cosh z$, the function class $\mathcal{S}^*_{\cosh z}$ was introduced and studied by Alotaibi et al. [6]. For $\phi(z) = e^z$, the class $\mathcal{S}^*_e$ was defined and studied by Mendiratta et al. [7]. For $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi)$ reduces to $\mathcal{S}^*_{\sin}$, as presented and examined by Cho et al. [8]. For $\phi(z) = 1 + z - \frac{1}{3}z^3$, we get the family $\mathcal{S}^*_{nep}$ that was examined by Wani and Swaminathan [9]. For $\phi(z) = 1 + \sinh^{-1}(z)$, the family $\mathcal{S}^*(\phi)$ was established and studied by Kumar and Arora [10] for more details see [11]. For $\phi(z) = \frac{2}{1+e^{-z}}$, the class $\mathcal{S}^*(\phi)$ reduces to $\mathcal{S}^*_{sig}$; see [12] and [13, 14]. For $\phi(z) = \sqrt{1+z}$, we obtain the family $\mathcal{S}^*\left(\sqrt{1+z}\right) = \mathcal{S}^*_L$ as studied by Sokol and Stankiewicz [15]. The class $\mathcal{S}^*_{\tanh z} = \mathcal{S}^*(\phi(z))$, for $\phi(z) = 1 + \tanh z$, was established by Ullah et al. [16] see also [17].

For the given parameters $n,\, r \in \mathbb{N}$, the rth Hankel determinant $\mathcal{H}_{r,\,n}$ was defined in [18] as follows:

$$\mathcal{H}_{r,\,n}(f) = \begin{vmatrix} \xi_n & \xi_{n+1} & . & . & . & \xi_{n+r-1} \\ \xi_{n+1} & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \xi_{n+r-1} & . & . & . & . & \xi_{n+2(r-1)} \end{vmatrix}.$$

For the given values of n, r and $\xi_1 = 1$ the second and third Hankel determinants are defined as follows:

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & \xi_2 \\ \xi_2 & \xi_3 \end{vmatrix} = \xi_3 - \xi_2^2,\ \mathcal{H}_{2,2}(f) = \begin{vmatrix} \xi_2 & \xi_3 \\ \xi_3 & \xi_4 \end{vmatrix} = \xi_2\xi_4 - \xi_3^2. \tag{1.4}$$

This technique has proven to be usful when examining power series with integral coefficients and singularities by taking the Hankel determinant into account; see [19]. Bounds of $\mathcal{H}_{r,\,n}(f)$ for several kinds of univalent functions have been examined recently. For a detailed study on the Hankel determinant, we refer the reader to [20–22].

Scholars in the field of geometric function theory of complex analysis are still motivated by the study of coefficient problems, which include the Fekete–Szegö and Hankel determinant problems. To encourage and motivate interested readers, we have included numerous recent works (see, e.g., [20–22]) on a variety of the Fekete–Szegö and Hankel determinant problems, along with ongoing applications of the q-calculus in the study of other analytic or meromorphic univalent and multivalent function classes. Motivated and inspired by the work mentioned above, in this article, we first define a new subclass of holomorphic convex functions that are related to the tangent functions. We then derive geometric properties like the necessary and sufficient conditions, radius of convexity, growth, and distortion estimates for our defined function class. Furthermore, the sharp coefficient bounds, sharp Fekete-Szegö inequality, sharp 2nd order Hankel determinant, and Krushkal inequalities are given. Moreover, we calculate the sharp coefficient bounds, sharp Fekete-Szegö inequality, and sharp second-order Hankel determinant for the functions whose coefficients are logarithmic.

We present the following subfamily of holomorphic functions.

Definition 1.1. Let $f \in \mathcal{A}$, be given in (1.1). Then $f \in C_{\tan}$ if the following condition holds true:

$$f \in C_{\tan} \iff f \in \mathcal{A} \text{ and } \frac{(zf'(z))'}{f'(z)} \prec 1 + \frac{\tan z}{2}, \qquad z \in \mathbb{D}. \tag{1.5}$$

Geometrically, the family $C_{\tan}$ comprises all of the functions f that lie within the image domain of $1 + \frac{\tan z}{2}$, for a specified radius.

2. Set of lemmas

We utilize the following lemmas in our major conclusion.

Let $\mathcal{P}$ stand for the family of all holomorphic functions p that have a positive real portion and are represented by the following series:

$$p(\kappa) = 1 + \sum_{j=1}^{\infty} c_j z^j,\ \kappa \in \Omega. \tag{2.1}$$

Lemma 2.1. *If $p \in \mathcal{P}$, then the following estimations hold:*

$$\begin{aligned} \left|c_j\right| &\leq 2, \; j \geq 1, && (2.2) \\ \left|c_{j+n} - \mu c_j c_n\right| &< 2, \; \; 0 < \mu \leq 1, && (2.3) \end{aligned}$$

and for $\eta \in \mathbb{C}$, we have

$$\left|c_2 - \eta c_1^2\right| < 2 \max\{1, |2\eta - 1|\}. \tag{2.4}$$

Regarding the inequalities (2.2)–(2.4) are detailed in [23].

Lemma 2.2. *[24] If $p \in \mathcal{P}$ and it has the form* (2.1)*, then*

$$|\alpha_1 c_1^3 - \alpha_2 c_1 c_2 + \alpha_3 c_3| \leq 2|\alpha_1| + 2|\alpha_2 - 2\alpha_1| + 2|\alpha_1 - \alpha_2 + \alpha_3|, \tag{2.5}$$

where α_1, α_2 and α_3 are real numbers.

Lemma 2.3. *[25] Let χ_1, σ_1, ψ_1 and ϱ_1 satisfy the inequalities for $\chi_1, \varrho_1 \in (0, 1)$ and*

$$\begin{aligned} &8\varrho_1 (1 - \varrho_1) \left[(\chi_1 \sigma_1 - 2\psi_1)^2 + (\chi_1 (\varrho_1 + \chi_1) - \sigma_1)^2\right] \\ &+ \chi_1 (1 - \chi_1) (\sigma_1 - 2\varrho_1 \chi_1)^2 \\ &\leq 4\chi_1^2 (1 - \chi_1)^2 \varrho_1 (1 - \varrho_1). \end{aligned}$$

If $h \in \mathcal{P}$ and is of the form (2.1)*, then*

$$\left|\psi_1 c_1^4 + \varrho_1 c_2^2 + 2\chi_1 c_1 c_3 - \frac{3}{2}\sigma_1 c_1^2 c_2 - c_4\right| \leq 2.$$

Lemma 2.4. *Let $p \in \mathcal{P}$ and x and z belong to Λ, then, we have*

$$\begin{aligned} 2c_2 &= c_1^2 + x\left(4 - c_1^2\right), \\ 4c_3 &= 2x\left(4 - c_1^2\right) c_1 - x^2 \left(4 - c_1^2\right) c_1 + 2z\left(1 - |x|^2\right)\left(4 - c_1^2\right) + c_1^3, \end{aligned}$$

where c_2 and c_3 are discussed in [26] and [27] respectively.

The goal of the current study was to derive the necessary and sufficient conditions, radius of convexity, growth and distortion estimates, sharp coefficient bounds, sharp Fekete-Szegö inequality, Krushkal inequality, and logarithmic coefficient estimates for the subclass $C_{\tan}$ of class $\mathcal{A}$ which is related to tangent functions.

3. Main results

Theorem 3.1. *Let $f \in C_{\tan}$ be as given in (1.1). Then*

$$\frac{1}{z}\left[f(z) * \left(\frac{z - Mz^2}{(1 - z)^3}\right)\right] \neq 0, \tag{3.1}$$

where

$$M = \frac{4 + \tanh\left(e^{i\theta}\right)}{2}. \tag{3.2}$$

Proof. Because $f \in C_{\tan}$ is analytic in $\mathbb{D}$, $\frac{1}{z}f(z) \neq 0$ for all z in $\mathbb{D}$ then, by using the definition of subordination and (1.5), we have

$$\frac{(zf'(z))'}{f'(z)} = 1 + \tanh \omega(z), \tag{3.3}$$

where $\omega(z)$ is the Schwarz function. Let $\omega(z) = e^{i\theta}$, $-\pi \leq \theta \leq \pi$. Then (3.3) becomes

$$\frac{zf''(z)}{f'(z)} \neq \frac{\tan\left(e^{i\theta}\right)}{2},$$

which implies that

$$z^2 f''(z) - zf'(z)\frac{\tan\left(e^{i\theta}\right)}{2} \neq 0. \tag{3.4}$$

It can be easily seen that

$$z^2 f''(z) + zf'(z) = f(z) * \frac{z(1+z)}{(1-z)^3} \text{ and } zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{3.5}$$

Using (3.5), and through some simple calculations (3.4) becomes

$$f(z) * \left(\frac{z - Mz^2}{(1-z)^3}\right) \neq 0. \tag{3.6}$$

From (3.6), we will obtain (3.1), where M is given in (3.2). □

Theorem 3.2. *Let $f \in \mathcal{A}$. Then $f \in C_{\tan}$ if*

$$\sum_{n=2}^{\infty}\left[\frac{2n\left(2+\tan\left(e^{i\theta}\right)\right) - 4n^2}{\tan\left(e^{i\theta}\right)}\right]\xi_n z^{n-1} - 1 \neq 0. \tag{3.7}$$

Proof. If $f \in C_{\tan}$ then from Theorem 3.1, we have

$$\frac{1}{z}\left[f(z) * \left(\frac{z - Mz^2}{(1-z)^3}\right)\right] \neq 0,$$

where M is given in (3.2). The above relation implies that

$$\frac{1}{z}\left[\left(f(z) * \frac{z}{(1-z)^3}\right) - \left(f(z) * \frac{Mz^2}{(1-z)^3}\right)\right] \neq 0.$$

Since $z^2 = z(1+z) - z$, so we have

$$\frac{1}{z}\left[\left(f(z) * \frac{z}{(1-z)^3}\right) - M\left(f(z) * \frac{z(1+z)}{(1-z)^3} - f(z) * \frac{z}{(1-z)^3}\right)\right] \neq 0. \tag{3.8}$$

Now applying (3.5) and some properties of convolution, (3.8), reduces to

$$\frac{1}{z}\left[\left(\frac{1}{2}z^2 f''(z) + zf'(z)\right) - M\left(z^2 f''(z)\right)\right] \neq 0.$$

Using (1.1) and after some simplification, we obtain (3.7). □

Theorem 3.3. *Let $f \in \mathcal{A}$ be as given in (1.1). Then $f \in C_{\tan}$ if*

$$\sum_{n=2}^{\infty}\left(\left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right|\right)|\xi_n| < 1. \tag{3.9}$$

Proof. To demonstrate the necessary outcome, we employ relation (3.7) as follows:

$$\begin{aligned}&\left|1-\sum_{n=2}^{\infty}\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\xi_n z^{n-1}\right|\\ >\ & 1-\sum_{n=2}^{\infty}\left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right||\xi_n|\,|z|^{n-1}.\end{aligned} \tag{3.10}$$

From (3.9), we have

$$1-\sum_{n=2}^{\infty}\left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right||\xi_n| > 0. \tag{3.11}$$

From (3.10) and (3.11), we obtain the intended outcome by applying Theorem 3.2. □

Theorem 3.4. *Let $f \in C_{\tan}$. Then f is convex and of order α, $0 \le \alpha < 1$ and $|z| < r_1$, where*

$$r_1 = \inf_{n\ge 2}\left(\frac{\left|4+n(n-3)\tan\left(e^{i\theta}\right)\right|}{|2\tan\left(e^{i\theta}\right)|}\frac{(1-\alpha)}{n(n-\alpha)}\right)^{\frac{1}{n-1}}. \tag{3.12}$$

Proof. It is sufficient to show that

$$\left|\frac{(zf'(z))'}{f'(z)} - 1\right| \le 1-\alpha. \tag{3.13}$$

From (1.1), we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)\xi_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} n\xi_n|z|^{n-1}}. \tag{3.14}$$

(3.14) is bounded above by $1-\alpha$, if

$$\sum_{n=2}^{\infty}\left[\frac{n(n-1)+n(1-\alpha)}{1-\alpha}\right]|\xi_n|\,|z|^{n-1} \le 1. \tag{3.15}$$

But by Theorem 3.1, the above inequality is true if

$$\sum_{n=2}^{\infty}\left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right||\xi_n| < 1. \tag{3.16}$$

Then the inequality (3.15), becomes

$$\left[\frac{n(n-\alpha)}{1-\alpha}\right]|z|^{n-1} \le \left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right|.$$

Simple math yields

$$r_1 = \inf_{n\geq 2}\left(\left|\frac{(1-\alpha)\left(4n-2\left(2+\tan\left(e^{i\theta}\right)\right)\right)}{(n-\alpha)\tan\left(e^{i\theta}\right)}\right|\right)^{\frac{1}{n-1}}.$$

The desired outcome is demonstrated. □

4. Growth and distortion estimates

Theorem 4.1. *Let $f \in C_{\tan}$ and $|z| = r$. Then*

$$r - \left|\frac{\tan\left(e^{i\theta}\right)}{8-4\tan\left(e^{i\theta}\right)}\right| r^2 \leq |f(z)| \leq r + \left|\frac{\tan\left(e^{i\theta}\right)}{8-4\tan\left(e^{i\theta}\right)}\right| r^2. \tag{4.1}$$

Proof. Consider that

$$\begin{aligned} |f(z)| &= \left|z + \sum_{n=2}^{\infty} \xi_n z^n\right| \\ &\leq r + \sum_{n=2}^{\infty} |\xi_n| r^n. \end{aligned}$$

Since $r^n \leq r^2$ for $n \geq 2$ and $r < 1$, we have

$$|f(z)| \leq r + r^2 \sum_{n=2}^{\infty} |\xi_n|. \tag{4.2}$$

Similarly

$$|f(z)| \geq r - r^2 \sum_{n=2}^{\infty} |\xi_n|. \tag{4.3}$$

Now, applying (3.9) implies that

$$\sum_{n=2}^{\infty} \left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right| |\xi_n| < 1.$$

Since

$$\left|\frac{16-4\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right| \sum_{n=2}^{\infty} |\xi_n| \leq \sum_{n=2}^{\infty} \left|\frac{4n^2 - 2n\left(2+\tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right| |\xi_n|,$$

we get

$$\left|\frac{8-4\tan\left(e^{i\theta}\right)}{\tan\left(e^{i\theta}\right)}\right| \sum_{n=2}^{\infty} |\xi_n| < 1,$$

One can easily write this as follows:

$$\sum_{n=2}^{\infty} |\xi_n| < \left|\frac{\tan\left(e^{i\theta}\right)}{16-4\left(2+\tan\left(e^{i\theta}\right)\right)}\right|,$$

Placing this value in (4.2) and (4.3) the necessary inequality is obtained. □

Theorem 4.2. *Let $f \in C_{\tan}$ and $|z| = r$. Then,*

$$1 - 2\left|\frac{\tan\left(e^{i\theta}\right)}{8 - 4\tan\left(e^{i\theta}\right)}\right| r \leq |f'(z)| \leq 1 + 2\left|\frac{\tan\left(e^{i\theta}\right)}{8 - 4\tan\left(e^{i\theta}\right)}\right| r.$$

Proof. Consider that

$$\begin{aligned} |f'(z)| &= \left|1 + \sum_{n=2}^{\infty} n\xi_n z^n\right| \\ &\leq 1 + \sum_{n=2}^{\infty} |\xi_n| r^{n-1}. \end{aligned}$$

Since $r^{n-1} \leq r$ for $n \geq 2$ and $r < 1$, we have

$$|f'(z)| \leq 1 + 2r \sum_{n=2}^{\infty} |\xi_n|. \tag{4.4}$$

Similarly

$$|f'(z)| \geq 1 - 2r \sum_{n=2}^{\infty} |\xi_n|. \tag{4.5}$$

Now, applying (3.9) implies that

$$\sum_{n=2}^{\infty} \left|\frac{4n^2 - 2n\left(2 + \tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right| |\xi_n| < 1.$$

Since

$$\left|\frac{16 - 4\left(2 + \tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right| \sum_{n=2}^{\infty} |\xi_n| \leq \sum_{n=2}^{\infty} \left|\frac{4n^2 - 2n\left(2 + \tan\left(e^{i\theta}\right)\right)}{\tan\left(e^{i\theta}\right)}\right| |\xi_n|,$$

we get

$$\left|\frac{8 - 4\tan\left(e^{i\theta}\right)}{\tan\left(e^{i\theta}\right)}\right| \sum_{n=2}^{\infty} |\xi_n| < 1,$$

one can easily write this as follows:

$$\sum_{n=2}^{\infty} |\xi_n| < \left|\frac{\tan\left(e^{i\theta}\right)}{8 - 4\tan\left(e^{i\theta}\right)}\right|.$$

Setting this value in (4.4) and (4.5) , we accomplish what is needed. □

Theorem 4.3. *For $f(z) \in C_{\tan}$, the coefficient bounds are given by*

$$|\xi_2| \leq \frac{1}{4}, \tag{4.6}$$

$$|\xi_3| \leq \frac{1}{12}, \tag{4.7}$$
$$|\xi_4| \leq \frac{1}{24}, \tag{4.8}$$
$$|\xi_5| \leq \frac{1}{24}. \tag{4.9}$$

and

$$\left|\xi_3 - \eta\xi_2^2\right| \leq \frac{1}{12}\max\left\{1, \left|\frac{3\eta-2}{4}\right|\right\}. \tag{4.10}$$

The above outcomes (4.6)–(4.9) *are sharp for the functions given below:*

$$f_1(z) = \int_0^z \exp\int_0^z \frac{\frac{\tan x}{2}}{x}dx = z + \frac{1}{4}z^2 + \frac{1}{24}z^3 + \cdots, \tag{4.11}$$

$$f_2(z) = \int_0^z \exp\int_0^z \frac{\frac{\tan x^2}{2}}{x}dx = z + \frac{z^3}{12} + \frac{z^5}{160} + \cdots, \tag{4.12}$$
$$f_3(z) = \int_0^z \exp\int_0^z \frac{\frac{\tan x^3}{2}}{x}dx = z + \frac{z^4}{24} + \frac{z^7}{504} + \cdots, \tag{4.13}$$
$$f_4(z) = \int_0^z \exp\int_0^z \frac{\frac{\tan x^4}{2}}{x}dx = z + \frac{z^5}{40} + \frac{z^9}{1152} + \cdots. \tag{4.14}$$

And the bound (4.10) *is extreme for the function defined in* (4.12).

Proof. Because $f(z) \in C_{\tan}$, we have the definition

$$\frac{(zf'(z))'}{f'(z)} \prec \frac{2+\tan(z)}{2},$$

which can be written as

$$\frac{(zf'(z))'}{f'(z)} = \frac{2+\tan(\omega(z))}{2},$$

where $\omega(z)$ is the holomorphic function with the following properties:

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1.$$

Now let

$$\frac{(zf'(z))'}{f'(z)} = 1 + 2\xi_2 z + \left(6\xi_3 - 4\xi_2^2\right)z^2 + \left(12\xi_4 - 18\xi_2\xi_3 + 8\xi_2^3\right)z^3 + \cdots, \tag{4.15}$$

and

$$1 + \frac{\tan(\omega(z))}{2} = 1 + \frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right)z^2 + \left(\frac{1}{12}c_1^3 - \frac{1}{4}c_2c_1 + \frac{1}{4}c_3\right)z^3$$

$$+\left(-\frac{1}{16}c_1^4+\frac{1}{4}c_1^2c_2-\frac{1}{4}c_3c_1-\frac{1}{8}c_2^2+\frac{1}{4}c_4\right)z^4+\ldots. \tag{4.16}$$

Comparing (4.15) and (4.16), we have

$$\xi_2 = \frac{1}{8}c_1, \tag{4.17}$$

$$\xi_3 = \frac{1}{24}c_2-\frac{1}{96}c_1^2, \tag{4.18}$$

$$\xi_4 = \frac{17}{4608}c_1^3-\frac{5}{384}c_2c_1+\frac{1}{48}c_3. \tag{4.19}$$

$$\xi_5 = -\frac{1}{80}\left(\frac{157}{1152}c_1^4-\frac{29}{48}c_1^2c_2+\frac{2}{3}c_3c_1+\frac{3}{8}c_2^2-c_4\right). \tag{4.20}$$

Then by applying (2.2) to (4.17) , we have

$$|\xi_2| \le \frac{1}{4}.$$

And applying (2.3) with $n = k = 1$ to (4.18) , we get

$$|\xi_3| \le \frac{1}{12}.$$

For (4.19), applying Lemma 2.2 yields

$$|\xi_4| \le \frac{1}{24}.$$

And for (4.20) , we have

$$\begin{aligned} |\xi_5| &= \left|-\frac{1}{80}\right|\left|\frac{157}{1152}c_1^4-\frac{29}{48}c_1^2c_2+\frac{2}{3}c_3c_1+\frac{3}{8}c_2^2-c_4\right| \\ &\le \frac{1}{40} \text{ (by Lemma 2.3)}. \end{aligned}$$

Now from (4.17) and (4.18) , we have

$$\left|\xi_3-\eta\xi_2^2\right| = \frac{1}{24}\left|c_2-\frac{3\eta-2}{4}c_1^2\right|.$$

And applying (2.4) to the above relation, we achieve our goals. □

The following outcome occurs if we set $\eta = 1$ in the above result.

Remark 4.4. If we set $\eta = 1$ in (4.10), we get the following result

$$\left|\xi_3-\xi_2^2\right| \le \frac{1}{12}.$$

The outcome is precise for the function defined in (4.12), and it cannot be further enhanced.

Theorem 4.5. *Let* $f(z) \in \mathcal{C}_{\tan}$. *Then*

$$|\xi_2\xi_3-\xi_4| \le \frac{1}{24}.$$

The outcome is sharp for the function defined in (4.13).

Proof. From (4.17)–(4.19) , we have

$$|\xi_2\xi_3 - \xi_4| = \left|\frac{23}{4608}c_1^3 + \frac{7}{384}c_2c_1 - \frac{1}{48}c_3\right|.$$

Applying Lemma 2.2, we achieve the intended outcomes. □

Theorem 4.6. *Let $f(z) \in C_{\tan}$. Then*

$$\left|\xi_2\xi_4 - \xi_3^2\right| \leq \frac{1}{144}.$$

The outcome is sharp for the function defined in (4.12) .

Proof. From (4.17)–(4.19), we have

$$\left|\xi_2\xi_4 - \xi_3^2\right| = \left|\frac{13}{36\,864}c_1^4 - \frac{7}{9216}c_1^2c_2 + \frac{1}{384}c_3c_1 - \frac{1}{576}c_2^2\right|.$$

Now using Lemma 2.4, with $c_1 = c$ and $|x| = y$, we have

$$\begin{aligned}\left|\xi_2\xi_4 - \xi_3^2\right| &\leq \frac{7}{36\,864}c^4 + \frac{1}{1536}c^2\left(4 - c^2\right)y^2 + \frac{1}{18\,432}c^2\left(4 - c^2\right)y \\ &\quad + \frac{1}{768}c\left(1 - y^2\right)\left(4 - c^2\right) + \frac{1}{2304}\left(4 - c^2\right)^2 y^2 \\ &= G(y, c) \text{ (say).}\end{aligned}$$

Further,

$$\frac{\partial G(y, c)}{\partial y} = \frac{1}{18\,432}\left(4 - c^2\right)\left(\left(64 + 8c^2 - 48c\right)y + c^2\right) > 0.$$

Clearly $\frac{\partial G(y,c)}{\partial y} > 0$ in $y \in [0, 1]$ so the maximum is attained at $y = 1$, i.e.,

$$G(1, c) = \frac{7}{36\,864}c^4 + \frac{1}{1536}c^2\left(4 - c^2\right) + \frac{1}{18\,432}c^2\left(4 - c^2\right) + \frac{1}{2304}\left(4 - c^2\right)^2 = H(c).$$

Further,

$$H'(c) = -\frac{1}{3072}c\left(c^2 + 4\right),$$

since $H'(c) = 0$ has three roots namely $c = 0, -2i$ and $2i$. The only root lying in the interval $[0, 2]$ is 0. Also, one may check easily that $H''(c) \leq 0$ for $c = 0$; thus, the maximum is attained at $c = 0$, that is

$$\left|\xi_2\xi_4 - \xi_3^2\right| \leq \frac{1}{144}.$$

□

5. Krushkal inequality

Here, we will provide direct evidence of the inequality

$$\left|\xi_n^p - \xi_2^{p(n-1)}\right| \leq 2^{p(n-1)} - n^p,$$

over the class $C_{\tan}$ for the choice of $n = 4$, $p = 1$, and for $n = 5$, $p = 1$. For a class of univalent functions as a whole, Krushkal introduced and demonstrated this inequality in [28]. For some recent investigations into the Krushkal inequality, we refer the readers to [14, 29].

Theorem 5.1. *For $f(z) \in \mathcal{C}_{\tan}$, we have*

$$\left|\xi_4 - \xi_2^3\right| \leq \frac{1}{24}.$$

The outcome is sharp for the function defined in (4.13) .

Proof. From (4.17) and (4.19) , we have

$$\left|\xi_4 - \xi_2^3\right| = \left|\frac{1}{576}c_1^3 - \frac{5}{384}c_2c_1 + \frac{1}{48}c_3\right|.$$

By applying Lemma 2.2, we get

$$\left|\xi_4 - \xi_2^3\right| \leq \frac{1}{24}.$$

□

Theorem 5.2. *For $f(z) \in \mathcal{C}_{\tan}$, we have*

$$\left|\xi_5 - \xi_2^4\right| \leq \frac{1}{40}.$$

The outcome is sharp for the function defined in (4.14) .

Proof. From (4.17) and (4.20), we have

$$\begin{aligned}\left|\xi_5 - \xi_2^4\right| &= \left|-\frac{1}{80}\right|\left|\frac{359}{2304}c_1^4 - \frac{29}{48}c_1^2c_2 + \frac{2}{3}c_3c_1 + \frac{3}{8}c_2^2 - c_4\right| \\ &\leq \frac{1}{40} \text{ (by Lemma 2.3).}\end{aligned}$$

□

6. Logarithmic coefficients for the family $\mathcal{C}_{\tan}$

The logarithmic coefficients of $f \in \mathcal{S}$ denoted by $\kappa_n = \kappa_n(f)$, are defined by the following series expansion:

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\kappa_n z^n.$$

For the function f given by (1.1) , the logarithmic coefficients are as follows:

$$\kappa_1 = \frac{1}{2}\xi_2, \tag{6.1}$$

$$\kappa_2 = \frac{1}{2}\left(\xi_3 - \frac{1}{2}\xi_2^2\right), \tag{6.2}$$

$$\kappa_3 = \frac{1}{2}\left(\xi_4 - \xi_2\xi_3 + \frac{1}{3}\xi_2^3\right), \tag{6.3}$$

$$\kappa_4 = \frac{1}{2}\left(\xi_5 - \xi_2\xi_4 - \xi_2^2\xi_3 - \frac{1}{2}\xi_3^2 - \frac{1}{4}\xi_2^4\right). \tag{6.4}$$

Theorem 6.1. *If f has the form* (1.1) *and belongs to* $\mathcal{C}_{\tan}$, *then*

$$\begin{aligned}|\kappa_1| &\le \frac{1}{8},\\ |\kappa_2| &\le \frac{1}{24},\\ |\kappa_3| &\le \frac{1}{48},\\ |\kappa_4| &\le \frac{1}{80}.\end{aligned}$$

The bounds of Theorem 6.1 are precise and cannot be improved further.

Proof. Now from (6.1) to (6.4) and (4.17) to (4.20) , we get

$$\kappa_1 = \frac{1}{16}c_1, \tag{6.5}$$

$$\kappa_2 = \frac{1}{48}c_2 - \frac{7}{768}c_1^2, \tag{6.6}$$

$$\kappa_3 = \frac{13}{4608}c_1^3 - \frac{7}{768}c_2c_1 + \frac{1}{96}c_3, \tag{6.7}$$

$$\kappa_4 = -\frac{1561}{1474\,560}c_1^4 + \frac{413}{92\,160}c_1^2c_2 - \frac{7}{1280}c_3c_1 - \frac{1}{360}c_2^2 + \frac{1}{160}c_4, \tag{6.8}$$

Applying (2.2) to (6.5), we get

$$|\kappa_1| \le \frac{1}{8}.$$

From (6.6), using (2.3), we get

$$|\kappa_2| \le \frac{1}{24}.$$

Applying Lemma 2.2 to (6.7), we get

$$|\kappa_3| \le \frac{1}{48}.$$

Also, applying Lemma 2.3 to (6.8) , we get

$$|\kappa_4| \le \frac{1}{80}.$$

Proof for sharpness: Since

$$\begin{aligned}\log\frac{f_1(z)}{z} &= 2\sum_{n=2}^{\infty}\kappa(f_1)z^n = \frac{1}{4}z + \cdots,\\ \log\frac{f_2(z)}{z} &= 2\sum_{n=2}^{\infty}\kappa(f_2)z^n = \frac{1}{12}z^2 + \cdots,\\ \log\frac{f_3(z)}{z} &= 2\sum_{n=2}^{\infty}\kappa(f_2)z^n = \frac{1}{24}z^3 + \cdots,\end{aligned}$$

$$\log \frac{f_4(z)}{z} = 2\sum_{n=2}^{\infty} \kappa(f_2) z^n = \frac{1}{40} z^4 + \cdots,$$

it follows that these inequalities can be obtained for the functions denoted by $f_n(z)$ for $n = 1, 2, 3$ and 4 as defined in (4.11) to (4.14). □

Theorem 6.2. *Let $f \in \mathcal{C}_{\tan}$. Then for a complex number λ, we have*

$$\left|\kappa_2 - \lambda\kappa_1^2\right| \le \frac{1}{24} \max\left\{1, \frac{|3\lambda - 1|}{8}\right\}.$$

The result is the best possible.

Proof. From (6.5) and (6.6), we have

$$\left|\kappa_2 - \lambda\kappa_1^2\right| = \frac{1}{48}\left|c_2 - \frac{7 + 3\lambda}{16} c_1^2\right|.$$

Applying (2.4) to the preceding equation yields the desired outcome. □

Theorem 6.3. *Let $f \in \mathcal{C}_{\tan}$. Then*

$$|\kappa_1\kappa_2 - \kappa_3| \le \frac{1}{48}.$$

The outcome is extremal.

Proof. From (6.5)–(6.7), we have

$$|\kappa_1\kappa_2 - \kappa_3| = \left|\frac{125}{36\,864} c_1^3 - \frac{1}{96} c_2 c_1 + \frac{1}{96} c_3\right|.$$

Applying Lemma 2.2, we achieve the intended outcomes. □

Theorem 6.4. *Let $f \in \mathcal{C}_{\tan}$. Then*

$$\left|\kappa_1\kappa_3 - \kappa_2^2\right| \le \frac{1}{576}.$$

The outcome is sharp.

Proof. From (6.5)–(6.7), we have

$$\left|\kappa_1\kappa_3 - \kappa_2^2\right| = \left|\frac{55}{589\,824} c_1^4 - \frac{7}{36\,864} c_1^2 c_2 + \frac{1}{1536} c_3 c_1 - \frac{1}{2304} c_2^2\right|.$$

Now using Lemma 2.4, with $c_1 = c$, $|z| = 1$ and $|x| = y$, we have

$$\begin{aligned}\left|\kappa_1\kappa_3 - \kappa_2^2\right| &\le \frac{31}{589\,824} c^4 + \frac{1}{6144} c^2\left(4 - c^2\right) y^2 + \frac{1}{73\,728} c^2\left(4 - c^2\right) y \\ &\quad + \frac{1}{3072} c\left(1 - y^2\right)\left(4 - c^2\right) + \frac{1}{9216}\left(4 - c^2\right)^2 y^2 \\ &= G(y, c) \ \text{(say)}.\end{aligned}$$

Further,

$$\frac{\partial G(y,c)}{\partial y} = \frac{1}{73\,728}\left(4-c^2\right)\left(64y+8c^2y-48cy+c^2\right).$$

Clearly, $\frac{\partial G(y,c)}{\partial y} > 0$ in $y \in [0,1]$ so the maximum is attained at $y = 1$, i.e.,

$$G(1,c) = \frac{31}{589\,824}c^4 + \frac{1}{6144}c^2\left(4-c^2\right) + \frac{1}{73\,728}c^2\left(4-c^2\right) + \frac{1}{9216}\left(4-c^2\right)^2 = H(c).$$

Further,

$$H'(c) = -\frac{1}{49\,152}c\left(3c^2+16\right),$$

since $H'(c) = 0$ has only one solution $c = 0$, that lies in the interval $[0,2]$. Also, one may check easily that $H''(c) \leq 0$ for $c = 0$; thus, the maximum can be attained at $c = 0$, that is

$$H(0) \leq \frac{1}{576}.$$

□

7. Conclusions

In this study, we were motivated by the recent research and the sharp bounds of Hankel inequalities, and have have defined a new subclass of holomorphic convex functions that are related to the tangent functions. We then derived geometric properties like the necessary and sufficient conditions, radius of convexity, growth, and distortion estimates for our defined function class. Furthermore, the sharp coefficient bounds, sharp Fekete-Szegö inequality, sharp 2nd order Hankel determinant, and Krushkal inequalities have been given. Moreover, we have calculated the sharp coefficient bounds, sharp Fekete-Szegö inequality, and sharp second-order Hankel determinant for the functions whose coefficients are logarithmic. Hopefully, this work will open new directions for those working in geometric function theory and related areas. One can extend the work here by replacing the ordinal derivative with a certain q-derivative operator.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest.

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