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Research article

Boundedness and higher integrability of minimizers to a class of two-phase free boundary problems under non-standard growth conditions

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Abstract: In this paper, we are concerned with the existence, boundedness, and integrability of minimizers of heterogeneous, two-phase free boundary problems $\mathcal{J}_\gamma(u) = \int_\Omega (f(x,\nabla u) + \lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma + gu)\,\mathrm{d}x \to \min$ under non-standard growth conditions. Included in such problems are heterogeneous jets and cavities of Prandtl-Batchelor type with $\gamma = 0$, chemical reaction problems with $0 < \gamma < 1$, and obstacle type problems with $\gamma = 1$, respectively.

Keywords: free boundary problem; two-phase; integrability; non-standard growth; minimizer
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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 2)$. Let $\psi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$ with $p, q \in C(\Omega; (1, +\infty))$ being certain given functions. The aim of this paper is to study heterogeneous, two-phase free boundary problems

$$\mathcal{J}_\gamma(u) = \int_\Omega \Big(f(x,\nabla u) + F_\gamma(u) + gu\Big)\,\mathrm{d}x \to \min \tag{1.1}$$

over the set $\mathcal{K} = \left\{u \in W^{1,p(\cdot)}(\Omega) : u - \psi \in W_0^{1,p(\cdot)}(\Omega)\right\}$ in the framework of Sobolev spaces with variable exponents, where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function having a form

$$L^{-1}|z|^{p(x)} \leq f(x,z) \leq L(1 + |z|^{p(x)}), \forall x \in \Omega, z \in \mathbb{R}^n \tag{1.2}$$

with $L \geq 1$ being a constant. The non-differentiable potential $F_\gamma(\cdot)$ is given by

$$F_\gamma(u) := \lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma,$$

where $\gamma \in [0,1]$ is a parameter, and $\lambda_+, \lambda_- \in \mathbb{R}$ are positive constants with $\lambda_+ > \lambda_-$. As usual, $u^\pm := \max\{\pm u, 0\}$, and by convention,

$$F_0(u) := \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u\leq 0\}}.$$

As is well known, the lower limiting case, i.e., $\gamma = 0$, relates to the jets and cavities problems. The upper case, i.e., $\gamma = 1$, relates to obstacle-type problems. The intermediary problem, i.e., $\gamma \in (0,1)$, can be used to model the density of certain chemical species in reaction with a porous catalyst pellet, and has intrigued a number of mathematicians in the past decades.

It should be mentioned that a large class of functionals and identical obstacle problems under non-standard growth conditions have been studied in [1,3–5,8,17], which provide the reference estimates, and suitable localization and freezing techniques, etc., to treat the non-standard growth exponents in the functional governed by (1.1). It is well known that the boundedness of minimizers plays a crucial role in getting regularity results. For more details about the history of free boundary problems of these types, we refer to the work [15], where the authors provided a complete description of regularity theory for the free boundary problems governed by (1.1) with $f(z) \equiv |z|^p$ and a constant $p \in [2, +\infty)$. Local and global higher integrability results for solutions or derivatives of the solutions to the obstacle problems, one may refer to [9–12, 19] and the references therein. The existence and asymptotic analysis of nontrivial solutions for some related double-phase problems under unbalanced growth conditions may be referred to [16, 18, 21] and the references therein.

In this paper, we would like to extend several known results to a larger class of free boundary problems governed by (1.1). We shall establish the existence, boundedness, and integrability of minimizers of $\mathcal{J}_\gamma(u)$. The results obtained in this paper are not only extensions of the one in the one-phase obstacle problems under non-standard growth conditions (see, e.g., [3,4]), but also a supplement to the one in the degenerate free boundary problems studied in [15], as we also consider the singular case $p \in (1,2)$.

In the rest of the paper, we first introduce some notations used in this paper. In Section 2, we state basic assumptions on the functions f, p, and q and main results on the existence, boundedness, and higher integrability of minimizers, which are proved in Sections 3 and 4, respectively.

Notation. Denote by $B_R(x)$ the open ball in $\mathbb{R}^n$ with center x and radius $R > 0$, and $|B_R(x)|$ is the Lebesgue measure of $B_R(x)$. For an integrable function u defined on $B_R(x)$, let $(u)_{x,R} := \frac{1}{|B_R(x)|}\int_{B_R(x)} u(x)\mathrm{d}x$. Without confusion, for $R > 0$, we will write B_R and $(u)_R$ instead of $B_R(x)$ and $(u)_{x,R}$ respectively. Let $C, c, C_1, C_2, C_3, ...$ denote constants that may be different from each other, but independent of γ.

The variable exponent of Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ u |\ u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}\mathrm{d}x < +\infty \right\}$$

with the norm $\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0;\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\}$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega);\ |\nabla u| \in L^{p(\cdot)}(\Omega)\right\} \tag{1.3}$$

with the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$. Define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If Ω is bounded and $p(\cdot)$ satisfies (2.5) specified in Section 2, then the spaces $L^{p(\cdot)}(\Omega), W^{1,p(\cdot)}(\Omega)$, and $W_0^{1,p(\cdot)}(\Omega)$ are all separable and reflexive Banach spaces. $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is an equivalent norm of $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined for $W_0^{1,p(\cdot)}(\Omega)$. We refer to [6,7,13] for the elementary properties and more details of the space $W^{1,p(\cdot)}(\Omega)$.

2. Main results

In this paper, we always propose the following growth, ellipticity, and continuity conditions on the function f:

$$f : \Omega \times \mathbb{R}^n \to \mathbb{R}, f(x,z) \text{ is convex in } z \text{ for every } x, \tag{2.1}$$

$$L^{-1}(\mu^2 + |z|^2)^{\frac{p(x)}{2}} \le f(x,z) \le L(\mu^2 + |z|^2)^{\frac{p(x)}{2}}, \forall x \in \Omega, z \in \mathbb{R}^n, \tag{2.2}$$

where $L \ge 1$ and $\mu \in [0,1]$ are constants. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing continuous function, vanishing at zero, which represents the modulus of $p \in C(\Omega; (1, +\infty))$:

$$|p(x) - p(y)| \le \omega(|x-y|) \text{ for all } x, y \in \overline{\Omega}, \tag{2.3}$$

and satisfies $\limsup_{R\to 0} \omega(R) \log\left(\frac{1}{R}\right) < +\infty$. Without loss of generality, we assume that

$$\omega(R) \le L|\log R|^{-1}, \forall R < 1. \tag{2.4}$$

Assume further that

$$1 < p_- = \inf_{x\in\Omega} p(x) \le p(x) \le \sup_{x\in\Omega} p(x) = p_+ < n \text{ for all } x \in \Omega \tag{2.5}$$

with

$$\frac{1}{p_-} - \frac{1}{p_+} < \frac{1}{n}. \tag{2.6}$$

Let $q \in C(\Omega; (1, +\infty))$ satisfy the conditions of the types (2.3) and (2.4) and

$$q(x) \ge q_- \text{ for all } x \in \Omega,\ q_- > \begin{cases} \frac{1}{p_- - 1} \frac{1}{\frac{1}{n} - \frac{1}{p_-} + \frac{1}{p_+}} > n, \text{ if } p_- < 2, \\ \frac{1}{\frac{1}{n} - \frac{1}{p_-} + \frac{1}{p_+}} \ge n, \text{ if } p_- \ge 2. \end{cases} \tag{2.7}$$

A function $u \in \mathcal{K}$ is said to be a minimizer of the functional $\mathcal{J}_\gamma(u)$ governed by (1.1) if $\mathcal{J}_\gamma(u) \le \mathcal{J}_\gamma(v)$ for all $v \in \mathcal{K}$.

The first result obtained in this paper is concerned with the existence and uniform (w.r.t. γ) boundedness of minimizers for the functional $\mathcal{J}_\gamma(u)$.

Theorem 2.1. *Assume that* (2.1)–(2.7) *hold. Then, for each* $\gamma \in [0,1]$, *there exists a minimizer* $u_\gamma \in \mathcal{K}$ *of the functional* $\mathcal{J}_\gamma(u)$. *Furthermore,* u_γ *is bounded. More precisely,*

$$\|u_\gamma\|_{L^\infty(\Omega)} \le C(n, L, q_-, p_\pm, \lambda_\pm, \Omega, \|\psi\|_{L^\infty(\partial\Omega)}, \|g\|_{L^{q(\cdot)}(\Omega)}).$$

The second result obtained in this paper is the following theorem, which indicates higher integrability of minimizers of the functional $\mathcal{J}_\gamma(u)$.

Theorem 2.2. *Assume that* (2.1)–(2.7) *hold and* $u_\gamma \in \mathcal{K}$ *is a minimizer of the functional* $\mathcal{J}_\gamma(u)$. *Then there exist two positive constants* C_0 *and* $\delta_0 < q_-\left(1-\frac{1}{p_-}\right)-1$, *both depending only on* $n, p_\pm, \lambda_\pm, q_-, L, M$, *and* Ω, *such thatf*

$$\left(\frac{1}{|B_{R/2}|}\int_{B_{R/2}}|\nabla u_\gamma|^{p(x)(1+\delta_0)}\mathrm{d}x\right)^{\frac{1}{1+\delta_0}} \leq \frac{C_0}{|B_R|}\int_{B_R}|\nabla u_\gamma|^{p(x)}\mathrm{d}x + C_0\left(\frac{1}{|B_R|}\int_{B_R}\left(1+|g|^{\frac{p_-}{p_--1}(1+\delta_0)}\right)\mathrm{d}x\right)^{\frac{1}{1+\delta_0}}, \forall B_R \subset\subset \Omega. \quad (2.8)$$

3. Existence and L^∞-boundedness of minimizers

In this section, we prove Theorem 2.1 in a similar way as in [15].

Proof of Theorem 2.1. Firstly, we prove the existence of a minimizer of the functional $\mathcal{J}_\gamma(u)$.

Let $I_0 := \min\{\mathcal{J}_\gamma(u) : u \in \mathcal{K}\}$. We claim that $I_0 > -\infty$. Indeed, for any $u \in \mathcal{K}$, by Poincaré's inequality, there exists a positive constant $C = C(n, p_\pm, \Omega)$ such that

$$\begin{aligned}\|u\|_{L^{p(\cdot)}(\Omega)} &\leq \|u-\psi\|_{L^{p(\cdot)}(\Omega)} + \|\psi\|_{L^{p(\cdot)}(\Omega)}\\ &\leq C\|\nabla u - \nabla\psi\|_{L^{p(\cdot)}(\Omega)} + \|\psi\|_{L^{p(\cdot)}(\Omega)}\\ &\leq C\left(\|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla\psi\|_{L^{p(\cdot)}(\Omega)} + \|\psi\|_{L^{p(\cdot)}(\Omega)}\right), \end{aligned} \quad (3.1)$$

which implies

$$\|\nabla u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \geq C_1\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} - \|\psi\|^{p_-}_{L^{p(\cdot)}(\Omega)} - \|\nabla\psi\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \quad (3.2)$$

and

$$\|\nabla u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \geq C_2\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} - \|\psi\|^{p_+}_{L^{p(\cdot)}(\Omega)} - \|\nabla\psi\|^{p_+}_{L^{p(\cdot)}(\Omega)}, \quad (3.3)$$

where C_1 and C_2 are positive constants depending only on $n, p_\pm$, and Ω.

Due to $q(x) \geq q_-$, we deduce from (2.7), Hölder's inequality, and Young's inequality with $\epsilon > 0$ that

$$\begin{aligned}\left|\int_\Omega gu\mathrm{d}x\right| &\leq C_3(p_+,p_-)\|g\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)}\\ &\leq C_4(p_+,p_-)\|g\|_{L^{q(\cdot)}(\Omega)}\|1\|_{L^{\frac{1}{1-\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}}}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)}\\ &\leq C_4(p_+,p_-)\left(1+|\Omega|^{1-\frac{1}{p_-}-\frac{1}{q_-}}\right)\|g\|_{L^{q(\cdot)}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)} \qquad (3.4)\\ &\leq \begin{cases}\varepsilon\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} + C_5(\varepsilon,p_\pm,\Omega)\|g\|^{\frac{p_-}{p_--1}}_{L^{q(\cdot)}(\Omega)}, \text{ or,}\\ \varepsilon\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} + C_6(\varepsilon,p_\pm,\Omega)\|g\|^{\frac{p_+}{p_+-1}}_{L^{q(\cdot)}(\Omega)},\end{cases} \qquad (3.5)\end{aligned}$$

where $\varepsilon \in (0, 1)$ will be chosen later.

Now we consider two cases: $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} > 1$ and $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le 1$.

Case 1: $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} > 1$. It follows from (2.2), (3.2), and (3.5) that

$$\begin{aligned} \mathcal{J}_\gamma(u) &\ge L^{-1}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - \left|\int_\Omega gu\mathrm{d}x\right| \qquad (3.6)\\ &\ge L^{-1}\|\nabla u\|^{p_-}_{L^{p(\cdot)}(\Omega)} - \left|\int_\Omega gu\mathrm{d}x\right|\\ &\ge L^{-1}C_1\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} - L^{-1}\left(\|\psi\|^{p_-}_{L^{p(\cdot)}(\Omega)} + \|\nabla\psi\|^{p_-}_{L^{p(\cdot)}(\Omega)}\right) - \varepsilon\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}\\ &\quad -C_5(\varepsilon, p_\pm, \Omega)\|g\|^{\frac{p_-}{p_--1}}_{L^{q(\cdot)}(\Omega)}. \qquad (3.7)\end{aligned}$$

Choose $\varepsilon \in (0, 1)$ such that $L^{-1}C_1 - \varepsilon > 0$, then, (3.7) yields

$$\mathcal{J}_\gamma(u) > -L^{-1}(\|\psi\|^{p_-}_{L^{p(\cdot)}(\Omega)} + \|\nabla\psi\|^{p_-}_{L^{p(\cdot)}(\Omega)}) - C_5(\varepsilon, p_\pm, \Omega)\|g\|^{\frac{p_-}{p_--1}}_{L^{q(\cdot)}(\Omega)} > -\infty.$$

Case 2: $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le 1$. We deduce from (2.2), (3.3), and (3.5) that

$$\begin{aligned} \mathcal{J}_\gamma(u) &\ge L^{-1}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - \left|\int_\Omega gu\mathrm{d}x\right|\\ &\ge L^{-1}\|\nabla u\|^{p_+}_{L^{p(\cdot)}(\Omega)} - \left|\int_\Omega gu\mathrm{d}x\right|\\ &\ge L^{-1}C_2\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} - L^{-1}\left(\|\psi\|^{p_+}_{L^{p(\cdot)}(\Omega)} + \|\nabla\psi\|^{p_+}_{L^{p(\cdot)}(\Omega)}\right) - \varepsilon\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}\\ &\quad -C_6(\varepsilon, p_\pm, \Omega)\|g\|^{\frac{p_+}{p_+-1}}_{L^{q(\cdot)}(\Omega)}. \qquad (3.8)\end{aligned}$$

Choose $\varepsilon \in (0, 1)$ such that $L^{-1}C_2 - \varepsilon > 0$, then, (3.8) gives

$$\mathcal{J}_\gamma(u) > -L^{-1}\left(\|\psi\|^{p_+}_{L^{p(\cdot)}(\Omega)} + \|\nabla\psi\|^{p_+}_{L^{p(\cdot)}(\Omega)}\right) - C_6(\varepsilon, p_\pm, \Omega)\|g\|^{\frac{p_+}{p_+-1}}_{L^{q(\cdot)}(\Omega)} > -\infty.$$

Now we prove the existence of a minimizer of $\mathcal{J}_\gamma(u)$. Let $u_j \in \mathcal{K}$ be a minimizing sequence. We will show that $\{u_j - \psi\}$ (up to a subsequence) is bounded in $W^{1,p(\cdot)}_0(\Omega)$. Without loss of generality, we assume that $\|\nabla u_j\|_{L^{p(\cdot)}(\Omega)} > 1$. For $j \gg 1$, we have $\mathcal{J}_\gamma(u_j) \le I_0 + 1$.

From (3.1), (3.4), (3.6), and Young's inequality with $\varepsilon > 0$, we obtain

$$\begin{aligned} \|\nabla u_j\|^{p_-}_{L^{p(\cdot)}(\Omega)} &\le \int_\Omega |\nabla u_j|^{p(x)}\mathrm{d}x\\ &\le L\mathcal{J}_\gamma(u_j) + L\left|\int_\Omega gu_j\mathrm{d}x\right|\\ &\le L(I_0 + 1) + LC_7(p_\pm, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)})\|u_j\|_{L^{p(\cdot)}(\Omega)},\\ &\le C_8(\|\nabla u_j\|_{L^{p(\cdot)}(\Omega)} + \|\nabla\psi\|_{L^{p(\cdot)}(\Omega)} + \|\psi\|_{L^{p(\cdot)}(\Omega)}) + L(I_0 + 1),\\ &\le \frac{1}{2}\|\nabla u_j\|^{p_-}_{L^{p(\cdot)}(\Omega)} + C_9\left(1 + \|\nabla\psi\|_{L^{p(\cdot)}(\Omega)} + \|\psi\|_{L^{p(\cdot)}(\Omega)}\right),\end{aligned}$$

where C_8 and C_9 depend only on $L, I_0, p_\pm, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$. Then, we get

$$\|\nabla u_j\|^{p_-}_{L^{p(\cdot)}(\Omega)} \le 2C_9\left(1 + \|\nabla\psi\|_{L^{p(\cdot)}(\Omega)} + \|\psi\|_{L^{p(\cdot)}(\Omega)}\right),$$

which, along with Poincaré's inequality, ensures that $\{u_j-\psi\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Therefore, there is a function $u\in\mathcal{K}$ such that, up to a subsequence,

$$u_j \rightharpoonup u \text{ weakly in } W^{1,p(\cdot)}(\Omega),\quad u_j\to u \text{ in } L^{p(\cdot)}(\Omega),\quad u_j\to u \text{ a.e. in } \Omega.$$

With a slight modification of the proof of [20, Theorem 1.6], we infer from (2.1) and (2.2) that

$$\int_\Omega f(x,|\nabla u|)\mathrm{d}x \le \liminf_{j\to\infty}\int_\Omega f(x,|\nabla u_j|)\mathrm{d}x. \tag{3.9}$$

For $\gamma\in(0,1]$, by the pointwise convergence, we have

$$\int_\Omega (F_\gamma(u)+gu)\mathrm{d}x \le \liminf_{j\to\infty}\int_\Omega (F_\gamma(u_j)+gu_j)\mathrm{d}x. \tag{3.10}$$

For $\gamma=0$, recalling that $\lambda_+>\lambda_->0$, we have

$$\int_\Omega \lambda_-\chi_{\{u\le0\}}\mathrm{d}x = \int_{\{u\le0\}}\lambda_-\chi_{\{u_j>0\}}\mathrm{d}x + \int_{\{u\le0\}}\lambda_-\chi_{\{u_j\le0\}}\mathrm{d}x \le \int_{\{u\le0\}}\lambda_+\chi_{\{u_j>0\}}\mathrm{d}x + \int_\Omega \lambda_-\chi_{\{u_j\le0\}}\mathrm{d}x,$$

which implies

$$\int_\Omega \lambda_-\chi_{\{u\le0\}}\mathrm{d}x \le \liminf_{j\to\infty}\left(\int_{\{u\le0\}}\lambda_+\chi_{\{u_j>0\}}\mathrm{d}x + \int_\Omega \lambda_-\chi_{\{u_j\le0\}}\mathrm{d}x\right).$$

In addition, since $u_j\to u$ a.e. in Ω, it follows from the Dominated Convergence Theorem that

$$\int_\Omega \lambda_+\chi_{\{u>0\}}\mathrm{d}x = \int_{\{u>0\}}\lambda_+\lim_{j\to\infty}\chi_{\{u_j>0\}}\mathrm{d}x = \lim_{j\to\infty}\int_{\{u>0\}}\lambda_+\chi_{\{u_j>0\}}\mathrm{d}x.$$

Therefore, it holds that

$$\int_\Omega (F_0(u)+gu)\mathrm{d}x \le \liminf_{j\to\infty}\int_\Omega (F_0(u_j)+gu_j)\mathrm{d}x. \tag{3.11}$$

From (3.9), (3.10), and (3.11), we conclude that

$$\mathcal{J}_\gamma(u)\le \liminf_{j\to\infty}\mathcal{J}_\gamma(u_j)=I_0, \forall\gamma\in[0,1], \tag{3.12}$$

which indicates the existence of a minimizer in $\mathcal{K}$.

Secondly, we establish the L^∞- boundedness of u_γ. Hereafter, in this proof, we will refer to u_γ as u.

Let $j_0 := [\sup_{\partial\Omega}|\psi|]$ be the smallest positive integer above $\sup_{\partial\Omega}|\psi|$. For each $j\ge j_0$, we define the truncated function $u_j:\Omega\to\mathbb{R}$ by

$$u_j=\begin{cases} j\cdot\mathrm{sgn}(u), & \text{if } |u|>j,\\ u, & \text{if } |u|\le j,\end{cases}$$

where $\mathrm{sgn}(u)=1$ if $u>0$ and $\mathrm{sgn}(u)=-1$ if $u\le0$. Define the set $A_j:=\{|u|>j\}$.

For $\gamma \in (0,1]$, in view of the minimality of u, we derive that

$$\begin{aligned}\int_{A_j} f(x,\nabla u)\mathrm{d}x &= \int_{\Omega}(f(x,\nabla u)-f(x,\nabla u_j)) + \int_{A_j} f(x,\nabla u_j)\mathrm{d}x \\ &\leq \int_{A_j} g(u_j-u)\mathrm{d}x + \int_{A_j}\lambda_+\left((u_j^+)^\gamma-(u^+)^\gamma\right)\mathrm{d}x \\ &\quad + \int_{A_j}\lambda_-\left((u_j^-)^\gamma-(u^-)^\gamma\right)\mathrm{d}x + L|A_j|. \qquad (3.13)\end{aligned}$$

Now we estimate each term on the right-hand side of (3.13).

$$\int_{A_j}\lambda_+\left((u_j^+)^\gamma-(u^+)^\gamma\right)\mathrm{d}x = \lambda_+\int_{A_j\cap\{u>0\}}(j^\gamma-|u|^\gamma)\,\mathrm{d}x + \lambda_+\int_{A_j\cap\{u\leq 0\}}(((-j)^+)^\gamma-(u^+)^\gamma)\,\mathrm{d}x \leq 0.$$

$$\int_{A_j}\lambda_-\left((u_j^-)^\gamma-(u^-)^\gamma\right)\mathrm{d}x = \lambda_-\int_{A_j\cap\{u\leq 0\}}(j^\gamma-|u|^\gamma)\mathrm{d}x + \lambda_-\int_{A_j\cap\{u>0\}}((j^-)^\gamma-(u^-)^\gamma)\,\mathrm{d}x \leq 0.$$

Then, we get

$$\int_{A_j}\left(F_\gamma(u_j)-F_\gamma(u)\right)\mathrm{d}x \leq 0. \qquad (3.14)$$

For the first term in the right-hand side of (3.13), we deduce that

$$\int_{A_j} g(u_j-u)\mathrm{d}x = \int_{A_j\cap\{u>0\}} g(j-u)\mathrm{d}x + \int_{A_j\cap\{u\leq 0\}} g(-u-j)\mathrm{d}x \leq 2\int_{A_j}|g|(|u|-j)\mathrm{d}x. \qquad (3.15)$$

For $\gamma = 0$, it suffices to notice that $u_j > 0$ and u have the same sign. From the choice of the truncated function, we know that $(|u|-j)^+ \in W_0^{1,p(\cdot)}(A_j)$. Applying Hölder's inequality and the embedding theorem, we have

$$\begin{aligned}\int_{A_j}|g|(|u|-j)^+\mathrm{d}x &\leq 2\|g\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(A_j)}\|(|u|-j)^+\|_{L^{p(\cdot)}(A_j)} \\ &\leq C\|g\|_{L^{q(\cdot)}(A_j)}\|1\|_{L^{t(\cdot)}(A_j)}\|(|u|-j)^+\|_{L^{p^*(\cdot)}(A_j)}\|1\|_{L^n(A_j)} \\ &\leq \begin{cases} C\|g\|_{L^{q(\cdot)}(\Omega)}|A_j|^{\frac{1}{t_-}+\frac{1}{n}}\|\nabla(|u|-j)^+\|_{L^{p(\cdot)}(A_j)}, & \text{if } |A_j|>1 \\ C\|g\|_{L^{q(\cdot)}(\Omega)}|A_j|^{\frac{1}{t_+}+\frac{1}{n}}\|\nabla(|u|-j)^+\|_{L^{p(\cdot)}(A_j)}, & \text{if } |A_j|\leq 1\end{cases} \\ &= \begin{cases} C|\Omega|^{\frac{1}{t_-}+\frac{1}{n}}\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{t_-}+\frac{1}{n}}\|\nabla(|u|-j)^+\|_{L^{p(\cdot)}(A_j)}, & \text{if } |A_j|>1 \\ C|\Omega|^{\frac{1}{t_+}+\frac{1}{n}}\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{t_+}+\frac{1}{n}}\|\nabla(|u|-j)^+\|_{L^{p(\cdot)}(A_j)}, & \text{if } |A_j|\leq 1\end{cases} \\ &\leq C(1+|\Omega|)^{\frac{1}{t_-}+\frac{1}{n}}\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{t_+}+\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)} \\ &= C\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{t_+}+\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)}, \qquad (3.16)\end{aligned}$$

where $t \in C(\Omega; (1, +\infty))$ satisfies $\frac{1}{t(\cdot)} = 1 - \frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}$, we denote

$$t_- := \inf_{x\in\Omega} t(x),\ t_+ := \sup_{x\in\Omega} t(x),\ p^*(\cdot) := \frac{np(\cdot)}{n - p(\cdot)},$$

and the constant C in the last inequality depends only on $p_\pm, q_-, n, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$.

From (3.13) to (3.16), we infer that

$$\int_{A_j} f(x, \nabla u)\mathrm{d}x \le C\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{t_+}+\frac{1}{n}} \|\nabla u\|_{L^{p(\cdot)}(A_j)} + L|A_j|, \tag{3.17}$$

where C depends only on $p_\pm, q_-, n, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$.

Now we consider two cases: $\|\nabla u\|_{L^{p(\cdot)}(A_j)} > 1$ and $\|\nabla u\|_{L^{p(\cdot)}(A_j)} \le 1$.

Case 1: $\|\nabla u\|_{L^{p(\cdot)}(A_j)} > 1$. We deduce from (2.2), (3.17), and Young's inequality with $\epsilon > 0$ that

$$\begin{aligned}
\|\nabla u\|^{p_-}_{L^{p(\cdot)}(A_j)} &\le \int_{A_j} |\nabla u|^{p(x)}\mathrm{d}x \\
&\le L\int_{A_j} f(x, \nabla u)\mathrm{d}x \\
&\le C\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{t_+}+\frac{1}{n}} \|\nabla u\|_{L^{p(\cdot)}(A_j)} + L^2|A_j| \\
&\le C\left(\frac{|A_j|}{|\Omega|}\right)^{\left(\frac{1}{t_+}+\frac{1}{n}\right)\frac{p_-}{p_--1}} + \frac{1}{2}\|\nabla u\|^{p_-}_{L^{p(\cdot)}(A_j)} + L^2|A_j|,
\end{aligned}$$

which implies

$$\|\nabla u\|^{p_-}_{L^{p(\cdot)}(A_j)} \le C\left(\frac{|A_j|}{|\Omega|}\right)^{\left(\frac{1}{t_+}+\frac{1}{n}\right)\frac{p_-}{p_--1}} + L^2|A_j| = C\left(\frac{|A_j|}{|\Omega|}\right)^{\left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{p_-}{p_--1}} + L^2|A_j|.$$

Therefore, we have

$$\|\nabla u\|_{L^{p(\cdot)}(A_j)} \le C\left(\frac{|A_j|}{|\Omega|}\right)^{\left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{1}{p_--1}} + C\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{p_-}}, \tag{3.18}$$

where C depends only on $L, p_\pm, q_-, n, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$.

Analogous to (3.16), we deduce that

$$\begin{aligned}
\int_{A_j} (|u| - j)^+\mathrm{d}x &\le 2\|1\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(A_j)} \|(|u| - j)^+\|_{L^{p(\cdot)}(A_j)} \\
&\le \begin{cases} C|A_j|^{1-\frac{1}{p_+}+\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)}, & \text{if } |A_j| > 1 \\ C|A_j|^{1-\frac{1}{p_-}+\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)}, & \text{if } |A_j| \le 1 \end{cases} \\
&\le C\left(\frac{|A_j|}{|\Omega|}\right)^{1-\frac{1}{p_-}+\frac{1}{n}} \|\nabla u\|_{L^{p(\cdot)}(A_j)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\frac{|A_j|}{|\Omega|}\right)^{1-\frac{1}{p_-}+\frac{1}{n}}\left(\left(\frac{|A_j|}{|\Omega|}\right)^{\left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{1}{p_--1}}+C\left(\frac{|A_j|}{|\Omega|}\right)^{\frac{1}{p_-}}\right)\\
&= C\left(\frac{|A_j|}{|\Omega|}\right)^{\left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{1}{p_--1}+\left(1-\frac{1}{p_-}+\frac{1}{n}\right)}+C\left(\frac{|A_j|}{|\Omega|}\right)^{1+\frac{1}{n}} \qquad (3.19)
\end{aligned}
$$

where in the last inequality we used (3.18), and the constant C depends only on $L, p_\pm, q_-, n, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$.

Case 2: $\|\nabla u\|_{L^{p(\cdot)}(A_j)} \leq 1$. Analogously, we may obtain

$$
\int_{A_j}(|u|-j)^+\mathrm{d}x \leq C\left(\frac{|A_j|}{|\Omega|}\right)^{\left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{1}{p_+-1}+\left(1-\frac{1}{p_-}+\frac{1}{n}\right)}+C\left(\frac{|A_j|}{|\Omega|}\right)^{1+\frac{1}{n}}, \qquad (3.20)
$$

where the constant C depends only on $L, p_\pm, q_-, n, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$.

Now, combining (3.19) and (3.20), we get

$$
\int_{A_j}(|u|-j)^+\mathrm{d}x \leq C\left(\frac{|A_j|}{|\Omega|}\right)^{1+\left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{1}{p_+-1}-\frac{1}{p_-}+\frac{1}{n}}+C\left(\frac{|A_j|}{|\Omega|}\right)^{1+\frac{1}{n}},
$$

where C depends only on $L, p_\pm, q_-, n, \Omega$, and $\|g\|_{L^{q(\cdot)}(\Omega)}$.

Notice that by (2.6) and (2.7) we have $\frac{1}{q_-} < \frac{1}{n} - \frac{1}{p_-} + \frac{1}{p_+}$ and $\frac{1}{p_+} - \frac{1}{p_-} + \frac{1}{n} > 0$, respectively, thus

$$
\begin{aligned}
\epsilon_0 &:= \min\left\{\frac{1}{n}, \left(1-\frac{1}{p_-}-\frac{1}{q_-}+\frac{1}{n}\right)\frac{1}{p_+-1}-\frac{1}{p_-}+\frac{1}{n}\right\}\\
&\geq \min\left\{\frac{1}{n}, \left(1-\frac{1}{p_-}-\frac{1}{n}+\frac{1}{p_-}-\frac{1}{p_+}+\frac{1}{n}\right)\frac{1}{p_+-1}-\frac{1}{p_-}+\frac{1}{n}\right\}\\
&= \min\left\{\frac{1}{n}, \frac{1}{p_+}-\frac{1}{p_-}+\frac{1}{n}\right\}\\
&>0.
\end{aligned}
$$

Notice also that $\|u\|_{L^1(A_{j_0})} \leq \left(1+|A_{j_0}|^{\frac{p_--1}{p_-}}\right)\|u\|_{L^{p(x)}(A_{j_0})} \leq C$. Then, applying [14, Lemma 5.1], we obtain the boundedness of minimizers. □

Remark 3.1. *Note that in [5], the assumption that* $\int_\Omega |\nabla u|^{p(x)}dx \leq M$ *with some* $M \geq 0$ *is proposed for establishing local regularity of minimizers of functionals having a form* $\int_\Omega f(x,u,\nabla u)dx$*, while in this paper, we are able to show that any minimizer* u_γ *of* $\mathcal{J}_\gamma(u)$ *is uniformly bounded w.r.t.* $\gamma \in [0,1]$ *in* $W^{1,p(\cdot)}(\Omega)$ *by using the* L^∞*- estimate of* u_γ*. Indeed, we have*

$$
\begin{aligned}
\int_\Omega |\nabla u_\gamma|^{p(x)}dx &\leq L\int_\Omega f(x,\nabla u_\gamma)dx\\
&\leq L\left(\mathcal{J}_\gamma(\psi)-\int_\Omega F(u_\gamma)dx+\int_\Omega |gu_\gamma|dx\right)\\
&\leq L\mathcal{J}_\gamma(\psi)+C\left(L,n,p_\pm,\lambda_\pm,\Omega,\|\psi\|_{L^\infty(\partial\Omega)},\|g\|_{L^{q(\cdot)}(\Omega)}\right)\\
&\leq M,
\end{aligned}
$$

where $M = M\left(L,n,q_-,p_\pm,\lambda_\pm,\Omega,\|\psi\|_{L^\infty(\partial\Omega)},\|g\|_{L^{q(\cdot)}(\Omega)}\right)$ *is a positive constant. Therefore, we conclude that* $u_\gamma - \psi \in W_0^{1,p(\cdot)}(\Omega)$ *with* $\|u_\gamma\|_{W^{1,p(\cdot)}(\Omega)} \leq C$*, where* C *is independent of* γ*.*

4. Higher integrability

In this section, we prove the higher integrability of minimizers of $\mathcal{J}_\gamma(u)$. We first recall some important lemmas that will be used in the proof.

Lemma 4.1 ([5]). *Let $\theta \in (0,1), A > 0$, and $B \geq 0$ be constants, and $h \in L^{p(\cdot)}(B_R)$. If $k \geq 0$ is a bounded function on (r, R) and satisfies*

$$k(t) \leq \theta k(s) + A \int_{B_R} \left|\frac{h(x)}{s-t}\right|^{p(x)} \mathrm{d}x + B,$$

for all $r \leq t < s \leq R$, there exists a constant $C \equiv C(\theta, p_+)$ such that

$$k(r) \leq C\left(A \int_{B_R} \left|\frac{h(x)}{R-r}\right|^{p(x)} \mathrm{d}x + B\right).$$

Lemma 4.2 (Gehring-type Lemma, [2]). *Let E be a closed subset of $\overline{\Omega}$. Consider two nonnegative functions $f, g \in L^1(\Omega)$ and $p \in (1, +\infty)$ such that there holds*

$$\frac{1}{|B_{\frac{\rho}{2}}(x) \cap \Omega|} \int_{B_{\frac{\rho}{2}}(x)\cap\Omega} |g|^p dx \leq b^p \left(\left(\frac{1}{|B_\rho(x) \cap \Omega|} \int_{B_\rho(x)\cap\Omega} |g| dx\right)^p + \frac{1}{|B_\rho(x) \cap \Omega|} \int_{B_\rho(x)\cap\Omega} |f|^p dx\right)$$

for almost all $x \in \Omega \backslash E$ with $B_\rho \cap E = \emptyset$, for some constant b. Then, there exist constants $C = C(n, p, q, b)$ and $\epsilon = \epsilon(n, p, b)$ such that

$$\left(\frac{1}{|\Omega|} \int_\Omega |\widetilde{g}|^q dx\right)^{\frac{1}{q}} \leq C\left(\left(\frac{1}{|\Omega|} \int_\Omega |g|^p dx\right)^{\frac{1}{p}} + \left(\frac{1}{|\Omega|} \int_\Omega |f|^q dx\right)^{\frac{1}{q}}\right)$$

holds true for all $q \in [p, p+\epsilon)$, where $\widetilde{g}(x) := \frac{|B_{d(x,E)}(x)\cap\Omega|}{|\Omega|} g(x)$.

Based on Lemma 4.2 and the technique of iteration, we can prove the higher integrability of minimizers of $\mathcal{J}_\gamma(u)$.

Proof of Theorem 2.2. Let $0 < R < R_0 \leq 1$ and $x_0 \in B_R$ with $\overline{B}_{R_0}(x_0) \subset \Omega$. Let $t, s \in \mathbb{R}$ with $\frac{R}{2} < t < s < R$. Let $\eta \in C_c^\infty(B_R), 0 \leq \eta \leq 1$, be a cut-off function with $\eta \equiv 1$ on $B_t, \eta \equiv 0$ outside B_s, and $|\nabla \eta| \leq \frac{2}{s-t}$.

In the sequel, we refer to u_γ as u. Let $z := u - \eta(u - (u)_R)$. We deduce from (2.2) and the minimality of u that

$$\begin{aligned} L^{-1} \int_{B_t} |\nabla u|^{p(x)} \mathrm{d}x &\leq \int_{B_t} f(x, \nabla u) \mathrm{d}x \\ &\leq \int_{B_s} f(x, \nabla u) \mathrm{d}x \\ &\leq \int_{B_s} \left(f(x, \nabla z) + \left(F_\gamma(z) - F_\gamma(u)\right) + g(z-u)\right) \mathrm{d}x \\ &\leq L \int_{B_s} \left(\mu^2 + |\nabla z|^2\right)^{\frac{p(x)}{2}} \mathrm{d}x + \int_{B_s} \left(F_\gamma(z) - F_\gamma(u)\right) \mathrm{d}x + \int_{B_s} g(z-u) \mathrm{d}x, \quad (4.1) \end{aligned}$$

where as in the last but one inequality, we used the fact that the inequality

$$\int_{\mathrm{spt}\ \varphi}\Big(f(x,\nabla u)+F_\gamma(u)+gu\Big)\,\mathrm{d}x\le\int_{\mathrm{spt}\ \varphi}\Big(f(x,\nabla u+\nabla\varphi)+F_\gamma(u+\varphi)+g(u+\varphi)\Big)\,\mathrm{d}x \tag{4.2}$$

holds true for all $\varphi\in W_0^{1,p(\cdot)}(\Omega)$ with spt $\varphi\subset\subset\Omega$. Indeed, it follows from the minimality of u that

$$\begin{aligned}
&\int_{\mathrm{spt}\ \varphi}\Big(f(x,\nabla u)+F_\gamma(u)+gu\Big)\,\mathrm{d}x+\int_{\Omega\setminus(\mathrm{spt}\ \varphi)}\Big(f(x,\nabla u)+F_\gamma(u)+gu\Big)\,\mathrm{d}x\\
\le\ &\int_{\mathrm{spt}\ \varphi}\Big(f(x,\nabla u+\nabla\varphi)+F_\gamma(u+\varphi)+g(u+\varphi)\Big)\,\mathrm{d}x\\
&+\int_{\Omega\setminus(\mathrm{spt}\ \varphi)}\Big(f(x,\nabla u+\nabla\varphi)+F_\gamma(u+\varphi)+g(u+\varphi)\Big)\,\mathrm{d}x\\
\le\ &\int_{\mathrm{spt}\ \varphi}\Big(f(x,\nabla u+\nabla\varphi)+F_\gamma(u+\varphi)+g(u+\varphi)\Big)\,\mathrm{d}x\\
&+\int_{\Omega\setminus(\mathrm{spt}\ \varphi)}\Big(f(x,\nabla u+\nabla\varphi)+F_\gamma(u+\varphi)+g(u+\varphi)\Big)\,\mathrm{d}x\\
=\ &\int_{\mathrm{spt}\ \varphi}\Big(f(x,\nabla u+\nabla\varphi)+F_\gamma(u+\varphi)+g(u+\varphi)\Big)\,\mathrm{d}x+\int_{\Omega\setminus(\mathrm{spt}\ \varphi)}\Big(f(x,\nabla u)+F_\gamma(u)+gu\Big)\,\mathrm{d}x.
\end{aligned}$$

Now we estimate each term at (4.1).

$$\begin{aligned}
\int_{B_s}|\nabla z|^{p(x)}\mathrm{d}x &\le \int_{B_s}|(1-\eta)\nabla u-\nabla\eta(u-(u)_R)|^{p(x)}\,\mathrm{d}x\\
&\le C\int_{B_s\setminus B_t}|\nabla u|^{p(x)}\mathrm{d}x+C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x, \qquad (4.3)
\end{aligned}$$

where $C=C(p_+,p_-)$ is a positive constant.

A direct calculus shows that

$$\int_{B_s}\Big(F_\gamma(z)-F_\gamma(u)\Big)\,\mathrm{d}x = \lambda_+\int_{B_s}((z^+)^\gamma-(u^+)^\gamma)\,\mathrm{d}x+\lambda_-\int_{B_s}((z^-)^\gamma-(u^-)^\gamma)\,\mathrm{d}x\le C\int_{B_s}|z-u|^\gamma\mathrm{d}x,$$

where $C=C(\lambda_+,\lambda_-)$ is a positive constant.

Then, by Young's inequality, we deduce that

$$\begin{aligned}
\int_{B_s}\Big(F_\gamma(z)-F_\gamma(u)\Big)\,\mathrm{d}x &\le C\int_{B_s}|u-(u)_R|^\gamma\mathrm{d}x=C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^\gamma|s-t|^\gamma\mathrm{d}x\\
&\le C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x+C\int_{B_s}|s-t|^{\frac{\gamma p(x)}{p(x)-\gamma}}\mathrm{d}x \qquad (4.4)\\
&= C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x+C|B_s|, \qquad (4.5)
\end{aligned}$$

where $C=C(p_\pm,\lambda_\pm)$ is a positive constant.

The Young's inequality also gives

$$\begin{aligned}\int_{B_s}|g(z-u)|\mathrm{d}x &\le \int_{B_s}|g||u-(u)_R|\mathrm{d}x\\ &\le C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x + C\int_{B_s}(|g||s-t|)^{\frac{p(x)}{p(x)-1}}\mathrm{d}x\\ &\le C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x + C\int_{B_s}|g|^{\frac{p(x)}{p(x)-1}}\mathrm{d}x\\ &\le C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x + C\int_{B_s}\left(1+|g|^{\frac{p_-}{p_--1}}\right)\mathrm{d}x, \qquad (4.6)\end{aligned}$$

where $C = C(p_+, p_-)$ is a positive constant.

Combining (4.1)–(4.6), we obtain

$$\int_{B_t}|\nabla u|^{p(x)}\mathrm{d}x \le C\int_{B_s\setminus B_t}|\nabla u|^{p(x)}\mathrm{d}x + C\int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x + C\int_{B_s}\left(1+|g|^{\frac{p_-}{p_--1}}\right)\mathrm{d}x, \qquad (4.7)$$

where the constant C depends only on L, $p_\pm$, and $\lambda_\pm$.

Now, "filling the hole," we get

$$\int_{B_t}|\nabla u|^{p(x)}\mathrm{d}x \le \frac{C}{1+C}\int_{B_s}|\nabla u|^{p(x)}\mathrm{d}x + \int_{B_s}\left|\frac{u-(u)_R}{s-t}\right|^{p(x)}\mathrm{d}x + \int_{B_s}\left(1+|g|^{\frac{p_-}{p_--1}}\right)\mathrm{d}x,$$

which, along with Lemma 4.1, implies that

$$\frac{1}{|B_{R/2}|}\int_{B_{R/2}}|\nabla u|^{p(x)}\mathrm{d}x \le C\frac{1}{|B_R|}\int_{B_R}\left|\frac{u-(u)_R}{R-R/2}\right|^{p(x)}\mathrm{d}x + C\frac{1}{|B_R|}\int_{B_R}\left(1+|g|^{\frac{p_-}{p_--1}}\right)\mathrm{d}x. \qquad (4.8)$$

Let $p_1 := \min_{x\in\overline{B}_R} p(x)$ and $p_2 := \max_{x\in\overline{B}_R} p(x)$. By Sobolev–Poincaré's inequality, we deduce that there exists $\nu \in (0,1)$ such that

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_R}\left|\frac{u-(u)_R}{R}\right|^{p(x)}\mathrm{d}x &\le 1 + \frac{1}{|B_R|}\int_{B_R}\left|\frac{u-(u)_R}{R}\right|^{p_2}\mathrm{d}x\\ &\le 1 + C\left(\int_{B_R}\left(1+|\nabla u|^{p(x)}\right)\mathrm{d}x\right)^{\frac{p_2-p_1}{p_1\nu}} R^{\frac{(p_1-p_2)n}{p_1\nu}}\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{p_1\nu}\mathrm{d}x\right)^{\frac{1}{\nu}}\\ &\le C\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{p(x)\nu}\mathrm{d}x\right)^{\frac{1}{\nu}} + C, \qquad (4.9)\end{aligned}$$

where in the last inequality we used the result stated in Remark 3.1.

Combining (4.8) and (4.9), we get

$$\frac{1}{|B_{R/2}|}\int_{B_{R/2}}|\nabla u|^{p(x)}\mathrm{d}x \le C\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{p(x)\nu}\mathrm{d}x\right)^{\frac{1}{\nu}} + C\frac{1}{|B_R|}\int_{B_R}\left(1+|g|^{\frac{p_-}{p_--1}}\right)\mathrm{d}x,$$

where $C = C(n, p_\pm, \lambda_\pm, L, M, \Omega)$.

Now applying Lemma 4.2, we conclude that there exists $\delta_0 \in \left(0, q_1\left(1-\frac{1}{p_-}\right)-1\right)$ such that (2.8) holds true. □

5. Conclusions

In this paper, we proved the existence, uniform boundedness, and a higher integrability of minimizers of the functional $J_\gamma(u)$ under the framework of Sobolev spaces with variable exponents. Based on the obtained results, we will further study the regularity such as Hölder continuity of minimizers of the functional $J_\gamma(u)$.

Author contributions

Jiayin Liu and Jun Zheng: Writing-original draft preparation; Jiayin Liu and Jun Zheng: writing-review and editing; All authors equally contributed to this work. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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