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Research article

Viscosity-type inertial iterative methods for variational inclusion and fixed point problems

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Abstract: In this paper, we have introduced some viscosity-type inertial iterative methods for solving fixed point and variational inclusion problems in Hilbert spaces. Our methods calculated the viscosity approximation, fixed point iteration, and inertial extrapolation jointly in the starting of every iteration. Assuming some suitable assumptions, we demonstrated the strong convergence theorems without computing the resolvent of the associated monotone operators. We used some numerical examples to illustrate the efficiency of our iterative approaches and compared them with the related work.

Keywords: viscosity method; inertial extrapolation; fixed point problem; variational inclusion; strong convergence
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1. Introduction

A fixed point problem (FPP) is a significant problem that provides a natural support for studying a broad range of nonlinear problems with applications. The fixed point problem of mapping T is defined as

$$\mathrm{Fix}(T) = \{s \in \mathbb{E} : T(s) = s\}, \tag{1.1}$$

where $\mathbb{E}$ is a real Hilbert space and $T : \mathbb{E} \to \mathbb{E}$ is a nonexpansive mapping.

For a single-valued monotone operator $Q : \mathbb{E} \to \mathbb{E}$ and a set-valued operator $G : \mathbb{E} \rightrightarrows \mathbb{E}$, the variational inclusion problem ($\mathrm{VI_sP}$) is to search $s^\star \in \mathbb{E}$ such that

$$0 \in Q(s^\star) + G(s^\star). \tag{1.2}$$

Several problems, such as image recovery, optimization, variational inequality, can be transformed into a FPP or $\mathrm{VI_sP}$. Due to such applicability, in the last decades, several iterative methods have been formulated to solve $\mathrm{FPP_s}$ and $\mathrm{VI_sP_s}$ in linear and nonlinear spaces, for example, [4, 8, 9, 12, 13, 15, 32].

Dauglas and Rachford [11] formulated the forward-backward splitting method for VI_sP:

$$s_{n+1} = R^G_{\mu_n}[I - \mu_n Q](s_n), \tag{1.3}$$

where $\mu_n > 0$, $R^G_{\mu_n} = [I + \mu_n G]^{-1}$ is the resolvent of G (also known as the backward operator), and $[I - \mu_n Q]$ is known as the forward operator. We can rewrite (1.3) as

$$\frac{s_n - s_{n+1}}{\mu_n} \in Q(s_n) + G(s_{n+1}), \tag{1.4}$$

which is studied by Ansari and Babu [2] in nonlinear space. If $Q = 0$, the monotone inclusion problem (MI_sP) is to search $s^\star \in \mathbb{E}$ such that

$$0 \in G(s^\star), \tag{1.5}$$

which was studied in [26]. The proximal point method, or the regularization method, is one of the renowned methods for MI_sP studied by Lions and Mercier [18]:

$$s_{n+1} = [I + \mu_n G]^{-1}(s_n). \tag{1.6}$$

Since the operator $R^G_{\mu_n}$ is nonexpansive appearing in backward step, the algorithms have been studied widely by numerous authors, see for example [7, 10, 15–17, 19, 23, 27].

An essential development in the field of nonlinear science is the inertial extrapolation, introduced by Polyak [22], for fast convergence of algorithms. Alvarez and Attouch [6] implemented the inertial extrapolation to acquire the inertial proximal point method to solve MI_sP. For $\mu_n > 0$, find $s_{n+1} \in \mathbb{E}$ such that

$$0 \in \mu_n G(s_{n+1}) + s_{n+1} - s_n - \beta_n(s_n - s_{n-1}), \tag{1.7}$$

and equivalently

$$s_{n+1} = R^G_{\mu_n}[s_n + \beta_n(s_n - s_{n-1})], \tag{1.8}$$

where $\beta_n \in [0, 1)$ is the extrapolation coefficient and $\beta_n(s_n - s_{n-1})$ is known as the inertial step. They proved the weak convergence of (1.8) assuming

$$\sum_{n=1}^{\infty} \beta_n \|s_n - s_{n-1}\|^2 < +\infty. \tag{1.9}$$

Inertial extrapolation has been demonstrated to have good convergence properties and a high convergence rate, therefore they have been improved and used in a variety of nonlinear problems, see [3, 5, 13, 14, 28, 29] and the references inside.

The following inertial proximal point approach was presented by Moudafi and Oliny in [21] to solve VI_sP:

$$\begin{cases} u_n = s_n + \beta_n(s_n - s_{n-1}), \\ s_n = [I + \mu_n G]^{-1}(u_n - \mu_n Q u_n), \end{cases} \tag{1.10}$$

where $\mu_n < 2/\kappa$, and κ is the Lipschitz constant of operator Q. They proved the weak convergence of (1.10) using the same assumption (1.9). Recently, Duang et al. [30] studied the VI_sP and FPP. They proposed the following viscosity inertial method (Algorithm 1.1) for estimating the common solution in Hilbert spaces.

Algorithm 1.1. *(Algorithm 3 of [30]) Viscosity inertial method (VIM)*

Choose arbitrary points s_0 and s_1 and set $n = 1$.

Step 1. *Compute*

$$\begin{aligned} u_n &= s_n + \theta_n(s_n - s_{n-1}), \\ v_n &= [I + \lambda G]^{-1}(I - \lambda Q)u_n. \end{aligned}$$

If $u_n = v_n$, then stop (s_n is a solution of $\mathrm{VI_sP}$) *. If not, proceed to Step 2.*

Step 2. *Compute*

$$s_{n+1} = \psi_n k(u_n) + (1 - \psi_n)T v_n.$$

Let $n = n + 1$ and proceed to Step 1.

In the above calculation Q is η-inverse strongly monotone (in short η-ism) and G is a maximal monotone operator, k is a contraction, T is a nonexpansive mapping, $\lambda \in (0, 2\eta)$, and the control sequence fulfills the requirements listed below:

(i) $\psi_n \in (0, 1), \lim\limits_{n\to\infty} \psi_n = 0, \quad \sum\limits_{n=1}^{\infty} \psi_n = \infty, \quad \lim\limits_{n\to\infty} \frac{\psi_{n-1}}{\psi_n} = 0,$

(ii) $\theta_n \in [0, \theta), \theta > 0, \lim\limits_{n\to\infty} \frac{\theta_n}{\psi_n} \|s_n - s_{n-1}\| = 0.$

Recently, Reich and Taiwo [24] investigated hybrid viscosity-type iterative schemes for solving variational inclusion problems in which viscosity approximation and inertial extrapolation were computed jointly. Ahmed and Dilshad [1] studied the Halpern-type iterative method for solving split common null point problems where the Halpern iteration and inertial iterations are computed simultaneously at the start of every iteration.

Motivated by the work in [1, 24, 30], we present two viscosity-type inertial iteration methods for common solutions of $\mathrm{VI_sP_s}$ and $\mathrm{FPP_s}$. In our algorithms, we implement the viscosity iteration, fixed point iteration, and inertial extrapolation at the first step of each iteration. Our methods do not need the inverse strongly monotone assumptions on the operators Q and G, which are considered in the literature. We prove the strong convergence of the presented methods without calculating the resolvent of the associated monotone operators Q and G.

We organize the paper as follows: In Section 2, we discuss some basic definitions and useful lemmas. In Section 3, we propose viscosity-type iterative methods for solving $\mathrm{VI_sP_s}$ and $\mathrm{FPP_s}$ and prove the strong convergence theorems. In Section 4, as a consequence of our methods, we present Halpern-type inertial iterative methods for $\mathrm{VI_sP_s}$ and $\mathrm{FPP_s}$. Sections 5 describes some applications for solving variational inequality and optimization problems. In Section 6, we show the efficiency of the suggested methods by comparing them with Algorithm 3 of [30].

2. Preliminaries

Let $\{s_n\}$ be a sequence in $\mathbb{E}$. Then $s_n \to s$ denotes strong convergence of $\{s_n\}$ to s and $s_n \rightharpoonup s$ denotes weak convergence. The weak w-limit of $\{s_n\}$ is defined by

$$w_w(s_n) = \{s \in H : y_{n_j} \rightharpoonup s \text{ as} j \to \infty \text{ where } s_{n_j} \text{ is a subsequnce of } s_n\}.$$

The following useful inequality is well-known in the Hilbert space $\mathbb{E}$:

$$\|s_1 \pm w_1\|^2 \leq \|s_1\|^2 \pm 2\langle s_1, w_1\rangle + \|w_1\|^2. \tag{2.1}$$

Definition 2.1. *A mapping $k : \mathbb{E} \to \mathbb{E}$ is called*

(i) a contraction, if $\|k(s_1) - k(w_1)\| \leq \tau\|s_1 - w_1\|, \ \forall \ s_1, w_1 \in \mathbb{E}, \tau \in (0, 1)$;
(ii) nonexpansive, if $\|k(s_1) - k(w_1)\| \leq \|s_1 - w_1\|, \forall \ s_1, w_1 \in \mathbb{E}$.

Definition 2.2. *Let $Q : \mathbb{E} \to \mathbb{E}$. Then*

(i) Q is called monotone, if $\langle Q(s_1) - Q(w_1), \ s_1 - w_1\rangle \geq 0, \forall \ s_1, w_1 \in \mathbb{E}$;
(ii) Q is called η-ism, if there exists $\eta > 0$ such that

$$\langle Q(s_1) - Q(w_1), \ s_1 - w_1\rangle \geq \eta\|Q(s_1) - Q(w_1)\|^2, \ \forall \ s_1, w_1 \in \mathbb{E};$$

(iii) Q is called δ-strongly monotone, if there exists $\delta > 0$ such that

$$\langle Q(s_1) - Q(w_1), \ s_1 - w_1\rangle \geq \delta\|s_1 - w_1\|^2, \ \forall \ s_1, w_1 \in \mathbb{E};$$

(iv) Q is called κ-Lipschitz continuous, if there exists $\kappa > 0$ such that

$$\|Q(s_1) - Q(w_1)\| \leq \kappa\|s_1 - w_1\|, \ \forall \ s_1, w_1 \in \mathbb{E}.$$

Definition 2.3. *Let $G : \mathbb{E} \to 2^{\mathbb{E}}$. Then*

(i) the graph of G is defined by $Graph(G) = \{(s_1, w_1) \in \mathbb{E} \times \mathbb{E} : w_1 \in G(s_1)\}$;
(ii) G is called monotone, if for all $(s_1, w_1), (s_2, w_2) \in Graph(G)$, $\langle w_1 - w_2, \ s_1 - s_2\rangle \geq 0$;
(iii) G is called maximal monotone, if G is monotone and $(I + \mu G)(\mathbb{E}) = \mathbb{E}$, for $\mu > 0$.

Lemma 2.1. *[31] Let $s_n \in \mathbb{R}$ be a nonnegative sequence such that*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\xi_n, \ \ n \geq n_0 \in \mathbb{N},$$

where $\lambda_n \in (0, 1)$ and $\xi_n \in \mathbb{R}$ fulfill the requirements given below:

$$\lim_{n\to\infty} \lambda_n = 0, \ \ \sum_{n=1}^{\infty} \lambda_n = \infty, \ \ \text{and} \ \ \limsup_{n\to\infty} \xi_n \leq 0.$$

Then $s_n \to 0$ as $n \to \infty$.

Lemma 2.2. *[20] Let $y_n \in \mathbb{R}$ be a sequence that does not decrease at infinity in the sense that there exists a subsequence y_{n_k} of y_n such that $y_{n_k} < y_{n_k+1}$ for all $k \geq 0$. Also consider the sequence of integers $\{\Upsilon(n)\}_{n\geq n_0}$ defined by*

$$\Upsilon(n) = \max\{k \leq n : y_k \leq y_{k+1}\}.$$

Then $\{\Upsilon(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \Upsilon(n) = \infty$, and for all $n \geq n_0$, the following inequalities hold:

$$\begin{aligned} y_{\Upsilon(n)} &\leq y_{\Upsilon(n)+1}, \\ y_{(n)} &\leq y_{\Upsilon(n)+1}. \end{aligned}$$

3. Main results

In the present section, we define our viscosity-type inertial iteration methods for solving FPP and $\mathrm{VI_sP}$. We symbolize the solution set of FPP by Λ and of $\mathrm{VI_sP}$ by Δ and assume that $\Lambda \cap \Delta \neq \emptyset$. We adopt the following assumptions in order to prove the convergence of the sequences obtained from the suggested methods:
($\mathbf{S_1}$) $k : \mathbb{E} \to \mathbb{E}$ is a τ-contraction with $0 < \tau < 1$;
($\mathbf{S_2}$) $Q : \mathbb{E} \to \mathbb{E}$ is a δ-strongly monotone and κ-Lipschitz continuous operator and $G : \mathbb{E} \rightrightarrows \mathbb{E}$ is a maximal monotone operator;
($\mathbf{S_3}$) μ_n is a sequence such that $0 < \bar{\mu} \leq \mu_n \leq \mu < 1/2\delta$ and $\kappa \leq 2\delta$;
($\mathbf{S_4}$) $\lambda_n \in (0, 1)$ satisfies $\lim\limits_{n\to\infty} \lambda_n = 0$ and $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$;
($\mathbf{S_5}$) σ_n is a positive sequence satisfying $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$ and $\lim\limits_{n\to\infty} \frac{\sigma_n}{\lambda_n} = 0$.

Theorem 3.1. *If the Assumptions* ($\mathbf{S_1}$)–($\mathbf{S_5}$) *are fulfilled then the sequences induced by the Algorithm 3.1 converge strongly to* $s^\star \in \Delta \cap \Lambda$, *which solve the following variational inequality:*

$$\langle k(s^\star) - s^\star,\ \ y - s^\star \rangle \leq 0, \ \ \forall\, y \in \Delta \cap \Lambda. \tag{3.1}$$

Algorithm 3.1. *Viscosity-type inertial iterative method-I (VIIM-I)*

Let $\beta \in [0, 1)$ *and* $\mu_n > 0$ *are given. Choose arbitrary points* s_0 *and* s_1 *and set* $n = 0$.
Iterative step. *For iterates* s_n, *and* s_{n-1}, *for* $n \geq 1$, *select* $0 < \beta_n < \bar{\beta}_n$, *where*

$$\bar{\beta}_n = \begin{cases} \min\{\frac{\sigma_n}{\|s_n - s_{n-1}\|}, \quad \beta\}, & \text{if } s_n \neq s_{n-1}, \\ \beta, & \text{otherwise}, \end{cases} \tag{3.2}$$

compute

$$u_n = \lambda_n k(s_n) + (1 - \lambda_n)[T(s_n) + \beta_n(s_n - s_{n-1})], \tag{3.3}$$

$$0 \in Q(u_n) + G(s_{n+1}) + \frac{s_{n+1} - u_n}{\mu_n}. \tag{3.4}$$

If $s_{n+1} = u_n$, *then stop. If not, set* $n = n + 1$ *and proceed to the iterative step.*

Remark 3.1. *From* (3.2), *we have* $\beta_n \leq \frac{\sigma_n}{\|s_n - s_{n-1}\|}$. *Since* $\beta_n > 0$ *and* σ_n *satisfies* $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$, *we obtain* $\lim\limits_{n\to\infty} \beta_n \|s_n - s_{n-1}\| = 0$ *and* $\lim\limits_{n\to\infty} \frac{\beta_n \|s_n - s_{n-1}\|}{\lambda_n} \leq \lim\limits_{n\to\infty} \frac{\sigma_n}{\lambda_n} = 0$.

Proof. Let $s^\star \in \Delta \cap \Lambda$, then $-Q(s^\star) \in G(s^\star)$ and using (3.4), we have $\frac{u_n - s_{n+1}}{\mu_n} - Q(u_n) \in G(s_{n+1})$. Since G is monotone, we get

$$\Big\langle \frac{u_n - s_{n+1}}{\mu_n} - Q(u_n) + Q(s^\star),\ s_{n+1} - s^\star \Big\rangle \geq 0. \tag{3.5}$$

Since Q is strongly monotone with constant $\delta > 0$, we have

$$\Big\langle Q(s_{n+1}) - Q(s^\star),\ s_{n+1} - s^\star \Big\rangle \geq \delta \|s_{n+1} - s^\star\|^2. \tag{3.6}$$

By adding (3.5) and (3.6), we get

$$\left\langle\frac{u_n - s_{n+1}}{\mu_n} + Q(s_{n+1}) - Q(u_n),\ s_{n+1} - s^\star\right\rangle \geq \delta\|s_{n+1} - s^\star\|^2 \tag{3.7}$$

or

$$\frac{1}{\mu_n}\left\langle u_n - s_{n+1},\ s_{n+1} - s^\star\right\rangle + \left\langle Q(s_{n+1}) - Q(u_n),\ s_{n+1} - s^\star\right\rangle \geq \delta\|s_{n+1} - s^\star\|^2. \tag{3.8}$$

By using the Cauchy Schwartz inequality and Lipschitz continuity of Q, we have

$$\begin{aligned}\left\langle Q(s_{n+1}) - Q(u_n),\ s_{n+1} - s^\star\right\rangle &\leq \|Q(s_{n+1}) - Q(u_n)\|\|s_{n+1} - s^\star\| \\ &\leq \kappa\|s_{n+1} - u_n\|\|s_{n+1} - s^\star\| \\ &= \frac{\kappa}{2}\{\|s_{n+1} - u_n\|^2 + \|s_{n+1} - s^\star\|^2\}.\end{aligned} \tag{3.9}$$

By using (2.1), we have

$$\|u_n - s^\star\|^2 = \|u_n - s_{n+1} + s_{n+1} - s^\star\|^2 = \|u_n - s_{n+1}\|^2 + \|s_{n+1} - s^\star\|^2 + 2\langle u_n - s_{n+1},\ s_{n+1} - s^\star\rangle. \tag{3.10}$$

Considering (3.8)–(3.10), we get

$$\|s_{n+1} - s^\star\|^2 \leq \|u_n - s^\star\|^2 - \|u_n - s_{n+1}\|^2 + \mu_n\kappa\left\{\|s_{n+1} - u_n\|^2 + \|s_{n+1} - s^\star\|^2\right\} - 2\mu_n\delta\|s_{n+1} - s^\star\|^2. \tag{3.11}$$

Since $\kappa \leq 2\delta$, we have

$$\|s_{n+1} - s^\star\|^2 \leq \|u_n - s^\star\|^2 - (1 - 2\delta\mu_n)\|s_{n+1} - u_n\|^2 \tag{3.12}$$

or

$$\|s_{n+1} - s^\star\|^2 \leq \|u_n - s^\star\|^2. \tag{3.13}$$

Since $\lim_{n\to\infty} \frac{\beta_n\|s_n - s_{n-1}\|}{\lambda_n} = 0$ (Remark 3.1), there exists $K_1 > 0$ such that $\frac{\beta_n\|s_n - s_{n-1}\|}{\lambda_n} \leq K_1$, that is $\beta_n\|s_n - s_{n-1}\| \leq \lambda_n K_1$. By using (3.13) and mathematical induction, bearing in mind that k is a contraction and T is nonexpansive, it follows from (3.3) that

$$\begin{aligned}\|u_n - s^\star\| &= \|\lambda_n k(s_n) + (1 - \lambda_n)[T(s_n) + \beta_n(s_n - s_{n-1}\|] - s^\star\| \\ &= \lambda_n\|k(s_n) - s^\star\| + (1 - \lambda_n)[\|(T(s_n) - s^\star + \beta_n(s_n - s_{n-1})\|] \\ &\leq \lambda_n\|k(s_n) - k(s^\star)\| + \lambda_n\|k(s^\star) - s^\star\| + (1 - \lambda_n)[\|T(s_n) - s^\star\| + \beta_n\|s_n - s_{n-1}\|] \\ &\leq \lambda_n\tau\|s_n - s^\star\| + \lambda_n\|k(s^\star) - s^\star\| + (1 - \lambda_n)\|s_n - s^\star\| + \lambda_n K_1 \\ &\leq [1 - \lambda_n(1 - \tau)]\|s_n - s^\star\| + \lambda_n(1 - \tau)\frac{\|k(s^\star) - s^\star\| + K_1}{1 - \tau} \\ &\leq \max\left\{\|s_n - s^\star\|,\quad \frac{\|k(s^\star) - s^\star\| + K_1}{1 - \tau}\right\} \\ &\leq \max\left\{\|u_{n-1} - s^\star\|,\quad \frac{\|k(s^\star) - s^\star\| + K_1}{1 - \tau}\right\} \\ &\vdots\end{aligned}$$

$$\leq \max\Big\{\|u_0 - s^\star\|, \quad \frac{\|k(s^\star) - s^\star\| + K_1}{1-\tau}\Big\},$$

meaning that $\{u_n\}$ is bounded and hence $\{s_n\}$ is also bounded. Let $v_n = T(s_n) + \beta_n(s_n - s_{n-1})$. Note that v_n is also bounded. By using (3.3), we get

$$\begin{aligned}\|u_n - s^\star\|^2 &= \|\lambda_n k(s_n) + (1-\lambda_n)v_n - s^\star\|^2 \\ &= \lambda_n^2\|k(s_n) - s^\star\|^2 + (1-\lambda_n)^2\|v_n - s^\star\|^2 + 2\lambda_n(1-\lambda_n)\langle k(s_n) - s^\star, v_n - s^\star\rangle. \quad (3.14)\end{aligned}$$

Now, we need to calculate

$$\begin{aligned}\|v_n - s^\star\|^2 &= \|T(s_n) + \beta_n(s_n - s_{n-1}) - s^\star\|^2 \\ &= \|T(s_n) - s^\star\|^2 + 2\beta_n\langle s_n - s_{n-1}, v_n - s^\star\rangle \\ &\leq \|T(s_n) - s^\star\|^2 + 2\beta_n\|s_n - s_{n-1}\|\|v_n - s^\star\| \\ &\leq \|s_n - s^\star\|^2 + 2\Theta_n\|v_n - s^\star\|, \quad (3.15)\end{aligned}$$

where $\Theta_n = \beta_n\|z_n - z_{n-1}\|$, and

$$\begin{aligned}\langle k(s_n) - s^\star, v_n - s^\star\rangle &= \langle k(s_n) - k(s^\star), v_n - s^\star\rangle + \langle k(s^\star) - s^\star, v_n - s^\star\rangle \\ &\leq \|k(s_n) - k(s^\star\|\|v_n - s^\star\| + \langle k(s^\star) - s^\star, v_n - s^\star\rangle \\ &\leq \frac{1}{2}\{\tau^2\|s_n - s^\star\|^2 + \|v_n - s^\star\|^2\} + \langle k(s^\star) - s^\star, \quad v_n - s^\star\rangle \quad (3.16)\end{aligned}$$

and

$$\begin{aligned}\langle k(s^\star) - s^\star, \quad v_n - s^\star\rangle &= \langle k(s^\star) - s^\star, \quad T(s_n) + \beta_n(s_n - s_{n-1}) - s^\star\rangle \\ &\leq \langle k(s^\star) - s^\star, \quad T(s_n) - s^\star\rangle + \langle k(s^\star) - s^\star, \quad \beta_n(s_n - s_{n-1})\rangle \\ &\leq \langle k(s^\star) - s^\star, \quad T(s_n) - s^\star\rangle + \beta_n\|k(s^\star) - s^\star\|\|s_n - s_{n-1}\| \\ &\leq \langle k(s^\star) - s^\star, \quad T(s_n) - s^\star\rangle + \Theta_n\|k(s^\star) - s^\star\|. \quad (3.17)\end{aligned}$$

By using (3.14)–(3.17), we get

$$\begin{aligned}\|u_n - s^\star\|^2 \leq\ & \lambda_n^2\|k(s_n) - s^\star\|^2 + (1-\lambda_n)^2\Big\{\|s_n - s^\star\|^2 + 2\Theta_n\|v_n - s^\star\|\Big\} \\ & + \lambda_n(1-\lambda_n)\tau^2\|s_n - s^\star\|^2 + \lambda_n(1-\lambda_n)\|v_n - s^\star\|^2 \\ & + 2\lambda_n(1-\lambda_n)\langle k(s^\star) - s^\star, \quad T(s_n) - s^\star\rangle + 2\lambda_n(1-\lambda_n)\Theta_n\|k(s^\star) - s^\star\| \\ \leq\ & [1 - \lambda_n(1-\tau^2)]\|s_n - s^\star\|^2\| \\ & + \lambda_n\Big\{\lambda_n\|k(s_n) - s^\star\|^2 + 2(1-\lambda_n)\langle k(s^\star) - s^\star, \quad T(s_n) - s^\star\rangle \\ & + \Theta_n\|k(s^\star) - s^\star\| + \frac{2\Theta_n}{\lambda_n}\|v_n - s^\star\|\Big\}. \quad (3.18)\end{aligned}$$

Let $\gamma_n = \lambda_n(1-\tau^2)$. Then it follows from (3.12) and (3.18) that

$$\|s_{n+1} - s^\star\|^2 \leq (1-\gamma_n)\|s_n - s^\star\|^2 + \gamma_n U_n - (1 - 2\delta\mu_n)\|s_{n+1} - u_n\|^2, \quad (3.19)$$

where

$$U_n = \frac{\lambda_n\|k(s_n) - s^\star\|^2 + 2(1-\lambda_n)\langle k(s^\star) - s^\star,\ \ T(s_n) - s^\star\rangle + \Theta_n\|k(s^\star) - s^\star\| + \frac{2\Theta_n}{\lambda_n}\|v_n - s^\star\|}{1-\tau^2}. \quad (3.20)$$

Now, we continue the proof in the following two cases:

Case I: If $\{\|s_n - s^\star\|\}$ is monotonically decreasing then there exists a number N_1 such that $\|s_{n+1} - s^\star\| \leq \|s_n - s^\star\|$ for all $n \geq N_1$. Hence, boundedness of $\{\|s_n - s^\star\|\}$ implies that $\{\|s_n - s^\star\|\}$ is convergent. Therefore, using (3.19), we have

$$(1-2\delta\mu_n)\|s_{n+1} - u_n\|^2 \leq \|s_{n+1} - s^\star\|^2 + \|s_n - s^\star\|^2 - \gamma_n\|s_n - s^\star\|^2 + \gamma_n U_n. \quad (3.21)$$

Since $2\delta\mu_n < 1$ and $\lim\limits_{n\to\infty}\gamma_n = 0$, we obtained

$$\lim_{n\to\infty}\|s_{n+1} - u_n\| = 0. \quad (3.22)$$

By using (3.22) and Remark 3.1, we get

$$\lim_{n\to\infty}\|v_n - T(s_n)\| = \lim_{n\to\infty}\beta_n\|s_n - s_{n-1}\| = 0. \quad (3.23)$$

Boundedness of $\{s_n\}$ and $\{v_n\}$ implies that there exist M_1 and M_2 such that $\|s_n - s^\star\| \leq M_1$ and $\|k(s^\star) - v_n\| \leq M_2$, hence

$$\begin{aligned}\|u_n - v_n\| &= \lambda_n\|k(s_n) - v_n\|\\ &\leq \lambda_n[\|k(s_n) - k(s^\star)\| + \|k(s^\star) - v_n\|]\\ &\leq \lambda_n[\tau\|s_n - s^\star\| + \|k(s^\star) - v_n\|]\\ &\leq \lambda_n[\tau M_1 + M_2] \to 0 \text{ as } n\to\infty. \end{aligned} \quad (3.24)$$

The following can be obtained easily by using (3.22) and (3.23):

$$\lim_{n\to\infty}\|Ts_n - s_n\| = \lim_{n\to\infty}\|s_n - v_n\| = 0. \quad (3.25)$$

Since $\{s_n\}$ is bounded, it guarantees the existence of subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \rightharpoonup \bar{s}$. As a consequence, from (3.22) and (3.25), it follows that $u_{n_k} \rightharpoonup \bar{s}$ and $v_{n_k} \rightharpoonup \bar{s}$. Now, we will show that $\bar{s} \in \Delta\cap\Lambda$. Since T is nonexpansive, hence by (3.25), we obtain $\bar{s} \in \mathrm{Fix}(T)$. From (3.4), we have

$$z_{n_k} = \frac{u_{n_k} - s_{n_{k+1}}}{\mu_{n_k}} - Q(u_{n_k}) \in G(s_{n_k+1}). \quad (3.26)$$

Since $0 < \bar{\mu} < \mu_n < \mu$ and from (3.22), we have $\|s_{n_k+1} - u_{n_k}\| \to 0$ and by the Lipschitz continuity of Q, we get

$$z_{n_k} \to -Q(\bar{y}) \text{ as } k\to\infty. \quad (3.27)$$

Taking $k \to \infty$, since the graph of the maximal monotone operator is weakly-strongly closed, we get $-Q(\bar{s}) \in G(\bar{s})$, that is $0 \in Q(\bar{s}) + G(\bar{s})$. Thus $\bar{s} \in \Delta\cap\Lambda$.

Next, we show that $\{s_n\}$ strongly converges to $s^\star$. From (3.19), it immediately follows that

$$\|s_{n+1} - s^\star\|^2 \quad \leq \quad (1-\gamma_n)\|\|\|s_n - s^\star\|^2\| + \gamma_n U_n \tag{3.28}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} U_n \\
&= \limsup_{n\to\infty} \frac{\lambda_n\|k(s_n) - s^\star\|^2 + 2(1-\lambda_n)\langle k(s^\star) - s^\star,\ \ T(s_n) - s^\star\rangle + \Theta_n\|k(s^\star) - s^\star\| + \frac{2\Theta_n}{\lambda_n}\|v_n - s^\star\|}{1-\tau^2} \\
&= \langle k(s^\star) - s^\star,\ \ \bar{s} - s^\star\rangle \leq 0.
\end{aligned}$$

By using Lemma 2.1, we deduce that $\{s_n\}$ converges strongly to $s^\star$, where $s^\star$ is the solution to the variational inequality (3.1). Further, it follows that $\|s_n - u_n\| \to 0$, $u_n \rightharpoonup \bar{y} \in \Delta \cap \Lambda$, and $s_n \to s^\star$ as $n \to \infty$, thus $\bar{y} = s^\star$. This completes the proof.

Case II: If Case I is false, then the function $\rho : \mathbb{N} :\to \mathbb{N}$ defined by $\rho(n) = \max\{n \geq m : \|s_m - s^\star\| \leq \|s_{m+1} - s^\star\|\}$ is an increasing sequence and $\rho(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \|s_{\rho(n)} - s^\star\| \leq \|s_{\rho(n)+1} - s^\star\|,\ \forall\ n \geq m. \tag{3.29}$$

For the same reasons as in the proof of Case I, we obtain $\|s_{\rho(n)} - s^\star\| \to 0$ and $\|s_{\rho(n)} - u_{\rho(n)}\| \to 0$ as $n \to \infty$. By using (3.19) and (3.29), we obtain

$$0 \leq \|s_{\rho(n)} - s^\star\| \leq U_n. \tag{3.30}$$

Thus, we get $\|s_{\rho(n)} - s^\star\| \to 0$ as $n \to \infty$. Keeping in mind Lemma 2.2, we have

$$0 \leq \|s_n - s^\star\| \leq \max\{\|s_n - s^\star\|, \|s_{\rho(n)} - s^\star\|\} \leq \|s_{\rho(n)+1} - s^\star\|. \tag{3.31}$$

Consequently, from (3.31), $\|s_n - s^\star\| \to 0$ as $n \to \infty$. Therefore, $s_n \to s^\star$ as $n \to \infty$, where $s^\star$ is a solution of the variational inequality (3.1). □

Theorem 3.2. *If the Assumptions* ($\mathbf{S_1}$)–($\mathbf{S_5}$) *are satisfied then the sequences induced by the Algorithm 3.2 converge strongly to* $s^\star \in \Lambda \cap \Delta$, *which solves the following variational inequality:*

$$\langle k(s^\star) - s^\star,\ \ w - s^\star\rangle \leq 0,\ \ \forall\ w \in \Lambda \cap \Delta.$$

Algorithm 3.2. *Viscosity-type inertial iterative method-II (VIIM-II)*

Let $\beta \in [0, 1)$ *and* $\mu_n > 0$ *are given. Choose arbitrary points* s_0 *and* s_1 *and set* $n = 0$.

Iterative step. *For iterates* s_n, *and* s_{n-1}, *for* $n \geq 1$, *select* $0 < \beta_n < \bar{\beta}_n$, *where*

$$\bar{\beta}_n = \begin{cases} \min\{\frac{\sigma_n}{\|s_n - s_{n-1}\|},\quad \beta\}, & \text{if } s_n \neq s_{n-1}, \\ \beta, & \text{otherwise,} \end{cases} \tag{3.32}$$

compute

$$u_n \ = \ \lambda_n k(s_n) + (1-\lambda_n)T(s_n) + \beta_n(s_n - s_{n-1}), \tag{3.33}$$

$$0 \ \in \ Q(u_n) + G(s_{n+1}) + \frac{s_{n+1} - u_n}{\mu_n}. \tag{3.34}$$

If $s_{n+1} = u_n$, *then stop. If not, set* $n = n + 1$ *and go back to the iterative step.*

Proof. Let $s^\star \in \Lambda \cap \Delta$, then by using (3.33), we obtain

$$
\begin{aligned}
\|u_n - s^\star\| &= \|\lambda_n k(s_n) + (1-\lambda_n)T(s_n) + \beta_n(s_n - s_{n-1}) - s^\star\| \\
&\le \lambda_n\|k(s_n) - s^\star\| + (1-\lambda_n)\|T(s_n) - s^\star\| + \beta_n\|s_n - s_{n-1}\| \\
&\le \lambda_n\|k(s_n) - k(s^\star)\| + \lambda_n\|k(s^\star) - s^\star\| + (1-\lambda_n)\|s_n - s^\star\| + \beta_n\|s_n - s_{n-1}\| \\
&= \lambda_n[\tau\|s_n - s^\star\| + \|k(s^\star) - s^\star\| + \frac{\beta_n}{\lambda_n}\|s_n - s_{n-1}\|] + (1-\lambda_n)\|s_n - s^\star\| \\
&= [1 - \lambda_n(1-\tau)]\|s_n - s^\star\| + \lambda_n(1-\tau)\frac{\|k(s^\star) - s^\star\| + M_1}{1-\tau} \\
&\le \max\left\{\|s_n - s^\star\|, \frac{\|k(s^\star) - s^\star\| + M_1}{1-\tau}\right\} \\
&\le \max\left\{\|u_{n-1} - s^\star\|, \frac{\|k(s^\star) - s^\star\| + M_1}{1-\tau}\right\} \\
&\vdots \\
&\le \max\left\{\|u_0 - s^\star\|, \frac{\|k(s^\star) - s^\star\| + M_1}{1-\tau}\right\}, \qquad (3.35)
\end{aligned}
$$

implying that $\{u_n\}$ is bounded and so is $\{s_n\}$. Let $x_n = \lambda_n k(s_n) + (1-\lambda_n)T(s_n)$, then by using (2.1), we get

$$
\begin{aligned}
\|u_n - s^\star\|^2 &= \|x_n + \beta_n(s_n - s_{n-1}) - s^\star\|^2 \\
&= \|x_n - s^\star\|^2 + 2\langle x_n - s^\star, \beta_n(s_n - s_{n-1})\rangle + \beta_n^2\|s_n - s_{n-1}\|^2, \qquad (3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_n - s^\star\|^2 &= \|\lambda_n k(s_n) + (1-\lambda_n)T(s_n) - s^\star\|^2 \\
&= \lambda_n^2\|k(s_n) - s^\star\|^2 + 2\lambda_n(1-\lambda_n)\langle k(s_n) - s^\star, T(s_n) - s^\star\rangle \\
&\quad + (1-\lambda_n)^2\|T(s_n) - s^\star\|^2 \\
&= \lambda_n^2\|k(s_n) - s^\star\|^2 + 2\lambda_n(1-\lambda_n)\langle k(s_n) - k(s^\star), T(s_n) - s^\star\rangle \\
&\quad + 2\lambda_n(1-\lambda_n)\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle + (1-\lambda_n)^2\|T(s_n) - s^\star\|^2 \\
&\le \lambda_n^2\|k(s_n) - s^\star\|^2 + (1-\lambda_n)^2\|T(s_n) - s^\star\|^2 + 2\lambda_n\langle k(s_n) - k(s^\star), T(s_n) - s^\star\rangle \\
&\quad + 2\lambda_n\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle + 2\lambda_n^2\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle \\
&\le \lambda_n^2\|k(s_n) - s^\star\|^2 + (1-\lambda_n)^2\|s_n - s^\star\|^2 + 2\lambda_n\tau\|s_n - s^\star\|\|s_n - s^\star\| \\
&\quad + 2\lambda_n\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle + 2\lambda_n^2\|k(s^\star) - s^\star\|\|s_n - s^\star\| \\
&\le [1 - 2\lambda_n(1-\tau)]\|s_n - s^\star\|^2 + \lambda_n\Big\{\lambda_n\|k(s_n) - s^\star\|^2 + \lambda_n\|s_n - s^\star\|^2 \\
&\quad + 2\lambda_n\|k(s_n) - s^\star\|\|s_n - s^\star\| + 2\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle\Big\} \qquad (3.37)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle x_n - s^\star,\ \beta_n(s_n - s_{n-1})\rangle &= \langle \lambda_n k(s_n) + (1-\lambda_n)T(s_n) - s^\star,\ \beta_n(s_n - s_{n-1})\rangle \\
&= \lambda_n\langle k(s_n) - s^\star,\ \beta_n(s_n - s_{n-1})\rangle \\
&\quad + (1-\lambda_n)\langle T(s_n) - s^\star,\ \beta_n(s_n - s_{n-1})\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda_n\|k(s_n) - s^\star\| \beta_n\|s_n - s_{n-1}\| \\
&\quad +(1-\lambda_n)\|T(s_n) - s^\star\| \beta_n\|s_n - s_{n-1}\| \\
&\leq \beta_n\|s_n - s_{n-1}\|\{\|k(s_n) - s^\star\| + \|s_n - s^\star\|\}. \qquad (3.38)
\end{aligned}
$$

From (3.36)–(3.38), we get

$$
\begin{aligned}
\|u_n - s^\star\|^2 \leq\ & [1-\lambda_n(1-\tau)]\|s_n - s^\star\|^2 + \lambda_n\Big\{\lambda_n\|k(s_n) - s^\star\|^2 + \lambda_n\|s_n - s^\star\|^2 \\
& +2\lambda_n\|k(s_n) - s^\star\|\|s_n - s^\star\| + 2\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle \\
& +\frac{2\beta_n}{\lambda_n}\|s_n - s_{n-1}\|\Big(\|k(s_n) - s^\star\| + \|s_n - s^\star\|\Big) + \frac{\beta_n^2\|s_n - s_{n-1}\|^2}{\lambda_n}\Big\}
\end{aligned}
$$

or

$$
\|u_n - s^\star\|^2 \leq (1-\varsigma_n)\|s_n - s^\star\|^2 + \varsigma_n V_n, \qquad (3.39)
$$

where $\varsigma_n = \lambda_n(1-\tau)$ and

$$
\begin{aligned}
V_n =\ & \frac{1}{(1-\tau)}\Big[\lambda_n\|k(s_n) - s^\star\|^2 + \lambda_n\|s_n - s^\star\|^2 + 2\lambda_n\|k(s_n) - s^\star\|\|s_n - s^\star\| \\
& +2\langle k(s^\star) - s^\star, T(s_n) - s^\star\rangle + \frac{2\beta_n}{\lambda_n}\|s_n - s_{n-1}\|\Big(\|k(s_n) - s^\star\| + \|s_n - s^\star\|\Big) + \frac{\beta_n^2\|s_n - s_{n-1}\|^2}{\lambda_n}\Big].
\end{aligned}
$$

By taking together (3.12) and (3.39), we obtain

$$
\|s_{n+1} - s^\star\|^2 \leq (1-\varsigma_n)\|s_n - s^\star\|^2 + \varsigma_n V_n - (1-2\delta\mu_n)\|s_{n+1} - u_n\|^2.
$$

We obtain the intended outcomes by following the same procedures as in the proof of Theorem 3.1. □

4. Consequences

Some Halpern-type inertial iterative methods for $\mathrm{VI_sP_s}$ and $\mathrm{FPP_s}$ are the consequences of our suggested methods.

Corollary 4.1. *Suppose that the assumptions* (S2)–(S5) *hold. The sequence* $\{s_n\}$ *induced by Algorithm 4.1 converges strongly to* $y^\star = \mathrm{P}_{\Lambda\cap\Delta}(z)$.

Algorithm 4.1. *Halpern-type inertial iteration method-1*

Let $\beta \in [0,1)$ *and* $\mu_n > 0$ *are given. Choose arbitrary points* s_0 *and* s_1, *and* $z \in \mathbb{E}$ *for* $n = 0$.
Iterative step. *For iterates* s_n, *and* s_{n-1}, *for* $n \geq 1$, *select* $0 < \beta_n < \bar{\beta}_n$, *where*

$$
\bar{\beta}_n = \begin{cases} \min\{\frac{\sigma_n}{\|s_n - s_{n-1}\|}, \quad \beta\}, & \text{if } s_n \neq s_{n-1}, \\ \beta, & \text{otherwise,} \end{cases}
$$

compute

$$
\begin{aligned}
u_n &= \lambda_n z + (1-\lambda_n)[T(s_n) + \beta_n(s_n - s_{n-1})], \\
0 &\in Q(u_n) + G(s_{n+1}) + \frac{s_{n+1} - u_n}{\mu_n}.
\end{aligned}
$$

If $s_{n+1} = u_n$, *then stop. If not, set* $n = n+1$ *and go back to the iterative step.*

Proof. By replacing $k(y)$ by z in Algorithm 3.1 and following the proof of Theorem 3.1, we get the desired result. □

Corollary 4.2. *Suppose that the assumptions* (S2)–(S5) *hold. The sequence* $\{s_n\}$ *induced by Algorithm 4.2 converges strongly to* $y^\star = P_{\Lambda\cap\Delta}(z)$.

Algorithm 4.2. *Halpern-type inertial iteration method-2*

Let $\beta \in [0, 1)$ *and* $\mu_n > 0$ *are given. Choose arbitrary points* s_0, s_1, *and* $z \in \mathbb{E}$ *for* $n = 0$.
Iterative step. *For iterates* s_n, *and* s_{n-1}, *for* $n \geq 1$, *select* $0 < \beta_n < \bar{\beta}_n$, *where*

$$\bar{\beta}_n = \begin{cases} \min\{\frac{\sigma_n}{\|s_n - s_{n-1}\|}, \quad \beta\}, & \text{if } s_n \neq s_{n-1}, \\ \beta, & \text{otherwise}, \end{cases}$$

compute

$$\begin{aligned} u_n &= \lambda_n z + (1-\lambda_n)T(s_n) + \beta_n(s_n - s_{n-1}), \\ 0 &\in Q(u_n) + G(s_{n+1}) + \frac{s_{n+1} - u_n}{\mu_n}. \end{aligned}$$

If $s_{n+1} = u_n$, *then stop. If not, set* $n = n + 1$ *and go back to the iterative step.*

Proof. By replacing $k(y)$ by z in Algorithm 3.2 and following the proof of Theorem 3.2, we get the result. □

5. Applications

Now, we present some theoretical applications of our methods for solving variational inequality and optimization problems together with the fixed point problem.

5.1. Variational inequality problem

Let $\Omega \subseteq \mathbb{E}$ and $Q : \mathbb{E} \to \mathbb{E}$ be a monotone operator. The variational inequality problem (VI_tP) is to find $s^\star \in \mathbb{E}$ such that

$$\langle Q(s^\star), \ w - s^\star\rangle \geq 0, \ \forall \ w \in \Omega. \tag{5.1}$$

The normal cone to Ω at z is defined by

$$N_\Omega(z) = \{u \in \mathbb{E} : \langle u, w - z\rangle \leq 0, \ \forall \ w \in \Omega\}. \tag{5.2}$$

It is know to us that $s^\star$ solves (VI_tP) if and only if

$$0 \in Q(s^\star) + N_\Omega(s^\star). \tag{5.3}$$

The indicator function of Ω is defined by

$$I_\Omega(w) = \begin{cases} 0, & \text{if } w \in \Omega, \\ +\infty, & \text{if } w \notin \Omega. \end{cases}$$

Since I_Ω is a proper lower semicontinuous convex function on $\mathbb{E}$, the subdifferential of I_Ω is defined as

$$\partial I_\Omega(z) = \{z \in \mathbb{E} : \langle u, w - z\rangle \leq 0, \ \forall\ w \in \Omega\}, \tag{5.4}$$

which is maximal monotone (see [26]). From (5.2) and (5.4), we can write (5.3) as

$$0 \in Q(s^\star) + \partial I_\Omega(s^\star).$$

By replacing G by ∂I_Ω in Algorithms 3.1 and 3.2, we get viscosity-type inertial iteration methods for common solutions to $\mathrm{VI_sP_s}$ and $\mathrm{FPP_s}$.

5.2. Optimization problem

Let $\Omega \subseteq \mathbb{E}$ be a nonempty closed convex subset and f_1, f_2 be proper, lower semicontinuous functions. Assume that f_1 is differentiable and ∇f_1 is δ-strongly monotone (hence, monotone) and κ-Lipschitz continuous. The subdifferential of f_2 is defined by

$$\partial f_2(y) = \{z \in \Omega : f_2(y) - f_1(w) \geq \langle y - w, z\rangle, \forall w \in \mathbb{E}\}$$

and is maximal monotone [25]. The following convex minimization problem (COP) is taken into consideration:

$$\min_{w\in\Omega}\{F(y)\} = \min_{y\in\Omega}\{f_1(y) + f_2(y)\}.$$

Therefore, by taking $Q = \nabla f_1$ and $G = \partial f_2$ in Algorithms 3.1 and 3.2, we get two viscosity-type inertial iteration methods for common solutions to $\mathrm{COP_s}$ and $\mathrm{FPP_s}$.

6. Numerical experiments

Example 6.1. *Let $\mathbb{E} = \mathbb{R}^3$. For $s = (s_1, s_2, s_3)$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, the usual inner product is defined by $\langle s,\ w\rangle = s_1w_1 + s_2w_2 + s_3w_3$ and $\|w\|^2 = |w_1|^2 + |w_2|^2 + |w_3|^2$. We define the operators Q and G by*

$$Q\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}1/2 & 0 & 0\\0 & 1/3 & 0\\0 & 0 & 1/4\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} \text{ and } G\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}1/6 & 0 & 0\\0 & 1/5 & 0\\0 & 0 & 1/4\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}.$$

It is trivial to show that the mapping Q is η-inverse strongly monotone with $\eta = 2$, δ-strongly monotone (hence monotone) with $\delta = \frac{1}{4}$, and κ-Lipschitz continuous with $\kappa = \frac{1}{2}$. The mapping G is maximal monotone. We define the mappings T and k as follows:

$$T\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} \text{ and } k\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}1/6 & 0 & 0\\0 & 1/6 & 0\\0 & 0 & 1/6\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}.$$

The mapping T is nonexpansive and k is s τ-contraction with $\tau = 1/6$. For Algorithms 3.1 and 3.2, we choose $\beta = 0.3, \lambda_n = \frac{1}{\sqrt{100+n}}, \sigma_n = \frac{1}{(1+n)^2}, \mu_n = \frac{3}{2} - \frac{1}{10+n}$, β_n is selected randomly from $(0, \bar{\beta}_n)$, and $\bar{\beta}_n$ is calculated by (3.2). *For Algorithm 1.1, we choose $\theta = 0.5$ and $\theta_n = \frac{1}{(1+n)^2} \in (0, \theta)$, $\lambda = 0.5 \in (0, 2\eta)$*

and $\psi_n = \frac{1}{(10+n)^{0.1}}$. *We compute the results of Algorithms 3.1 and 3.2 and then compare them with Algorithm 1.1. The stopping criteria for our calculation is* $Tol_n < 10^{-15}$, *where* $Tol_n = \|s_{n+1} - s_n\|$. *We select some different cases of initial values as given below:*
Case (a): $w_0 = (1, 7, -9)$ $w_1 = (1, -3, 4)$;
Case (b): $w_0 = (30, 53, 91)$ $w_1 = (1/2, -3/4, -4/11)$;
Case (c): $w_0 = (1/2, -14, 0)$ $w_1 = (0, -23, 1/4)$;
Case (d): $w_0 = (0.1, -10, 200)$ $w_1 = (100, -2, 1/4)$.

The experimental findings are shown in Table 1 and Figures 1–4.

Table 1. Comparison table of VIIM-1, VIIM-2, and VIM by using Cases (a)–(d).

Case		VIIM-1	VIIM-2	VIM
(a)	Iterations	22	22	31
	Time in seconds	8.7e-006	1.3e-005	1.06e-005
(b)	Iterations	21	23	31
	Time in seconds	8.6e-006	8.9e-006	8.1e-006
(c)	Iterations	17	19	28
	Time in seconds	8.9e-006	1.22e-005	1.08e-005
(d)	Iterations	23	26	35
	Time in seconds	8.8e-006	9.5e-006	1.08e-005

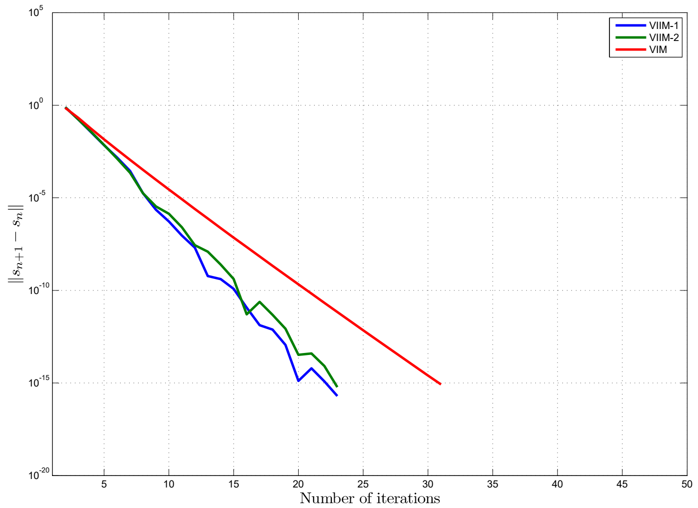

Figure 1. Graphical behavior of $\|s_{n+1} - s_n\|$ from VIIM-1, VIIM-2, and VIM by choosing Case (a).

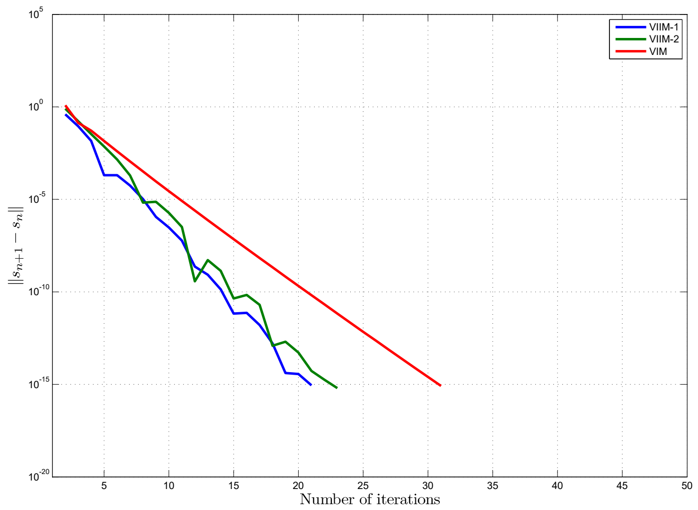

Figure 2. Graphical behavior of $\|s_{n+1} - s_n\|$ from VIIM-1, VIIM-2, and VIM by choosing Case (b).

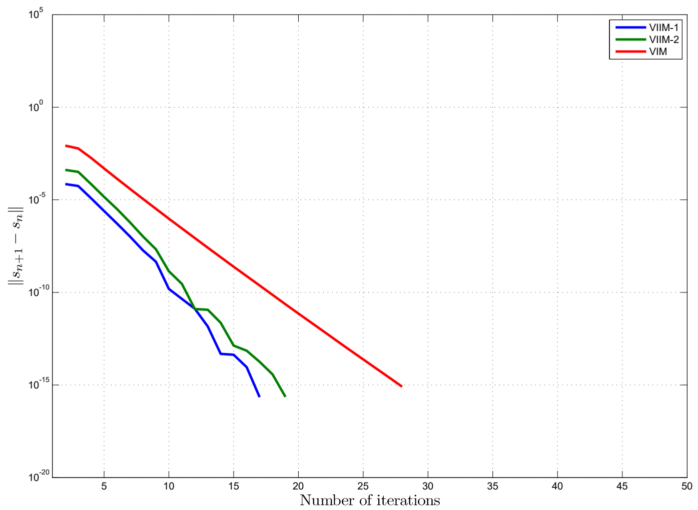

Figure 3. Graphical behavior of $\|s_{n+1} - s_n\|$ from VIIM-1, VIIM-2, and VIM by choosing Case (c).

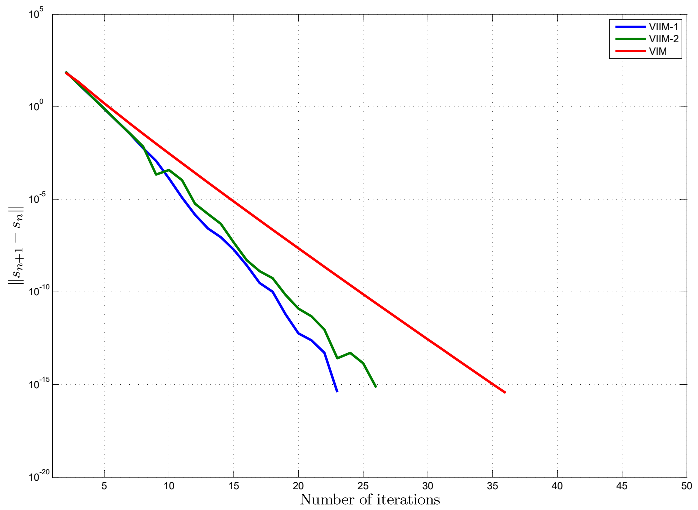

Figure 4. Graphical behavior of $\|s_{n+1} - s_n\|$ from VIIM-1, VIIM-2, and VIM by choosing Case (d).

Example 6.2. *Let us consider the infinite dimensional real Hilbert space* $\mathbb{E}_1 = \mathbb{E}_2 = l_2 := \{u := (u_1, u_2, u_3, \cdots, u_n, \cdots), u_n \in \mathbb{R} : \sum_{n=1}^{\infty} |u_n| < \infty\}$ *with inner product* $\langle u, v\rangle = \sum_{n=1}^{\infty} u_n v_n$ *and the norm is given by* $\|u\| = \left(\sum_{n=1}^{\infty} |u_n|^2\right)^{1/2}$. *We define the monotone mappings by* $Q(u) := \frac{u}{5} = (\frac{u_1}{5}, \frac{u_2}{5}, \frac{u_3}{5}, \cdots, \frac{u_n}{5}, \cdots)$ *and* $G(u) := u = (u_1, u_2, u_3, \cdots, u_n, \cdots)$. *Let* $k(u) := \frac{u}{15}$ *be the contraction and the nonexpansive map* T *is defined by* $T(u) := \frac{u}{3} = (\frac{u_1}{3}, \frac{u_2}{3}, \frac{u_3}{3}, \cdots, \frac{u_n}{3}, \cdots)$.

It can be seen that Q *is* δ*-strongly monotone with* $\delta = \frac{1}{5}$ *and* κ*-Lipschitz continuous with* $\kappa = \frac{1}{5}$ *and also* η*-inverse strongly monotone with* $\eta = 5$*;* G *is maximal monotone;* k *be the* τ*-contraction with* $\tau = \frac{1}{15}$. *We choose* $\beta = 0.4$, $\lambda_n = \frac{1}{(n+200)^{0.25}}$, $\sigma_n = \frac{1}{(10+n)^3}$, $\mu_n = \frac{4}{3} - \frac{1}{n+50}$, β_n *is selected randomly, and* $\bar{\beta}_n$ *by* (3.2). *We choose* $\theta = 0.4$ *and* $\theta_n = \frac{1}{(10+n)^3} \in (0, \theta)$, $\lambda = 0.7 \in (0, 2\eta)$, *and* $\psi_n = \frac{1}{(200+n)^{0.25}}$. *We compute the results of Algorithms 3.1 and 3.2, then compare with Algorithm 1.1. The stopping criteria for our computation is* $Tol_n < 10^{-15}$, *where* $Tol_n = \frac{1}{2}\|s_{n+1} - s_n\|$. *We compute the results of the Algorithms 3.1 and 3.2, and then compare them with Algorithm 1.1. We consider the following four cases of initial values:*

Case (a'): $w_0 = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, $w_1 = \left\{\frac{1}{1+n^2}\right\}_{n=0}^{\infty}$;

Case (b'): $w_0 = \begin{cases} \frac{1}{n+1}, & if\ n\ is\ odd, \\ 0, & if\ n\ is\ even, \end{cases}$ $w_1 = \left\{\frac{1}{1+n^3}\right\}_{n=1}^{\infty}$;

Case (c'): $w_0 = (0, 0, 0, 0, \cdots)$, $w_1 = (1, 2, 3, 4, 0, 0, 0, \cdots)$;

Case (d'): $w_0 = \left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$, $w_1 = \begin{cases} 0, & if\ n\ is\ odd, \\ \frac{1}{n^2}, & if\ n\ is\ even. \end{cases}$

The experimental findings are shown in Table 2 and Figures 5–8.

Table 2. Comparison table of VIIM-1, VIIM-2, and VIM by using Case (a′)–Case (d′).

Case		VIIM-1	VIIM-2	VIM
(a′)	Iterations	37	40	90
	Time in seconds	9.4e-006	1.2e-005	8.5e-006
(b′)	Iterations	36	39	90
	Time in seconds	1.39e-005	1.02e-005	1.06e-005
(c′)	Iterations	32	34	39
	Time in seconds	9.6e-006	9.7e-006	1.66e-005
(d′)	Iterations	22	32	50
	Time in seconds	1.35e-005	2.1e-005	1.41e-005

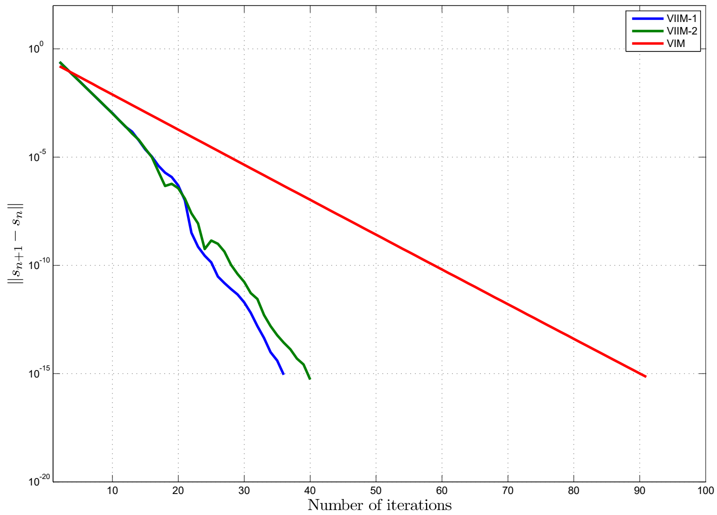

Figure 5. Graphical behavior of $\|s_{n+1} - s_n\|$ from VIIM-1, VIIM-2, and VIM by choosing Case (a′).

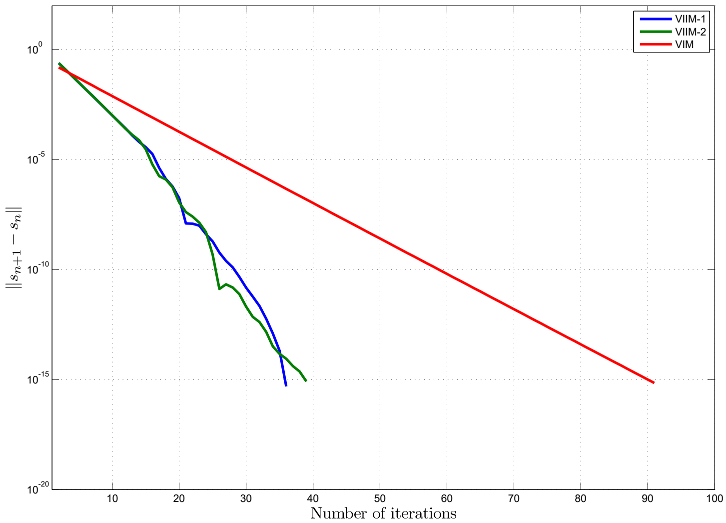

Figure 6. Graphical behavior of $\|s_{n+1} - s_n\|$ from VIIM-1, VIIM-2, and VIM by choosing Case (b′).

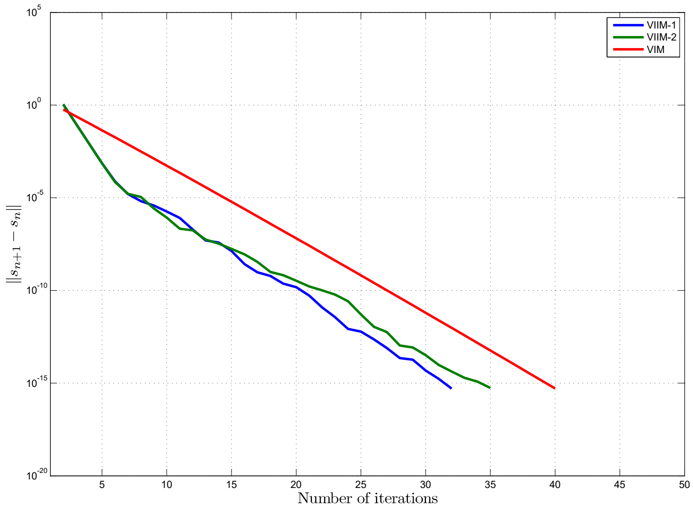

Figure 7. Graphical behavior of $||s_{n+1} - s_n||$ from VIIM-1, VIIM-2, and VIM by choosing Case (c′).

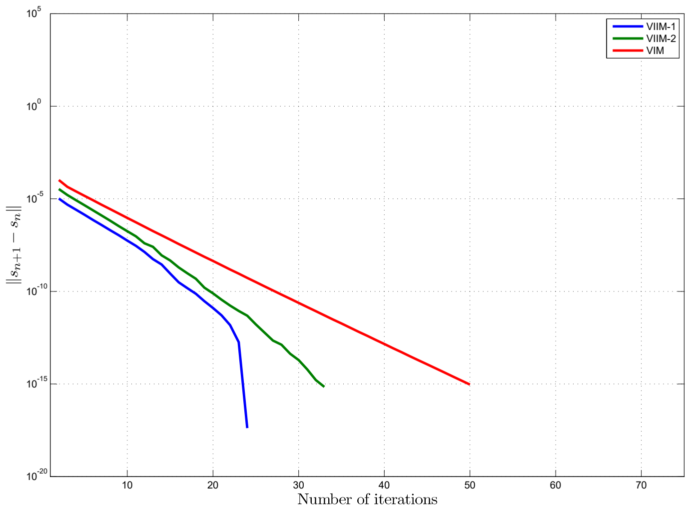

Figure 8. Graphical behavior of $||s_{n+1} - s_n||$ from VIIM-1, VIIM-2, and VIM by choosing Case (d′).

7. Conclusions

We suggested two viscosity-type inertial iteration methods for solving VI_sP and FPP in Hilbert spaces. Our methods calculated the viscosity approximation, fixed point iteration, and inertial extrapolation simultaneously at the beginning of each iteration. We proved the strong convergence of the proposed methods without calculating the resolvent of the associated monotone operators. Some consequences and theoretical applications were also discussed. Finally, we illustrated the proposed methods by using some suitable numerical examples. It has been deduced from the numerical examples that our algorithms performed well in the sense of time acquired by the CPU and the number of iterations.

Author contributions

M. Dilshad: Conceptualization, Methodology, Formal analysis, Investigation, Writing-original draft, Software, Writing-review & editing; A. Alamer: Conceptualization, Methodology, Formal analysis, Software, Writing-review & editing; Maryam G. Alshahri: Conceptualization, Methodology, Formal analysis, Software, Writing-review & editing; Esmail Alshaban: Investigation, Writing-original draft; Fahad M. Alamrani: Investigation, Writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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