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Research article

On an asymmetric multivariate stochastic difference volatility: structure and estimation

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Abstract: In this study, we explored an asymmetric multivariate stochastic difference volatility model that extends various probabilistic and statistical properties previously discussed in the literature. We rigorously established that the model exhibits periodic stationarity and periodic ergodicity. Additionally, we delved into the robust consistency and asymptotic normality of the Quasi-Maximum Likelihood Estimator (QMLE), providing a comprehensive analysis of its theoretical underpinnings. Finally, we demonstrated the practical applicability of our major findings through a series of pertinent applications. This work not only contributes to the existing body of knowledge on stochastic volatility modeling, but also opens new avenues for further research in this domain.

Keywords: periodicity; multivariate asymmetric $GARCH$; stationarity; asymptotic properties
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1. Introduction

Since Engle's groundbreaking work on the univariate ARCH model [12], and the subsequent generalization to the GARCH model by Bollerslev [9], as well as the introduction of the threshold GARCH by Glosten et al. [20], these models have become fundamental tools for capturing the key stylized facts of financial time series. The capability to model the co-movements of multiple series is of paramount practical importance. Analyzing them as components of a multivariate process is of particular benefit, especially in the presence of temporal or contemporaneous dependencies. The spectrum of multivariate (A)GARCH models is expanding, with a substantial body of literature already established [3, 15, 26, 28]. This includes significant probabilistic, structural, and asymptotic findings that support various estimation methods for a wide range of multivariate GARCH models and their extensions.

Engle and Kroner [13] set the groundwork for covariance stationarity in multivariate generalized ARCH models. The existence of a 4th-moment structure in the CCC-GARCH model was validated by He and Terasvirta [22] and Aue et al. [1]. Jeantheau [23] demonstrated the robust consistency of the Quasi-Maximum Likelihood Estimator (QMLE) in multivariate ARCH models. Ling and McAleer [25] furthered this by establishing the asymptotic properties of the QMLE for vector ARMA-GARCH models, contingent on the finiteness of the 4th-moment of the observed process. Notably, while asymptotic results for some asymmetric models were known, the works of [2] extended these to a broader class of multidimensional causal processes, including asymmetries. McAleer et al. [27] contributed to the structure and asymptotic theory of the VARMA-AGARCH model, a direction further expanded by Francq and Zakoïan [14, 15], who provided results for a variety of multivariate (A)GARCH models without relying on moment assumptions for the observed process. Extensions of the EGARCH model to the multivariate context were also proposed by Koutmos and Booth [24]. In terms of more recent developments, stationary multivariate GARCH models have evolved from time-invariant to time-varying coefficients. These models allow coefficients to depend on an unobservable, time-homogeneous Markov chain, as demonstrated in the multivariate MS-GARCH model by Haas and Liu [21].

Francq and Zakoïan (2010, [14]) investigated the quasi-maximum likelihood estimation of a class of symmetric multivariate GARCH models without imposing moment conditions for the observed process. They extended this study to multivariate asymmetric GARCH models in their subsequent paper (Francq and Zakoïan, 2012, [15]). In [5], Bibi (2018) extended the symmetric multivariate GARCH models studied by Francq and Zakoïan (2010, [14]) by investigating the asymptotic properties of QML estimation for symmetric multivariate periodic CCC-GARCH models. Maïnassara et al. (2022, [26]) made additions that can be considered an extension of Francq and Zakoïan's models by studying the estimation of multivariate asymmetric power GARCH models.

We extend the work undertaken in these studies. On the one hand, we extend the standard models by introducing coefficients related to time-varying and periodicity (for more details, see Ghezal et al. [16, 17]); on the other, we extend Bibi's paper by incorporating the effect of asymmetry. More accurately, in our paper, we expanded the standard models to include the effects of temporal and periodic factors on the model coefficients. Additionally, we incorporate the effects of asymmetry in multivariate GARCH models, allowing for cross-leverage effects, which was not considered in Bibi's paper, which focused solely on symmetric models. Moreover, our paper illustrates our major findings by application to a bivariate exchange rates series, while Bibi's paper focused solely on the theoretical aspects of periodic multivariate CCC-GARCH models. Furthermore, our primary contribution lies in providing asymptotic results for this model category without imposing moment assumptions for the observed process, a topic further explored by Ghezal et al. [7, 16]. These differences highlight the unique contribution of our research to the development of multivariate GARCH models.

In this context, the following symbols are utilized:

- $I_{(n)}$ represents the $n \times n$ identity matrix and $O_{(k,l)}$ represents a matrix of size $k \times l$ with all entries being zeros. For clarity, let us consider: $\underline{O}_{(k)} := O_{(k,1)}$ and $O_{(k)} := O_{(k,k)}$.
- $A^{\otimes s}$ signifies $A \otimes A \otimes ... \otimes A$ repeated s times, where $\otimes$ represents the usual Kronecker product of matrices. $Vec(A)$ denotes the vector stacking operator. If $(A(k), k \in K)$ is a sequence of $n \times n$ matrices, it is important to note that for any integers m and t, $\prod_{i=m}^{t} A(i) = A(m)A(m+1)\ldots A(t)$ if

$m \le t$, and $I_{(n)}$ otherwise.

- $\rho(A)$ represents the spectral radius of the squared matrix A, $\det(A)$ (resp. $Tr(A)$) denotes the determinant (resp. trace) of A. Moreover, for any matrices A, B, and C with suitable dimensions, $Tr(B'A) = Vec(B)' Vec(A)$ and $Vec(BAC) = (C' \otimes B) Vec(A)$. Let $\|.\|$ represent any induced matrix norm on the set of $r \times p$ and $r \times 1$ matrices. This norm verifies, for any $p \times r$ matrix A and $r \times p$ matrix B, that $|Tr(AB)| \le \sqrt{nm} \|A\| \|B\|$, $\|A\|^2 \le Tr(A'A) \le m \|A\|^2$, and $|A'A| \le \|A\|^{2m}$.
- $\nabla_{\underline{\theta}}$ represents the vector of first-order partial derivatives with respect to $\underline{\theta}$, $\nabla^2_{\underline{\theta}}$ represents the matrix of second-order partial derivatives with respect to $\underline{\theta}$, and $\rightsquigarrow$ represents convergence in the distribution.
- Recall that if $g(B)$ is a real-valued function of a matrix B depending on some parameter $\underline{\theta}$, then $\nabla_{\underline{\theta}} g(B) = Tr\left(\frac{\partial g(B)}{\partial B'} \nabla_{\underline{\theta}} B\right)$. Furthermore, assuming the invertibility of matrix A, the following holds true for any vector $\underline{x}$ and matrices B and C with suitable dimensions:

$$\nabla_{\underline{\theta}} A^{-1} = -A^{-1}\left(\nabla_{\underline{\theta}} A\right) A^{-1}$$

and

$$\begin{aligned} \frac{\partial \underline{x}' A \underline{x}}{\partial A'} = \underline{x}\underline{x}', \ \frac{\partial \log|\det(A)|}{\partial A'} &= A^{-1}, \\ \frac{\partial Tr(BAC)}{\partial A'} &= CB, \\ \frac{\partial Tr\left(BA^{-1}C\right)}{\partial A'} &= -A^{-1}CBA^{-1}, \\ \frac{\partial Tr(BA'CA')}{\partial A'} &= B'AC' + C'AB'. \end{aligned}$$

2. The multivariate periodic asymmetric *GARCH* model

We begin with the *CCC-PAGARCH* models defined for a $r-$dimensional $\left\{\underline{X}_t = (X_{1t}, ..., X_{rt})'\right\}$ as follows

$$\begin{cases} \underline{X}_t = V_t^{1/2} \underline{\eta}_t, \\ Var\left\{\underline{X}_t | \mathcal{F}_{t-1}\right\} = V_t = H_t M H_t, \ H_t = diag\left(\underline{\sigma}_t^{1/2}\right), \\ \underline{\sigma}_t = \underline{a}_0(t) + \sum\limits_{k=1}^{q} \left(A_k(t) \underline{X}_{t-k}^{+} + B_k(t) \underline{X}_{t-k}^{-}\right) + \sum\limits_{j=1}^{p} C_j(t) \underline{\sigma}_{t-j}. \end{cases}, \quad (2.1)$$

where $\underline{\sigma}_t^{\tau} := \left(\sigma_{1t}^{\tau}, ..., \sigma_{rt}^{\tau}\right)'$, $\tau \in \mathbb{R}_+^*$, $\underline{X}_t^{+} := \left(X_{1t}^{+2}, ..., X_{rt}^{+2}\right)'$, $\underline{X}_t^{-} := \left(X_{1t}^{-2}, ..., X_{rt}^{-2}\right)'$, $z^{-} = (-z)^{+} = (-z) \vee 0$, $z^{\pm 2} = (z^{\pm})^2$, $\mathcal{F}_t := \sigma\left\{\underline{X}_{t-i}, i \ge 0\right\}$, M represents a constant conditional correlation matrix, $\underline{a}_0(.)$ represents a $r \times 1$ vector whose entries are strictly positive, the $A_i(.)$, $B_i(.)$ and $C_j(.)$ for all i, j are $r \times r$ matrices with positive entries, the functions $\underline{a}_0(t)$, $A_i(t)$, $B_i(t)$ and $C_j(t)$ exhibit periodicity in t with a period of s, and $\left(\underline{\eta}_t\right)$ is an i.i.d. random vector with a mean and covariance matrix $\left(\underline{O}_{(r)}, I_{(r)}\right)$. Now, it is important to emphasize that this model is limited to the periodic case, i.e., by setting $t = ns + v$, where $v = 1, ..., s$ and $n \in \mathbb{Z}$. Consequently, the model (2.1) can be expressed in the following form

$$\begin{cases} \underline{X}_{ns+v} = V_{ns+v}^{1/2}\underline{\eta}_{ns+v}, \\ V_{ns+v} = H_{ns+v} M H_{ns+v},\ H_{ns+v} = diag\left(\underline{\sigma}_{ns+v}^{1/2}\right), \\ \underline{\sigma}_{ns+v} = \underline{a}_0(v) + \sum\limits_{k=1}^{q}\left(A_k(v)\underline{X}_{ns+v-k}^{+} + B_k(v)\underline{X}_{ns+v-k}^{-}\right) + \sum\limits_{j=1}^{p} C_j(v)\underline{\sigma}_{ns+v-j}. \end{cases} \tag{2.2}$$

CCC-PAGARCH is a comprehensive framework that encompasses several models previously explored in the literature. This includes standard *CCC-AGARCH* models when $s = 1$, as discussed in [15], conventional symmetric *CCC*−Periodic*GARCH* models when $A_{.}(.) = B_{.}(.)$, as explored in [5], and periodic asymmetric *GARCH* models when $r = 1$, as studied by Ghezal et al. [6]. Now, such as seen in several time series models, it is usually beneficial to express (2.1) as an analogous stochastic recurrence equation to facilitate a more streamlined analysis (see [17–19]). For this reason, we write $\underline{X}_t = H_t\underline{\widetilde{\eta}}_t$, $\underline{X}_t^{+} = G_t^{+}\underline{\sigma}_t$ and $\underline{X}_t^{-} = G_t^{-}\underline{\sigma}_t$, where $\underline{\widetilde{\eta}}_t = M^{1/2}\underline{\eta}_t$ and $G_t^{\pm} = diag\left(\underline{\widetilde{\eta}}_t^{\pm 2}\right)$, and we consider the $r(2q+p)\times r(2q+p)$ matrix

$$\Delta_t := \begin{pmatrix} \overline{\left(G^{+}A_q\right)_t} & \left(G_q^{+}B_q\right)_t & \left(G_q^{+}C_p\right)_t \\ \left(G_q^{-}C_q\right)_t & \overline{\left(G^{-}A_q\right)_t} & \left(G_q^{-}C_p\right)_t \\ \left(I_pA_q\right)_t & \left(I_pB_q\right)_t & \overline{\left(IA_p\right)_t} \end{pmatrix},$$

where

$$\left(\Phi_n^{+}\Psi_m\right)_t = \begin{pmatrix} \Phi_t^{+}\Psi_1(t) & \cdots & \Phi_t^{+}\Psi_m(t) \\ O_{(r(n-1),rm)} & & \end{pmatrix}, \overline{\left(\Phi^{+}\Psi_m\right)_t} = \begin{pmatrix} \Phi_t^{+}\Psi_1(t) & \cdots & \Phi_t^{+}\Psi_m(t) \\ I_{(r(m-1))} & & O_{(r(m-1),r)} \end{pmatrix}.$$

Consequently, we have $\underline{X}_t = G_t\underline{\sigma}_t$ and $\underline{\sigma}_t = F'\underline{\Lambda}_t$, such that

$$\underline{\Lambda}_t = \Delta_t\underline{\Lambda}_{t-1} + \underline{\Pi}_t, \tag{2.3}$$

where

$$\underline{\Lambda}_t' := \left(\left(\underline{X}_t^{+}\right)', \ldots \left(\underline{X}_{t-q+1}^{+}\right)', \left(\underline{X}_t^{-}\right)', \ldots, \left(\underline{X}_{t-q+1}^{-}\right)', \underline{\sigma}_t', \ldots, \underline{\sigma}_{t-p+1}'\right),$$

$$F' := \left(O_{(r,2rq)} \vdots I_{(r)} \vdots O_{(r,r(p-1))}\right),$$

and

$$\underline{\Pi}_t' := \left(\underline{a}_0'(t)\, G_t^{+\prime}, \underline{O}_{(r(q-1))}', \underline{a}_0'(t)\, G_t^{-\prime}, \underline{O}_{(r(q-1))}', \underline{a}_0'(t), \underline{O}_{(r(p-1))}'\right).$$

Now, iterating (2.3) $s$times, we get

$$\underline{\Gamma}_t := \underline{\Lambda}_{st} = \Omega_t\underline{\Gamma}_{t-1} + \underline{\Xi}_t, \tag{2.4}$$

in which $\Omega_t = \prod\limits_{j=0}^{s-1}\Delta_{st-j}$ and $\underline{\Xi}_t = \sum\limits_{i=0}^{s-1}\left\{\prod\limits_{l=0}^{i-1}\Delta_{st-l}\right\}\underline{\Pi}_{st-i}$ where, as is customary, emptied products are defined to be equal to $I_{(r(2q+p))}$. Note here that (Ω_t) (resp. $\left(\underline{\Xi}_t\right)$) represents a sequence of i.i.d. positive random matrices (resp. vectors), independent of $\underline{\Gamma}_n$, $n < t$. Therefore, a causal, periodically ergodic, and strictly periodical solution (short form: PE and SPS) for Eq (2.3), with an equivalent representation in (2.2), holds true if Eq (2.4) has a causal, ergodic, and strictly stationary (short form: SS) solution.

Observe that according to (2.3) and (2.4) are correct for all integer t, through recursive substitution, we derive formal series solutions denoted, respectively, by

$$\underline{\Gamma}_t(1) = \sum_{i\geq 0}\left\{\prod_{l=0}^{i-1}\Delta_{t-l}\right\}\underline{\Pi}_{t-i}$$

and

$$\underline{\Gamma}_t(2) = \sum_{i\geq 0}\left\{\prod_{l=0}^{i-1}\Omega_{t-j}\right\}\underline{\Xi}_{t-i}.$$

3. Probabilistic properties of periodic CCC-$AGARCH$ models

The uniqueness of the non-anticipative SS solution $(\underline{\Gamma}_t)$ to the model (2.4) appears in a form comparable to that of Bougerol and Picard [10] through the use of the top−Lyapunov exponent γ_Ω linked to the sequence of stochastic matrices $\Omega = (\Omega_t)$ defined by

$$\begin{aligned}\gamma_\Omega &:= \inf_{m\geq 1}\left\{\frac{1}{m}E\left\{\log\|\Omega_m\Omega_{m-1}...\Omega_1\|\right\}\right\}\\ &= \lim_{m\to\infty}\frac{1}{m}\log\|\Omega_m\Omega_{m-1}...\Omega_1\| \text{ a.s.}\end{aligned}$$

Furthermore, since $E\left\{\log^+\|\Omega_1\|\right\} \leq E\left\{\|\Omega_1\|\right\} < +\infty$ where $\log^+ z$ represents $\log z \vee 0$, $\forall z > 0$, this ensures the existence of γ_Ω. Building upon the preceding discussion, we are ready to unveil the following outcomes.

Theorem 1. *Consider the process* $\left(\underline{X}_t\right)_{t\in\mathbb{Z}}$ *defined by the model* (2.1) *with a periodic vector representation* (2.3)*, and suppose that* $E\left\{\left\|\underline{\eta}_t\right\|\right\} < +\infty$*. The following statements are thus equivalent:*

- *The Eq* (2.4) *having a unique, strictly stationary, causal and ergodic solution provided by* $\underline{\Gamma}_t(2)$*.*
- *The Eq* (2.3) *having a unique,* SPS*, causal and PE solution provided by* $\underline{\Gamma}_t(1)$*.*
- $\gamma_\Omega < 0$.

Furthermore, for all $t \in \mathbb{Z}$*, the series* $\underline{\Gamma}_t(1)$ *and* $\underline{\Gamma}_t(2)$ *converge absolutely almost surely,* $\underline{\Gamma}_t(1) \overset{a.s}{=} \underline{\Gamma}_t(2)$*.*

We also need the subsequent simple outcome which characterizes a necessary condition for SS.

Corollary 1. *If the* CCC-$PAGARCH(p, q)$ *model described in* (2.1) *satisfies* $\gamma_\Omega < 0$*, thus* $\rho\left(\prod_{v=0}^{s-1}C_{s-v}\right) < 1$*, where* $C_t = \begin{pmatrix} C_1(t) & \cdots & C_p(t) \\ I_{(r(p-1))} & & O_{(r(p-1),r)} \end{pmatrix}$.

Now, the subsequent proposition introduces conditions that are slightly more robust, guaranteeing the negativity of γ_Ω.

Proposition 1. *Consider the* CCC-$PAGARCH(p, q)$ *model, then*

a. $\rho\left(E\left\{\Omega_1\right\}\right) < 1$ *implies that* $\gamma_\Omega < 0$.

b. $E\left\{\log\left\|\prod_{i=0}^{t-1}\Omega_{t-i}\right\|\right\}<0$ *or* $E\left\{\left\|\prod_{i=0}^{t-1}\Omega_{t-i}\right\|\right\}<1$ *for some* $t\geq 1\Longrightarrow \gamma_{\Omega}<0$.

Moreover, if $\gamma_{\Omega}<0$*, then there is* $\varsigma>0$ *such that* $\forall t$, $E\left\{\left\|\underline{\sigma}_t\right\|^{\varsigma}\right\}<+\infty$, $E\left\{\left\|\underline{X}_t\right\|^{\varsigma}\right\}<+\infty$.

The following results establish the geometric ergodicity and $\beta-$mixing of Markov chain $\left(\underline{\Gamma}_t\right)$ with state space $\mathbb{R}^{r(2q+p)}$ whose $n-$step transition probability is given by $P^n_{\underline{x}_0}(D)=P\left(\underline{\Gamma}_t\in D\middle|\underline{\Gamma}_0=\underline{x}_0\right)$ for any Borel set $D\in\mathcal{B}_{\mathbb{R}^{r(2q+p)}}$.

Theorem 2. *If* $\left(\underline{X}_t\right)_{t\in\mathbb{Z}}$ *is the process governed by* (2.1)*, characterized by a state-space representation as described in* (2.4)*, then*

i. $\rho\left(E\left\{\Omega_1\right\}\right)<1$ *entails the existence of only one invariant distribution of* $P_{\underline{x}}(.)$, $\forall\underline{x}$*, a non-negative Borel measurable function* $h\geq 1$ *defined on* $\mathbb{R}^{r(2q+p)}$*, a compact set* W *in* $\mathcal{B}_{\mathbb{R}^{r(2q+p)}}$*, a constant* κ *and* $\delta\in\,]0,1[$ *such that for any* $\underline{x}\in W^c$, $E\left\{h\left(\underline{\Gamma}_{n+1}\right)\middle|\underline{\Gamma}_n=x\right\}\leq\delta h\left(\underline{x}\right)-\kappa$ *and* $\sup_{\underline{x}\in W}E\left\{h\left(\underline{\Gamma}_{n+1}\right)\middle|\underline{\Gamma}_n=x\right\}<+\infty$.

ii. The chain $\left(\underline{\Gamma}_n\right)$ *is geometrically ergodic and* $\beta-$*mixing property with exponential decay rate with* $E\left\{\underline{\Gamma}_n\right\}<+\infty$ *if, and only if,* $\rho\left(E\left\{\Omega_1\right\}\right)<1$.

iii. $\sup_{1\leq v\leq s}\rho\left(\sum_{i=1}^{q}\left(A_i(v)+B_i(v)\right)+\sum_{j=1}^{p}C_j(v)\right)<1$ *implies that the chain* $\left(\underline{\Gamma}_n\right)$ *is geometrically ergodic.*

4. Estimation

In this section, we consider the *QMLE* for estimating the parameters of the *CCC-PAGARCH* (p,q) model assembled in

$$\underline{\theta}=(\theta_1,...,\theta_l)':=\left(\underline{\theta}'_1,...,\underline{\theta}'_s,\underline{m}'\right)\in\Theta\subset\left(]0;+\infty[^r\times[0;+\infty[^{r^2(p+2q)}\right)^s\times]-1;1[^{r(r-1)/2}$$

where

$$l=s\widetilde{r}+r(r-1)/2,\underline{\theta}'_v:=\left(\theta_1(v),...,\theta_{\widetilde{r}}(v),1\leq v\leq s\right):=\left(\underline{a}'_0(v),\underline{\alpha}'(v),\underline{\beta}'(v),\underline{\gamma}'(v)\right)$$

and

$$\widetilde{r}=r+r^2(2q+p)$$

with

$$\underline{\alpha}'(v):=\left(Vec'\left(A_1(v)\right),...,Vec'\left(A_q(v)\right)\right),\underline{\beta}'(v):=\left(Vec'\left(B_1(v)\right),...,Vec'\left(B_q(v)\right)\right),$$
$$\underline{\gamma}'(v):=\left(Vec'\left(C_1(v)\right),...,Vec'\left(C_p(v)\right)\right),$$

for all $1\leq v\leq s$, $\theta_i(v)=a_{0,i}(v)$, $i=1,...,r$, $\theta_{r+ijk}(v)=\alpha_{ijk}(v)$, $i,j=1,...,r,k=1,...,q$, $\theta_{r+r^2q+ijk}(v)=\beta_{ijk}(v)$, $i,j=1,...,r,k=1,...,q$, $\theta_{r+2r^2q+ijk}(v)=\gamma_{ijk}(v)$, $i,j=1,...,r,k=1,...p$ and $M=\left(m_{ij}\right)$ is the symmetric matrix with $m_{ii}=1$ (i.e., $\underline{m}':=\left(m_{21},...,m_{r1},m_{32},...,m_{r2},...,m_{rr-1}\right)$). The true parameter value, symbolized by $\underline{\theta}_0$ and belonging to the parameter space Θ, is unknown and needs to be estimated.

In pursuit of this, consider that the observations $\underline{X}_1, ..., \underline{X}_{n=sN}$ constitute a time series from the unique, SPS and causal solution of (2.1). A $QMLE$ of $\underline{\theta}$ is defined as any measurable solution $\widehat{\underline{\theta}}_n$ of

$$\widehat{\underline{\theta}}_n = Arg\max_{\underline{\theta}\in\Theta}\widetilde{L}_{Ns}\left(\underline{\theta}\right) = Arg\min_{\underline{\theta}\in\Theta}\left(-\widetilde{L}_{Ns}\left(\underline{\theta}\right)\right). \tag{4.1}$$

For initial values $\underline{X}_0, ..., \underline{X}_{1-q}, \underline{\widetilde{\sigma}}_0\left(\underline{\theta}\right), ..., \underline{\widetilde{\sigma}}_{1-p}\left(\underline{\theta}\right)$, the Gaussian log −likelihood function for $\underline{\theta} \in \Theta$ is given up to an additive constant by

$$\widetilde{L}_{Ns}\left(\underline{\theta}\right) = -\left(Ns\right)^{-1}\sum_{t=1}^{N}\sum_{v=0}^{s-1}\widetilde{l}_{st+v}\left(\underline{\theta}\right), \tag{4.2}$$

with $\widetilde{l}_t\left(\underline{\theta}\right) = \log\det\left(\widetilde{V}_t\right) + \underline{X}_t'\widetilde{V}_t^{-1}\underline{X}_t$, where $\widetilde{V}_t$ is recursively defined for $t \geq 1$ by

$$\begin{cases} \widetilde{V}_{ns+v} = \widetilde{H}_{ns+v}M\widetilde{H}_{ns+v},\ \widetilde{H}_{ns+v} = diag\left(\underline{\widetilde{\sigma}}_{ns+v}^{1/2}\right), \\ \underline{\widetilde{\sigma}}_{ns+v} = \underline{a}_0\left(v\right) + \sum\limits_{i=1}^{q}\left(A_i\left(v\right)\underline{X}_{ns+v-i}^{+} + B_i\left(v\right)\underline{X}_{ns+v-i}^{-}\right) + \sum\limits_{j=1}^{p}C_j\left(v\right)\underline{\widetilde{\sigma}}_{ns+v-j}, \end{cases}.$$

Due to the strong dependency of $\widetilde{V}_t\left(\underline{\theta}\right)$ on initial values, the process $\left(\widetilde{l}_t\left(\underline{\theta}\right)\right)$ is neither an SPS nor PE process. Hence, it would be appropriate to replace the process $\left(\widetilde{l}_t\left(\underline{\theta}\right)\right)$ with its SPS and PE version. Therefore, we introduce an approximate version $L_{Ns}\left(\underline{\theta}\right) = -\left(Ns\right)^{-1}\sum_{t=1}^{N}\sum_{v=0}^{s-1}l_{st+v}\left(\underline{\theta}\right)$ of the likelihood (4.2) with $l_t\left(\underline{\theta}\right) = \log\det\left(V_t\right) + \underline{X}_t'V_t^{-1}\underline{X}_t$. The upcoming findings in this paper confirm the robust consistency of $\widehat{\underline{\theta}}_{Ns}$ and its asymptotic normality.

4.1. Robust consistency of QMLE

Take into account the regularity assumptions listed below:

A0 $\underline{\theta}_0 \in \Theta$ and Θ is compact.

A1 Consider the polynomials $\mathcal{A}_{0,v}(z) = \sum_{i=1}^{q}A_{0,i}(v)z^i$, $\mathcal{B}_{0,v}(z) = \sum_{i=1}^{q}B_{0,i}(v)z^i$ and $C_{0,v}(z) = I_{(r)} - \sum_{j=1}^{p}C_{0,j}(v)z^j$ adheres to the convention $\mathcal{A}_{0,v}(z) = \mathcal{B}_{0,v}(z) = O_{(r)}$ if $q = 0$ and $C_{0,v}(z) = I_{(r)}$ if $p = 0$, for all $v \in \{1, ..., s\}$. If $p > 0$, $\mathcal{A}_{0,v}(1) + \mathcal{B}_{0,v}(1) \neq O$, $\mathcal{A}_{0,v}(z)$, $\mathcal{B}_{0,v}(z)$ and $C_{0,v}(z)$ are left-coprime and $\left[A_{0,q}(v)\ B_{0,q}(v)\ \ C_{0,p}(v)\right]$ has full rank r for all $v \in \{1, ..., s\}$.

A2 $\gamma_{\Omega^0} < 0$ with Ω^0 in place of Ω to confirm that the unknown parameter is $\underline{\theta}_0$ and $\rho\left(\prod\limits_{v=0}^{s-1}C_{0,s-v}\right) < 1$.

A3 The distribution of the components of $\widetilde{\underline{\eta}}_t$ isn't focused on two points and $P\left(\widetilde{\eta}_{it} \in (0, +\infty)\right) \in (0, 1)$ for $i = 1, \ldots, r$.

A4 M represents a correlation matrix that is positive-definite.

Currently, we have the capability to articulate our initial finding.

Theorem 3. *Assuming the specified Assumptions,* ***A0–A4****,* $\widehat{\underline{\theta}}_{Ns}$ *exhibits robust consistency, meaning that*

$$P\left(\widehat{\underline{\theta}}_{Ns} \underset{N\to\infty}{\longrightarrow} \underline{\theta}_0\right) = 1.$$

To establish the proof of Theorem 3, we articulate the subsequent technical statements encapsulated in the following lemma.

Lemma 1. *Assuming the specified Assumptions, **A0–A4**, we obtain*

I. $\sup_{\underline{\theta}\in\Theta}\left|\left(\widetilde{L}_{Ns}-L_{Ns}\right)\left(\underline{\theta}\right)\right| \xrightarrow[N]{a.s.} 0.$

II. There exists an integer $t\in\mathbb{Z}$ such that $\underline{\sigma}_t\left(\underline{\theta}\right)\overset{a.s.}{=}\underline{\sigma}_t\left(\underline{\theta}_0\right)$ and $M=M_0\Rightarrow\underline{\theta}=\underline{\theta}_0$.

III. $\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}_0\right)\right\}<\infty$ and $\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}\right)\right\}$ is minimized at $\underline{\theta}=\underline{\theta}_0$.

IV. $\forall\underline{\theta}\neq\underline{\theta}_0$, there exists a neighborhood $\mathcal{V}\left(\underline{\theta}\right)$ such that $\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}_0\right)\right\}<\liminf_{N\to\infty}\inf_{\underline{\widetilde{\theta}}\in\Theta}\left(\widetilde{L}_{Ns}\left(\underline{\widetilde{\theta}}\right)\right)$ a.s.

4.2. Asymptotic normality (AN) of QMLE

To demonstrate the AN of $\widehat{\underline{\theta}}_{Ns}$, the analysis relies on additional assumptions.

A5 $\underline{\theta}_0\in\mathring{\Theta}$, where $\mathring{\Theta}$ represents the interior of the parameter space Θ.

A6 $E\left\{\left\|\underline{\eta}_t\underline{\eta}_t'\right\|^2\right\}<\infty$.

These supplementary assumptions are essential to establishing the required conditions under which the estimator $\widehat{\underline{\theta}}_{Ns}$ exhibits asymptotic normality. The second principal outcome is encapsulated in the following theorem

Theorem 4. *Assuming that* $\left(\underline{X}_t, t\in\mathbb{Z}\right)$ *is formed by the model referenced in* (2.1)*, under the conditions specified in **A0–A6** we obtain*

$$(Ns)^{\frac{1}{2}}\left(\widehat{\underline{\theta}}_{Ns}-\underline{\theta}_0\right)\rightsquigarrow\mathcal{N}\left(\underline{O},K^{-1}UK^{-1}\right)\ \text{as}\ N\to\infty,$$

where the matrix $K=\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\nabla_{\underline{\theta}}^2 l_{st+v}\left(\underline{\theta}_0\right)\right\}$ *is positive-definite and the matrix*

$$U=\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\nabla_{\underline{\theta}}l_{st+v}\left(\underline{\theta}_0\right)\nabla_{\underline{\theta}}'l_{st+v}\left(\underline{\theta}_0\right)\right\}.$$

The demonstration of Theorem 4 relies on the classical technique of a Taylor series expansion of $\nabla_{\underline{\theta}}L_{Ns}\left(\underline{\theta}\right)$ around $\underline{\theta}_0$, and this expansion is expressed as

$$0=(Ns)^{-\frac{1}{2}}\sum_{t=1}^{Ns}\nabla_{\underline{\theta}}\widetilde{l}_t\left(\widehat{\underline{\theta}}_{Ns}\right)=(Ns)^{-\frac{1}{2}}\sum_{t=1}^{Ns}\nabla_{\underline{\theta}}\widetilde{l}_t\left(\underline{\theta}_0\right)+\left((Ns)^{-1}\sum_{t=1}^{Ns}\nabla_{\underline{\theta}}^2\widetilde{l}_t\left(\widetilde{\underline{\theta}}\right)\right)(Ns)^{\frac{1}{2}}\left(\widehat{\underline{\theta}}_{Ns}-\underline{\theta}_0\right).$$

Here, the coordinates of $\widetilde{\underline{\theta}}$ lie between $\widehat{\underline{\theta}}_{Ns}$ and $\underline{\theta}_0$. The theorem will consequently follow straightforwardly. To achieve this, we will establish the subsequent intermediate outcomes, consolidated in the following lemma

Lemma 2. *Assuming **A0–A6**, we obtain*

a. $\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\times}\left\|\nabla_{\underline{\theta}}l_{st+v}\left(\underline{\theta}_0\right)\nabla_{\underline{\theta}}'l_{st+v}\left(\underline{\theta}_0\right)\right\|\right\}<\infty$ *and* $\sum_{v=1}^{s}E\left\{\sup_{\underline{\theta}\in\times}\left\|\nabla_{\underline{\theta}}^2 l_{st+v}\left(\underline{\theta}_0\right)\right\|\right\}<\infty.$

b. *K is invertible and the existence of the matrix U.*

c. $\exists\mathcal{V}\left(\underline{\theta}_0\right)$ *of* $\underline{\theta}_0$ *such that* $\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}\left(\underline{\theta}_0\right)}\left|\frac{\partial^3 l_{st+v}\left(\underline{\theta}\right)}{\partial\theta_i(v)\,\partial\theta_j(v)\,\partial\theta_k(v)}\right|\right\}<\infty$ *for all* $1\leq i,j,k\leq l.$

d. $(Ns)^{-\frac{1}{2}} \sum_{t=1}^{Ns} \nabla_{\underline{\theta}} l_t \left(\underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, U\right)$ *as* $N \to \infty$.

Remark 4.1. *The extension of the existing models to accommodate asymmetries, as highlighted in the symmetric CCC-PGARCH paper by Bibi [5], addresses a crucial aspect of volatility modeling in financial time series data. Asymmetries arise when positive and negative past values of the process have different impacts on the current volatility, thereby affecting the current volatility differently. This phenomenon introduces complexities in characterizing the usual leverage effect, particularly when considering the possibility of positive or negative volatility values. To address these asymmetries, researchers have proposed extensions to existing models, such as asymmetric specifications of GARCH models, and incorporating correlations between volatility process noise and observation series noise in the stochastic volatility framework. The aim of these efforts is to enhance the models' ability to capture the nuanced dynamics of financial markets, including asymmetric responses to market shocks. Therefore, the proposed extension to the class of CCC-PGARCH models fills an important gap in the literature and contributes to advancing our understanding of volatility dynamics in financial time series data.*

Remark 4.2. *In our study, we introduce the concept of an expanded model derived from our primary model, presenting an open problem that invites scientific inquiry and exploration. This extended model, termed the double-switching CCC-(A) GARCH model, combines the principles of a CCC-(A) GARCH model with Markov switching and a periodic sequence (see., Zerari et al. [30] for further ideas). The aim of integrating these elements is to establish a robust analytical framework to comprehend volatility within financial markets. By enhancing the predictive capabilities of volatility models and refining risk management strategies, this approach not only contributes to more effective financial analysis but also lays the groundwork for future research endeavors in the field of financial data analysis.*

Remark 4.3. *The extension of Bibi's theories in our paper offers substantial value-added contributions to this field. While Bibi's work primarily focused on extending symmetric multivariate GARCH models to include periodic features and investigate the asymptotic properties of QML estimation, our paper goes further by incorporating the effects of asymmetry. This extension is crucial as it allows for a more comprehensive understanding of the dynamics of financial time series data, particularly in capturing cross-leverage effects and asymmetric behavior. Additionally, our paper provides asymptotic results for a class of generally nonstationary multivariate GARCH models, a topic that has not been extensively explored in the existing literature. By doing so without imposing moment assumptions on the observed process, we contribute to a deeper understanding of the underlying processes governing financial time series data, thereby enhancing the applicability and robustness of multivariate GARCH models in real-world scenarios.*

5. Simulation results

Herein, we present a simulation study to assess the effectiveness of the $QMLE$, where we simulated 500 independent trajectories, each with lengths of $n = 1000$ and $n = 3000$, from a bivariate periodic stationary *CCC-AGARCH* $(1, 1)$ model, with $\mathcal{N}(\underline{O}_{(2)}, I_{(2)})$ and t_{15} as innovations distributions with a $\underline{O}_{(2)}$mean and a $I_{(2)}$covariance matrix. The parameter $\underline{\theta}_0$ is selected to meet the SPS condition

$\gamma_\Omega < 0$. The Root Mean Square Errors (RMSE) of $\widehat{\underline{\theta}}(v)$, where $v = 1, \ldots, s$, are computed to assess the performance of the estimators.

5.1. *Bivariate standard CCC-AGARCH (1, 1) model*

Our initial Monte Carlo study involves the estimation of a bivariate standard $CCC-AGARCH\,(1,1)$ model, with the parameters associated with this model gathered in vector $\underline{\theta} = \left(\underline{a}_0', Vec'(A_1), Vec'(B_1), Vec'(C_1), m_{21}\right)'$, where C_1 is a diagonal matrix, implying that the volatility of each component is associated with its own lagged value, as well as the lagged values of the squared observations of both components. The results of the simulation, reported in Table 1, are in accordance with the consistency.

Table 1. Results of a simulation study for a bivariate standard *CCC-AGARCH* with a variety of sample sizes n and a number of replications equal to 500.

		$\mathcal{N}(\underline{O}_{(2)}, I_{(2)})$				t_{15}			
	n	1000		3000		1000		3000	
	True value	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
$\underline{a}_0$	1.50	1.4998	0.0002	1.4999	0.0001	1.4997	0.0003	1.4999	0.0001
	1.00	0.9910	0.0090	0.9924	0.0076	0.9901	0.0099	0.9918	0.0082
$Vec(A_1)$	0.15	0.1445	0.0055	0.1449	0.0051	0.1443	0.0057	0.1446	0.0054
	0.10	0.1014	0.0014	0.1010	0.0010	0.1020	0.0020	0.1012	0.0012
	0.05	0.0533	0.0033	0.0524	0.0024	0.0538	0.0038	0.0531	0.0031
	0.24	0.2434	0.0066	0.2432	0.0032	0.2437	0.0037	0.2433	0.0033
$Vec(B_1)$	0.50	0.4978	0.0022	0.4983	0.0017	0.4976	0.0024	0.4979	0.0021
	0.07	0.0710	0.0010	0.0707	0.0007	0.0721	0.0021	0.0709	0.0009
	0.06	0.0618	0.0018	0.0610	0.0010	0.0671	0.0071	0.0625	0.0025
	0.20	0.2031	0.0031	0.2019	0.0019	0.2034	0.0034	0.2028	0.0028
$Vec(C_1)$	0.10	0.0982	0.0018	0.0984	0.0016	0.0981	0.0019	0.0983	0.0017
	0.76	0.7663	0.0063	0.7616	0.0016	0.7686	0.0086	0.7648	0.0048
m_{21}	0.80	0.8057	0.0057	0.8052	0.0052	0.8061	0.0061	0.8059	0.0059

5.2. *Bivariate periodic CCC-AGARCH (1, 1) model*

Our second Monte Carlo study involved the estimation of a bivariate periodic $CCC-AGARCH\,(1,1)$ model, with the parameters associated with this model gathered in vector $\underline{\theta} = \left(\underline{\theta}'(1), \underline{\theta}'(2), m_{21}\right)'$ with $\underline{\theta}(v) = \left(\underline{a}_0'(v), Vec'(A_1(v)), Vec'(B_1(v)), Vec'(C_1(v))\right)'$, is chosen to ensure the SPS condition, where the matrix $C_1(1)$ (resp. $C_1(2)$) is diagonal, implying that the volatility of each component is associated with its own lagged value, as well as the lagged values of the squared observations of both components. The first results of the simulation presented in Table 2 align with the consistency.

From Tables 1 and 2, and as expected, the $QMLE$ estimates for the coefficients of the periodic $CCC-AGARCH\,(1,1)$ model exhibit a decrease in $RMSE$ as the sample size increases. Additionally, it is observed that the estimates based on the normal distribution $\mathcal{N}(0,1)$ generally demonstrate greater efficiency compared to those corresponding to the $t-$distribution with 15 degrees of freedom t_{15}.

Table 2. Results of a simulation study for a bivariate periodic CCC-AGARCH (1, 1) model, with a variety of sample sizes n and a number of replications equal to 500.

			$\mathcal{N}(\underline{O}_{(2)}, I_{(2)})$				t_{15}			
		n	1000		3000		1000		3000	
	$\underline{\theta}$	True value	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
	$\underline{a}_0(1)$	1.50	1.4958	0.0042	1.4966	0.0034	1.4950	0.0050	1.4961	0.0039
		1.50	1.5016	0.0016	1.5012	0.0012	1.5018	0.0018	1.5014	0.0014
	$Vec(A_1(1))$	0.50	0.4930	0.0070	0.4939	0.0061	0.4926	0.0074	0.4938	0.0062
		0.15	0.1407	0.0093	0.1415	0.0085	0.1402	0.0098	0.1414	0.0086
		0.15	0.1523	0.0023	0.1518	0.0018	0.1524	0.0024	0.1517	0.0017
$\underline{\theta}(1)$		0.25	0.2535	0.0035	0.2528	0.0028	0.2537	0.0037	0.2528	0.0028
	$Vec(B_1(1))$	0.20	0.1916	0.0084	0.1976	0.0024	0.1919	0.0081	0.1949	0.0051
		0.05	0.0030	0.0030	0.0013	0.0013	0.0063	0.0063	0.0018	0.0018
		0.05	0.0573	0.0073	0.0545	0.0045	0.0579	0.0079	0.0549	0.0049
		0.35	0.3557	0.0057	0.3523	0.0023	0.3563	0.0063	0.3528	0.0028
	$Vec(C_1(1))$	0.10	0.1019	0.0019	0.1010	0.0010	0.1034	0.0034	0.1017	0.0017
		0.60	0.6440	0.0440	0.6378	0.0378	0.6485	0.0485	0.6391	0.0391
	$\underline{a}_0(2)$	1.50	1.4980	0.0020	1.4985	0.0015	1.4978	0.0022	1.4983	0.0017
		1.50	1.5013	0.0013	1.5011	0.0011	1.5016	0.0016	1.5013	0.0013
	$Vec(A_1(2))$	0.45	0.4393	0.0107	0.4408	0.0092	0.4390	0.0110	0.4407	0.0093
		0.19	0.1833	0.0067	0.1848	0.0052	0.1820	0.0080	0.1847	0.0053
		0.20	0.2085	0.0085	0.2075	0.0075	0.2096	0.0096	0.2078	0.0078
$\underline{\theta}(2)$		0.45	0.4534	0.0034	0.4517	0.0017	0.4548	0.0048	0.4521	0.0021
	$Vec(B_1(2))$	0.25	0.2397	0.0103	0.2416	0.0016	0.2384	0.0116	0.2414	0.0014
		0.10	0.0968	0.0032	0.0996	0.0004	0.0951	0.0049	0.0996	0.0004
		0.10	0.1141	0.0141	0.1126	0.0126	0.1158	0.0158	0.1127	0.0127
		0.35	0.3564	0.0064	0.3542	0.0042	0.3588	0.0088	0.3545	0.0045
	$Vec(C_1(2))$	0.40	0.4062	0.0062	0.4009	0.0009	0.4068	0.0068	0.4015	0.0015
		0.34	0.3478	0.0078	0.3460	0.0060	0.3483	0.0083	0.3473	0.0073
	m_{21}	0.80	0.8168	0.0168	0.8106	0.0106	0.8259	0.0259	0.8154	0.0154

6. Empirical application

This section is devoted to modeling, with periodic $CCC\text{-}AGARCH(1,1)$ model, two datasets, namely the daily time series $(X_{1t})_{t\geq 1}$ for the exchange rate Euro/Algerian dinar (EUR/DZD), and the daily time series $(X_{2t})_{t\geq 1}$ for the exchange rate U.S. dollar/ Algerian dinar (USD/DZD).

We first removed all the days when the market was closed, including holidays and weekends. The observations extended over the period from $2000-01-03$ to $2011-09-29$. In order to do so, we collected the two exchange rates at time t in the vector $\underline{X}'_t = (X_{1t}, X_{2t}) = (EUR/DZD_t, USD/DZD_t)$, and their corresponding log –return series in the vector $\underline{r}'_t = (r_{1t}, r_{2t})$, where $r_{kt} = \log(X_{kt}/X_{kt-1})$, for $k = 1, 2$. The plots of prices (X_{1t}, X_{2t}) and the daily returns series of prices (r_{1t}, r_{2t}), squad and absolute returns are plotted in Figures 1 and 2.

The findings indicate that for the EUR/DZD (resp. USD/DZD), the lowest returns (-0.0233) (resp. -0.0191) and the highest returns (0.0497) (resp. 0.0335). The skewness for two log –return series is positive. Moreover, one of the features which prominently stands out from Tables 3 and 4 is that the kurtosis for two log −returns is much larger than 3, suggesting that models dependent on the Gaussian assumption may not adequately describe the data. Now, we suggest the 5-periodic *CCC-AGARCH* (1, 1) model that permits for the description of the intraweek effect in the daily exchange rate, where the parameters are allowed to vary with the day of the week, $v = 1$ corresponds to Monday, $v = 2$ to Tuesday, and so forth. The estimated parameters of the 5-periodic *CCC*-(*A*)*GARCH* (1, 1) model and their *RMSE* are reported in Tables 5 and 6.

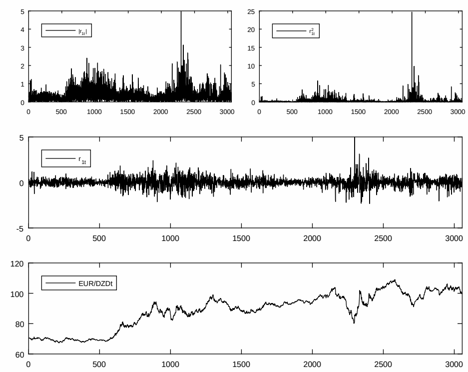

Figure 1. The plots of the price series (X_{1t}), (r_{1t}), $\left(r_{1t}^2\right)$ and $(|r_{1t}|)$.

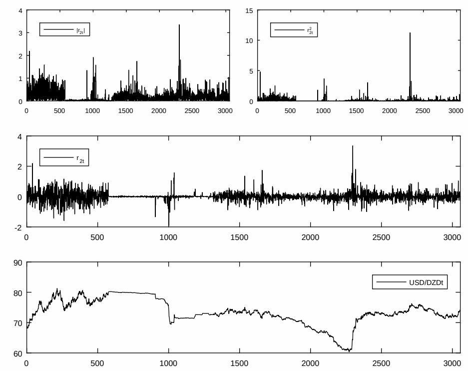

Figure 2. The plots of the price series (X_{2t}), (r_{2t}), $\left(r_{2t}^2\right)$ and $(|r_{2t}|)$.

Several rudimentary descriptive statistics are provided for the two log-return series in Tables 3 and 4.

Table 3. Summary statistics for daily exchange rate series $(X_{1t})_{t\geq1}$, their returns, $(r_{1t})_{t\geq1}$, $(|r_{1t}|)_{t\geq1}$ and $\left(r_{1t}^2\right)_{t\geq1}$.

Series	mean	Std. Dev	Median	Skewness	Kurtosis	Min	Max	Arch(300)	J. Bera	LBtest
(X_{1t})	88.612	11.575	91.099	−0.5181	2.1329	67.204	109.07	100%	2.324×10^2	100%
(r_{1t})	11.816×10^{-5}	50.427×10^{-4}	12.346×10^{-5}	0.3536	8.9678	−0.0233	0.0497	100%	4.597×10^3	24.20%
$(\|r_{1t}\|)$	35.753×10^{-4}	35.575×10^{-4}	0.2554×10^{-2}	2.6956	18.431	0.0000	0.0497	100%	3.401×10^4	100%
$\left(r_{1t}^2\right)$	2.5434×10^{-5}	7.1925×10^{-5}	6.523×10^{-6}	16.103	464.37	0.0000	0.0025	30, 67%	2.723×10^7	100%

Table 4. Summary statistics for daily exchange rate series $(X_{2t})_{t\geq1}$, their returns, $(r_{2t})_{t\geq1}$, $(|r_{2t}|)_{t\geq1}$ and $\left(r_{2t}^2\right)_{t\geq1}$.

Series	mean	Std. Dev	Median	Skewness	Kurtosis	Min	Max	Arch(300)	J. Bera	LBtest
(X_{2t})	73.451	4.2424	73.126	−0.6005	3.7642	60.345	81.282	100%	2.580×10^{2}	100%
(r_{2t})	2.199×10^{-5}	30.251×10^{-4}	-6.431×10^{-6}	0.7162	13.007	−0.0191	0.0335	100%	1.301×10^{4}	43.29%
$(\|r_{2t}\|)$	18.78×10^{-4}	23.713×10^{-4}	10.606×10^{-4}	3.0114	20.804	0.0000	0.0335	100%	4.497×10^{4}	100%
$\left(r_{2t}^2\right)$	9.149×10^{-6}	3.173×10^{-5}	1.1249×10^{-6}	17.678	531.45	0.0000	0.0011	00%	3.571×10^{7}	100%

In Tables 5 and 6, we unveil the outcomes from the QML parameter estimation conducted for the periodic $CCC\text{-}AGARCH(1,1)$ vis-à-vis periodic $CCC\text{-}GARCH(1,1)$ models. The results distinctly showcase that the periodically estimated models exhibit periodic stationarity. Notably, the measurement invariance estimates of the periodic $CCC\text{-}AGARCH(1,1)$ model are significantly smaller than those obtained from the periodic $CCC\text{-}GARCH(1,1)$ model fitted by Bibi (2018, [5]). Furthermore, the empirical coverages of the prediction intervals based on the periodic $CCC\text{-}AGARCH(1,1)$ model tend to closely align with the nominal coverages, unlike those derived from the periodic $CCC\text{-}GARCH(1,1)$ model. These observations suggest that the periodic $CCC\text{-}AGARCH(1,1)$ model, fitted to daily $(EUR/DZD, USD/DZD)$ log returns time series, evinces enhanced precision and superior forecasting efficacy relative to its $CCC\text{-}GARCH(1,1)$ counterpart. This comparative analysis underscores the significant contribution of Patton (2011, [29]) to developing volatility forecasting models, offering improved efficiency and accuracy.

Table 5. QMLE estimate and their RMSE for 5-periodic $CCC\text{-}AGARCH\,(1,1)$ model.

Days	$\widehat{\underline{a}}_0$	$\widehat{A}_1$	$\widehat{B}_1$	$\widehat{C}_1$	$\widehat{m}_{21}$
Monday	$\begin{pmatrix} 0.6021 \\ (0.0001) \\ 1.0649 \\ (0.0012) \end{pmatrix}$	$\begin{pmatrix} 0.0532 & 0.0071 \\ (0.0156) & (0.0092) \\ 0.0081 & 0.5563 \\ (0.0752) & (0.0154) \end{pmatrix}$	$\begin{pmatrix} 0.0838 & 0.0001 \\ (0.0157) & (0.0000) \\ 0.5436 & 1.0154 \\ (0.0047) & (0.0026) \end{pmatrix}$	$diag\left(\begin{pmatrix} 0.8036 \\ (0.0000) \\ 0.0879 \\ (0.0583) \end{pmatrix}'\right)$	0.6438 (0.0325)
Tuesday	$\begin{pmatrix} 2.8023 \\ (0.1725) \\ 0.5018 \\ (0.0034) \end{pmatrix}$	$\begin{pmatrix} 0.0878 & 0.5136 \\ (0.0135) & (0.0468) \\ 0.0632 & 0.5051 \\ (0.0076) & (0.0085) \end{pmatrix}$	$\begin{pmatrix} 0.1266 & 0.2072 \\ (0.0149) & (0.0185) \\ 0.7687 & 0.0327 \\ (0.0624) & (0.0619) \end{pmatrix}$	$diag\left(\begin{pmatrix} 0.4485 \\ (0.0285) \\ 0.1596 \\ (0.0094) \end{pmatrix}'\right)$	
Wednesday	$\begin{pmatrix} 1.5257 \\ (0.2452) \\ 0.3766 \\ (0.0651) \end{pmatrix}$	$\begin{pmatrix} 0.0713 & 0.0711 \\ (0.0138) & (0.0652) \\ 0.1133 & 0.5386 \\ (0.0215) & (0.1123) \end{pmatrix}$	$\begin{pmatrix} 0.1264 & 0.1864 \\ (0.0096) & (0.0207) \\ 0.3381 & 0.6061 \\ (0.0234) & (0.0175) \end{pmatrix}$	$diag\left(\begin{pmatrix} 0.6931 \\ (0.0355) \\ 0.2743 \\ (0.0739) \end{pmatrix}'\right)$	
Thursday	$\begin{pmatrix} 1.3665 \\ (0.5006) \\ 0.0000 \\ (0.0014) \end{pmatrix}$	$\begin{pmatrix} 0.0506 & 0.2765 \\ (0.0153) & (0.0319) \\ 0.0597 & 0.4648 \\ (0.0079) & (0.1140) \end{pmatrix}$	$\begin{pmatrix} 0.0672 & 0.1151 \\ (0.0114) & (0.0145) \\ 0.3216 & 0.9808 \\ (0.0233) & (0.0526) \end{pmatrix}$	$diag\left(\begin{pmatrix} 0.7134 \\ (0.0814) \\ 0.0391 \\ (0.0899) \end{pmatrix}'\right)$	
Friday	$\begin{pmatrix} 0.4513 \\ (0.0000) \\ 0.2405 \\ (0.0084) \end{pmatrix}$	$\begin{pmatrix} 0.0549 & 0.0611 \\ (0.0169) & (0.0528) \\ 0.0739 & 0.2680 \\ (0.0055) & (0.0438) \end{pmatrix}$	$\begin{pmatrix} 0.1169 & 0.4327 \\ (0.0143) & (0.0352) \\ 0.5524 & 0.4693 \\ (0.0647) & (0.0462) \end{pmatrix}$	$diag\left(\begin{pmatrix} 0.9261 \\ (0.0000) \\ 0.2717 \\ (0.0452) \end{pmatrix}'\right)$	

Table 6. QMLE estimate and their RMSE for 5- periodic $CCC\text{-}GARCH\,(1,1)$ model.

Days	$\widehat{\underline{a}}_0$	$\widehat{A}_1$	$\widehat{B}_1$	$\widehat{C}_1$	$\widehat{m}_{21}$
Monday	$\begin{pmatrix} 0.8139 \\ (0.0065) \\ 1.0927 \\ (0.0096) \end{pmatrix}$	$\begin{pmatrix} 0.0758 & 0.0360 \\ (0.0173) & (0.0128) \\ 0.0213 & 0.5801 \\ (0.0778) & (0.0179) \end{pmatrix}$	–	$diag\left(\begin{pmatrix} 0.8158 \\ (0.0070) \\ 0.1005 \\ (0.0637) \end{pmatrix}'\right)$	0.8162 (0.0578)
Tuesday	$\begin{pmatrix} 3.1603 \\ (0.2136) \\ 0.5601 \\ (0.0111) \end{pmatrix}$	$\begin{pmatrix} 0.0905 & 0.5419 \\ (0.0153) & (0.0497) \\ 0.0684 & 0.5148 \\ (0.0148) & (0.0099) \end{pmatrix}$	–	$diag\left(\begin{pmatrix} 0.4042 \\ (0.0323) \\ 0.1388 \\ (0.0139) \end{pmatrix}'\right)$	
Wednesday	$\begin{pmatrix} 1.7182 \\ (0.2907) \\ 0.4154 \\ (0.0749) \end{pmatrix}$	$\begin{pmatrix} 0.0720 & 0.0843 \\ (0.0142) & (0.0810) \\ 0.1352 & 0.5395 \\ (0.0283) & (0.1204) \end{pmatrix}$	–	$diag\left(\begin{pmatrix} 0.6475 \\ (0.0383) \\ 0.2658 \\ (0.0721) \end{pmatrix}'\right)$	
Thursday	$\begin{pmatrix} 1.4214 \\ (0.5815) \\ 0.0053 \\ (0.0081) \end{pmatrix}$	$\begin{pmatrix} 0.0674 & 0.3020 \\ (0.0298) & (0.0526) \\ 0.0708 & 0.4969 \\ (0.0156) & (0.1282) \end{pmatrix}$	–	$diag\left(\begin{pmatrix} 0.6959 \\ (0.0802) \\ 0.0407 \\ (0.0926) \end{pmatrix}'\right)$	
Friday	$\begin{pmatrix} 0.5244 \\ (0.0099) \\ 0.2792 \\ (0.0135) \end{pmatrix}$	$\begin{pmatrix} 0.0576 & 0.0664 \\ (0.0185) & (0.0551) \\ 0.0763 & 0.2715 \\ (0.0074) & (0.0460) \end{pmatrix}$	–	$diag\left(\begin{pmatrix} 0.9726 \\ (0.0033) \\ 0.2879 \\ (0.0496) \end{pmatrix}'\right)$	

7. Conclusions

This research marks a significant advancement in asymmetric multivariate stochastic difference volatility modeling. Through meticulous analysis, we have successfully extended and enhanced various probabilistic and statistical properties that have been a focal point in the prior literature. A key achievement of this study is the establishment of periodic stationarity and periodic ergodicity within the model, underscoring its dynamic and adaptable nature. Furthermore, our in-depth exploration of the Quasi-Maximum Likelihood Estimator (QMLE) reveals its robust consistency and asymptotic normality. This rigorous examination not only fortifies the theoretical foundation of QMLE but also underscores its reliability and efficiency in practical scenarios. The analytical insights gained here significantly bolster the estimator's credibility in complex stochastic modeling contexts.

The practical implications of our findings are illustrated through a series of applications, demonstrating the model's versatility and relevance in real-world situations. These applications validate our theoretical results and showcase the model's potential in addressing a range of challenging problems in stochastic volatility modeling.

In summary, this study not only enriches the theoretical framework of stochastic volatility modeling, but also paves the way for future investigations. By pushing the boundaries of current understanding and application, it opens new avenues for research and development in this ever-evolving domain. The implications of this work extend beyond the immediate scope of this study, offering valuable insights and tools for both researchers and practitioners in the field of financial econometrics and beyond.

Author contributions

Omar Alzeley: Methodology, Validation, Formal analysis, Investigation, Visualization, Writing-review and Editing; Ahmed Ghezal: Software, Validation, Resources, Data curation, Writing-original draft preparation, Supervision, Project administration. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix

Proof of Theorem 1. The proof of the sufficient condition is analogous to that provided by Bougerol and Picard [10, 11], since $E\left\{\underline{\widetilde{e}}_t^{\otimes 2}\right\} < \infty$, where

$$\underline{\widetilde{e}}_t := \left(\left(\underline{\widetilde{\eta}}_{st}^{+2}\right)', \ldots, \left(\underline{\widetilde{\eta}}_{st+s-1}^{+2}\right)', \left(\underline{\widetilde{\eta}}_{st}^{-2}\right)', \ldots, \left(\underline{\widetilde{\eta}}_{st+s-1}^{-2}\right)'\right)',$$

hence $E\{\log^+ \|\Omega_t\|\} < \infty$. By employing Cauchy's root test when $\gamma_\Omega < 0$, it follows that the series $\underline{\Gamma}_t$ (2) converges almost surely for all t. Consequently, an SPS solution to the model (2.1) is attained, denoted as $\underline{X}_t = \left\{diag\left(\underline{\Delta}_{2q+1,t}\right)\right\}^{1/2} M^{1/2}\underline{\eta}_t$, where $\underline{\Delta}_{2q+1,t}$ is the subvector of $\underline{\Delta}_t$ corresponding to the $(2q+1)^{th}$ element. This solution is thus nonanticipative and PE. To prove the uniqueness, let $(\underline{\widetilde{\Gamma}}_t)$ be another non-negative and SS solution of (2.4). For all positive integers $n \geq 0$, $\underline{\Gamma}_t = \left\{\prod_{j=0}^{n} \Omega_{t-j}\right\} \underline{\Gamma}_{t-n-1} + \underline{\widetilde{\Gamma}}_t(n)$, where $\underline{\widetilde{\Gamma}}_t(n) := \underline{\Xi}_t + \sum_{i=0}^{n} \left\{\prod_{j=0}^{i} \Omega_{t-j}\right\} \underline{\Xi}_{t-i-1}$. Then $\left\|\underline{\Gamma}_t - \underline{\widetilde{\Gamma}}_t\right\| \leq \left\|\prod_{j=0}^{n} \Omega_{t-j}\right\| \left\|\underline{\Gamma}_{t-n-1}\right\| + \left\|\underline{\widetilde{\Gamma}}_t(n) - \underline{\widetilde{\Gamma}}_t\right\|$. The second term on the right-hand side converges to 0 almost surely as n approaches infinity. Moreover, we have $\left\|\prod_{j=0}^{n} \Omega_{t-j}\right\| \longrightarrow 0$ with a probability of 1 when $n \longrightarrow \infty$ and $\left\|\prod_{j=0}^{n} \Omega_{t-j}\right\| \left\|\underline{\Gamma}_{t-n-1}\right\| \longrightarrow 0$ in probability as $n \longrightarrow \infty$. We have shown that $\underline{\Gamma}_t = \underline{\widetilde{\Gamma}}_t$ for any t, a.s. Now, we give a simple direct proof. Since the necessary part is easy, see Bougerol and Picard [10], we prove only $\left\|\prod_{j=0}^{n} \Delta_{t-j}\right\| \longrightarrow 0$ as $n \longrightarrow \infty$. For this, it will suffice to show that

$$\left\{\prod_{j=0}^{n} \Delta_{t-j}\right\} \varphi_k \longrightarrow O \text{ a.s. when } n \longrightarrow \infty, \tag{A.1}$$

for all $k = 1, \ldots, 2q+p$, where $\varphi_k = \underline{\psi}_k \otimes I_{(r)}$ and $\underline{\psi}_k$ represents the k^{th} element of the canonical basis of $\mathbb{R}^{2q+p}$. Indeed, assume that model (2.3) admits a causal, SPS solution $\left(\underline{\Delta}_t\right)$. We exploit the non-negativity of the coefficients of Δ_t and $\underline{\Delta}_t$, from which it follows that for all $n \geq 2$, $\underline{\Delta}_t \geq \underline{\Pi}_t + \sum_{k=0}^{n}\left\{\prod_{j=0}^{k}\Delta_{t-j}\right\}\underline{\Pi}_{t-k-1}$ $\forall n$ implies that $\left\{\prod_{j=0}^{n}\Delta_{t-j}\right\}\underline{\Pi}_{t-n-1} \longrightarrow \underline{O}$ a.s. as $n \longrightarrow \infty$. Using the relationship

$$\underline{\Pi}_{t-n-1} = \varphi_1 G^{+}_{t-n-1}\underline{a}_0\left(t-n-1\right) + \varphi_{q+1}G^{-}_{t-n-1}\underline{a}_0\left(t-n-1\right) + \varphi_{2q+1}\underline{a}_0\left(t-n-1\right),$$

we have

$$\left\{\prod_{j=0}^{n}\Delta_{t-j}\right\}\varphi_1 G^{+}_{t-n-1}\underline{a}_0\left(t-n-1\right) \longrightarrow \underline{O}, \left\{\prod_{j=0}^{n}\Delta_{t-j}\right\}\varphi_{q+1}G^{-}_{t-n-1}\underline{a}_0\left(t-n-1\right) \longrightarrow \underline{O}$$

and

$$\left\{\prod_{j=0}^{n}\Delta_{t-j}\right\}\varphi_{2q+1}\underline{a}_0\left(t-n-1\right) \longrightarrow \underline{O}\ a.s.\ as\ n \longrightarrow \infty.$$

Since the components of $\underline{a}_0\left(t-n-1\right)$ are strictly positive, $(A.1)$ thus holds for $n = 2q+1$. Using

$$\Delta_{t-n}\varphi_{2q+k} = G^{+}_{t-n}C_k\left(t-n\right)\varphi_1 + G^{-}_{t-n}C_k\left(t-n\right)\varphi_{q+1} + C_k\left(t-n\right)\varphi_{2q+1} + \varphi_{2q+k+1}, k = 1, ..., p$$

with the convention that

$$\varphi_{2q+p+1} = O, for k = 1 we obtain O = \lim_{n\longrightarrow\infty}\left\{\prod_{j=0}^{n}\Delta_{t-j}\right\}\varphi_{2q+1} \geq \lim_{n\longrightarrow\infty}\left\{\prod_{j=0}^{n-1}\Delta_{t-j}\right\}\varphi_{2q+2} \geq O.$$

In this context, the inequalities are considered componentwise. Moreover, $(A.1)$ is valid for $k = q+2$, and, by induction, for $k = 2q+l$, $l = 1, \ldots, p$. Moreover, since

$$\begin{aligned}\Delta_{t-n}\varphi_{2q} &= G^{+}_{t-n}B_q\left(t-n\right)\varphi_1 + G^{-}_{t-n}B_q\left(t-n\right)\varphi_{q+1} + B_q\left(t-n\right)\varphi_{2q+1}(resp. \Delta_{t-n}\varphi_q \\ &= G^{+}_{t-n}A_q\left(t-n\right)\varphi_1 + G^{-}_{t-n}A_q\left(t-n\right)\varphi_{q+1} + A_q\left(t-n\right)\varphi_{2q+1}),\end{aligned}$$

$(A.1)$ holds for $k = 2q$ (resp. $k = q$), we reach a similar finding for the remaining values of k through an upward recursion. □

Proof of Corollary 1. Since Δ_t is non-negative, given that all its elements are non-negative, it is evident that γ_Ω corresponding to the nonrandom sequence of matrices $\left(\Omega^0_t\right)$ attained through substitution of the matrices $A_j\left(t\right)$ and $B_j\left(t\right)$ with a null matrix in Δ_t. In a different sense, it implies $\log\rho\left(\prod_{v=0}^{s-1}C_{s-v}\right) \leq \gamma_\Omega < 0$. Consequently, the conclusion stated in the corollary is affirmed. □

Proof of Proposition 1. To prove **(a)**. By Kingman's subadditive ergodic theorem, we get

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\log\left\|\prod_{k=0}^{t-1}\Omega_{t-k}\right\| \leq \log\left(\rho\left(E\left\{\Omega_1\right\}\right)\right),$$

which completes the proof of **(a)**. We now prove **(b)**. By Jensen's inequality, we get

$$\gamma_\Omega := \inf_{t\geq 1}\left\{E\left\{\log\left\|\prod_{i=0}^{t-1}\Omega_{t-i}\right\|^{1/t}\right\}\right\} \leq E\left\{\log\left\|\prod_{i=0}^{t-1}\Omega_{t-i}\right\|\right\} \leq \log E\left\{\left\|\prod_{i=0}^{t-1}\Omega_{t-i}\right\|\right\}a.s.$$

Furthermore, the rest is analogous to that of Lemma 2.3 of Berkes et al. [4] in the univariate case, and [5] in the multivariate case that the SPS solution conforms to $E\left\{\left\|\underline{\Delta}_t\right\|^{\varsigma}\right\} < \infty$ for some $\varsigma > 0$. The result is a natural consequence of $\left\|\underline{X}_t\right\| \leq \left\|\underline{\Delta}_t\right\|$ and $\left\|\underline{\sigma}_t\right\| \leq \left\|\underline{\Delta}_t\right\|$. □

Proof of Theorem 2. The arguments in the proof closely resemble those found in [5]. □

Proof of Lemma 1. Rewrite (2.2) in a vector form as

$$\underline{\Sigma}_t\left(\underline{\theta}\right) = C_t\underline{\Sigma}_{t-1}\left(\underline{\theta}\right) + \underline{\pi}_t, \tag{A.2}$$

where

$$\underline{\Sigma}_t'\left(\underline{\theta}\right) = \left(\underline{\sigma}_t'\left(\underline{\theta}\right), ..., \underline{\sigma}_{t-p+1}'\left(\underline{\theta}\right)\right)$$

and

$$\underline{\pi}_t' = \left(\left(\underline{a}_0\left(t\right) + \sum_{i=1}^{q}\left(A_i\left(t\right)\underline{X}_{t-i}^{+} + B_i\left(t\right)\underline{X}_{t-i}^{-}\right)\right)', \underline{O}_{(r)}', ..., \underline{O}_{(r)}'\right).$$

Proof of (**I**)*: Initial values are asymptotically irrelevant.* In view of the last part imposed in assumption **A2** and Corollary 1, we have

$$\sup_{\underline{\theta}\in\Theta}\rho\left(\prod_{v=0}^{s-1}C_{s-v}\right) < 1. \tag{A.3}$$

Iteratively using (A.2), as in the univariate case, we have

$$\underline{\Sigma}_t\left(\underline{\theta}\right) = \left\{\prod_{j=0}^{t}C_{t-j}\right\}\underline{\Sigma}_0\left(\underline{\theta}\right) + \sum_{i=0}^{t-1}\left\{\prod_{j=0}^{i-1}C_{t-j}\right\}\underline{\pi}_{t-i}, t \in \mathbb{Z}.$$

If we represent the vectors attained from certain initial values as $\widetilde{\underline{\Sigma}}_t\left(\underline{\theta}\right)$ and $\widetilde{\underline{\pi}}_t$, then we obtain

$$\widetilde{\underline{\Sigma}}_t\left(\underline{\theta}\right) = \left\{\prod_{j=0}^{t}C_{t-j}\right\}\widetilde{\underline{\Sigma}}_0\left(\underline{\theta}\right) + \sum_{i=0}^{t-q-1}\left\{\prod_{j=0}^{i-1}C_{t-j}\right\}\underline{\pi}_{t-i} + \sum_{i=t-q}^{t-1}\left\{\prod_{j=0}^{i-1}C_{t-j}\right\}\widetilde{\underline{\pi}}_{t-i}, t \in \mathbb{Z}.$$

Hence, we deduce that a.s.

$$\sup_{\underline{\theta}\in\Theta}\left\|\widetilde{\underline{\Sigma}}_t\left(\underline{\theta}\right) - \underline{\Sigma}_t\left(\underline{\theta}\right)\right\| \leq \lambda\omega^t, \forall t,$$

where $\lambda > 0$ and $\omega \in (0, 1)$, which implies that $\sup_{\underline{\theta}\in\Theta}\left\|\widetilde{V}_t\left(\underline{\theta}\right) - V\left(\underline{\theta}\right)\right\| \leq \lambda\omega^t\ \forall t$. Since $H_t^{-1}\left(\underline{\theta}\right) \leq \left\{\min_j \sigma_{t,j}^2\right\}^{-1}$, we have

$$\sup_{\underline{\theta}\in\Theta}\left\|\widetilde{V}_t^{-1}\left(\underline{\theta}\right)\right\| \leq \sup_{\underline{\theta}\in\Theta}\left\|\widetilde{H}_t^{-1}\left(\underline{\theta}\right)\right\|^2\left\|M^{-1}\right\| \leq \sup_{\underline{\theta}\in\Theta}\left\{\min_j a_{0,j}\left(t\right)\right\}^{-2}\left\|M^{-1}\right\| \leq \lambda,$$

Similarly, we obtain $\sup_{\underline{\theta}\in\Theta}\left\|V_t^{-1}\left(\underline{\theta}\right)\right\| \leq \lambda$.

Now,

$$\sup_{\underline{\theta}\in\Theta}\left|\left(\widetilde{L}_{Ns} - L_{Ns}\right)\left(\underline{\theta}\right)\right| \leq \frac{1}{N}\sum_{t=1}^{Ns}\sup_{\underline{\theta}\in\Theta}\left|\underline{X}_t'\left(\widetilde{V}_t^{-1} - V_t^{-1}\right)\left(\underline{\theta}\right)\underline{X}_t\right| + \frac{1}{N}\sum_{t=1}^{Ns}\sup_{\underline{\theta}\in\Theta}\left|\log\left(\left|\widetilde{V}_t\left(\underline{\theta}\right)\right|\Big/\left|V_t\left(\underline{\theta}\right)\right|\right)\right|.$$

The first sum can be written as

$$\begin{aligned}\frac{1}{N}\sum_{t=1}^{Ns}\sup_{\underline{\theta}\in\Theta}\left|\underline{X}_t'\left(\widetilde{V}_t^{-1}\left(\widetilde{V}_t-V_t\right)V_t^{-1}\right)\left(\underline{\theta}\right)\underline{X}_t\right| &= \frac{1}{N}\sum_{t=1}^{Ns}\sup_{\underline{\theta}\in\Theta}\left|Tr\left(\left(\widetilde{V}_t^{-1}\left(\widetilde{V}_t-V_t\right)V_t^{-1}\right)\left(\underline{\theta}\right)\underline{X}_t\underline{X}_t'\right)\right|\\ &\leq \frac{\lambda}{N}\sum_{t=1}^{Ns}\sup_{\underline{\theta}\in\Theta}\left\|\widetilde{V}_t^{-1}\left(\underline{\theta}\right)\right\|\left\|\left(\widetilde{V}_t-V_t\right)\left(\underline{\theta}\right)\right\|\left\|V_t^{-1}\left(\underline{\theta}\right)\right\|\left\|\underline{X}_t\underline{X}_t'\right\|\\ &\leq \frac{\lambda}{N}\sum_{t=1}^{Ns}\omega^t\left\|\underline{X}_t\underline{X}_t'\right\|\longrightarrow 0 \text{ as } N\longrightarrow\infty,\end{aligned}$$

Using the Cesaro lemma, the Borel–Cantelli lemma, the Markov inequality, and by applying Proposition 1, we have

$$\sum_{t\geq 1}P\left(\omega^t\underline{X}_t\underline{X}_t'>\varepsilon\right)\leq\sum_{t\geq 1}\frac{\omega^{t\tau}E\left\{\left\|\underline{X}_t\right\|^{\tau}\right\}}{\varepsilon^{\tau}}<\infty,$$

which implies $\omega^t\underline{X}_t'\underline{X}_t\xrightarrow{a.s.}0$. Now, using the inequality for $y+1\geq 0$, $y\geq\log(1+y)$, we have

$$\log\left(\left|V_t\left(\underline{\theta}\right)\right|\Big/\left|\widetilde{V}_t\left(\underline{\theta}\right)\right|\right)=\log\left|I_{(r)}+\left(\left(V_t-\widetilde{V}_t\right)\widetilde{V}_t^{-1}\right)\left(\underline{\theta}\right)\right|\leq r\left\|\left(V_t-\widetilde{V}_t\right)\left(\underline{\theta}\right)\right\|\left\|\widetilde{V}_t^{-1}\left(\underline{\theta}\right)\right\|,$$

and, by symmetry,

$$\log\left|\widetilde{V}_t\left(\underline{\theta}\right)\right|-\log\left|V_t\left(\underline{\theta}\right)\right|\leq r\left\|V_t\left(\underline{\theta}\right)-\widetilde{V}_t\left(\underline{\theta}\right)\right\|\left\|V_t^{-1}\left(\underline{\theta}\right)\right\|,$$

so we deduce that the second sum tends to 0. Therefore, we have demonstrated that

$$\sup_{\underline{\theta}\in\Theta}\left|\left(\widetilde{L}_{Ns}-L_{Ns}\right)\left(\underline{\theta}\right)\right|\underset{N\longrightarrow\infty}{\overset{a.s.}{\longrightarrow}}0.$$

□

Proof of (**II**)*: Parameter identifiability.* Suppose that, for some $\underline{\theta}\neq\underline{\theta}_0$, $\underline{\sigma}_t\left(\underline{\theta}\right)=\underline{\sigma}_t\left(\underline{\theta}_0\right)$ a.s. and $M=M_0$.

Consequently, it immediately follows that $\underline{m}=\underline{m}_0$, and by leveraging the invertibility of the polynomial $\left(C_{0,v}(z)\right)_{1\leq v\leq s}$ under the assumption **A2**, then we have a.s. $\forall v=1,...,s$,

$$\begin{aligned}C_{0,v}^{-1}(L)\,\underline{a}_{0,0}(v)-C_v^{-1}(L)\,\underline{a}_0(v) &= \left(C_v^{-1}(L)\,\mathcal{A}_v(L)-C_{0,v}^{-1}(L)\,\mathcal{A}_{0,v}(L)\right)\underline{X}_{ns+v}^{+}\\ &\quad+\left(C_v^{-1}(L)\,\mathcal{B}_v(L)-C_{0,v}^{-1}(L)\,\mathcal{B}_{0,v}(L)\right)\underline{X}_{ns+v}^{-}\\ &:=\mathcal{P}_v(L)\,\underline{X}_{ns+v}^{+}+\mathcal{Q}_v(L)\,\underline{X}_{ns+v}^{-},\end{aligned}$$

where L denotes the lag operator,

$$\mathcal{P}_v(L)=\sum\nolimits_{k\geq 1}P_k(v)\,L^k,\mathcal{Q}_v(L)=\sum\nolimits_{k\geq 1}Q_k(v)\,L^k$$

with

$$\mathcal{P}_v(0)=P_0(v)=\mathcal{Q}_v(0)=Q_0(v)=O_{(r)},$$

and isolating the terms that are functions of $\underline{\eta}_{t-1}$,

$$P_1(t)\,G^{+}_{t-1}\underline{\sigma}_{t-1} + Q_1(t)\,G^{-}_{t-1}\underline{\sigma}_{t-1} \in \sigma\left(\underline{\eta}_{t-2}, \underline{\eta}_{t-3}, ...\right).$$

Since $P_1(t)\,G^{+}_{t-1} + Q_1(t)\,G^{-}_{t-1}$ is independent of this $\sigma-$field and $\underline{\sigma}_{t-1}$ and $\underline{\sigma}_{t-1} > 0$, we have

$$P_1(t)\,G^{+}_{t-1} + Q_1(t)\,G^{-}_{t-1} = T$$

for some constant matrix T, the matrix equality for the element (k,l) writes $(P_1(t))(k,l)\widetilde{\underline{\eta}}^{+2}_{t-1}(l) + (Q_1(t))(k,l)\widetilde{\underline{\eta}}^{-2}_{t-1}(l) = T(k,l)$. If $(P_1(t))(k,l)(Q_1(t))(k,l) \neq 0$. The contradiction with Assumption **A3** arises from the fact that $\widetilde{\underline{\eta}}_{t-1}(l)$ takes at most two distinct values.

If $(P_1(t))(k,l) \neq 0$ and $(Q_1(t))(k,l) = 0$, thus $(P_1(t))(k,l)\widetilde{\underline{\eta}}^{+2}_{t-1}(l) = T(k,l)$, which entails $T(k,l) = 0$, because $P\left(\widetilde{\underline{\eta}}^{-}_{t-1}(l) > 0\right) \neq 0$, and thus $\widetilde{\underline{\eta}}^{-}_{t-1}(l) \overset{a.s.}{=} 0$, which contradicts **A3**. Therefore, we have $P_1(t) = Q_1(t) = 0$. Similarly in this way, we show that $\mathcal{P}_v(L) = \mathcal{Q}_v(L) = 0$ for all $v = 1, \ldots, s$. Therefore, in view of **A4**, we have $\underline{a}_0(v) = \underline{a}_{0,0}(v)$, $\underline{\alpha}(v) = \underline{\alpha}_0(v)$, $\underline{\beta}(v) = \underline{\beta}_0(v)$ and $\underline{\gamma}(v) = \underline{\gamma}_0(v)$ for all $v = 1, ..., s$. Hence $\underline{\theta} = \underline{\theta}_0$. Thus, we have successfully established (**II**). □

Proof of (**III**)*: The limit criterion attains its minimum at* $\underline{\theta}_0$. At $\underline{\theta}_0$, Jensen's inequality and Proposition 1 entail that

$$\begin{aligned}
\sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log\left|V_{st+v}\left(\underline{\theta}_0\right)\right|\right\} &= \frac{r}{\tau}\sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log\left|V_{st+v}\left(\underline{\theta}_0\right)\right|^{\frac{\tau}{r}}\right\} \\
&\leq \frac{r}{\tau}\sum\nolimits_{v=1}^{s} \log E_{\underline{\theta}_0}\left\{\left\|V_{st+v}\left(\underline{\theta}_0\right)\right\|^{\tau}\right\} \\
&\leq \frac{r}{\tau}\sum\nolimits_{v=1}^{s} \log E_{\underline{\theta}_0}\left\{\|M\|^{\tau}\left\|H_{st+v}\left(\underline{\theta}_0\right)\right\|^{2\tau}\right\} \\
&\leq \lambda + \frac{r}{\tau}\sum\nolimits_{v=1}^{s} \log E_{\underline{\theta}_0}\left\{\left\|H_{st+v}\left(\underline{\theta}_0\right)\right\|^{2\tau}\right\} \\
&\leq \lambda + \frac{r}{\tau}\sum\nolimits_{v=1}^{s} \log E_{\underline{\theta}_0}\left\{\max_i\left\{\sigma_{i\,st+v}\left(\underline{\theta}_0\right)\right\}^{\tau}\right\} \\
&\leq \lambda + \frac{r}{\tau}\sum\nolimits_{v=1}^{s} \log E_{\underline{\theta}_0}\left\{\left\|\underline{\sigma}_{st+v}\left(\underline{\theta}_0\right)\right\|^{\tau}\right\} \\
&< \infty.
\end{aligned}$$

As a result,

$$\begin{aligned}
\sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}_0\right)\right\} &= \sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\underline{\eta}'_{st+v} V^{1/2\prime}_{st+v}\left(\underline{\theta}_0\right) V^{-1}_{st+v}\left(\underline{\theta}_0\right) V^{1/2}_{st+v}\left(\underline{\theta}_0\right)\underline{\eta}_{st+v} + \log\left|V_{st+v}\left(\underline{\theta}_0\right)\right|\right\} \\
&= r + \sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log\left|V_{st+v}\left(\underline{\theta}_0\right)\right|\right\} < \infty.
\end{aligned}$$

We now have

$$\sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}\right) - l_{st+v}\left(\underline{\theta}_0\right)\right\} = \sum\nolimits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log\frac{\left|V_{st+v}\left(\underline{\theta}\right)\right|}{\left|V_{st+v}\left(\underline{\theta}_0\right)\right|}\right\}$$

$$+\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\underline{\eta}'_{st+v}\left(V^{1/2}_{st+v}\left(\underline{\theta}_0\right)V^{-1}_{st+v}\left(\underline{\theta}\right)V^{1/2}_{st+v}\left(\underline{\theta}_0\right)-I_{(r)}\right)\underline{\eta}_{st+v}\right\}$$
$$=\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log\left|V_{st+v}\left(\underline{\theta}\right)\right|\left|V^{-1}_{st+v}\left(\underline{\theta}_0\right)\right|\right\}$$
$$+\sum_{v=1}^{s} Tr\left(E_{\underline{\theta}_0}\left\{\left(V^{1/2}_{st+v}\left(\underline{\theta}_0\right)V^{-1}_{st+v}\left(\underline{\theta}\right)V^{1/2}_{st+v}\left(\underline{\theta}_0\right)-I_{(r)}\right)E\left\{\underline{\eta}_{st+v}\underline{\eta}'_{st+v}\right\}\right\}\right)$$
$$=\sum_{v=1}^{s}\left(E_{\underline{\theta}_0}\left\{\log\left|V_{st+v}\left(\underline{\theta}\right)\right|\left|V^{-1}_{st+v}\left(\underline{\theta}_0\right)\right|\right\}+E_{\underline{\theta}_0}\left\{Tr\left(V_{st+v}\left(\underline{\theta}_0\right)V^{-1}_{st+v}\left(\underline{\theta}\right)-I_{(r)}\right)\right\}\right)$$
$$=\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\sum_{j=1}^{r}\left(\delta_j(v)-1-\log\delta_j(v)\right)\right\},$$

where $\delta_j(v)$ is the positive eigenvalues of $V_{st+v}\left(\underline{\theta}\right)V^{-1}_{st+v}\left(\underline{\theta}_0\right)$, because $y-1\geq\log y$, $\forall y>0$, where equality holds iff $y=1$, thus we obtain $\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}\right)\right\}-\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}_0\right)\right\}\geq 0$, and hence the inequality is strict unless if, for all j and v, $\delta_j(v)\overset{a.s.}{=}1$, meaning that $V_{st+v}\left(\underline{\theta}\right)\overset{a.s.}{=}V_{st+v}\left(\underline{\theta}_0\right)$. Therefore, we have $\underline{\sigma}_{st+v}\left(\underline{\theta}\right)\overset{a.s.}{=}\underline{\sigma}_{st+v}\left(\underline{\theta}_0\right)$, $v=1,...,s$ and $M=M_0$ and thus $\underline{\theta}=\underline{\theta}_0$. □

Proof. To demonstrate (**IV**), $\forall\underline{\theta}\in\Theta$ and every integer k, let $\mathcal{V}_k\left(\underline{\theta}\right)$ be an open sphere of centre $\underline{\theta}$ and radius $\frac{1}{k}$. Utilizing (**I**), we obtain

$$\underline{\lim}_{N\to\infty}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}\left(\widetilde{L}_{Ns}\left(\underline{\theta}^*\right)\right)\geq\underline{\lim}_{N\to\infty}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}\left(L_{Ns}\left(\underline{\theta}^*\right)\right)-\overline{\lim}_{N\to\infty}\sup_{\underline{\theta}\in\Theta}\left|\left(L_{Ns}-\widetilde{L}_{Ns}\right)\left(\underline{\theta}\right)\right|$$
$$\geq\underline{\lim}_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}l_{st+v}\left(\underline{\theta}^*\right).$$

By applying the ergodic theorem to the i.i.d. sequence $\left(\sum_{v=1}^{s}l_{st+v}\left(\underline{\theta}^*\right)\right)_t$, where $E\left\{\sum_{v=1}^{s}l_{st+v}\left(\underline{\theta}^*\right)\right\}\in\mathbb{R}\cup\{\infty\}$ (see., [8]), we have

$$\lim_{N\to\infty}\inf\frac{1}{N}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}l_{st+v}\left(\underline{\theta}^*\right)=\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}l_{st+v}\left(\underline{\theta}^*\right)\right\}$$

and according to the Beppo-Levi theorem (see., [8]), we get

$$\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}l_{st+v}\left(\underline{\theta}^*\right)\right\}\longrightarrow\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}\right)\right\}$$

as $k\to\infty$. This concludes the proof of the lemma. □

Proof of Theorem 3. The proof is now complete by an argument of compactness of Θ and Lemma 1. Now, for all neighborhood $\mathcal{V}\left(\underline{\theta}_0\right)$, we obtain

$$\overline{\lim}_{N\to\infty}\inf_{\underline{\theta}^*\in\mathcal{V}(\underline{\theta}_0)}\left(\widetilde{L}_{Ns}\left(\underline{\theta}^*\right)\right)\leq\lim_{N\to\infty}\widetilde{L}_{Ns}\left(\underline{\theta}_0\right)=\lim_{N\to\infty}L_{Ns}\left(\underline{\theta}_0\right)=\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{l_{st+v}\left(\underline{\theta}_0\right)\right\}.$$

The compact set Θ is reconstructed as the union of a neighborhood $\mathcal{V}\left(\underline{\theta}_0\right)$ and a set of neighborhoods $\mathcal{V}\left(\underline{\theta}\right)$, where $\underline{\theta} \in \Theta \diagdown \mathcal{V}\left(\underline{\theta}_0\right)$. Consequently, there exists a finite sub-covering of Θ by $\mathcal{V}\left(\underline{\theta}_0\right), \mathcal{V}\left(\underline{\theta}_1\right), ..., \mathcal{V}\left(\underline{\theta}_k\right)$ such that

$$\inf_{\underline{\theta}^* \in \mathcal{V}\left(\underline{\theta}_0\right)} \left(\widetilde{L}_{Ns}\left(\underline{\theta}^*\right)\right) = \min_{j \in \{1,...,k\}} \inf_{\underline{\theta}^* \in \Theta \cap \mathcal{V}\left(\underline{\theta}_j\right)} \left(\widetilde{L}_{Ns}\left(\underline{\theta}^*\right)\right).$$

The latter relation indicates that $\widehat{\underline{\theta}}_{Ns} \in \mathcal{V}\left(\underline{\theta}_0\right)$ for N is sufficiently large, thereby completing the proof of Theorem 3. □

Proof of Lemma 2. The demonstration closely follows that of [15] for the standard case, employing the SPS and PE arguments in place of the stationarity and ergodicity considerations, respectively. First and foremost, we can write the first-and second-order derivatives of $l_t\left(\underline{\theta}\right)$ as follows:

$$\begin{aligned}
\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial \theta_i(v)} &= Tr\left(\underline{X}_v \underline{X}'_v \frac{\partial H_v^{-1} M^{-1} H_v^{-1}}{\partial \theta_i(v)}\right) + 2\frac{\partial \log|\det H_v|}{\partial \theta_i(v)} \\
&= Tr\left(\left(I_{(r)} - M^{-1} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1}\right) \frac{\partial H_v}{\partial \theta_i(v)} H_v^{-1} + \left(I_{(r)} - H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1} M^{-1}\right) H_v^{-1} \frac{\partial H_v}{\partial \theta_i(v)}\right), \\
&\qquad 1 \leq i \leq \widetilde{r}, \\
\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial \theta_i} &= Tr\left(\left(I_{(r)} - M^{-1} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1}\right) M^{-1} \frac{\partial M}{\partial \theta_i}\right), \widetilde{r} + 1 \leq i \leq l, \\
\frac{\partial^2 l_{st+v}\left(\underline{\theta}_0\right)}{\partial \theta_i(v) \partial \theta_j(v)} &= \sum_{k=1}^{3} Tr\left(L_k(v)\right), 1 \leq i, j \leq \widetilde{r}, \frac{\partial^2 l_{st+v}\left(\underline{\theta}_0\right)}{\partial \theta_i(v) \partial \theta_j(v)} = \sum_{k=1}^{2} Tr\left(L_{k+3}(v)\right), 1 \leq i \leq \widetilde{r}, \widetilde{r} + 1 \leq j \leq l, \\
\frac{\partial^2 l_{st+v}\left(\underline{\theta}_0\right)}{\partial \theta_i(v) \partial \theta_j(v)} &= Tr\left(L_6(v)\right), \widetilde{r} + 1 \leq i, j \leq l,
\end{aligned}$$

for all $1 \leq v \leq s$, where

$$\begin{aligned}
2^{-1} L_1(v) &= H_v^{-1}\left(\frac{\partial^2 H_v}{\partial \theta_i(v) \partial \theta_j(v)} - \frac{\partial H_v}{\partial \theta_i(v)} H_v^{-1} \frac{\partial H_v}{\partial \theta_j(v)}\right), \\
L_2(v) &= H_v^{-1}\left(\frac{\partial H_v}{\partial \theta_i(v)} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1} M^{-1} + M^{-1} H_v^{-1} \frac{\partial H_v}{\partial \theta_i(v)} H_v^{-1} \underline{X}_v \underline{X}'_v\right) H_v^{-1} \frac{\partial H_v}{\partial \theta_j(v)} \\
&\quad - 2^{-1} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1} M^{-1} L_1(v), \\
L_4(v) &= M^{-1} H_v^{-1} \frac{\partial H_v}{\partial \theta_i(v)} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1} M^{-1} \frac{\partial M}{\partial \theta_j}, \\
L_6(v) &= M^{-1} \frac{\partial M}{\partial \theta_i} M^{-1} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1} M^{-1} \frac{\partial M}{\partial \theta_j} + M^{-1} H_v^{-1} \underline{X}_v \underline{X}'_v H_v^{-1} M^{-1}\left(\frac{\partial M}{\partial \theta_i} M^{-1} \frac{\partial M}{\partial \theta_j} - \frac{\partial^2 M}{\partial \theta_i \partial \theta_j}\right) \\
&\quad - M^{-1} \frac{\partial^2 M}{\partial \theta_i \partial \theta_j} - M^{-1} \frac{\partial M}{\partial \theta_i} M^{-1} \frac{\partial M}{\partial \theta_j},
\end{aligned}$$

where $L_3(v)$ (resp. $L_5(v)$) is obtained by permuting $\underline{X}_v\underline{X}_v'$ and M^{-1} (resp. $\frac{\partial H_v}{\partial\theta_i(v)}$) in $L_4(v)$. To prove (**a**) it would be appropriate to use the Cauchy–Schwarz inequality, from which we obtain

$$E_{\underline{\theta}_0}\left\{\left\|\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_j(v)}\right\|\right\}\leq\omega\left(E_{\underline{\theta}_0}\left\{\left\|H_v^{-1}\left(\underline{\theta}_0\right)\frac{\partial H_v\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\right\|^2\right\}E_{\underline{\theta}_0}\left\{\left\|H_v^{-1}\left(\underline{\theta}_0\right)\frac{\partial H_v\left(\underline{\theta}_0\right)}{\partial\theta_j(v)}\right\|^2\right\}\right)^{1/2},1\leq i,j\leq\widetilde{r},$$

$$E_{\underline{\theta}_0}\left\{\left\|\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_j(v)}\right\|\right\}\leq\omega E_{\underline{\theta}_0}\left\{\left\|H_v^{-1}\left(\underline{\theta}_0\right)\frac{\partial H_v\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\right\|\right\},1\leq i\leq\widetilde{r},\widetilde{r}+1\leq j\leq l,$$

$$E_{\underline{\theta}_0}\left\{\left\|\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\frac{\partial l_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_j(v)}\right\|\right\}\leq\omega,\widetilde{r}+1\leq i,j\leq l.$$

Now, letting $\sigma_{i,t}$ (resp. $\Sigma_{i,st+v}$) denote the i^{th} component of $\underline{\sigma}_t$ (resp. $\underline{\Sigma}_{st+v}$), it is thus sufficient to prove that

$$E_{\underline{\theta}_0}\left\{\left|\frac{1}{\sigma_{j,st+v}}\frac{\partial\sigma_{j,st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\right|^2\right\}<\omega,1\leq i\leq\widetilde{r},,1\leq j\leq r.$$

Then, we have a.s.

$$\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial\underline{\Sigma}_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\right\|<\infty,1\leq i\leq r,\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial\underline{\Sigma}_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}\right\|<\omega\left\|\underline{X}_{st+v}\right\|^2,r+1\leq i\leq\widetilde{r}-p.$$

Furthermore, we have

$$\begin{aligned}\underline{\Sigma}_{st+v}&=\sum_{j=1}^{\infty}\left\{\prod_{k=0}^{j-1}C_{st+v-k}\right\}\underline{\pi}_{st+v-j}+\underline{\pi}_{st+v}\\&=\sum_{j=0}^{\infty}\left\{\prod_{k=0}^{j}C_{s(t-k)+v}C(k)\right\}\underline{\pi}_{s(t-j)+v-m}+\underline{\pi}_{st+v},0\leq m\leq s-1,\end{aligned}$$

where

$$C(m)=\prod_{k=sm+1}^{s(m+1)-1}C_{st+v-k},0\leq m\leq j-1andC(j)=\prod_{k=sj+1}^{sj+m-1}C_{st+v-k},0\leq m\leq s-1.$$

So, we get a.s.

$$\frac{\partial\underline{\Sigma}_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)}=\sum_{j=1}^{\infty}\sum_{m=0}^{j}\left\{\prod_{k=0}^{m-1}C_{s(t-k)+v}C(k)\right\}C_{(i)}C(m)\left\{\prod_{k=m+1}^{j}C_{s(t-k)+v}C(k)\right\}\underline{\pi}_{s(t-j)+v-n},$$

where $0\leq n\leq s-1$ and $C_{(i)}$ is a matrix whose entries are all zero except for a 1 situated in the same location as $\theta_i(v)$ in C_{st+v}. Applying the inequality $y^{\tau}\geq\frac{y}{1+y}$ for $y>0$ and any $\tau\in[0;1]$ and the fact that, almost surely $\log\left\|\prod_{k=0}^{j-1}C_{st+v-k}\right\|\leq\omega+j\gamma_{(s)}(C)$, we can obtain the inequalities

$$\theta_i(v)\frac{\partial\Sigma_{st+v}\left(\underline{\theta}_0\right)}{\partial\theta_i(v)} \le \sum_{j=1}^{\infty} j\left\{\prod_{k=0}^{j-1} C_{st+v-k}\right\}\underline{\pi}_{st+v-j},$$
$$\theta_i(v)\frac{\partial\Sigma_{n,st+v}}{\partial\theta_i(v)} \le \sum_{j=1}^{\infty} j\sum_{m=1}^{r}\left\{\prod_{k=0}^{j-1} C_{st+v-k}\right\}(n,m)\,\underline{\pi}_{st+v-j}(m),$$

and, setting

$$a_0(v) = \inf\left\{\underline{a}_{i,0}(v), 1 \le i \le r\right\}, \Sigma_{i,st+v} \ge a_0(v) + \sum_{m=1}^{r}\left\{\prod_{k=0}^{j-1} C_{st+v-k}\right\}(i,m)\,\underline{\pi}_{st+v-j}(m)\,\forall j,$$

we obtain

$$\left|\frac{\theta_i(v)}{\Sigma_{n,st+v}}\frac{\partial\Sigma_{n,st+v}}{\partial\theta_i(v)}\right| < \omega.$$

Using the same technique as Francq and Zakoïan [15], we can demonstrate a result that is even more robust than the one initially stated: $\forall 1 \le i, j, k \le \widetilde{r}$, all $1 \le n \le r$ and all $m \ge 0$, $\exists\vartheta\left(\underline{\theta}_0\right)$ of $\underline{\theta}_0$, such that

$$E\left\{\sup_{\underline{\theta}\in\vartheta\left(\underline{\theta}_0\right)}\left|\frac{1}{\sigma_{n,st+v}}\frac{\partial\sigma_{n,st+v}\left(\underline{\theta}\right)}{\partial\theta_i(v)}\right|^m\right\} < \infty,$$
$$E\left\{\sup_{\underline{\theta}\in\vartheta\left(\underline{\theta}_0\right)}\left|\frac{1}{\sigma_{n,st+v}}\frac{\partial^2\sigma_{n,st+v}\left(\underline{\theta}\right)}{\partial\theta_i(v)\,\partial\theta_j(v)}\right|^m\right\} < \infty$$

and

$$E\left\{\sup_{\underline{\theta}\in\vartheta\left(\underline{\theta}_0\right)}\left|\frac{1}{\sigma_{n,st+v}}\frac{\partial^3\sigma_{n,st+v}\left(\underline{\theta}\right)}{\partial\theta_i(v)\,\partial\theta_j(v)\,\partial\theta_k(v)}\right|^m\right\} < \infty.$$

We now prove (**b**), clearly, $\nabla_{\underline{\theta}} l_t\left(\underline{\theta}_0\right)$ is SPS and measurable with respect to $\mathcal{F}_t = \sigma\left(\underline{\eta}_n, n \le t\right)$. The property stated in proof (**a**) ensures the existence of matrices U and K in Theorem 4. Moreover, it can be seen that $\forall\underline{\lambda} \in \mathbb{R}^l$, the sequence $\left(\underline{\lambda}'\nabla_{\underline{\theta}} l_t\left(\underline{\theta}_0\right), \mathcal{F}_t\right)$ is an SPS, PE, and square-integrable martingale difference. The $C.L.T$ of Bllingsley implies

$$(Ns)^{-\frac{1}{2}}\sum_{t=1}^{N}\sum\nolimits_{v=1}^{s}\nabla_{\underline{\theta}} l_{st+v}\left(\underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, U\right)\ as\ N \to \infty.$$

□