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Research article

Weighted Milne-type inequalities through Riemann-Liouville fractional integrals and diverse function classes

Areej A Almoneef[1], **Abd-Allah Hyder**[2,*] **and Hüseyin Budak**[3]

[1] Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabi

[2] Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia

[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce,Türkiye

* **Correspondence:** Email: abahahmed@kku.edu.sa.

Abstract: This research paper investigated weighted Milne-type inequalities utilizing Riemann-Liouville fractional integrals across diverse function classes. A key contribution lies in the establishment of a fundamental integral equality, facilitated by the use of a nonnegative weighted function, which is pivotal for deriving the main results. The paper systematically proved weighted Milne-type inequalities for various function classes, including differentiable convex functions, bounded functions, Lipschitzian functions, and functions of bounded variation. The obtained results not only contribute to the understanding of Milne-type inequalities but also offer insights that pave the way for potential future research in the considered topics. Furthermore, it is evident that the results obtained encompass numerous findings that were previously presented in various studies as special cases.

Keywords: weighted Milne-type inequalities; Riemann-Liouville fractional integrals; nonnegative weighted function; differentiable convex functions
Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

Exploring error upper limits through numerical integration formulas and various methodologies has been an area of study for numerous mathematicians. The investigation into error bounds for these formulas involves the analysis of mathematical inequalities via different types of functions, involving convex, Lipschitzian, and bounded functions. This paper specifically focuses on establishing bounds for functions where either the derivatives or second derivatives exhibit convex behavior.

First, let's present a few numerical integration methods along with their respective upper error bounds:

(i) The expression below represents Simpson's quadrature formula, commonly known as Simpson's 1/3 rule:

$$\int_{\eta}^{\epsilon} \mathcal{P}(x)\,dx \approx \frac{\epsilon-\eta}{6}\left[\mathcal{P}(\eta)+4\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{P}(\epsilon)\right]. \tag{1.1}$$

(ii) The characterization of Simpson's second formula, also known as the Newton-Cotes quadratic formula or Simpson's 3/8 rule (see [1]), is as follows:

$$\int_{\eta}^{\epsilon} \mathcal{P}(x)\,dx \approx \frac{\epsilon-\eta}{8}\left[\mathcal{P}(\eta)+3\mathcal{P}\left(\frac{2\eta+\epsilon}{3}\right)+3\mathcal{P}\left(\frac{\eta+2\epsilon}{3}\right)+\mathcal{P}(\epsilon)\right]. \tag{1.2}$$

Formulas (1.1) and (1.2) are applicable to any function $\mathcal{P}$ that has a continuous fourth derivative over the interval $[\eta, \epsilon]$.

The classical Simpson inequality [2] is expressed as follows:

Theorem 1.1. *Assume $\mathcal{P} : [\eta, \epsilon] \to \mathbb{R}$ is a function with continuous fourth derivative on (η, ϵ), and $\left\|\mathcal{P}^{(4)}\right\|_{\infty} = \sup_{x\in(\eta,\epsilon)} \left|\mathcal{P}^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[\mathcal{P}(\eta)+4\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(x)dx\right| \leq \frac{1}{2880}\left\|\mathcal{P}^{(4)}\right\|_{\infty}(\epsilon-\eta)^4. \tag{1.3}$$

Sarikaya et al. initially established the Simpson type inequality using convex functions in [3]. Regarding Riemann-Liouville fractional integrals, there exist three distinct types of Simpson inequalities characterized by the representation of fractional integrals. The proofs for these inequalities are documented in the papers [4–6]. Additionally, some papers are devoted to Simpson-type inequalities specifically tailored for twice-differentiable functions, as demonstrated [7–11].

The classical Newton inequality is expressed as follows:

Theorem 1.2. *(See [1]) Suppose $\mathcal{P} : [\eta, \epsilon] \to \mathbb{R}$ is a function with four continuous derivatives on the open interval (η, ϵ), and $\left\|\mathcal{P}^{(4)}\right\|_{\infty} = \sup_{x\in(\eta,\epsilon)} \left|\mathcal{P}^{(4)}(x)\right| < \infty$. In such a case, the subsequent inequality is satisfied:*

$$\left|\frac{1}{8}\left[\mathcal{P}(\eta)+3\mathcal{P}\left(\frac{2\eta+\epsilon}{3}\right)+3\mathcal{P}\left(\frac{\eta+2\epsilon}{3}\right)+\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(x)dx\right| \leq \frac{1}{6480}\left\|\mathcal{P}^{(4)}\right\|_{\infty}(\epsilon-\eta)^4. \tag{1.4}$$

In the works [12–14], several Newton-type inequalities were presented, utilizing convex functions for local fractional integrals. In the paper [15], the authors were the first to establish Newton-type inequalities for Riemann-Liouville fractional integrals. Subsequent to this development, several papers have been devoted to deriving Newton-type inequalities for Riemann-Liouville fractional integrals [16, 17]. Gao and Shi demonstrated Newton-type inequalities for functions with two derivatives [18].

Djenaoui and Meftah were the first to establish Milne-type inequalities employing the concept of convexity [19]. Subsequently, Budak et al. extended these inequalities to Riemann-Liouville fractional integrals [20]. In the same work, the authors introduced several Milne-type inequalities applicable to diverse function classes, including bounded functions, Lipschitz functions, and functions of bounded variation. Furthermore, in recent publications such as [11, 21], new fractional versions of Milne-type inequalities have been derived using differentiable convex functions and various function classes like bounded functions, Lipschitz functions, and functions of bounded variation. For additional Milne-type inequalities, please see [22–25].

The objective of this paper is to derive some fractional Milne-type inequalities for functions whose second derivatives exhibit convexity. First, we introduce a fundamental integral equality enabled by a nonnegative weighted function, essential for deriving the main results. Second, we systematically establish weighted Milne-type inequalities across various function classes, encompassing differentiable convex functions, bounded functions, Lipschitzian functions, and functions of bounded variation. These results not only advance the understanding of Milne-type inequalities but also provide insights for potential future research in the considered topics. Moreover, the findings of this study incorporate numerous results previously presented in various studies as special cases, thus consolidating and extending the existing knowledge in the field.

To achieve this goal, we will rely on the following definition of Riemann-Liouville fractional integrals.

The commonly known Riemann-Liouville fractional integrals are defined as follows:

Definition 1.1. *(See [26, 27]) The Riemann-Liouville integrals* $\mathcal{I}^{\delta}_{\eta+}\mathcal{P}$ *and* $\mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}$ *of order* $\delta > 0$ *are expressed as*

$$\mathcal{I}^{\delta}_{\eta+}\mathcal{P}(x) = \frac{1}{\Gamma(\delta)} \int_{\eta}^{x} (x-t)^{\delta-1} \mathcal{P}(t)dt, \quad x > \eta, \tag{1.5}$$

and

$$\mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}(x) = \frac{1}{\Gamma(\delta)} \int_{x}^{\epsilon} (t-x)^{\delta-1} \mathcal{P}(t)dt, \quad x < \epsilon, \tag{1.6}$$

respectively. Here, $\mathcal{P}$ *belongs to* $L_1[\eta, \epsilon]$*, and* $\Gamma(\delta)$ *denotes the Gamma function, which is defined as*

$$\Gamma(\delta) := \int_{0}^{\infty} e^{-u} u^{\delta-1} du. \tag{1.7}$$

It is noteworthy that the fractional integral converges to the classical integral when δ *approaches* 1.

This paper is structured into seven parts, starting with an introduction and preliminary concepts. Section 2 focuses on proving a pivotal integral equality, crucial for establishing the main results, utilizing a nonnegative weighted function. Section 3 presents weighted Milne-type inequalities for various function classes employing Riemann-Liouville fractional integrals. Also, Section 3 addresses weighted Milne-type inequalities for differentiable convex functions. In Section 4, several weighted Milne-type inequalities for bounded functions through fractional integrals are provided. Section 5 establishes weighted fractional Milne-type inequalities for Lipschitzian functions. Additionally, Section 6 proves weighted fractional Milne-type inequalities for functions of bounded variation. Finally, Section 7 offers insights into Milne-type inequalities and discuss potential avenues for future research in this domain.

2. An essential equality

Throughout the paper, we make the assumption that $w : [\eta, \epsilon] \to \mathbb{R}$ is both nonnegative and continuous over the interval $[\eta, \epsilon]$, exhibiting symmetry with respect to $\frac{\eta+\epsilon}{2}$ (i.e., $w(x) = w(\eta + \epsilon - x)$ for all $x \in [\eta, \epsilon]$). Now, let us introduce the functions W_1 and W_2 as follows:

$$W_1(\delta, t) = \int_t^{\frac{\eta+\epsilon}{2}} \left(\frac{\eta+\epsilon}{2} - u\right)^{\delta-1} w(u)\, du, \tag{2.1}$$

and

$$W_2(\delta, t) = \int_{\frac{\eta+\epsilon}{2}}^{t} \left(u - \frac{\eta+\epsilon}{2}\right)^{\delta-1} w(u)\, du. \tag{2.2}$$

Given the symmetry of the function w with respect to $\frac{\eta+\epsilon}{2}$, we can establish the following equalities, which will be consistently employed in the subsequent sections:

$$W(\delta) := W_1(\delta, \eta) = W_2(\delta, \epsilon) = \Gamma(\delta)\mathcal{I}^{\delta}_{\epsilon-} w\left(\frac{\eta+\epsilon}{2}\right) = \Gamma(\delta)\mathcal{I}^{\delta}_{\eta+} w\left(\frac{\eta+\epsilon}{2}\right). \tag{2.3}$$

Especially, we also have

$$W(1) = \int_{\eta}^{\frac{\eta+\epsilon}{2}} w(x)\, dx = \int_{\frac{\eta+\epsilon}{2}}^{\epsilon} w(x)\, dx = \frac{1}{2}\int_{\eta}^{\epsilon} w(x)\, dx. \tag{2.4}$$

Lemma 2.1. *Let $f : [\eta, \epsilon] \to \mathbb{R}$ be an absolutely continuous function on the interval (η, ϵ) such that $\mathcal{P}' \in L_1[\eta, \epsilon]$. Hence, the following equality holds:*

$$\frac{1}{3}\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right] W(\delta) - \frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right) + \mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right] = \frac{1}{4}[I_2 - I_1]. \tag{2.5}$$

Here, Γ represents the Euler Gamma function, and

$$\begin{cases} I_1 = \int_{\eta}^{\frac{\eta+\epsilon}{2}} \left[W_1(\delta, t) + \frac{1}{3}W(\delta)\right] \left[\mathcal{P}'(t) - \mathcal{P}'(\eta + \epsilon - t)\right] dt, \\ I_2 = \int_{\frac{\eta+\epsilon}{2}}^{\epsilon} \left[W_2(\delta, t) + \frac{1}{3}W(\delta)\right] \left[\mathcal{P}'(t) - \mathcal{P}'(\eta + \epsilon - t)\right] dt. \end{cases}$$

Proof. Applying the principles of integration by parts, we can easily derive:

$$\begin{aligned} I_1 &= \int_{\eta}^{\frac{\eta+\epsilon}{2}} \left[W_1(\delta, t) + \frac{1}{3}W(\delta)\right] \left[\mathcal{P}'(t) - \mathcal{P}'(\eta + \epsilon - t)\right] dt \qquad (2.6) \\ &= \left[W_1(\delta, t) + \frac{1}{3}W(\delta)\right] \left[\mathcal{P}(t) + \mathcal{P}(\eta + \epsilon - t)\right]\Big|_{\eta}^{\frac{\eta+\epsilon}{2}} \end{aligned}$$

$$+\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(\frac{\eta+\epsilon}{2}-t\right)^{\delta-1}w(t)\left[\mathcal{P}(t)+\mathcal{P}(\eta+\epsilon-t)\right]dt$$
$$=\frac{2}{3}W(\delta)\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)-\left[W_1(\delta,\eta)+\frac{1}{3}W(\delta)\right]\left[\mathcal{P}(\eta)+\mathcal{P}(\epsilon)\right]$$
$$+\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(\frac{\eta+\epsilon}{2}-t\right)^{\delta-1}w(t)\mathcal{P}(t)\,dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left(\frac{\eta+\epsilon}{2}-t\right)^{\delta-1}w(t)\mathcal{P}(t)\,dt$$
$$=\frac{2}{3}W(\delta)\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)-\frac{4}{3}W(\delta)\left[\mathcal{P}(\eta)+\mathcal{P}(\epsilon)\right]+\Gamma(\delta)\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right].$$

Similar to the previous process, we have

$$I_2=\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left[\mathcal{P}'(t)-\mathcal{P}'(\eta+\epsilon-t)\right]dt \tag{2.7}$$
$$=\frac{4}{3}W(\delta)\left[\mathcal{P}(\eta)+\mathcal{P}(\epsilon)\right]-\frac{2}{3}W(\delta)\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)-\Gamma(\delta)\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right].$$

By combining the equalities (2.6) and (2.7), we can easily derive

$$I_2-I_1=\frac{8}{3}W(\delta)\left[\mathcal{P}(a)+\mathcal{P}(\epsilon)\right]-\frac{4}{3}W(\delta)\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) \tag{2.8}$$
$$-2\Gamma(\delta)\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right].$$

Hence, the proof of Lemma 2.1 is concluded by multiplying both sides of (2.8) by $\frac{1}{4}$. □

3. Weighted fractional Milne-type inequalities for convex functions

In this segment, we derive weighted fractional Milne-type inequalities for differentiable convex functions by applying the modulus of the established equality (2.5). Additionally, we formulate certain weighted fractional Milne-type inequalities using Hölder and power-mean inequalities.

Theorem 3.1. *Assuming that all the conditions of Lemma 2.1 are satisfied and $|\mathcal{P}'|$ represents a convex function on the interval $[\eta,\epsilon]$, we can establish the subsequent weighted fractional Milne-type inequality.*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_1(\delta,t)\,dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_2(\delta,t)\,dt+\frac{\epsilon-\eta}{3}W(\delta)\right]\left[\left|\mathcal{P}'(\eta)\right|+\left|\mathcal{P}'(\epsilon)\right|\right].$$

Proof. By Lemma 2.1, we have

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right| \tag{3.1}$$

$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left|W_1(\delta,t)+\frac{1}{3}W(\delta)\right|\left[\left|\mathcal{P}'(t)\right|+\left|\mathcal{P}'(\eta+\epsilon-t)\right|\right]dt\right.$$

$$\left.+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left|W_2(\delta,t)+\frac{1}{3}W(\delta)\right|\left[\left|\mathcal{P}'(t)\right|+\left|\mathcal{P}'(\eta+\epsilon-t)\right|\right]dt\right].$$

Since the function $|\mathcal{P}'|$ is convex, we have

$$\left|\mathcal{P}'(t)\right|=\left|\mathcal{P}'\left(\frac{\epsilon-t}{\epsilon-\eta}\eta+\frac{t-\eta}{\epsilon-\eta}\epsilon\right)\right|\leq\frac{\epsilon-t}{\epsilon-\eta}\left|\mathcal{P}'(\eta)\right|+\frac{t-\eta}{\epsilon-\eta}\left|\mathcal{P}'(\epsilon)\right|,$$

and

$$\left|\mathcal{P}'(\eta+\epsilon-t)\right|=\left|\mathcal{P}'\left(\frac{\epsilon-t}{\epsilon-\eta}\epsilon+\frac{t-\eta}{\epsilon-\eta}\eta\right)\right|\leq\frac{\epsilon-t}{\epsilon-\eta}\left|\mathcal{P}'(\epsilon)\right|+\frac{t-\eta}{\epsilon-\eta}\left|\mathcal{P}'(\eta)\right|.$$

Then, it follows that

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$

$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]dt\right]\left[\left|\mathcal{P}'(\eta)\right|+\left|\mathcal{P}'(\epsilon)\right|\right]$$

$$=\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_1(\delta,t)\,dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_2(\delta,t)\,dt+\frac{\epsilon-\eta}{3}W(\delta)\right]\left[\left|\mathcal{P}'(\eta)\right|+\left|\mathcal{P}'(\epsilon)\right|\right],$$

which completes the proof. □

Remark 3.1. *By selecting $w(t)=1$ for every t in the interval $[\eta,\epsilon]$ in Theorem 3.1, we obtain the ensuing fractional Milne-type inequality, a result previously demonstrated by Budak et al. [20]:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\epsilon-\eta)^{\delta}}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$

$$\leq\frac{\epsilon-\eta}{12}\left(\frac{\delta+4}{\delta+1}\right)\left[\left|\mathcal{P}'(\eta)\right|+\left|\mathcal{P}'(\epsilon)\right|\right].$$

Corollary 3.1. *Upon setting $\delta=1$ in Theorem 3.1, we derive the subsequent weighted Milne-type inequality.*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right| \tag{3.2}$$

$$\leq\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(t-\frac{7\eta-\epsilon}{6}\right)w(t)dt\right)\left[\left|\mathcal{P}'(\eta)\right|+\left|\mathcal{P}'(\epsilon)\right|\right].$$

Proof. Due to the symmetry of the function w with respect to $\frac{\eta+\epsilon}{2}$, the proof (3.2) is based on the fact that

$$\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(\int_{t}^{\frac{\eta+\epsilon}{2}}w(u)du\right)dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left(\int_{\frac{\eta+\epsilon}{2}}^{t}w(u)du\right)dt+\frac{\epsilon-\eta}{6}\int_{\eta}^{\epsilon}w(t)dt \tag{3.3}$$

$$\begin{aligned}
&= \int_{\eta}^{\frac{\eta+\epsilon}{2}}\int_{\eta}^{u}w(u)dtdu+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\int_{u}^{\epsilon}w(u)dtdu+\frac{\epsilon-\eta}{6}\int_{\eta}^{\epsilon}w(t)dt\\
&= \int_{\eta}^{\frac{\eta+\epsilon}{2}}(u-\eta)\,w(u)du+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}(b-u)\,w(u)du+\frac{\epsilon-\eta}{6}\int_{\eta}^{\epsilon}w(t)dt\\
&= \int_{\eta}^{\frac{\eta+\epsilon}{2}}(u-\eta)\,w(u)du+\int_{\eta}^{\frac{\eta+\epsilon}{2}}(u-\eta)\,w(u)du+\frac{\epsilon-\eta}{3}\int_{\eta}^{\frac{\eta+\epsilon}{2}}w(t)\,dt\\
&= 2\int_{\eta}^{\frac{\eta+\epsilon}{2}}(t-\eta)\,w(t)dt+\frac{\epsilon-\eta}{3}\int_{\eta}^{\frac{\eta+\epsilon}{2}}w(t)dt\\
&= 2\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(t-\frac{7\eta-\epsilon}{6}\right)w(t)dt.
\end{aligned}$$

Conversely, we also obtain

$$\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(\int_{t}^{\frac{\eta+\epsilon}{2}}w(u)du\right)dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left(\int_{\frac{\eta+\epsilon}{2}}^{t}w(u)du\right)dt+\frac{\epsilon-\eta}{6}\int_{\eta}^{\epsilon}w(t)dt=2\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(\frac{7\epsilon-\eta}{6}-t\right)w(t)dt. \tag{3.4}$$

Corollary 3.2. *Under the stipulations of Theorem 3.1, the subsequent weighted fractional Milne-type inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|\\
\leq&\frac{(\epsilon-\eta)^{\delta+1}\left\|w\right\|_{\infty}}{2^{\delta+2}}\left(\frac{\delta+4}{\delta(\delta+1)}\right)\left[\left|\mathcal{P}'(\eta)\right|+\left|\mathcal{P}'(\epsilon)\right|\right].
\end{aligned}$$

Corollary 3.3. *By selecting $\delta = 1$ in Corollary 3.2, we arrive at the following weighted Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|$$
$$\leq\frac{5(\epsilon-\eta)^2\|w\|_\infty}{24}\left[|\mathcal{P}'(\eta)|+|\mathcal{P}'(\epsilon)|\right].$$

Remark 3.2. *By choosing $w(t) = 1$ for every t in the interval $[\eta, \epsilon]$ in Corollary 3.3, we obtain the following Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(t)dt\right|\leq\frac{5(\epsilon-\eta)}{24}\left[|\mathcal{P}'(\eta)|+|\mathcal{P}'(\epsilon)|\right],$$

which is obtained by Budak et al. [20].

Theorem 3.2. *Assuming the fulfillment of all conditions stated in Lemma 2.1, if $|\mathcal{P}'|^q$ exhibits convexity on the interval $[\eta, \epsilon]$, where $q > 1$, then the following Milne-type inequality is established:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\left(\frac{\epsilon-\eta}{2}\right)^{\frac{1}{q}}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}\right]$$
$$\times\left[\left(\frac{3|\mathcal{P}'(\eta)|^q+|\mathcal{P}'(\epsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{P}'(\epsilon)|^q+|\mathcal{P}'(\eta)|^q}{4}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying the well-known Hölder inequality to (3.1), we obtain

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}|\mathcal{P}'(t)|^q dt\right)^{\frac{1}{q}}+\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}|\mathcal{P}'(\eta+\epsilon-t)|^q dt\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}|\mathcal{P}'(t)|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}|\mathcal{P}'(\eta+\epsilon-t)|^q dt\right)^{\frac{1}{q}}\right]\right].$$

Due to the convexity of the function $|\mathcal{P}'|^q$, we get

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right| \\
\leq&\frac{1}{4}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}\right. \\
&\times\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\frac{\epsilon-t}{\epsilon-\eta}|\mathcal{P}'(\eta)|^q+\frac{t-\eta}{\epsilon-\eta}|\mathcal{P}'(\epsilon)|^q\right]dt\right)^{\frac{1}{q}}+\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\frac{\epsilon-t}{\epsilon-\eta}|\mathcal{P}'(\epsilon)|^q+\frac{t-\eta}{\epsilon-\eta}|\mathcal{P}'(\eta)|^q\right]dt\right)^{\frac{1}{q}}\right] \\
&+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}} \\
&\times\left.\left[\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[\frac{\epsilon-t}{\epsilon-\eta}|\mathcal{P}'(\eta)|^q+\frac{t-\eta}{\epsilon-\eta}|\mathcal{P}'(\epsilon)|^q\right]dt\right)^{\frac{1}{q}}+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[\frac{\epsilon-t}{\epsilon-\eta}|\mathcal{P}'(\epsilon)|^q+\frac{t-\eta}{\epsilon-\eta}|\mathcal{P}'(\eta)|^q\right]dt\right)^{\frac{1}{q}}\right]\right] \\
=&\frac{1}{4}\left(\frac{\epsilon-\eta}{2}\right)^{\frac{1}{q}}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}\times\left[\left(\frac{3|\mathcal{P}'(\eta)|^q+|\mathcal{P}'(\epsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{P}'(\epsilon)|^q+|\mathcal{P}'(\eta)|^q}{4}\right)^{\frac{1}{q}}\right]\right. \\
&+\left.\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]^p dt\right)^{\frac{1}{p}}\times\left[\left(\frac{|\mathcal{P}'(a)|^q+3|\mathcal{P}'(\epsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{P}'(\epsilon)|^q+3|\mathcal{P}'(\eta)|^q}{4}\right)^{\frac{1}{q}}\right]\right].
\end{aligned}
$$

Hence, the proof of Theorem 3.2 is now concluded. □

Remark 3.3. *If we set $w(t) = 1$ for every $t \in [\eta, \epsilon]$ in Theorem 3.2, the theorem simplifies to [20, Theorem 2].*

Corollary 3.4. *Setting $\delta = 1$ in Theorem 3.2 yields the subsequent weighted Milne-type inequality:*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right| \\
\leq&\frac{1}{2}\left(\frac{\epsilon-\eta}{2}\right)^{\frac{1}{q}}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\int_{t}^{\frac{\eta+\epsilon}{2}}w(u)du+\frac{1}{6}\int_{\eta}^{\epsilon}w(u)du\right]^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[\int_{\frac{\eta+\epsilon}{2}}^{t}w(u)du+\frac{1}{6}\int_{\eta}^{\epsilon}w(u)du\right]^p dt\right)^{\frac{1}{p}}\right] \\
&\times\left[\left(\frac{3|\mathcal{P}'(\eta)|^q+|\mathcal{P}'(\epsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{P}'(\epsilon)|^q+|\mathcal{P}'(\eta)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 3.5. *Under the conditions specified in Theorem 3.2, the following weighted fractional Milne-type inequality is established:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right| \tag{3.5}$$

$$\leq\frac{\|w\|_{\infty}}{2\delta}\left(\frac{\epsilon-\eta}{2}\right)^{\frac{1}{q}}\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\left(\frac{\eta+\epsilon}{2}-t\right)^{\delta}+\frac{1}{3}\left(\frac{\epsilon-\eta}{2}\right)^{\delta}\right]^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{3\left|\mathcal{P}'(\eta)\right|^{q}+\left|\mathcal{P}'(\epsilon)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{P}'(\epsilon)\right|^{q}+\left|\mathcal{P}'(\eta)\right|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

Proof. The proof of (3.5) is obvious from the fact that

$$\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\left(\frac{\eta+\epsilon}{2}-t\right)^{\delta}+\frac{1}{3}\left(\frac{\epsilon-\eta}{2}\right)^{\delta}\right]^{p}dt=\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[\left(t-\frac{\eta+\epsilon}{2}\right)^{\delta}+\frac{1}{3}\left(\frac{\epsilon-\eta}{2}\right)^{\delta}\right]^{p}dt.$$

Corollary 3.6. *If we choose $\delta=1$ in Corollary 3.5, then we have the following weighted Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|$$

$$\leq\frac{\|w\|_{\infty}(\epsilon-\eta)^{2}}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{P}'(\eta)\right|^{q}+\left|\mathcal{P}'(\epsilon)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{P}'(\epsilon)\right|^{q}+\left|\mathcal{P}'(\eta)\right|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

Remark 3.4. *By selecting $w(t)=1$ for every $t\in[\eta,\epsilon]$ in Corollary 3.6, we obtain the following Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(t)dt\right|$$

$$\leq\frac{\epsilon-\eta}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{P}'(a)\right|^{q}+\left|\mathcal{P}'(\epsilon)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{P}'(\epsilon)\right|^{q}+\left|\mathcal{P}'(\eta)\right|^{q}}{4}\right)^{\frac{1}{q}}\right],$$

which is given by Budak et al. [20].

Theorem 3.3. *Assuming all conditions specified in Lemma 2.1 are met, if $|\mathcal{P}'|^{q}$ is convex on $[\eta,\epsilon]$ with $q\geq 1$, then the following Milne-type inequality holds:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$

$$\leq\frac{1}{4(\epsilon-\eta)^{\frac{1}{q}}}\left\{(\Omega_{1}(\delta))^{1-\frac{1}{q}}\left[\left(\Omega_{2}(\delta)\left|\mathcal{P}'(\eta)\right|^{q}+\Omega_{3}(\delta)\left|\mathcal{P}'(\epsilon)\right|^{q}\right)^{\frac{1}{q}}+\left(\Omega_{2}(\delta)\left|\mathcal{P}'(\epsilon)\right|^{q}+\Omega_{3}(\delta)\left|\mathcal{P}'(\eta)\right|^{q}\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\Upsilon_1(\delta)\right)^{1-\frac{1}{q}}\left[\left(\Upsilon_2(\delta)\left|\mathcal{P}'(\eta)\right|^q+\Upsilon_3(\delta)\left|\mathcal{P}'(\epsilon)\right|^q\right)^{\frac{1}{q}}+\left(\Upsilon_2(\delta)\left|\mathcal{P}'(\epsilon)\right|^q+\Upsilon_3(\delta)\left|\mathcal{P}'(\eta)\right|^q\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Here,

$$\begin{aligned}
\Omega_1(\delta) &= \int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]dt=\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_1(\delta,t)\,dt+\frac{\epsilon-\eta}{6}W(\delta),\\
\Omega_2(\delta) &= \int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right](\epsilon-t)\,dt=\int_{\eta}^{\frac{\eta+\epsilon}{2}}(\epsilon-t)\,W_1(\delta,t)\,dt+\frac{(\epsilon-\eta)^2}{8}W(\delta),\\
\Omega_3(\delta) &= \int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right](t-\eta)\,dt=\int_{\eta}^{\frac{\eta+\epsilon}{2}}(t-\eta)\,W_1(\delta,t)\,dt+\frac{(\epsilon-\eta)^2}{24}W(\delta),
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_1(\delta) &= \int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]dt=\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_2(\delta,t)\,dt+\frac{\epsilon-\eta}{6}W(\delta),\\
\Upsilon_2(\delta) &= \int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right](\epsilon-t)\,dt=\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}(\epsilon-t)\,W_2(\delta,t)\,dt+\frac{(\epsilon-\eta)^2}{24}W(\delta),\\
\Upsilon_3(\delta) &= \int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right](t-\eta)\,dt=\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}(t-\eta)\,W_2(\delta,t)\,dt+\frac{(\epsilon-\eta)^2}{8}W(\delta).
\end{aligned}$$

Proof. Utilizing the power mean inequality in (3.1), it consequently follows that:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|\\
\leq&\frac{1}{4}\left\{\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(t)\right|^q dt\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left|W_1(\delta,t)+\frac{1}{3}W(\delta)\right|\left|\mathcal{P}'(\eta+\epsilon-t)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(t)\right|^q dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(\eta+\epsilon-t)\right|^q dt\right)^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Since the function $|\mathcal{P}'|$ is convex, we have

$$\left|\mathcal{P}'(t)\right|^q \leq \frac{\epsilon-t}{\epsilon-\eta}\left|\mathcal{P}'(\eta)\right|^q+\frac{t-\eta}{\epsilon-\eta}\left|\mathcal{P}'(\epsilon)\right|^q$$

and

$$\left|\mathcal{P}'(\eta+\epsilon-t)\right|^q \leq \frac{\epsilon-t}{\epsilon-\eta}\left|\mathcal{P}'(\epsilon)\right|^q+\frac{t-\eta}{\epsilon-\eta}\left|\mathcal{P}'(\eta)\right|^q.$$

Then, it follows that

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|\\
\leq&\frac{1}{4(\epsilon-\eta)^{\frac{1}{q}}}\left\{(\Omega_1(\delta))^{1-\frac{1}{q}}\left[\left(\Omega_2(\delta)\left|\mathcal{P}'(\eta)\right|^q+\Omega_3(\delta)\left|\mathcal{P}'(\epsilon)\right|^q\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\Omega_2(\delta)\left|\mathcal{P}'(\epsilon)\right|^q+\Omega_3(\delta)\left|\mathcal{P}'(a)\right|^q\right)^{\frac{1}{q}}\right]+(\Upsilon_1(\delta))^{1-\frac{1}{q}}\left[\left(\Upsilon_2(\delta)\left|\mathcal{P}'(\eta)\right|^q+\Upsilon_3(\delta)\left|\mathcal{P}'(\epsilon)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\Upsilon_2(\delta)\left|\mathcal{P}'(\epsilon)\right|^q+\Upsilon_3(\delta)\left|\mathcal{P}'(\eta)\right|^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

This completes the proof. □

Remark 3.5. *By selecting $w(t)=1$ for every $t\in[\eta,\epsilon]$ in Theorem 3.3, the theorem simplifies to [20, Theorem 3].*

Corollary 3.7. *Under the assumptions of Theorem 3.3, the following weighted fractional Milne-type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|\\
\leq&\frac{\|w\|_\infty}{2\delta}\left(\frac{\epsilon-\eta}{2}\right)^{\delta+1}\left(\frac{\delta+4}{3(\delta+1)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\left(\frac{2\delta+3}{2(\delta+1)(\delta+2)}+\frac{1}{4}\right)\left|\mathcal{P}'(\eta)\right|^q+\left(\frac{1}{2(\delta+1)(\delta+2)}+\frac{1}{12}\right)\left|\mathcal{P}'(\epsilon)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(\frac{2\delta+3}{2(\delta+1)(\delta+2)}+\frac{1}{4}\right)\left|\mathcal{P}'(\epsilon)\right|^q+\left(\frac{1}{2(\delta+1)(\delta+2)}+\frac{1}{12}\right)\left|\mathcal{P}'(a)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 3.8. *Setting $\delta=1$ in Corollary 3.7 results in the subsequent weighted Milne-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|\\
\leq&\frac{5\|w\|_\infty(\epsilon-\eta)^2}{24}\left[\left(\frac{4\left|\mathcal{P}'(\eta)\right|^q+\left|\mathcal{P}'(\epsilon)\right|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{4\left|\mathcal{P}'(\epsilon)\right|^q+\left|\mathcal{P}'(\eta)\right|^q}{5}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 3.6. *By choosing $w(t) = 1$ for all $t \in [\eta, \epsilon]$ in Corollary 3.8, we obtain the following Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right] - \frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(t)dt\right|$$
$$\leq \frac{5(\epsilon-\eta)}{24}\left[\left(\frac{4|\mathcal{P}'(\eta)|^q + |\mathcal{P}'(\epsilon)|^q}{5}\right)^{\frac{1}{q}} + \left(\frac{4|\mathcal{P}'(\epsilon)|^q + |\mathcal{P}'(\eta)|^q}{5}\right)^{\frac{1}{q}}\right],$$

which is given by Budak et al. [20].

4. Weighted fractional Milne-type inequalities for bounded functions

Within this portion, we present a set of weighted fractional Milne-type inequalities designed for functions that are bounded.

Theorem 4.1. *Let us assume that the conditions outlined in Lemma 2.1 are satisfied. If there exist real numbers m and M such that $m \leq \mathcal{P}'(t) \leq M$ for $t \in [\eta, \epsilon]$, then it results in*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right]W(\delta) - \frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right) + \mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq \frac{M-m}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}} W_1(\delta,t)\,dt + \int_{\frac{\eta+\epsilon}{2}}^{\epsilon} W_2(\delta,t)\,dt + \frac{\epsilon-\eta}{3}W(\delta)\right].$$

Proof. By using Lemma 2.1, we have

$$\frac{1}{3}\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right]W(\delta) - \frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right) + \mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right] \tag{4.1}$$
$$= \frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t) + \frac{1}{3}W(\delta)\right]\left[\frac{m+M}{2} - \mathcal{P}'(t)\right]dt\right.$$
$$+ \int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t) + \frac{1}{3}W(\delta)\right]\left[\mathcal{P}'(\eta+\epsilon-t) - \frac{m+M}{2}\right]dt$$
$$+ \int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_1(\delta,t) + \frac{1}{3}W(\delta)\right]\left[\mathcal{P}'(t) - \frac{m+M}{2}\right]dt$$
$$\left.+ \int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_1(\delta,t) + \frac{1}{3}W(\delta)\right]\left[\frac{m+M}{2} - \mathcal{P}'(\eta+\epsilon-t)\right]dt\right].$$

By taking the absolute value of (4.1), we derive

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\frac{m+M}{2}-\mathcal{P}'(t)\right|dt\right.$$
$$+\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(\eta+\epsilon-t)-\frac{m+M}{2}\right|dt$$
$$+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(t)-\frac{m+M}{2}\right|dt$$
$$\left.+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\frac{m+M}{2}-\mathcal{P}'(\eta+\epsilon-t)\right|dt\right].$$

Given that $m\leq\mathcal{P}'(t)\leq M$ for $t\in[\eta,\epsilon]$, we can deduce

$$\left|\mathcal{P}'(t)-\frac{m+M}{2}\right|\leq\frac{M-m}{2} \tag{4.2}$$

and

$$\left|\mathcal{P}'(\eta+\epsilon-t)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}. \tag{4.3}$$

If we consider (4.2) and (4.3), then we get

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]dt\right](M-m)$$
$$=\frac{M-m}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_1(\delta,t)\,dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_2(\delta,t)\,dt+\frac{\epsilon-\eta}{3}W(\delta)\right].$$

This completes the proof. □

Remark 4.1. *Selecting $w(t)=1$ for all $t\in[\eta,\epsilon]$ in Theorem 4.1, we arrive at the following fractional Milne-type inequality, as demonstrated by Budak et al. [20]:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\epsilon-\eta)^{\delta}}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{\epsilon-\eta}{12}\left(\frac{\delta+4}{\delta+1}\right)(M-m).$$

Corollary 4.1. *When $\delta = 1$ is chosen in Theorem 4.1, the resulting weighted Milne-type inequality is as follows:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|$$
$$\leq\left(\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(t-\frac{7\eta-\epsilon}{6}\right)w(t)dt\right)(M-m).$$

Proof. The proof is obvious from the equality (3.3). □

Corollary 4.2. *Under the assumptions outlined in Theorem 4.1, we arrive at the following weighted fractional Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{(\epsilon-\eta)^{\delta+1}\|w\|_{\infty}}{2^{\delta+2}}\left(\frac{\delta+4}{\delta(\delta+1)}\right)(M-m).$$

Corollary 4.3. *When $\delta = 1$ is selected in Corollary 4.2, we obtain the following weighted Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|\leq\frac{5(\epsilon-\eta)^{2}\|w\|_{\infty}}{24}(M-m).$$

Remark 4.2. *When $w(t) = 1$ for all $t \in [\eta, \epsilon]$ in Corollary 4.3, the ensuing Milne-type inequality is as follows:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(t)dt\right|\leq\frac{5(\epsilon-\eta)}{24}(M-m),$$

which is given by Budak et al. [20].

Corollary 4.4. *Given the conditions stipulated in Theorem 4.1, if there exists $M \in \mathbb{R}^+$ such that $|\mathcal{P}(t)| \leq M$ for all $t \in [\eta, \epsilon]$, then we obtain*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{M}{2}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_{1}(\delta,t)\,dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_{2}(\delta,t)\,dt+\frac{\epsilon-\eta}{3}W(\delta)\right].$$

5. Weighted fractional Milne-type inequalities for Lipschitzian functions

Next, we present several weighted fractional Milne-type inequalities specifically for Lipschitzian functions.

Theorem 5.1. *Suppose the conditions of Lemma 2.1 are satisfied. If $\mathcal{P}'$ is an L-Lipschitzian function on $[\eta,\epsilon]$, then the ensuing inequality holds:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{L}{2}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_1(\delta,t)\left(\frac{\eta+\epsilon}{2}-t\right)dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_2(\delta,t)\left(t-\frac{\eta+\epsilon}{2}\right)dt+\frac{(\epsilon-\eta)^2}{12}W(\delta)\right].$$

Proof. Using the fact that $\mathcal{P}'$ is L-Lipschitzian function, by Lemma 2.1, we have

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(t)-\mathcal{P}'(\eta+\epsilon-t)\right|dt\right.$$
$$\left.+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]\left|\mathcal{P}'(t)-\mathcal{P}'(\eta+\epsilon-t)\right|dt\right]$$
$$\leq\frac{1}{4}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]L\left|2t-(a+\epsilon)\right|dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]L\left|2t-(a+\epsilon)\right|dt\right]$$
$$=\frac{L}{2}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[W_1(\delta,t)+\frac{1}{3}W(\delta)\right]\left(\frac{\eta+\epsilon}{2}-t\right)dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[W_2(\delta,t)+\frac{1}{3}W(\delta)\right]\left(t-\frac{\eta+\epsilon}{2}\right)dt\right]$$
$$=\frac{L}{2}\left[\int_{\eta}^{\frac{\eta+\epsilon}{2}}W_1(\delta,t)\left(\frac{\eta+\epsilon}{2}-t\right)dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}W_2(\delta,t)\left(t-\frac{\eta+\epsilon}{2}\right)dt+\frac{(\epsilon-\eta)^2}{12}W(\delta)\right].$$

This completes the proof. □

Remark 5.1. *When we set $w(t)=1$ for all $t\in[\eta,\epsilon]$ in Theorem 5.1, the resulting fractional Milne-type inequality, proven by Budak et al. [20], is as follows:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\epsilon-\eta)^{\delta}}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{(\epsilon-\eta)^2}{24}\left(\frac{\delta+8}{\delta+2}\right)L.$$

Corollary 5.1. *If we select $\delta = 1$ in Theorem 5.1, we obtain the subsequent weighted Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon} w(t)dt-\int_{\eta}^{\epsilon} w(t)\mathcal{P}(t)dt\right| \tag{5.1}$$
$$\leq\frac{L}{4}\left(\int_{\eta}^{\epsilon}\left[\frac{(\epsilon-\eta)^2}{3}-\left(\frac{\eta+\epsilon}{2}-t\right)^2\right]w(t)dt\right).$$

Proof. As the function w is symmetric with respect to $\frac{\eta+\epsilon}{2}$, the proof of (5.1) follows straightforwardly from the fact that

$$\begin{aligned}
&\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left(\int_{t}^{\frac{\eta+\epsilon}{2}} w(u)du\right)\left(\frac{\eta+\epsilon}{2}-t\right)dt+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left(\int_{\frac{\eta+\epsilon}{2}}^{t} w(u)du\right)\left(t-\frac{\eta+\epsilon}{2}\right)dt+\frac{(\epsilon-\eta)^2}{24}\int_{\eta}^{\epsilon} w(t)dt\\
=&\int_{\eta}^{\frac{\eta+\epsilon}{2}}\int_{\eta}^{u} w(u)\left(\frac{\eta+\epsilon}{2}-t\right)dtdu+\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\int_{u}^{\epsilon} w(u)\left(t-\frac{\eta+\epsilon}{2}\right)dtdu+\frac{(\epsilon-\eta)^2}{24}\int_{\eta}^{\epsilon} w(t)dt\\
=&\frac{1}{2}\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\left(\frac{\epsilon-\eta}{2}\right)^2-\left(\frac{\eta+\epsilon}{2}-u\right)^2\right]w(u)du+\frac{1}{2}\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[\left(\frac{\epsilon-\eta}{2}\right)^2-\left(u-\frac{\eta+\epsilon}{2}\right)^2\right]w(u)du\\
&+\frac{(\epsilon-\eta)^2}{24}\int_{\eta}^{\epsilon} w(t)dt\\
=&\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\left(\frac{\epsilon-\eta}{2}\right)^2-\left(\frac{\eta+\epsilon}{2}-t\right)^2\right]w(t)dt+\frac{(\epsilon-\eta)^2}{12}\int_{\eta}^{\frac{\eta+\epsilon}{2}} w(t)dt\\
=&\int_{\eta}^{\frac{\eta+\epsilon}{2}}\left[\frac{(\epsilon-\eta)^2}{3}-\left(\frac{\eta+\epsilon}{2}-t\right)^2\right]w(t)dt\\
=&\int_{\frac{\eta+\epsilon}{2}}^{\epsilon}\left[\frac{(\epsilon-\eta)^2}{3}-\left(\frac{\eta+\epsilon}{2}-t\right)^2\right]w(t)dt\\
=&\frac{1}{2}\int_{\eta}^{\epsilon}\left[\frac{(\epsilon-\eta)^2}{3}-\left(\frac{\eta+\epsilon}{2}-t\right)^2\right]w(t)dt.
\end{aligned}$$

Corollary 5.2. *Under the conditions specified in Theorem 5.1, the following weighted fractional Milne-type inequality holds:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{(\epsilon-\eta)^{\delta+2}L\|w\|_{\infty}}{2^{\delta+3}}\left(\frac{\delta+8}{3\delta(\delta+2)}\right).$$

Corollary 5.3. *If we choose $\delta = 1$ in Corollary 5.2, then we have the following weighted Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|\leq\frac{(\epsilon-\eta)^{2}\|w\|_{\infty}}{8}L.$$

Remark 5.2. *If we choose $w(t) = 1$ for all $t \in [\eta, \epsilon]$ in Corollary 5.3, then we have the following Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(t)dt\right|\leq\frac{(\epsilon-\eta)^{2}}{8}L,$$

which is given by Budak et al. [20].

6. Weighted fractional Milne-type inequalities for functions of bounded variation

Now, we present some weighted fractional Milne-type inequalities that are applicable to functions of bounded variation.

Theorem 6.1. *Let us assume that $\mathcal{P} : [\eta, \epsilon] \to \mathbb{R}$ is a function of bounded variation on $[\eta, \epsilon]$. Then, we deduce*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{2}{3}W(\delta)\bigvee_{\eta}^{\epsilon}(f).$$

Here, $\bigvee_{\eta}^{\epsilon}(f)$ denotes the total variation of $\mathcal{P}$ on $[\eta, \epsilon]$.

Proof. Define the function $K_{\delta}(x)$ by

$$K_{\delta}(x)=\begin{cases}-W_{1}(\delta,t)-\frac{1}{3}W(\delta), & \eta\leq x<\frac{\eta+\epsilon}{2},\\ W_{2}(\delta,t)+\frac{1}{3}W(\delta), & \frac{\eta+\epsilon}{2}\leq x<\epsilon.\end{cases}$$

With the help of the integrating by parts, we obtain

$$\int_{\eta}^{\epsilon} K_{\delta}(x)\, d\mathcal{P}(x) = \int_{\eta}^{\frac{\eta+\epsilon}{2}} \left[-W_1(\delta,t) - \frac{1}{3}W(\delta)\right] d\mathcal{P}(x) + \int_{\frac{\eta+\epsilon}{2}}^{\epsilon} \left[W_2(\delta,t) + \frac{1}{3}W(\delta)\right] d\mathcal{P}(x)$$
$$= \left[-W_1(\delta,t) - \frac{1}{3}W(\delta)\right]\mathcal{P}(x)\Big|_{\eta}^{\frac{\eta+\epsilon}{2}} - \int_{\eta}^{\frac{\eta+\epsilon}{2}} \left(\frac{\eta+\epsilon}{2} - x\right)^{\delta-1} w(x)\mathcal{P}(x)dx$$
$$+ \left[W_2(\delta,t) + \frac{1}{3}W(\delta)\right]\mathcal{P}(x)\Big|_{\frac{\eta+\epsilon}{2}}^{\epsilon} - \int_{\frac{\eta+\epsilon}{2}}^{\epsilon} \left(x - \frac{\eta+\epsilon}{2}\right)^{\delta-1} w(x)\mathcal{P}(x)dx$$
$$= -\frac{1}{3}W(\delta)\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + \frac{4}{3}W(\delta)\mathcal{P}(\eta) - \Gamma(\delta)\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)$$
$$+ \frac{4}{3}W(\delta)\mathcal{P}(\eta) - \frac{1}{3}W(\delta)\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) - \Gamma(\delta)\mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)$$
$$= \frac{1}{3}W(\delta)\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right] - \Gamma(\delta)\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right) + \mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right].$$

Thus, we have

$$\frac{1}{2}\int_{\eta}^{\epsilon} K_{\delta}(x)\, d\mathcal{P}(x) \tag{6.1}$$
$$= \frac{1}{3}\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right]W(\delta) - \frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right) + \mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right].$$

It is a known fact that if g and $\mathcal{P}$ are functions mapping from $[\eta,\epsilon]$ to $\mathbb{R}$ such that g is continuous on $[\eta,\epsilon]$ and $\mathcal{P}$ is of bounded variation on $[\eta,\epsilon]$, then $\int_{\eta}^{\epsilon} g(t), d\mathcal{P}(t)$ exists and

$$\left|\int_{\eta}^{\epsilon} g(t)d\mathcal{P}(t)\right| \leq \sup_{t\in[\eta,\epsilon]} |g(t)| \bigvee_{\eta}^{\epsilon}(f). \tag{6.2}$$

By using (6.1), we have

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta) - \mathcal{P}\left(\frac{\eta+\epsilon}{2}\right) + 2\mathcal{P}(\epsilon)\right]W(\delta) - \frac{\Gamma(\delta)}{2}\left[\mathcal{I}^{\delta}_{\eta+}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right) + \mathcal{I}^{\delta}_{\epsilon-}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right| \tag{6.3}$$
$$= \frac{1}{2}\left|\int_{\eta}^{\epsilon} K_{\delta}(x)d\mathcal{P}(x)\right|.$$

By applying (6.2), we have

$$\left|\int_{\eta}^{\epsilon} K_{\delta}(x)d\mathcal{P}(x)\right| \leq \left|\int_{\eta}^{\frac{\eta+\epsilon}{2}} \left[-W_1(\delta,t) - \frac{1}{3}W(\delta)\right] d\mathcal{P}(x)\right| + \left|\int_{\frac{\eta+\epsilon}{2}}^{\epsilon} \left[W_2(\delta,t) + \frac{1}{3}W(\delta)\right] d\mathcal{P}(x)\right| \tag{6.4}$$

$$\leq \sup_{x\in\left[a,\frac{\eta+\epsilon}{2}\right]}\left|-W_1(\delta,t)-\frac{1}{3}W(\delta)\right|\bigvee_{\eta}^{\frac{\eta+\epsilon}{2}}(f)+\sup_{x\in\left[\frac{\eta+\epsilon}{2},b\right]}\left|W_2(\delta,t)+\frac{1}{3}W(\delta)\right|\bigvee_{\frac{\eta+\epsilon}{2}}^{\epsilon}(f)$$

$$=\frac{4}{3}W(\delta)\bigvee_{\eta}^{\frac{\eta+\epsilon}{2}}(f)+\frac{4}{3}W(\delta)\bigvee_{\frac{\eta+\epsilon}{2}}^{\epsilon}(f)$$

$$=\frac{4}{3}W(\delta)\bigvee_{\eta}^{\epsilon}(f).$$

If we substitute inequalities (6.4) into (6.3), then we get the desired result. □

Remark 6.1. *When we set $w(t) = 1$ for all $t \in [\eta, \epsilon]$ in Theorem 6.1, the resultant fractional Milne-type inequality, established by Budak et al. [20], is as follows:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\epsilon-\eta)^{\delta}}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)\right]\right|\leq\frac{2}{3}\bigvee_{\eta}^{\epsilon}(f).$$

Corollary 6.1. *When $\delta = 1$ is chosen in Theorem 6.1, the resulting weighted Milne-type inequality is as follows:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|\leq\frac{2}{3}\left(\int_{\eta}^{\epsilon}w(t)dt\right)\bigvee_{\eta}^{\epsilon}(f). \tag{6.5}$$

Corollary 6.2. *Under the assumptions stated in Theorem 6.1, we derive the following weighted fractional Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]W(\delta)-\frac{\Gamma(\delta)}{2}\left[\mathcal{I}_{\eta+}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)+\mathcal{I}_{\epsilon-}^{\delta}\mathcal{P}w\left(\frac{\eta+\epsilon}{2}\right)\right]\right|$$
$$\leq\frac{(\epsilon-\eta)^{\delta}\|w\|_{\infty}}{3\cdot2^{\delta-1}}\bigvee_{\eta}^{\epsilon}(f).$$

Corollary 6.3. *If we choose $\delta = 1$ in Corollary 6.2, then we have the following weighted Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]\int_{\eta}^{\epsilon}w(t)dt-\int_{\eta}^{\epsilon}w(t)\mathcal{P}(t)dt\right|\leq\frac{2(\epsilon-\eta)\|w\|_{\infty}}{3}\bigvee_{\eta}^{\epsilon}(f).$$

Remark 6.2. *If we choose $w(t) = 1$ for all $t \in [\eta, \epsilon]$ in Corollary 6.3, then we have the following Milne-type inequality:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\eta)-\mathcal{P}\left(\frac{\eta+\epsilon}{2}\right)+2\mathcal{P}(\epsilon)\right]-\frac{1}{\epsilon-\eta}\int_{\eta}^{\epsilon}\mathcal{P}(t)dt\right|\leq\frac{2}{3}\bigvee_{\eta}^{\epsilon}(f),$$

which is given by Alomari [28].

7. Concluding remarks

In conclusion, this research paper has investigated weighted Milne-type inequalities utilizing Riemann-Liouville fractional integrals across diverse function classes. A key contribution has been the establishment of a fundamental integral equality, facilitated by the use of a nonnegative weighted function, which has been pivotal for deriving the main results. The paper has systematically proven weighted Milne-type inequalities for various function classes, including differentiable convex functions, bounded functions, Lipschitzian functions, and functions of bounded variation. The obtained results have not only contributed to the understanding of Milne-type inequalities but have also offered insights that have paved the way for potential future research in the considered topics. Moreover, Remaks 3.1–3.6, 4.1, 4.2, 5.1, 5.2, 6.1, and 6.2 show that the acquired results may be compared with a variety of earlier findings in the literature, several of which are included as special cases.

Author contributions

Areej A Almoneef, Abd-Allah Hyder and Hüseyin Budak: Conceptualization, Formal analysis, Investigation, Methodology, Writing-original draft, Writing-review & editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declarationt

We have used the AI, especially ChatGPT 3.5, in rephrasing some statements to provide a clear meaning for these statements and avoid the similarity in the used words.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this article.

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