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Research article

Local stability of isometries on 4-dimensional Euclidean spaces

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Abstract: In 1982, Fickett attempted to prove the Hyers-Ulam stability of isometries defined on a bounded subset of $\mathbb{R}^n$. In this paper, we applied an intuitive and efficient approach to prove the Hyers-Ulam stability of isometries defined on the bounded subset of $\mathbb{R}^4$, and we significantly improved Fickett's theorem for the four-dimensional case.

Keywords: isometry; ε-isometry; Hyers-Ulam stability; Fickett's theorem
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1. Introduction

Assume that $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ are real Hilbert spaces and D is a nonempty subset of E. Let $\varepsilon > 0$ be a given real number. According to Hyers and Ulam [1], a function $f : D \to F$ is called an *ε-isometry* if f satisfies the inequality

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \leq \varepsilon,$$

for all $x, y \in D$. If there exists a constant $K > 0$ that depends only on D and F (independent of f and ε) such that for every ε-isometry $f : D \to F$, there is an isometry $U : D \to F$ satisfying $\|f(x) - U(x)\| \leq K\varepsilon$ for all $x \in D$, then the functional equation, $\|f(x) - f(y)\| = \|x - y\|$, is said to have the *Hyers-Ulam stability*.

As is well-known, Hyers and Ulam were the first mathematicians to begin the study of the Hyers-Ulam stability of isometries (see [1]). Indeed, Hyers and Ulam proved the Hyers-Ulam stability of surjective isometry defined on the whole space by using properties of the inner product of Hilbert spaces:

Theorem 1.1. (Hyers and Ulam) *For any surjective ε-isometry $f : E \to E$ satisfying $f(0) = 0$, there* *exists a surjective isometry $U : E \to E$ satisfying $\|f(x) - U(x)\| \leq 10\varepsilon$ for all $x \in E$.*

Readers interested in more literature on similar subjects are referred to [2–8] and the references cited therein.

To the best of our knowledge, Fickett [9] was the first mathematician who studied the Hyers-Ulam stability of isometries whose domains are bounded.

Theorem 1.2. (Fickett) *Given an integer $n \geq 2$, let D be a bounded subset of $\mathbb{R}^n$ and let $\varepsilon > 0$ be given. If a function $f : D \to \mathbb{R}^n$ is an ε-isometry, then there exists an isometry $U : D \to \mathbb{R}^n$ such that*

$$\|f(x) - U(x)\| \leq 27\varepsilon^{1/2^n}, \tag{1.1}$$

for any $x \in D$.

Comparing Fickett's theorem with the definition of Hyers-Ulam stability mentioned at the beginning, it is obvious that although Fickett did not prove the Hyers-Ulam stability of isometries in the strict sense, his goal was to prove the Hyers-Ulam stability of isometries on the bounded domain.

We are much more interested in the rate at which the upper bound of inequality (1.1) decreases as the value of ε decreases to 0. One obvious weakness of Fickett's theorem is that the upper bound of inequality (1.1) decreases very slowly to 0 as ε approaches 0. Roughly speaking, the problem is that the speed of convergence to 0 is too slow. Since the purpose of this paper is to improve Fickett's theorem, which has the shortcoming pointed out previously, Fickett's theorem is a great motivation for writing this paper.

Since Fickett attempted to prove the Hyers-Ulam stability of isometries whose domains are bounded subsets of $\mathbb{R}^n$, over the past 40 years, several mathematicians have made steady attempts to improve Fickett's result. For example, see Alestalo et al. [10], Väisälä [11], Vestfrid [12], Jung [13], and Choi and Jung [14].

In [13, 14], the first author has already proven the local stability of isometries in the n-dimensional Euclidean space $\mathbb{R}^n$. However, if we look closely at the local stability results in low-dimensional spaces such as the three-dimensional or four-dimensional space in [13, 14], we can see that the error term appears much more inflated than the actual one. For this reason, we now try to reduce the error term that occurs when studying the local stability of isometries in four-dimensional Euclidean space. We note that the local stability of isometries in two-dimensional and three-dimensional Euclidean spaces has been addressed in [15].

In this paper, by applying the analytic method used in [13] and allowing the values for c_{ij} to be real numbers, as well as by constraining ε to have small values, we improve Fickett's theorem in the case of the four-dimensional Euclidean space. The analytic method applied in this paper is completely different from those used in [10–12]. In other words, we will prove the Hyers-Ulam stability of isometries whose domain is a bounded subset of $\mathbb{R}^4$ by using an analytic method that is completely different from the conventional method.

2. QR decomposition

An orthogonal matrix $\mathbf{Q}$ is a real square matrix whose columns and rows are orthonormal vectors. In other words, a real square matrix $\mathbf{Q}$ is orthogonal if its transpose is equal to its inverse: $\mathbf{Q}^{tr} = \mathbf{Q}^{-1}$.

As a linear transformation, an orthogonal matrix preserves the inner product of vectors, and therefore acts as an isometry of Euclidean space.

Throughout this paper, we assume that $\{e_1, e_2, e_3, e_4\}$ is the standard basis for the four-dimensional Euclidean space $\mathbb{R}^4$. In addition, let D be a subset of $\mathbb{R}^4$ that includes the set $\{0, e_1, e_2, e_3, e_4\}$. Whether D is bounded or not has no influence on the results in Sections 2 and 3 of this paper.

We now introduce the real version of QR decomposition, which was presented in [13, Theorem 2.1].

Theorem 2.1. (QR decomposition) *Every real square matrix* $\mathbf{A}$ *can be decomposed as* $\mathbf{A} = \mathbf{QR}$*, where* $\mathbf{Q}$ *is an orthogonal matrix and* $\mathbf{R}$ *is an upper triangular matrix whose entries are real numbers. In particular, every diagonal entry of* $\mathbf{R}$ *is nonnegative.*

Given a function $f : D \to \mathbb{R}^4$, we define a 4×4 matrix $\mathbf{A}$ by

$$\mathbf{A} = (f(e_1)\ \ f(e_2)\ \ f(e_3)\ \ f(e_4)),$$

where each $f(e_i)$ is written in column vector. By Theorem 2.1, there is an orthogonal matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$ whose entries are real numbers and whose diagonal entries are all nonnegative such that $\mathbf{A} = \mathbf{QR}$ or $\mathbf{Q}^{tr}\mathbf{A} = \mathbf{R}$. Hence, it holds that $\mathbf{Q}^{tr}\mathbf{A}e_i = \mathbf{R}e_i$ or $\mathbf{Q}^{tr}f(e_i) = \mathbf{R}e_i$ for every $i \in \{1, 2, 3, 4\}$. In other words, if we explicitly express the upper triangular matrix $\mathbf{R}$ as

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{21} & r_{31} & r_{41} \\ 0 & r_{22} & r_{32} & r_{42} \\ 0 & 0 & r_{33} & r_{43} \\ 0 & 0 & 0 & r_{44} \end{pmatrix},$$

where each r_{ij} is a real number and r_{11}, r_{22}, r_{33}, r_{44} are nonnegative, then we have

$$\begin{aligned} \mathbf{Q}^{tr}f(e_1) &= \begin{pmatrix} r_{11} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Q}^{tr}f(e_2) = \begin{pmatrix} r_{21} \\ r_{22} \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{Q}^{tr}f(e_3) &= \begin{pmatrix} r_{31} \\ r_{32} \\ r_{33} \\ 0 \end{pmatrix}, \quad \mathbf{Q}^{tr}f(e_4) = \begin{pmatrix} r_{41} \\ r_{42} \\ r_{43} \\ r_{44} \end{pmatrix}. \end{aligned} \tag{2.1}$$

If we change the standard basis $\{e_1, e_2, e_3, e_4\}$ to the new basis $Q = \{\mathbf{Q}e_1, \mathbf{Q}e_2, \mathbf{Q}e_3, \mathbf{Q}e_4\}$ for $\mathbb{R}^4$, then the *change-of-coordinates matrix* from Q to the standard basis in $\mathbb{R}^4$ is the 4×4 nonsingular matrix defined by

$$\mathbf{C}_Q = (\mathbf{Q}e_1\ \ \mathbf{Q}e_2\ \ \mathbf{Q}e_3\ \ \mathbf{Q}e_4) = \mathbf{Q},$$

where each column $\mathbf{Q}e_i$ is written in column vector.

The *Q-coordinates* of x are the weights c_1, c_2, c_3, c_4 such that

$$x = c_1\mathbf{Q}e_1 + c_2\mathbf{Q}e_2 + c_3\mathbf{Q}e_3 + c_4\mathbf{Q}e_4, \tag{2.2}$$

where c_1, c_2, c_3, c_4 are uniquely determined real numbers that depend only on the choice of $x \in \mathbb{R}^4$. We use the symbol $[x]_Q$ to denote the Q-coordinates of x. More precisely,

$$[x]_Q = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

It then follows from (2.2) that

$$x = \mathbf{C}_Q [x]_Q, \tag{2.3}$$

for all $x \in \mathbb{R}^4$.

We now put

$$[f(e_1)]_Q = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix},$$

and use (2.1) and (2.3) to get

$$[f(e_1)]_Q = \mathbf{Q}^{-1} f(e_1) = \mathbf{Q}^{tr} f(e_1) = \begin{pmatrix} r_{11} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, it follows that $f_1 = r_{11}$ and $f_2 = f_3 = f_4 = 0$. Similarly, we can obtain

$$[f(e_2)]_Q = \begin{pmatrix} r_{21} \\ r_{22} \\ 0 \\ 0 \end{pmatrix}, \quad [f(e_3)]_Q = \begin{pmatrix} r_{31} \\ r_{32} \\ r_{33} \\ 0 \end{pmatrix}, \quad [f(e_4)]_Q = \begin{pmatrix} r_{41} \\ r_{42} \\ r_{43} \\ r_{44} \end{pmatrix}.$$

Consequently, we can assume without loss of generality that

$$\begin{aligned} f(e_1) &= (r_{11}, 0, 0, 0), \\ f(e_2) &= (r_{21}, r_{22}, 0, 0), \\ f(e_3) &= (r_{31}, r_{32}, r_{33}, 0), \\ f(e_4) &= (r_{41}, r_{42}, r_{43}, r_{44}) \end{aligned} \tag{2.4}$$

written in row vectors for convenience, where each r_{ii} is nonnegative. The effect of the action of the orthogonal matrix $\mathbf{Q}$ or $\mathbf{Q}^{tr}$ appears as a rotation, which will of course be taken into account by introducing an appropriate isometry later in this paper.

3. A preliminary theorem

In the following theorem, let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for the four-dimensional Euclidean space $\mathbb{R}^4$, where we set $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, and $e_4 = (0, 0, 0, 1)$.

As already mentioned before, we are interested in the decreasing rate of the upper bound of inequality (1.1) at small values of ε than the decreasing rate at relatively large values of ε. Therefore, it is not at all strange that we constrain the value of ε to be less than $\frac{1}{1000}$ in the following theorem.

Theorem 3.1. *Assume that a subset D of the four-dimensional Euclidean space $\mathbb{R}^4$ includes the origin 0 as well as the standard basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$ and that a function $f : D \to \mathbb{R}^4$ satisfies $f(0) = 0$ and the inequality*

$$\big| \|f(x) - f(y)\| - \|x - y\| \big| \leq \varepsilon, \tag{3.1}$$

for all $x, y \in \{0, e_1, e_2, e_3, e_4\}$ and for some constant ε with $0 < \varepsilon < \frac{1}{1000}$. According to (2.4), it can be assumed that $f(e_1) = (r_{11}, 0, 0, 0)$, $f(e_2) = (r_{21}, r_{22}, 0, 0)$, $f(e_3) = (r_{31}, r_{32}, r_{33}, 0)$, and $f(e_4) = (r_{41}, r_{42}, r_{43}, r_{44})$, where $r_{11} \geq 0$, $r_{22} \geq 0$, $r_{33} \geq 0$, and $r_{44} \geq 0$. Then, there exist positive real numbers c_{ij}, $i, j \in \{1, 2, 3, 4\}$ with $j \leq i$, such that the inequalities

$$\begin{cases} -c_{ij}\varepsilon \leq r_{ij} \leq c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \leq r_{ii} \leq 1 + \varepsilon & (\text{for } i = j) \end{cases} \tag{3.2}$$

are true. In particular, $c_{11} = 1.00000$, $c_{21} = 3.41814$, $c_{22} = 1.00585$, $c_{31} = 3.41814$, $c_{32} = 3.42985$, $c_{33} = 1.01174$, $c_{41} = 3.41814$, $c_{42} = 3.42985$, $c_{43} = 3.44165$, and $c_{44} = 1.01767$.

Proof. It follows from inequality (3.1) and our assumption, $f(0) = 0$, that

$$\big|\|f(e_1)\| - 1\big| \leq \varepsilon, \quad \big|\|f(e_2)\| - 1\big| \leq \varepsilon, \quad \big|\|f(e_3)\| - 1\big| \leq \varepsilon, \quad \big|\|f(e_4)\| - 1\big| \leq \varepsilon,$$
$$\big|\|f(e_1) - f(e_2)\| - \sqrt{2}\,\big| \leq \varepsilon, \quad \big|\|f(e_2) - f(e_3)\| - \sqrt{2}\,\big| \leq \varepsilon, \quad \big|\|f(e_3) - f(e_4)\| - \sqrt{2}\,\big| \leq \varepsilon,$$
$$\big|\|f(e_4) - f(e_1)\| - \sqrt{2}\,\big| \leq \varepsilon, \quad \big|\|f(e_1) - f(e_3)\| - \sqrt{2}\,\big| \leq \varepsilon, \quad \big|\|f(e_2) - f(e_4)\| - \sqrt{2}\,\big| \leq \varepsilon,$$

for any ε with $0 < \varepsilon < \frac{1}{1000}$. Therefore, from the inequalities above, we obtain the following inequalities:

$$1 - \varepsilon \leq r_{11} \leq 1 + \varepsilon, \tag{3.3}$$
$$(1 - \varepsilon)^2 \leq r_{21}^2 + r_{22}^2 \leq (1 + \varepsilon)^2, \tag{3.4}$$
$$(\sqrt{2} - \varepsilon)^2 \leq (r_{11} - r_{21})^2 + r_{22}^2 \leq (\sqrt{2} + \varepsilon)^2, \tag{3.5}$$
$$(1 - \varepsilon)^2 \leq r_{31}^2 + r_{32}^2 + r_{33}^2 \leq (1 + \varepsilon)^2, \tag{3.6}$$
$$(\sqrt{2} - \varepsilon)^2 \leq (r_{21} - r_{31})^2 + (r_{22} - r_{32})^2 + r_{33}^2 \leq (\sqrt{2} + \varepsilon)^2, \tag{3.7}$$
$$(\sqrt{2} - \varepsilon)^2 \leq (r_{31} - r_{11})^2 + r_{32}^2 + r_{33}^2 \leq (\sqrt{2} + \varepsilon)^2, \tag{3.8}$$
$$(1 - \varepsilon)^2 \leq r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2 \leq (1 + \varepsilon)^2, \tag{3.9}$$
$$(\sqrt{2} - \varepsilon)^2 \leq (r_{31} - r_{41})^2 + (r_{32} - r_{42})^2 + (r_{33} - r_{43})^2 + r_{44}^2 \leq (\sqrt{2} + \varepsilon)^2, \tag{3.10}$$
$$(\sqrt{2} - \varepsilon)^2 \leq (r_{41} - r_{11})^2 + r_{42}^2 + r_{43}^2 + r_{44}^2 \leq (\sqrt{2} + \varepsilon)^2, \tag{3.11}$$

$$(\sqrt{2}-\varepsilon)^2 \le (r_{21}-r_{41})^2 + (r_{22}-r_{42})^2 + r_{43}^2 + r_{44}^2 \le (\sqrt{2}+\varepsilon)^2. \tag{3.12}$$

In view of (3.3), (3.4), and (3.5), by the same way as [15, Lemma 3.1], we get $c_{11} = 1$ and

$$\frac{-(4+2\sqrt{2})\varepsilon + \varepsilon^2}{2r_{11}} \le r_{21} \le \frac{(4+2\sqrt{2})\varepsilon + \varepsilon^2}{2r_{11}}. \tag{3.13}$$

Since $0 < \varepsilon < \frac{1}{1000}$, it follows from (3.3) that

$$\frac{1}{2r_{11}} \le \frac{1}{2(1-\varepsilon)} < \frac{500}{999}.$$

Hence, it follows from the last two inequalities that

$$r_{21} \le \frac{500}{999}(4+2\sqrt{2}+\varepsilon)\varepsilon < \frac{500}{999}\left(4+2\sqrt{2}+\frac{1}{1000}\right)\varepsilon.$$

Comparing the first condition in (3.2) and the last inequality, we can choose c_{21} as follows:

$$\frac{500}{999}\left(4+2\sqrt{2}+\frac{1}{1000}\right) < 3.41814 =: c_{21}. \tag{3.14}$$

By (3.4), we have

$$(1-\varepsilon)^2 - r_{21}^2 \le r_{22}^2 \le (1+\varepsilon)^2 - r_{21}^2.$$

If we apply the first condition in (3.2) to the previous inequality, we still get

$$(1-\varepsilon)^2 - c_{21}^2\varepsilon^2 \le r_{22}^2 \le (1+\varepsilon)^2.$$

Considering the second condition of (3.2) and the previous inequality, we will choose the constant c_{22} that satisfies

$$(1-c_{22}\varepsilon)^2 \le (1-\varepsilon)^2 - c_{21}^2\varepsilon^2,$$

and we have

$$c_{22} \ge \frac{1}{\varepsilon} - \sqrt{\left(\frac{1}{\varepsilon}-1\right)^2 - c_{21}^2},$$

whose righthand term increases strictly as ε increases in the vicinity of 0. Thus, we can set $\varepsilon = \frac{1}{1000}$ in the last inequality to determine

$$c_{22} = 1.00585.$$

By (3.3), (3.6), (3.8), and by the similar method introduced in [15, Lemma 3.3], we have

$$\frac{-(4+2\sqrt{2})\varepsilon + \varepsilon^2}{2r_{11}} \le r_{31} \le \frac{(4+2\sqrt{2})\varepsilon + \varepsilon^2}{2r_{11}},$$

which has the same formula as inequality (3.13) for r_{21}. Thus, it follows from (3.14) that

$$c_{31} = c_{21} = 3.41814.$$

On account of the first condition in (3.2), (3.4), (3.6), and (3.7), and by applying a similar method to [15, Lemma 3.3], we obtain

$$-(4+2\sqrt{2})\varepsilon + \varepsilon^2 \le 2r_{21}r_{31} + 2r_{22}r_{32} \le (4+2\sqrt{2})\varepsilon + \varepsilon^2,$$

and since $c_{21} = c_{31}$, we further have

$$\frac{-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}^2\varepsilon^2}{2r_{22}} \le r_{32} \le \frac{(4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}^2\varepsilon^2}{2r_{22}}. \tag{3.15}$$

In view of the both conditions in (3.2) and (3.15) and by using our assumption that $0 < \varepsilon < \frac{1}{1000}$, we will determine c_{32} that satisfies

$$\frac{4+2\sqrt{2}+(2c_{21}^2+1)\varepsilon}{2r_{22}} \le \frac{4+2\sqrt{2}+(2c_{21}^2+1)\varepsilon}{2(1-c_{22}\varepsilon)} \le \frac{4001+2000\sqrt{2}+2c_{21}^2}{2000-2c_{22}} \le c_{32}.$$

So, we can set $c_{32} = 3.42985$.

Moreover, using the first condition in (3.2) and (3.6), we have

$$(1-\varepsilon)^2 - c_{31}^2\varepsilon^2 - c_{32}^2\varepsilon^2 \le r_{33}^2 \le (1+\varepsilon)^2.$$

Referring to the second condition in (3.2), we solve the following inequality for the unknown c_{33}:

$$(1-c_{33}\varepsilon)^2 \le (1-\varepsilon)^2 - (c_{31}^2 + c_{32}^2)\varepsilon^2,$$

and we obtain

$$c_{33} \ge \frac{1}{\varepsilon} - \sqrt{\left(\frac{1}{\varepsilon} - 1\right)^2 - c_{31}^2 - c_{32}^2},$$

where the righthand term increases strictly as ε increases in the vicinity of 0. We put $\varepsilon = \frac{1}{1000}$ in the last inequality and we determine

$$c_{33} = 1.01174.$$

Now, it follows from (3.11) that

$$\begin{aligned} &(\sqrt{2}-\varepsilon)^2 - r_{11}^2 - (r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2) \\ \le\ & -2r_{11}r_{41} \\ \le\ & (\sqrt{2}+\varepsilon)^2 - r_{11}^2 - (r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2). \end{aligned}$$

By (3.3) and (3.9), we have

$$\frac{-(4+2\sqrt{2})\varepsilon + \varepsilon^2}{2r_{11}} \le r_{41} \le \frac{(4+2\sqrt{2})\varepsilon + \varepsilon^2}{2r_{11}},$$

which has the same formula as inequality (3.13) for r_{21}. Therefore, we can take the value of c_{41} to be the same value of c_{21}, i.e., according to (3.14) we have

$$c_{41} = c_{21} = 3.41814.$$

On account of (3.12), we obtain

$$\begin{aligned} & (\sqrt{2}-\varepsilon)^2 - (r_{21}^2 + r_{22}^2) - (r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2) \\ \leq \ & -2r_{21}r_{41} - 2r_{22}r_{42} \\ \leq \ & (\sqrt{2}+\varepsilon)^2 - (r_{21}^2 + r_{22}^2) - (r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2). \end{aligned}$$

By (3.4) and (3.9), we have

$$-(4+2\sqrt{2})\varepsilon + \varepsilon^2 \leq 2r_{21}r_{41} + 2r_{22}r_{42} \leq (4+2\sqrt{2})\varepsilon + \varepsilon^2$$

or

$$-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2r_{21}r_{41} \leq 2r_{22}r_{42} \leq (4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2r_{21}r_{41}.$$

Moreover, it follows from (3.2) that

$$-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}c_{41}\varepsilon^2 \leq 2r_{22}r_{42} \leq (4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}c_{41}\varepsilon^2.$$

Due to the last inequality, it holds that

$$\frac{-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}c_{41}\varepsilon^2}{2r_{22}} \leq r_{42} \leq \frac{(4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}c_{41}\varepsilon^2}{2r_{22}}.$$

In view of inequality (3.15), together with the fact $c_{21} = c_{41}$, we can take the value of c_{42} to be the same value of c_{32}, i.e., $c_{42} = c_{32} = 3.42985$.

From (3.10), it follows that

$$\begin{aligned} & (\sqrt{2}-\varepsilon)^2 - (r_{31}^2 + r_{32}^2 + r_{33}^2) - (r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2) \\ \leq \ & -2r_{31}r_{41} - 2r_{32}r_{42} - 2r_{33}r_{43} \\ \leq \ & (\sqrt{2}+\varepsilon)^2 - (r_{31}^2 + r_{32}^2 + r_{33}^2) - (r_{41}^2 + r_{42}^2 + r_{43}^2 + r_{44}^2). \end{aligned}$$

By (3.6) and (3.9), we obtain

$$\begin{aligned} & -(4+2\sqrt{2})\varepsilon - \varepsilon^2 + 2r_{31}r_{41} + 2r_{32}r_{42} \\ \leq \ & -2r_{33}r_{43} \\ \leq \ & (4+2\sqrt{2})\varepsilon - \varepsilon^2 + 2r_{31}r_{41} + 2r_{32}r_{42}. \end{aligned}$$

Thus, it follows from (3.2) that

$$\begin{aligned} & -(4+2\sqrt{2})\varepsilon - (2c_{31}c_{41} + 2c_{32}c_{42} + 1)\varepsilon^2 \\ \leq \ & -2r_{33}r_{43} \\ \leq \ & (4+2\sqrt{2})\varepsilon + (2c_{31}c_{41} + 2c_{32}c_{42} - 1)\varepsilon^2. \end{aligned}$$

Furthermore, using (3.2) again, we have

$$\begin{aligned} & \frac{-(4+2\sqrt{2})\varepsilon-(2c_{31}c_{41}+2c_{32}c_{42}-1)\varepsilon^2}{2(1-c_{33}\varepsilon)} \\ \le\ & r_{43} \\ \le\ & \frac{(4+2\sqrt{2})\varepsilon+(2c_{31}c_{41}+2c_{32}c_{42}+1)\varepsilon^2}{2(1-c_{33}\varepsilon)}. \end{aligned}$$

In view of (3.2) and the last inequality, we determine the minimum value of the positive constant c_{43}, which satisfies the following inequality:

$$\begin{aligned} -c_{43}\varepsilon &\le \frac{-(4+2\sqrt{2})\varepsilon-(2c_{31}c_{41}+2c_{32}c_{42}-1)\varepsilon^2}{2(1-c_{33}\varepsilon)} \\ &\le r_{43} \\ &\le \frac{(4+2\sqrt{2})\varepsilon+(2c_{31}c_{41}+2c_{32}c_{42}+1)\varepsilon^2}{2(1-c_{33}\varepsilon)} \\ &\le c_{43}\varepsilon. \end{aligned}$$

We divide by ε the righthand side of the last inequality and put $\varepsilon = \frac{1}{1000}$ to get

$$c_{43} = 3.44165.$$

Finally, it follows from (3.9) that

$$(1-\varepsilon)^2 - r_{41}^2 - r_{42}^2 - r_{43}^2 \le r_{44}^2 \le (1+\varepsilon)^2 - r_{41}^2 - r_{42}^2 - r_{43}^2.$$

If c_{44} is a solution of the inequality

$$(1-c_{44}\varepsilon)^2 \le (1-\varepsilon)^2 - (c_{41}^2 + c_{42}^2 + c_{43}^2)\varepsilon^2,$$

it then follows from the last two inequalities that

$$(1-c_{44}\varepsilon)^2 \le r_{44}^2 \le (1+\varepsilon)^2,$$

which is consistent with the second condition in (3.2). Hence, we obtain

$$c_{44} \ge \frac{1}{\varepsilon} - \sqrt{\left(\frac{1}{\varepsilon}-1\right)^2 - c_{41}^2 - c_{42}^2 - c_{43}^2},$$

whose righthand term increases strictly as ε increases in the vicinity of 0. We put $\varepsilon = \frac{1}{1000}$ in the last inequality and we determine

$$c_{44} = 1.01767,$$

which completes the proof. □

4. Hyers-Ulam stability of isometries

In the following theorem, let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, and $e_4 = (0,0,0,1)$. We denote by $B_d(0)$ the closed ball of radius d and centered at the origin of $\mathbb{R}^4$, i.e., $B_d(0) = \{x \in \mathbb{R}^4 : \|x\| \leq d\}$. The values of c_{ij} in the following theorem have already been presented in Theorem 3.1.

Theorem 4.1. *Let D be a subset of the four-dimensional Euclidean space $\mathbb{R}^4$ such that $\{0, e_1, e_2, e_3, e_4\} \subset D \subset B_d(0)$ for some $d \geq 1$, and let $f : D \to \mathbb{R}^4$ be a function that satisfies $f(0) = 0$ and inequality (3.1) for all $x, y \in D$ and for some constant ε with $0 < \varepsilon < \frac{1}{1000}$. Then, there exists an isometry $U : D \to \mathbb{R}^4$ such that*

$$\|f(x) - U(x)\| \leq \left(\sum_{i=1}^{4} \left(\left(d + \frac{1}{1000} \right) \left(2 + \sum_{j=1}^{i} c_{ij} \right) + 2 \right)^2 \right)^{1/2} \varepsilon,$$

for all $x \in D$.

Proof. We note that $\{e_1, e_2, e_3, e_4\}$ is the standard basis for $\mathbb{R}^4$. According to (2.4), it may be assumed that

$$f(e_i) = (r_{i1}, r_{i2}, \ldots, r_{ii}, 0, \ldots, 0),$$

where $r_{ii} \geq 0$ for every $i \in \{1,2,3,4\}$. For any point $x = (x_1, x_2, x_3, x_4)$ of D, let $f(x) = (x_1', x_2', x_3', x_4')$.

First, it follows from (3.1) that

$$\big| \|f(x)\| - \|x\| \big| \leq \varepsilon \quad \text{and} \quad \big| \|f(x) - f(e_j)\| - \|x - e_j\| \big| \leq \varepsilon,$$

and, hence, we have

$$\left| \left(\sum_{i=1}^{4} x_i'^2 \right)^{1/2} - \left(\sum_{i=1}^{4} x_i^2 \right)^{1/2} \right| \leq \varepsilon, \tag{4.1}$$

$$\left| \left(\sum_{i=1}^{j} (x_i' - r_{ji})^2 + \sum_{i=j+1}^{4} x_i'^2 \right)^{1/2} - \left(\sum_{i=1}^{4} x_i^2 - 2x_j + 1 \right)^{1/2} \right| \leq \varepsilon, \tag{4.2}$$

for all $x \in D$ and $j \in \{1,2,3,4\}$.

Now, by (4.1), we have

$$\begin{aligned} \left| \sum_{i=1}^{4} x_i'^2 - \sum_{i=1}^{4} x_i^2 \right| &= \left| \left(\sum_{i=1}^{4} x_i'^2 \right)^{1/2} + \left(\sum_{i=1}^{4} x_i^2 \right)^{1/2} \right| \left| \left(\sum_{i=1}^{4} x_i'^2 \right)^{1/2} - \left(\sum_{i=1}^{4} x_i^2 \right)^{1/2} \right| \\ &\leq \left(2d + \frac{1}{1000} \right) \varepsilon, \end{aligned} \tag{4.3}$$

since $\sqrt{x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2} \leq d + \varepsilon$, $\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \leq d$, and $0 < \varepsilon < \frac{1}{1000}$. Similarly, since $0 < \varepsilon < \frac{1}{1000}$,

$$\left(\sum_{i=1}^{j} (x_i' - r_{ji})^2 + \sum_{i=j+1}^{4} x_i'^2 \right)^{1/2} = \|f(x) - f(e_j)\| \leq \|x - e_j\| + \varepsilon \leq d + 1 + \varepsilon,$$

and

$$\left(\sum_{i=1}^{4} x_i^2 - 2x_j + 1\right)^{1/2} = \|x - e_j\| \le d + 1,$$

we use (4.2) to show that

$$\left|\left(\sum_{i=1}^{j}(x_i' - r_{ji})^2 + \sum_{i=j+1}^{4} x_i'^2\right) - \left(\sum_{i=1}^{4} x_i^2 - 2x_j + 1\right)\right| \le \left(2d + \frac{2001}{1000}\right)\varepsilon, \tag{4.4}$$

for all $j \in \{1, 2, 3, 4\}$.

It then follows from (4.4) that

$$\left|\left(\sum_{i=1}^{4} x_i'^2 - \sum_{i=1}^{4} x_i^2\right) - \sum_{i=1}^{j-1} 2r_{ji}x_i' - 1 + \sum_{i=1}^{j} r_{ji}^2 + (2x_j - 2r_{jj}x_j')\right| \le \left(2d + \frac{2001}{1000}\right)\varepsilon,$$

i.e.,

$$\begin{aligned} &-\left(2d + \frac{2001}{1000}\right)\varepsilon - \left(\sum_{i=1}^{4} x_i'^2 - \sum_{i=1}^{4} x_i^2\right) + \sum_{i=1}^{j-1} 2r_{ji}x_i' + 1 - \sum_{i=1}^{j} r_{ji}^2 \\ \le\ & 2x_j - 2r_{jj}x_j' \\ \le\ & \left(2d + \frac{2001}{1000}\right)\varepsilon - \left(\sum_{i=1}^{4} x_i'^2 - \sum_{i=1}^{4} x_i^2\right) + \sum_{i=1}^{j-1} 2r_{ji}x_i' + 1 - \sum_{i=1}^{j} r_{ji}^2, \end{aligned} \tag{4.5}$$

for any $j \in \{1, 2, 3, 4\}$.

Since $|x_i'| \le \|f(x)\| \le \|x\| + \varepsilon < d + \frac{1}{1000}$ and by (3.2), we get

$$-2\left(d + \frac{1}{1000}\right)\sum_{i=1}^{j-1} c_{ji}\varepsilon \le \sum_{i=1}^{j-1} 2r_{ji}x_i' \le 2\left(d + \frac{1}{1000}\right)\sum_{i=1}^{j-1} c_{ji}\varepsilon.$$

Moreover, since $1 - \sum_{i=1}^{j} r_{ji}^2 = 1 - \|f(e_j)\|^2$ and $1 - \varepsilon \le \|f(e_j)\| \le 1 + \varepsilon$, we have

$$-\frac{2001}{1000}\varepsilon \le -2\varepsilon - \varepsilon^2 \le 1 - \sum_{i=1}^{j} r_{ji}^2 \le 2\varepsilon - \varepsilon^2 \le 2\varepsilon.$$

Therefore, it follows from (4.3) and (4.5) that

$$-\left(\left(2d + \frac{1}{500}\right)\left(2 + \sum_{i=1}^{j-1} c_{ji}\right) + 4\right)\varepsilon \le 2x_j - 2r_{jj}x_j' \le \left(\left(2d + \frac{1}{500}\right)\left(2 + \sum_{i=1}^{j-1} c_{ji}\right) + 4\right)\varepsilon,$$

for all $j \in \{1, 2, 3, 4\}$.

We note that $|x'_j| < d + \frac{1}{1000}$ and $-c_{jj}\varepsilon \le -\varepsilon \le 1 - r_{jj} \le c_{jj}\varepsilon$ by Theorem 3.1, and since $x_j - r_{jj}x'_j = (x_j - x'_j) + (1 - r_{jj})x'_j$, we can see that

$$\begin{aligned} |x_j - x'_j| &= |(x_j - r_{jj}x'_j) - (1 - r_{jj})x'_j| \\ &\le |x_j - r_{jj}x'_j| + |1 - r_{jj}||x'_j| \\ &\le \left(\left(d + \frac{1}{1000}\right)\left(2 + \sum_{i=1}^{j} c_{ji}\right) + 2\right)\varepsilon, \end{aligned} \tag{4.6}$$

for $j \in \{1, 2, 3, 4\}$.

Since we can select an isometry $U : D \to \mathbb{R}^4$ defined by $U(x) = x = (x_1, x_2, x_3, x_4)$, it follows from (4.6) that

$$\begin{aligned} \|f(x) - U(x)\| &= \|(x'_1 - x_1, x'_2 - x_2, x'_3 - x_3, x'_4 - x_4)\| \\ &= \left(\sum_{j=1}^{4}(x'_j - x_j)^2\right)^{1/2} \\ &\le \left(\sum_{j=1}^{4}\left(\left(d + \frac{1}{1000}\right)\left(2 + \sum_{i=1}^{j} c_{ji}\right) + 2\right)^2\right)^{1/2}\varepsilon, \end{aligned}$$

for all $x \in D$. □

We now put $d = 1$ in Theorem 4.1 and use the values for c_{ij} given in Theorem 3.1 to prove the following corollary.

Corollary 4.2. *Let D be a subset of the four-dimensional Euclidean space $\mathbb{R}^4$ such that $\{0, e_1, e_2, e_3, e_4\} \subset D \subset B_1(0)$, and let $f : D \to \mathbb{R}^4$ be a function that satisfies $f(0) = 0$ and inequality (3.1) for all $x, y \in D$ and for some constant ε with $0 < \varepsilon < \frac{1}{1000}$. Then, there exists an isometry $U : D \to \mathbb{R}^4$ such that*

$$\begin{aligned} \|f(x) - U(x)\| &\le \left(\sum_{i=1}^{4}\left(\frac{4002}{1000} + \frac{1001}{1000}(c_{i1} + c_{i2} + \cdots + c_{ii})\right)^2\right)^{1/2}\varepsilon \\ &\le 21.71890\varepsilon, \end{aligned}$$

for all $x \in D$.

5. Discussion and conclusions

Theorem 1.2 provided an important motivation to explore this topic. Unfortunately, the upper bound of inequality (1.1) (in Fickett's theorem) decreases slowly to 0 as ε approaches 0. This is obviously a weak point of Fickett's theorem. Therefore, it would be meaningful to eliminate this weakness of Fickett's theorem.

The paper [13] allowed only natural numbers as the values of c_{ij}, while the paper [15] allowed the values of c_{ij} to be real numbers. In this way, the latter further improves the former result for the case $n = 3$, as can be seen from the following table.

In this paper, the results of [14] and [13] were improved when $n = 4$ by allowing real numbers as the values of c_{ij}. For example, according to Corollary 4.2 of this paper, there exists an isometry $U : D \to \mathbb{R}^4$ that satisfies inequality $\|f(x) - U(x)\| \leq 22\varepsilon$ for all $x \in D$, while the upper bound 22ε has increased to 128ε and 57ε in the previous papers [13] and [14], respectively.

We compare the result of this paper with those of notable previously published works and present them in the Table 1.

Table 1. Comparison of the result of this paper with those of existing papers.

	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$	$\cdots$
in [11]	4	> 79	> 799	> 7990	> 79900	$\cdots$
in [12]	27	54	81	108	135	$\cdots$
in [13]	–	–	< 84	128	< 179	$\cdots$
in [14]	–	–	< 37	< 57	< 79	$\cdots$
in [15]	–	12	21	–	–	–
in this paper	–	–	–	22	–	–

The values in the first row of Table 1 were obtained by substituting $c = 1$ in the formula presented in the proof of [11, Theorem 4.1]. The values of the third, fourth, and fifth rows are due to the formulas presented in [13], [14], and [15] with $d = 1$, respectively. Analyzing the numbers presented in the table above, we see that the result of this paper far exceeds other existing results in the four-dimensional case. Moreover, the result of this paper, along with the results of [15], improves and complements the results of papers [13] and [14].

The first author was notified recently that Vestfrid had obtained the result similar to that of [13]. In fact, Vestfrid [12] improved the existing results for large dimensions and demonstrated the local (Hyers-Ulam) stability of the isometry by proving the existence of a linear isometry $U : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|f(x) - U(x)\| \leq 27\frac{R}{r}n\varepsilon, \tag{5.1}$$

for all $x \in D$. We analyzed the main theorem of his paper for the case $r = R = 1$ and presented the results in the second row of the table above. Reducing the upper bound of inequality (5.1) will be an interesting task that we will pursue in the near future.

Author contributions

All authors contributed to the writing, review, and editing of this paper. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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