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Research article

Solution of the SIR epidemic model of arbitrary orders containing Caputo-Fabrizio, Atangana-Baleanu and Caputo derivatives

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Abstract: The main aim of this study was to apply an analytical method to solve a nonlinear system of fractional differential equations (FDEs). This method is the Adomian decomposition method (ADM), and a comparison between its results was made by using a numerical method: Runge-Kutta 4 (RK4). It is proven that there is a unique solution to the system. The convergence of the series solution is given, and the error estimate is also proven. After that, the susceptible-infected-recovered (SIR) model was taken as an real phenomenon with such systems. This system is discussed with three different fractional derivatives (FDs): the Caputo-Fabrizio derivative (CFD), the Atangana-Baleanu derivative (ABD), and the Caputo derivative (CD). A comparison between these three different derivatives is given. We aimed to see which one of the new definitions (ABD and CFD) is close to one of the most important classical definitions (CD).

Keywords: Caputo-Fabrizio; Caputo and Atangana-Baleanu derivatives; SIR epidemic model of arbitrary orders; Adomian method; unique solution; error estimation
Mathematics Subject Classification: 34A12, 34A34, 45G05

1. Introduction

To show the dynamics of an epidemic, the susceptible-infected-recovered (SIR) fractional model was proposed. Despite the fact that the SIR model is approaching its centenary and its wholly analytical solutions have just been computed, the ADM represents one of them. The SIR model gives details and predictions on the spread of a virus in societies that recorded data alone cannot. This model is a system of FDEs. FDEs have numerous applications in science and engineering: for example, electrical networks [1, 2], control theory, fluid flow, fractal theory [3, 4], electromagnetic theory [5–7], chemistry, optical and neural network systems, potential theory, and biology. The ADM [8, 9] was used in this study to solve a critical model of arbitrary orders: the SIR epidemic model. This technique has several benefits: It efficiently solves various classes of equations, whether they are linear or nonlinear, in stochastic and deterministic areas, and provides an analytical solution with no discretization or linearization [10–13].

In recent works, the SIR model involving the Riemann-Liouville derivative and CD for childhood disease has been proposed [14]. There, the authors used the homotopy analysis method to derive a numerical solution for their model. In [15], the solution for the SIR model involving time fractional CD by utilizing the Laplace ADM (LADM) was illustrated graphically for unequal values of order α, which showed that the recovered population increases with a decreasing rate of infection in the population. In [13], the author proposed an epidemic model and used the fractional interpolated variation iteration method to find an efficient approximate solution. In [16], numerical solutions for the SIR model involving CD were obtained by using the Adams-type predictor-corrector method. In [17], the authors considered the fractional-order SIR epidemic model involving conformable FDs and used the conformable differential transform method (CFDTM) to calculate an approximate solution that is in the form of a rapidly convergent series. In [18], they found the numerical solutions for the epidemic model involving fractional CD using the Euler method. In [19], the author considered the time fractional epidemic model of childhood disease involving fractional CD and presented the numerical results by using the Adams-type implicit fractional linear multistep method. In [20], the author presented an epidemic model of non-fatal disease involving fractional CD and computed the analytical solution for the corresponding system of nonlinear FDEs by applying the LADM. The SIR model in CD was solved using the residual power series in [21]. In [22], the SIR epidemic model with the Mittag-Leffler fractional derivative was discussed. In [23], a fractional SIR model with delay in the context of the generalized Liouville-Caputo fractional derivative was given. The stability and equilibrium points of this model when containing CD were discussed before in many works, such as [24–28].

In this research, the SIR model was used to solve problems involving three different FDs: CFD, ABD, and CD derivatives. A comparison between the solutions of the SIR system, containing these three derivatives, is given. Important applications of ABD and CFD can be found in [29, 30].

This research is organized as follows: In Section 2, the main definitions are given. In Section 3, the first definition (CFD) is discussed. Then, in Section 4, the second definition (CD) is given. In Section 5, the third definition (ABD) is discussed. In Section 6, a comparison between these three different definitions is discussed. Finally, in Section 7, a conclusive summary to this research is given. All the results of the SIR model are obtained and compared to all of these using the fourth order Runge-Kutta 4 (RK4) solution from the Mathematica 5.2 package.

2. Main definitions

The definitions of the three fractional derivatives used in this research and the main properties and advantages of using each definition are given here.

(1) The definition of the CD of order υ is [31]

$$ {}_{0}^{C}D_{t}^{\upsilon}f(t) = \frac{1}{\Gamma(n-\upsilon)} \int_{0}^{t} \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\upsilon-n+1}}, \quad n-1<\upsilon<n. \tag{2.1} $$

Its corresponding fractional integral (FI) is [31]

$$ {}^{C}I^{\upsilon}F(t) = \frac{1}{\Gamma(\upsilon)} \int_{0}^{t} F(\tau)(t-\tau)^{\upsilon-1}\, d\tau, \quad 0<\upsilon<1. \tag{2.2} $$

Moreover,

$$ \left({}^{C}I^{\upsilon}\right)\left({}^{C}D^{\upsilon}\right)F(t) = F(t) - F(a). \tag{2.3} $$

This definition is considered one of the classical FDs and is one of the most well known and famous FDs, as follows [31]:

i) Its main advantage is that the initial conditions for FDEs containing CD take an identical form as for integer-order DEs, such as

$$ f^{'}(a), f^{''}(a), \cdots, \tag{2.4} $$

and they have well-known physical meaning.

ii) The derivative of a constant by using the Caputo definition is equal to zero, whereas the Riemann-Liouville derivative of a constant is not equal to zero, as follows:

$$ {}_{0}^{RL}D_{t}^{\upsilon}(C) = \frac{Ct^{-\upsilon}}{\Gamma(1-\upsilon)}. \tag{2.5} $$

So, the properties of this definition coincide with the properties of the integer-order derivative definition.

(2) The definition of the CFD of order υ is [30]

$$ {}^{CF}D_{a}^{\upsilon}F(t) = \frac{B(\upsilon)}{1-\upsilon} \int_{a}^{t} \exp\left(\frac{-\upsilon(t-\tau)}{1-\upsilon}\right) F'(s)ds, \tag{2.6} $$

where the normalization function $B(\upsilon) > 0$ satisfies $B(0) = B(1) = 1$. Its corresponding FI is [32]

$$ {}^{CF}I_{a}^{\upsilon}F(t) = \frac{1-\upsilon}{B(\upsilon)}F(t) + \frac{\upsilon}{B(\upsilon)} \int_{a}^{t} F(s)ds, \quad \upsilon \in (0,1), \tag{2.7} $$

where

$$ \left({}^{CF}I_{a}^{\upsilon}\right)\left({}^{CF}D_{a}^{\upsilon}\right)F(t) = F(t) - F(a). \tag{2.8} $$

The main advantage of using this definition is that there is no singularity in the definition, as shown in (2.6) and (2.7).

(3) The ABD of order υ of $F(t)$ is [32]

$$ {}^{AB}D^{\upsilon}F(t) = \frac{B(\upsilon)}{1-\upsilon} \int_{0}^{t} E_{\upsilon}\left(\frac{-\upsilon(t-s)}{1-\upsilon}\right)^{\upsilon} F'(s)ds, \tag{2.9} $$

where E_υ is the Mittag-Leffler function of one variable, and its reduced FI is [9]

$$^{AB}I^{\upsilon}F(t) = \frac{1-\upsilon}{B(\upsilon)}F(t) + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t F(s)(t-s)^{\upsilon-1}\,ds, \quad 0<\upsilon<1. \tag{2.10}$$

Also,

$$\left(^{AB}I^{\upsilon}\right)\left(^{AB}D^{\upsilon}\right)F(t) = F(t) - F(a). \tag{2.11}$$

In this research, we aim to see which one of the two new FDs (CFD and ABD) is closer to the classical fractional CD because we can deduce the properties of the integer derivatives from the properties of the CD, as we can see from the relations (2.4) and (2.5).

3. First definition: Caputo-Fabrizio derivative (CFD)

The general form of the nonlinear FDE system with the CFD takes the following form:

$$^{CF}D_t^{\upsilon}y_k(t) + h_k(t)\ f_k(\bar{y}) = \chi_k(t), \tag{3.1}$$

subject to

$$y_k^{(j-1)}(0) = c_k, \quad k,j = 1,2,\ldots,n, \tag{3.2}$$

and

$$\bar{y} = \{y_1(t), y_2(t), \ldots, y_n(t)\}, \quad 0<\upsilon<1.$$

The SIR epidemic model is a special case of this system. Now, taking the FI (2.7) of order υ, and letting $a=0$ with the system (3.1)-(3.2), we get

$$y_k(t) = c_k + \frac{1-\upsilon}{B(\upsilon)}\chi_k(t) + \frac{\upsilon}{B(\upsilon)}\int_0^t \chi_k(s)ds - \frac{1-\upsilon}{B(\upsilon)}h_k(t)\,f_k(\bar{y}) - \frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\,f_k(\bar{y})\,ds. \tag{3.3}$$

Let $\chi_k(t)$ be bounded $\forall t \in J = [0,T]$, $T \in R^+$, $|h_k(\tau)| \leq M_k$ for all $0 \leq \tau \leq t \leq T$, M_k be finite constants, and $f_k(\bar{y})$ satisfy the Lipschitz condition, having Lipschitz constants L_k as

$$|f_k(\bar{y}) - f_k(\bar{z})| \leq L_k\,|\bar{y} - \bar{z}|. \tag{3.4}$$

Furthermore, it has Adomian polynomials (APs) presented as

$$f_k(\bar{y}) = \sum_{m=0}^{\infty} A_{km}(y_{k0}, y_{k1}, \ldots, y_{kn}), \tag{3.5}$$

where

$$A_{km} = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[f_k\left(\sum_{j=0}^{\infty}\lambda^j y_j\right)\right]_{\lambda=0}. \tag{3.6}$$

Substituting (3.5) into (3.3), we get

$$y_k(t) = c_k + \frac{1-\upsilon}{B(\upsilon)}\chi_k(t) + \frac{\upsilon}{B(\upsilon)}\int_0^t \chi_k(s)ds - \frac{1-\upsilon}{B(\upsilon)}h_i(t)\sum_{i=0}^{\infty}A_{ki} - \frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\sum_{i=0}^{\infty}A_{ki}\,ds. \tag{3.7}$$

Let $y_k(t) = \sum_{i=0}^{\infty} y_{ki}(t)$ in (3.7). Then,

$$
\begin{aligned}
y_{k0}(t) &= c_k + \frac{1-\upsilon}{B(\upsilon)}\chi_k(t) + \frac{\upsilon}{B(\upsilon)}\int_0^t \chi_k(s)ds, \qquad (3.8)\\
y_{ki}(t) &= -\frac{1-\upsilon}{B(\upsilon)}h_k(t)A_{k(i-1)} - \frac{\upsilon}{B(\upsilon)}\int_0^t h_i(s)A_{k(i-1)}ds, \quad k \geq 1. \qquad (3.9)
\end{aligned}
$$

The final solution will be

$$
y_k(t) = \sum_{i=0}^{\infty} y_{ki}(t). \tag{3.10}
$$

3.1. Convergence

3.1.1. Existence of a unique solution

Take the mapping $\Psi : \Omega \to \Omega$, where Ω is the Banach space $(C^{(n)}(J), \|\cdot\|)$. $C^{(n)}(J)$ is a class of continuous column vectors $Y = (\overline{y})'$ taking norm $\|Y\| = \sum_{k=1}^{n} \max_{t \in J} |y_k(t)|$, and $(.)'$ is the matrix transpose.

Theorem 3.1. *There exists a unique solution to the system (3.1)-(3.2) at* $0 < \phi < 1$, $\phi = \frac{LMT}{B(\upsilon)}$, *where* $L = \sum_{m=1}^{n} L_m$, $M = \max\{M_1, M_2, \dots, M_n\}$.

Proof. Rewrite Eq (3.7) as

$$
Y(t) = C + \frac{1-\upsilon}{B(\upsilon)}\chi(t) + \frac{\upsilon}{B(\upsilon)}\int_0^t \chi(s)ds - \frac{1-\upsilon}{B(\upsilon)}H(t)F(\overline{y}) - \frac{\upsilon}{B(\upsilon)}\int_0^t H(s)F(\overline{y})ds,
$$

where

$$
\begin{aligned}
C &= (c_1, c_2, \dots, c_n)', \\
\chi(t) &= (\chi_1, \chi_2, \dots, \chi_n)', \\
H(t) &= diag\{h_1, h_2, \dots, h_n\}, \\
F(\overline{y}(t)) &= (f_1(\overline{y}), f_2(\overline{y}), \dots, f_n(\overline{y}))'.
\end{aligned}
$$

The mapping $\Psi : \Omega \to \Omega$ is defined as

$$
\Psi Y(t) = C + \frac{1-\upsilon}{B(\upsilon)}\chi(t) + \frac{\upsilon}{B(\upsilon)}\int_0^t \chi(s)\,ds - \frac{1-\upsilon}{B(\upsilon)}H(t)F(\overline{y}) - \frac{\upsilon}{B(\upsilon)}\int_0^t H(s)F(\overline{y})ds.
$$

Let $Y, Z \in \Omega$.

$$
\begin{aligned}
\|\Psi Y(t) - \Psi Z(t)\| &= \left\| -\frac{1-\upsilon}{B(\upsilon)}H(t)(F(\overline{y}) - F(\overline{z})) - \frac{\upsilon}{B(\upsilon)}\int_0^t H(s)(F(\overline{y}) - F(\overline{z}))ds \right\| \\
&\leq \left\| \frac{1-\upsilon}{B(\upsilon)}H(t)(F(\overline{y}) - F(\overline{z})) \right\| + \left\| \frac{\upsilon}{B(\upsilon)}\int_0^t H(s)(F(\overline{y}) - F(\overline{z}))ds \right\| \\
&\leq \frac{1-\upsilon}{B(\upsilon)}\|H(t)\|\,\|F(\overline{y}) - F(\overline{z})\| + \frac{\upsilon}{B(\upsilon)}\int_0^t \|H(s)\|\,\|F(\overline{y}) - F(\overline{z})\|ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(1-\upsilon)\,M}{B(\upsilon)}\left(\sum_{m=1}^{n}\max_{t\in J}|f_m(\overline{y})-f_m(\overline{z})|\right)+\frac{\upsilon M}{B(\upsilon)}\int_0^t\left(\sum_{m=1}^{n}\max_{t\in J}|f_m(\overline{y})-f_m(\overline{z})|\right)ds\\
&\leq \frac{M}{B(\upsilon)}\left(\sum_{m=1}^{n}\max_{t\in J}|f_m(\overline{y})-f_m(\overline{z})|\right)\left[(1-\upsilon)+\upsilon\int_0^t ds\right]\\
&\leq \frac{M}{B(\upsilon)}\left[1-\upsilon+\upsilon T\right]\sum_{m=1}^{n}L_m\,\|Y-Z\|\\
&\leq \frac{M\left[1+\upsilon\,(T-1)\right]}{B(\upsilon)}\sum_{m=1}^{n}L_m\,\|Y-Z\|\\
&\leq \frac{MT}{B(\upsilon)}\sum_{m=1}^{n}L_m\,\|Y-Z\|\leq\frac{LMT}{B(\upsilon)}\,\|Y-Z\|\\
&\leq \phi\,\|Y-Z\|\,.
\end{aligned}
$$

If $0<\phi<1$, the mapping Ψ will be a contraction, so, there is a unique solution to the system (3.1)-(3.2). □

3.1.2. Convergence proof

Theorem 3.2. *The series solution (3.10) will be convergent if* $|y_{k1}|<\infty$ *and* $0<\phi<1$, $\phi=\frac{LMT}{B(\upsilon)}$, *as* $L=\sum_{k=1}^{n}L_k$, $M=\max\{M_1,M_2,\ldots,M_n\}$.

Proof. Take a sequence $\{S_{kr}\}$ such that $S_{kr}=\sum_{i=0}^{r}y_{ki}(t)$ is partial sums sequence of $\sum_{i=0}^{\infty}y_{ki}(t)$. We have

$$
f(S_{kr})=\sum_{i=0}^{r}A_{ki}(y_{k0},y_{k1},\ldots,y_{kr}).
$$

If S_{kr} and S_{kw} are two partial sums where $r>w$, our goal is to prove that $\{S_{ir}\}$ is a Cauchy sequence in this Banach space.

$$
\begin{aligned}
\|S_{kr}-S_{kw}\| &= \sum_{i=1}^{n}\max_{t\in J}|S_{ir}-S_{iw}|=\sum_{i=1}^{n}\max_{t\in J}\left|\sum_{j=w+1}^{r}y_{ij}(t)\right|\\
&\leq \sum_{i=1}^{n}\max_{t\in J}\left|\sum_{j=w+1}^{r}\frac{1-\upsilon}{B(\upsilon)}h_k(t)\,A_{k(i-1)}+\frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\,A_{k(i-1)}ds\right|\\
&\leq \sum_{i=1}^{n}\max_{t\in J}\left|\frac{1-\upsilon}{B(\upsilon)}h_k(t)\sum_{j=w+1}^{r}A_{i(j-1)}+\frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\sum_{j=w+1}^{r}A_{i(j-1)}ds\right|\\
&\leq \sum_{i=1}^{n}\max_{t\in J}\left|\frac{1-\upsilon}{B(\upsilon)}h_k(t)\sum_{j=w}^{r-1}A_{ij}+\frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\sum_{j=w}^{r-1}A_{ij}ds\right|\\
&\leq \sum_{i=1}^{n}\max_{t\in J}\left|\frac{1-\upsilon}{B(\upsilon)}h_k(t)\left[f(S_{i(r-1)})-f(S_{i(w-1)})\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\left[f(S_{i(r-1)})-f(S_{i(w-1)})\right]ds\Bigg] \\
\leq\ & \sum_{i=1}^{n}\max_{t\in J}\left[\left|\frac{1-\upsilon}{B(\upsilon)}h_k(t)\left[f(S_{i(r-1)})-f(S_{i(w-1)})\right]\right|\right. \\
& \left.+\left|\frac{\upsilon}{B(\upsilon)}\int_0^t h_k(s)\left[f(S_{i(r-1)})-f(S_{i(w-1)})\right]ds\right|\right] \\
\leq\ & \sum_{i=1}^{n}\max_{t\in J}\left[\frac{1-\upsilon}{B(\upsilon)}\left|h_k(t)\right|\left|f(S_{i(r-1)})-f(S_{i(w-1)})\right|\right. \\
& \left.+\frac{\upsilon}{B(\upsilon)}\int_0^t\left|h_k(s)\right|\left|f(S_{i(r-1)})-f(S_{i(w-1)})\right|]ds\right] \\
\leq\ & \sum_{i=1}^{n}\max_{t\in J}\left[\frac{1-\upsilon}{B(\upsilon)}\left|h_k(t)\right|\left(L_i\sum_{j=1}^{n}\left|S_{j(r-1)}-S_{j(w-1)}\right|\right)\right. \\
& \left.+\frac{\gamma}{B(\gamma)}\int_0^t\left|h_k(s)\right|\left(L_i\sum_{j=1}^{n}\left|S_{j(r-1)}-S_{j(w-1)}\right|\right)ds\right] \\
\leq\ & \sum_{k=1}^{n}\max_{t\in J}\left(L_i\sum_{j=1}^{n}\left|S_{j(r-1)}-S_{j(w-1)}\right|\right)\left[\frac{1-\upsilon}{B(\upsilon)}M+\frac{\upsilon M}{B(\upsilon)}\int_0^t ds\right] \\
\leq\ & \sum_{k=1}^{n}\max_{t\in J}\left(L_i\sum_{j=1}^{n}\left|S_{j(r-1)}-S_{j(w-1)}\right|\right)\left[\frac{1-\upsilon}{B(\upsilon)}M+\frac{\upsilon M}{B(\upsilon)}\int_0^t ds\right] \\
\leq\ & \frac{LMT}{B(\upsilon)}\left\|S_{k(r-1)}-S_{k(w-1)}\right\| \\
\leq\ & \phi\left\|S_{k(r-1)}-S_{k(w-1)}\right\|.
\end{aligned}
$$

Let $r = w + 1$. Then,

$$
\left\|S_{k(w+1)}-S_{kw}\right\|\leq\phi\left\|S_{kw}-S_{k(w-1)}\right\|\leq\phi^2\left\|S_{k(w-1)}-S_{k(w-2)}\right\|\leq\cdots\leq\phi^w\left\|S_{k1}-S_{k0}\right\|.
$$

From the triangle inequality, we get

$$
\begin{aligned}
\left\|S_{kr}-S_{kw}\right\| &\leq \left\|S_{k(w+1)}-S_{kw}\right\|+\left\|S_{k(w+2)}-S_{k(w+1)}\right\|+\cdots+\left\|S_{kr}-S_{k(r-1)}\right\| \\
&\leq \left[\phi^w+\phi^{w+1}+\cdots+\phi^{r-1}\right]\left\|S_{k1}-S_{k0}\right\| \\
&\leq \phi^w\left[1+\phi+\cdots+\phi^{r-w-1}\right]\left\|S_{k1}-S_{k0}\right\| \\
&\leq \phi^q\left[\frac{1-\phi^{r-w}}{1-\phi}\right]\left\|y_{k1}(t)\right\|.
\end{aligned}
$$

Since $0<\phi<1$ and $r>w$, $(1-\phi^{r-w})\leq 1$. Hence,

$$
\left\|S_{kr}-S_{kw}\right\|\leq\frac{\phi^w}{1-\beta}\left\|y_{k1}(t)\right\|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}\left|y_{k1}(t)\right|.
$$

If $|y_{k1}(t)|<\infty$ and $q\to\infty$, then $\|S_{kr}-S_{kw}\|\to 0$. So, $\{S_{kr}\}$ will be a Cauchy sequence in this Banach space, and the series $\sum_{i=0}^{\infty} y_{ki}(t)$ will converge. □

3.1.3. Error estimation

Theorem 3.3. *For the series solution (3.10), the maximum absolute error can be estimated as*

$$\max_{t\in J}\left|y_k(t)-\sum_{i=0}^{w}y_{ki}(t)\right|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|.$$

Proof. Using Theorem 3.2, we get

$$\|S_{kr}-S_{kw}\|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|.$$

However, $S_{ir}=\sum_{i=0}^{r}y_{ki}(t)$ as $r\to\infty$. Then, $S_{kr}\to y_k(t)$ and we get

$$\|y_k(t)-S_{kw}\|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|.$$

So, the maximum absolute error will be

$$\max_{t\in J}\left|y_k(t)-\sum_{i=0}^{w}y_{ki}(t)\right|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|.$$

3.2. SIR epidemic model of arbitrary orders with CFD

The simple epidemic model of arbitrary orders of a dangerous sickness in a wide range of populations, known as the SIR model, was presented in [14, 33–37], by considering that the populace consists of three types of individuals: susceptible (S), referring to individuals who are not infected although they can be severely affected in an easy way; infected (I), the individual individuals who carry the diseases and are able to transmit the sickness to the susceptible; and recovered (R). The formal SIR model is presented as

$$\begin{aligned}\frac{dS}{dt}&=(1-\eta)\pi-\varphi SI-\pi S,\\ \frac{dI}{dt}&=\varphi SI-(\gamma+\pi)I,\\ \frac{dR}{dt}&=\eta\pi+\gamma I-\pi R.\end{aligned}\tag{3.11}$$

This model assumes that immunization is completely effective. Individuals are added to the population with a constant birth rate, and there is an extremely low youth sickness death rate. Every year, the proportion of citizens immunized at childbirth is expressed as η. Infected individuals are approximated by constant rate φ and recover at a rate γ from infection.

The SIR epidemic model of arbitrary orders containing CFDs is

$$\begin{aligned}{}^{CF}D_0^{\alpha}S(t)&=(1-p)\pi-\beta SI-\pi S,\\ {}^{CF}D_0^{\alpha}I(t)&=\beta SI-(\gamma+\pi)I,\\ {}^{CF}D_0^{\alpha}R(t)&=p\pi+\gamma I-\pi R,\end{aligned}\tag{3.12}$$

subject to

$$S(0) = N_1,\ I(0) = N_2,\ R(0) = N_3.$$

Using the ADM on system (3.11), we get the ADM recurrence relation as

$$S_0 = N_1 + {}^{CF}I^{\alpha}(1-p)\pi, \quad S_{j+1} = -\beta\, {}^{CF}I^{\alpha}(A_j) - \pi\, {}^{CF}I^{\alpha}\left(S_j\right), \tag{3.13}$$

$$I_0 = N_2, \quad I_{j+1} = \beta\, {}^{CF}I^{\alpha}\left(A_j\right) - {}^{CF}I^{\alpha}\left[(\gamma+\pi) I_j\right], \tag{3.14}$$

$$R_0 = N_3 + {}^{CF}I^{\alpha}(p\pi) +, \quad R_{j+1} = \gamma\, {}^{CF}I^{\alpha}\left(I_j\right) - \pi\, {}^{CF}I^{\alpha}\left(R_j\right). \tag{3.15}$$

The series solution is

$$S(t) = \sum_{k=0}^{n} S_k(t), I(t) = \sum_{k=0}^{n} I_k(t), \text{ and } R(t) = \sum_{k=0}^{n} R_k(t). \tag{3.16}$$

Figures 1–3 show the ADM solution of the SIR system when $\alpha = 1, 0.95, 0.85, 0.75$ and $n = 5$.

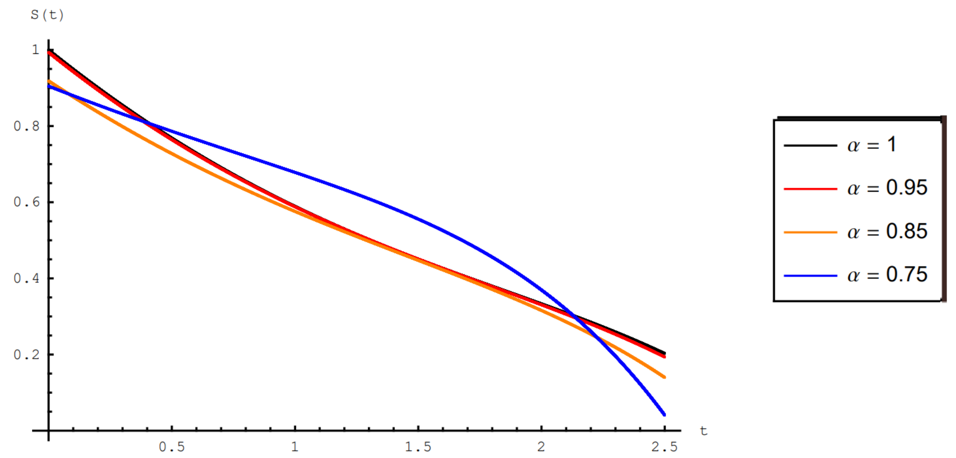

Figure 1. ADM solution of $S(t)$ with CFD.

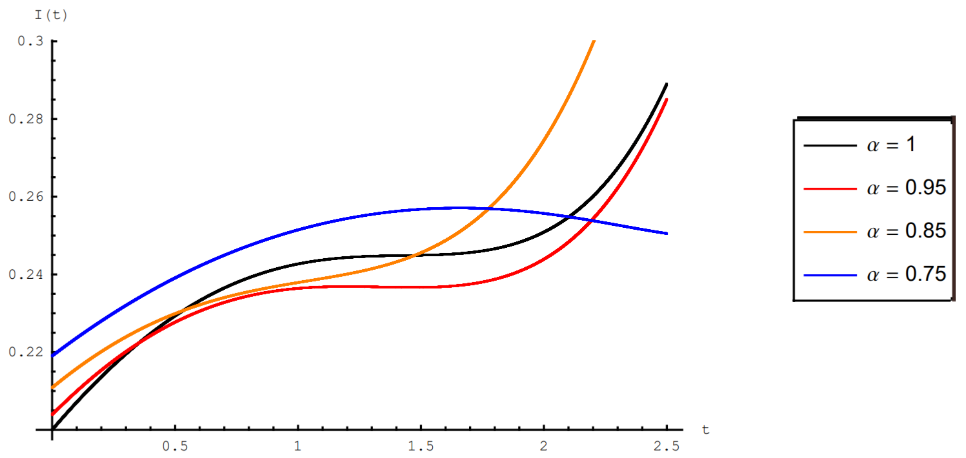

Figure 2. ADM solution of $I(t)$ with CFD.

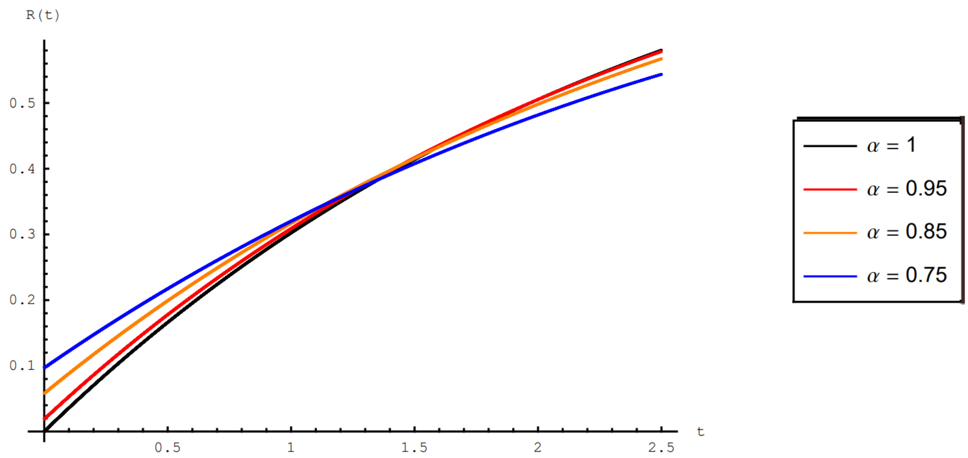

Figure 3. ADM solution of $R(t)$ with CFD.

Tables 1–3 show the absolute differences (ADs) between the ADM and RK4 solutions for the SIR system at $\alpha = 1$, respectively.

Table 1. ADM and RK4 solutions of $S(t)$.

t	Solution of ADM	Solution of RK4	AD
0	1	1	0
0.1	0.949	0.949	0.00004
0.2	0.901	0.901	4×10^{-6}
0.3	0.854	0.854	0.00001
0.4	0.810	0.810	0.00004
0.5	0.768	0.768	0.00003
0.6	0.729	0.729	0.00006
0.7	0.691	0.691	4×10^{-6}
0.8	0.655	0.655	0.00008
0.9	0.621	0.621	0.0003

Table 2. ADM and RK4 solutions of $I(t)$.

t	Solution of ADM	Solution of RK4	AD
0	0.2	0.2	0
0.1	0.207	0.207	0.00001
0.2	0.214	0.214	0.00003
0.3	0.2195	0.2195	0.00002
0.4	0.225	0.225	0.00005
0.5	0.22933	0.229	0.00003
0.6	0.233	0.233	0.00004
0.7	0.237	0.237	4×10^{6}
0.8	0.239	0.244	0.005
0.9	0.241	0.239	0.002

Table 3. ADM and RK4 solutions of $R(t)$.

t	Solution of ADM	Solution of RK4	AD
0	0	0	0
0.1	0.03589	0.0359	0.00001
0.2	0.0704	0.0704	0.00001
0.3	0.1036	0.1036	0.00004
0.4	0.1354	0.1354	0.00004
0.5	0.1661	0.1661	0.00002
0.6	0.1955	0.1955	0.00004
0.7	0.2239	0.2229	0.00096
0.8	0.2511	0.2511	0.00003
0.9	0.2772	0.2772	0.00002

From Tables 1–3, for $\alpha = 1$, ADM and RK4 have close values as shown from the values of ADs between them.

Figure 4 shows the ADM solution of the SIR system at $\alpha = 0.5$ and $n = 5$.

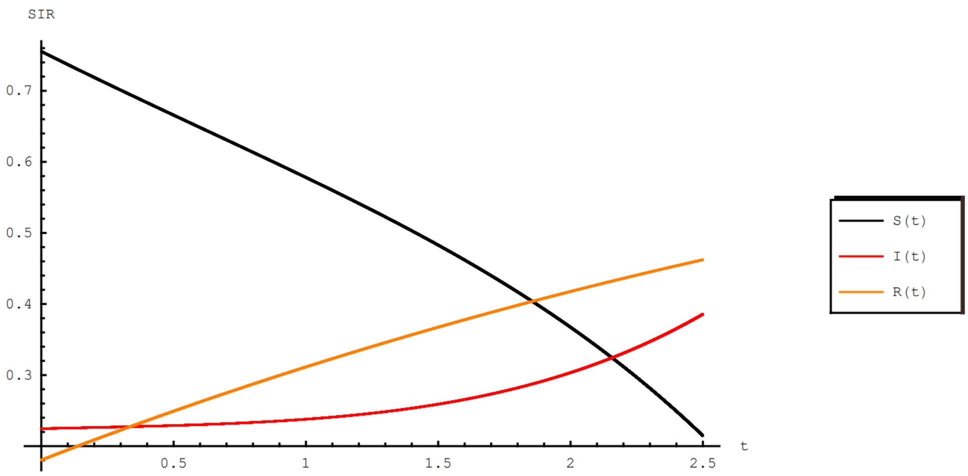

Figure 4. ADM solution of the SIR Model with CFD.

Figure 4 shows that the susceptible population decreases, whereas the infected population and the recovered population increase for a long time. In Figures 1–3, we see the effect of using different values of α on the SIR system.

4. Second definition: Caputo derivative (CD)

The general form of the nonlinear FDE system with a CD is

$$^{c}D_t^{\upsilon} y_k(t) + h_k(t) f_k(\overline{y}) = \chi_k(t), \tag{4.1}$$

with

$$y_k^{(j-1)}(0) = c_{kj}, \qquad k, j = 1, 2, \dots, n, \tag{4.2}$$

as

$$\overline{y} = \{y_1(t), y_2(t), \ldots, y_n(t)\}, \quad 0 < \upsilon < 1. \tag{4.3}$$

The SIR epidemic model is a special case of this system. Applying fractional integration of order υ, this minimizes (4.1)-(4.2) in the system of fractional integral equations (FIEs).

$$y_k(t) = \sum_{j=1}^{m} \frac{c_{kj}}{\Gamma(\upsilon)} t^{\upsilon-1} + \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} \chi_k(\tau) d\tau - \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) f_k(\overline{y}) d\tau. \tag{4.4}$$

Assume that $\chi_k(t)$ is bounded, $\forall\ t \in J = [0, T]$, $T \in R^+$, $|h_k(\tau)| \le K_k$, $\forall\ 0 \le \tau \le t \le T$, K_k are finite constants, and $f_k(\overline{y})$ satisfy the Lipschitz condition with Lipschitz constants L_k as

$$|f_k(\overline{y}) - f_k(\overline{z})| \le L_k |\overline{y} - \overline{z}|. \tag{4.5}$$

Substituting (3.5) into (4.4), we have

$$y_k(t) = \sum_{j=1}^{m} \frac{c_{kj}}{\Gamma(\upsilon)} t^{\upsilon-1} + \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} \chi_k(\tau) d\tau - \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) \sum_{m=0}^{\infty} A_{km} d\tau. \tag{4.6}$$

Let $y_k(t) = \sum\limits_{m=0}^{\infty} x_{km}(t)$ substitute in (4.6) and we get

$$\begin{aligned} y_{k0}(t) &= \sum_{j=1}^{m} \frac{c_{kj}}{\Gamma(\upsilon)} t^{\upsilon-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\upsilon-1} \chi_k(\tau) d\tau, \\ y_{km}(t) &= -\frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) \sum_{m=0}^{\infty} A_{k(m-1)} d\tau, \quad m \ge 1. \end{aligned} \tag{4.7}$$

The final solution will be

$$y_k(t) = \sum_{m=0}^{\infty} y_{km}(t). \tag{4.8}$$

5. Convergence

5.1. Existence of a unique solution

Take the mapping $\Psi : \Omega \to \Omega$. Ω is the Banach space $(C^{(n)}(J), \|\cdot\|)$, where $C^{(n)}(J)$ is the class of continuous column vectors $Y = (\overline{y})'$ with $\|Y\| = \sum\limits_{m=1}^{n} \max\limits_{t \in J} |y_m(t)|$, and $(.)'$ denotes the matrix transpose.

Theorem 5.1. *The system (4.1)-(4.2) has a unique solution when* $0 < \phi < 1$, $\phi = \frac{KLT^{\upsilon}}{\upsilon\Gamma(\upsilon)}$, *where* $L = \sum\limits_{m=1}^{n} L_m$, $K = \max\{K_1, K_2, \ldots, K_n\}$.

Proof. Equation (4.4) can be described as

$$Y(t) = A + \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} G(\tau) d\tau - \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} H(\tau) F(\overline{y}) d\tau,$$

where

$$\begin{aligned} A &= (a_1, a_2, \ldots, a_n)', \\ Y(t) &= (y_1, y_2, \ldots, y_n)', \\ H(t) &= diag\{h_1, h_2, \ldots, h_n\}, \\ F(\overline{y}(t)) &= (f_1(\overline{y}), f_2(\overline{y}), \ldots, f_n(\overline{y}))'. \end{aligned}$$

Let $X, Z \in \Omega$.

$$\begin{aligned} \|\Psi Y(t) - \Psi Z(t)\| &= \left\| -\frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} H(\tau)(F(\overline{y}) - F(\overline{z}))\, d\tau \right\| \\ &\le \left\| \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} H(\tau)(F(\overline{y}) - F(\overline{z}))\, d\tau \right\| \\ &\le \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} \|H(\tau)\| \|F(\overline{y}) - F(\overline{z})\|\, d\tau \\ &\le \frac{K}{\Gamma(\upsilon)} \int_0^t (t-\tau)^{\upsilon-1} \left(\sum_{m=1}^{n} \max_{t\in J} |f_m(\overline{y}) - f_m(\overline{z})| \right) d\tau \\ &\le \frac{K}{\Gamma(\upsilon)} \left(\sum_{m=1}^{n} \max_{t\in J} |f_m(\overline{y}) - f_m(\overline{z})| \right) \int_0^t (t-\tau)^{\upsilon-1}\, d\tau \\ &\le \frac{KT^{\upsilon}}{\Gamma(\upsilon)} \sum_{m=1}^{n} L_m \|Y - Z\| \\ &\le \frac{KLT^{\upsilon}}{\upsilon\Gamma(\upsilon)} \|Y - Z\| \\ &\le \phi \|Y - Z\|. \end{aligned}$$

With the condition $0 < \phi < 1$, the mapping Ψ is a contraction, and then there exists a unique solution $Y \in C^{(n)}(J)$. □

5.2. Convergence

Theorem 5.2. *The series solution of the system (4.1)-(4.2) using ADM converges if* $|y_{i1}| < \infty$, $0 < \phi < 1$, *and* $\phi = \frac{LKT^{\upsilon}}{\upsilon\Gamma(\upsilon)}$, *where* $L = \sum_{m=1}^{n} L_m$, $K = \max\{K_1, K_2, \ldots, K_n\}$.

Proof. Take a sequence $\{S_{kr}\}$ such that $S_{kr} = \sum_{m=0}^{r} y_{km}(t)$ is the partial sums sequence from the series solution $\sum_{m=0}^{\infty} y_{km}(t)$. We get

$$f(S_{kr}) = \sum_{m=0}^{r} A_{km}(y_{k0}, y_{k1}, \ldots, y_{kr}).$$

Let S_{kr} and S_{kw} be two partial sums where $r > w$. Our goal is to show that $\{S_{kr}\}$ is a Cauchy sequence in this Banach space.

$$
\begin{aligned}
\left\|S_{kp}-S_{kq}\right\| &= \sum_{m=1}^{n}\max_{t\in J}\left|S_{mr}-S_{mw}\right| = \sum_{m=1}^{n}\max_{t\in J}\left|\sum_{j=w+1}^{r} y_{mj}(t)\right| \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\left|\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) A_{m(j-1)} d\tau\right| \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\left|\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) \sum_{j=w+1}^{r} A_{m(j-1)} d\tau\right| \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\left|\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) \sum_{j=w}^{r-1} A_{mj} d\tau\right| \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\left|\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} h_k(\tau) \left[f(S_{m(r-1)})-f(S_{m(w-1)})\right] d\tau\right| \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} \left|h_k(\tau)\right| \left|\left(f(S_{m(r-1)})-f(S_{m(w-1)})\right)\right| d\tau \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} \left|h_i(\tau)\right| \left(L_m \sum_{j=1}^{n}\left|S_{j(r-1)}-S_{j(w-1)}\right|\right) d\tau \\
&\leq \sum_{m=1}^{n}\max_{t\in J}\frac{LK}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} d\tau \left\|S_{j(r-1)}-S_{j(w-1)}\right\| \\
&\leq \frac{LKT^{\upsilon}}{\Gamma(\upsilon)}\left\|S_{j(r-1)}-S_{j(w-1)}\right\| \\
&\leq \phi\left\|S_{j(r-1)}-S_{j(w-1)}\right\|.
\end{aligned}
$$

Let $p = q + 1$. Then,

$$
\left\|S_{k(q+1)}-S_{kq}\right\| \leq \phi\left\|S_{kq}-S_{k(q-1)}\right\| \leq \phi^2\left\|S_{k(q-1)}-S_{k(q-2)}\right\| \leq \cdots \leq \phi^q \left\|S_{k1}-S_{k0}\right\|.
$$

Using the triangle inequality,

$$
\begin{aligned}
\left\|S_{kr}-S_{kw}\right\| &\leq \left\|S_{k(w+1)}-S_{kw}\right\| + \left\|S_{k(w+2)}-S_{k(w+1)}\right\| + \cdots + \left\|S_{kr}-S_{k(r-1)}\right\| \\
&\leq \left[\phi^w + \phi^{w+1} + \cdots + \phi^{r-1}\right]\left\|S_{k1}-S_{k0}\right\| \\
&\leq \phi^w\left[1 + \phi + \cdots + \phi^{r-w-1}\right]\left\|S_{k1}-S_{k0}\right\| \\
&\leq \phi^w\left[\frac{1-\phi^{r-w}}{1-\phi}\right]\left\|y_{k1}(t)\right\|,
\end{aligned}
$$

where $0 < \phi < 1$, and $r > w$. Consequently, $(1-\phi^{r-w}) \leq 1$. Then,

$$
\left\|S_{kr}-S_{kw}\right\| \leq \frac{\phi^w}{1-\phi}\left\|y_{k1}(t)\right\| \leq \frac{\phi^w}{1-\phi}\max_{t\in J}\left|y_{k1}(t)\right|. \tag{5.1}
$$

However, $|y_{k1}(t)| < \infty$, and as $w \to \infty$, $\|S_{kr} - S_{kw}\| \to 0$. Therefore, $\{S_{kr}\}$ is a Cauchy sequence in this Banach space. □

5.3. Error estimation

Theorem 5.3. *The maximum absolute error of system (4.1)-(4.2) can be estimated as*

$$\max_{t\in J}\left|y_k(t)-\sum_{m=0}^{w}y_{km}(t)\right|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|\,.$$

Proof. From the convergence theorem inequality (5.1), we have

$$\|S_{kr}-S_{kw}\|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|\,.$$

However, $S_{kp}=\sum_{m=0}^{r}x_{km}(t)$ as $r\to\infty$. Then, $S_{kr}\to y_k(t)$, so

$$\left\|y_k(t)-S_{kq}\right\|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|\,.$$

So, the maximum absolute error will be

$$\max_{t\in J}\left|y_k(t)-\sum_{m=0}^{w}y_{km}(t)\right|\leq\frac{\phi^w}{1-\phi}\max_{t\in J}|y_{k1}(t)|\,.$$

5.4. SIR epidemic model with arbitrary order containing CD

The SIR epidemic model of arbitrary orders with CD is

$$\begin{aligned}{}^{C}D_0^{\alpha}S(t) &= (1-\eta)\pi-\varphi SI-\pi S,\\ {}^{C}D_0^{\alpha}I(t) &= \varphi SI-(\gamma+\pi)I,\\ {}^{C}D_0^{\alpha}R(t) &= \eta\pi+\gamma I-\pi R,\end{aligned}\tag{5.2}$$

subject to

$$S(0)=N_1,\ I(0)=N_2,\ R(0)=N_3.$$

Applying the ADM to the system (5.2), we get the following algorithm:

$$\begin{aligned}S_0 &= N_1+{}^{C}I^{\alpha}(1-\eta)\pi,\ \ S_{j+1}=-\varphi\,{}^{C}I^{\alpha}(A_j)-\pi\,{}^{C}I^{\alpha}\left(S_j\right),\\ I_0 &= N_2,\ \ I_{j+1}=\varphi\,{}^{C}I^{\alpha}\left(A_j\right)-{}^{C}I^{\alpha}\left[(\gamma+\pi)I_j\right],\\ R_0 &= N_3+{}^{C}I^{\alpha}(\eta\pi),\ \ R_{j+1}=\gamma\,{}^{C}I^{\alpha}\left(I_j\right)-\pi\,{}^{C}I^{\alpha}\left(R_j\right).\end{aligned}\tag{5.3}$$

From the relations (5.3), the solution of the system (5.2) will be

$$S(t)=\sum_{m=0}^{n}S_m(t),\ I(t)=\sum_{m=0}^{n}I_m(t),\text{ and }R(t)=\sum_{m=0}^{n}R_m(t).$$

Figures 5–7 show ADM solutions of the SIR system with different values of α ($\alpha=1,0.9,0.8,0.7$). It is essential here to note that all the Parameters depend on the fractional order α of the model. The model consists of three variables, subject to time and $n = 5$. The parameters can be identified as follows: Parameters (N_1, N_2, and N_3) are the initial conditions of the SIR system, taking $N_1 = 1$, $N_2 = 0.2$, $N_3 = 0$. Also, $\eta = 0.9$ is the therapy rate, $\pi = 0.4$ is the birth rate $\varphi = 0.8$ is the infected individual rate, and $\gamma = 0.03$ is the recovery from infection rate.

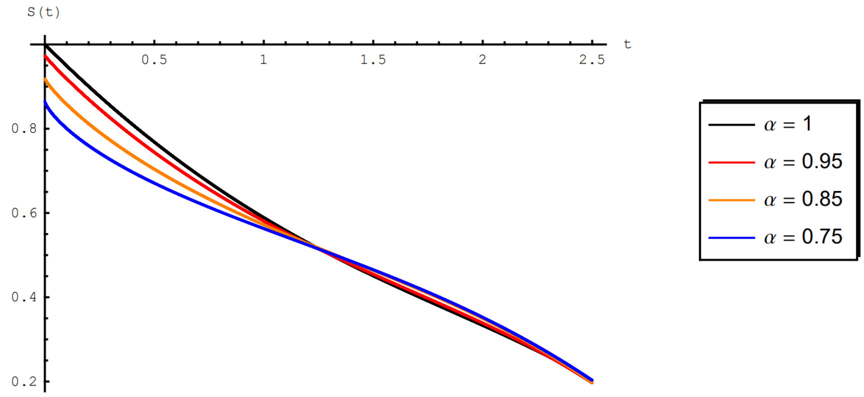

Figure 5. ADM solution of $S(t)$ with CD.

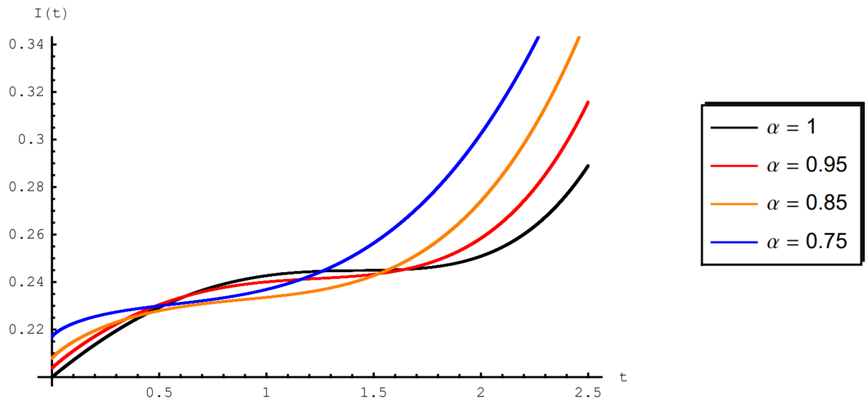

Figure 6. ADM solution of $I(t)$ with CD.

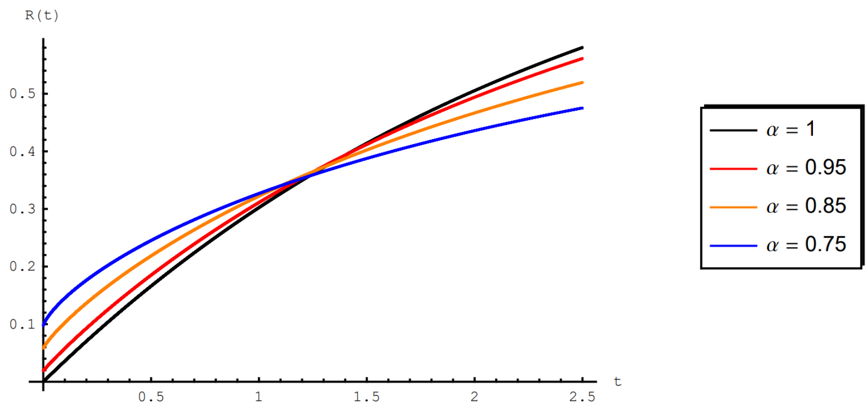

Figure 7. ADM solution of $R(t)$ with CD.

Tables 4–6 give ADs between the ADM and RK4 solutions of the SIR system at $\alpha = 1$, respectively.

Table 4. ADM and RK4 solutions of $S(t)$.

t	ADM solution	RK4 solution	AD
0	1	1	0
0.1	0.949	0.949	4.4×10^{-5}
0.2	0.901	0.901	4×10^{-6}
0.3	0.854	0.854	1.2×10^{-5}
0.4	0.810	0.810	3.7×10^{-5}
0.5	0.768	0.768	3.3×10^{-5}
0.6	0.729	0.729	5.9×10^{-5}
0.7	0.691	0.691	4×10^{-6}
0.8	0.655	0.655	0.00008
0.9	0.621	0.621	0.0003

Table 5. ADM and RK4 solutions of $I(t)$.

t	ADM solution	RK4 solution	AD
0	0.2	0.2	0
0.1	0.2071	0.207	0.00001
0.2	0.2136	0.214	0.00003
0.3	0.2195	0.2195	0.00002
0.4	0.225	0.225	0.00005
0.5	0.229	0.229	0.00003
0.6	0.233	0.233	0.00004
0.7	0.237	0.237	4×10^{-6}
0.8	0.239	0.244	0.005
0.9	0.241	0.239	0.002

Table 6. ADM and RK4 solutions of $R(t)$.

t	ADM solution	RK4 solution	AD
0	0	0	0
0.1	0.036	0.036	0.00001
0.2	0.070	0.070	0.00001
0.3	0.104	0.104	0.00004
0.4	0.135	0.135	0.00004
0.5	0.166	0.166	0.00002
0.6	0.196	0.196	0.00004
0.7	0.224	0.223	0.00096
0.8	0.251	0.251	0.00003
0.9	0.277	0.277	0.00002

From Tables 4–6, for $\alpha = 1$, ADM and RK4 are given enclosed values, as shown by the values of ADs between them.

Figure 8 shows the ADM solution of the SIR system at $\alpha = 0.5$ and $n = 5$.

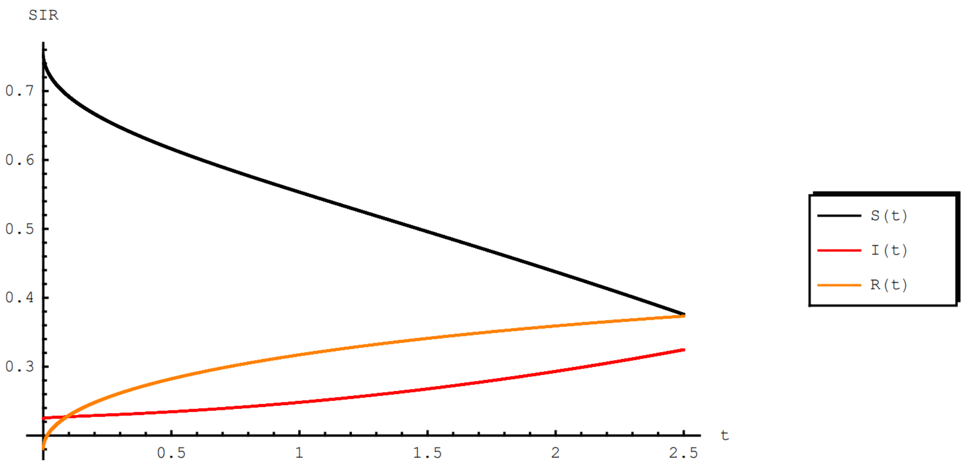

Figure 8. ADM Solution of the SIR model in the Caputo sense at $\alpha = 0.5$.

Figure 8 shows that the susceptible population decreases, whereas the infected population and the recovered population increase for a long time. In Figures 5–7, we see the effect of using different values of α on the SIR system.

6. Third definition: Atangana-Baleanu derivative (ABD)

The general form of the nonlinear FDE system with the ABD is

$$^{AB}D_t^{\upsilon} y_k(t) + h_k(t) f_k(\overline{y}) = \chi_k(t), \tag{6.1}$$

subject to

$$x_k^{(j-1)}(0) = c_k, \qquad k, j = 1, 2, \ldots, n, \tag{6.2}$$

as

$$\overline{y} = \{y_1(t), y_2(t), \ldots, y_n(t)\}, \qquad 0 < \upsilon < 1.$$

The SIR epidemic model is a special case of this system. Now, applying the FI of order υ, this reduces the system (6.1)-(6.2) to the system of FIEs,

$$\begin{aligned} x_k(t) &= c_k + \frac{1-\upsilon}{B(\upsilon)}\chi_k(t) + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} \chi_k(s) ds \\ &\quad - \frac{1-\upsilon}{B(\upsilon)} h_k(t) f_k(\overline{y}) - \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} h_k(s) f_k(\overline{y}) ds. \end{aligned} \tag{6.3}$$

Assume that $\chi_k(t)$ is bounded $\forall t \in J = [0, T]$, $T \in R^+$, $|h_k(\tau)| \leq M_k, \forall 0 \leq \tau \leq t \leq T$, M_k are finite constants, and $f_k(\overline{y})$ satisfy Lipschitz condition with the Lipschitz constants L_k such as

$$|f_k(\overline{y}) - f_k(\overline{z})| \leq L_k |\overline{y} - \overline{z}|. \tag{6.4}$$

Substituting Eq (3.5) into Eq (6.3), we get

$$\begin{aligned} y_k(t) &= c_k + \frac{1-\upsilon}{B(\upsilon)}\chi_k(t) + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1}\chi_k(s)ds \\ &\quad -\frac{1-\upsilon}{B(\upsilon)}h_k(t)\sum_{i=0}^{\infty}A_{ki} - \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1} h_k(s)\sum_{i=0}^{\infty}A_{ki}\,ds. \end{aligned} \tag{6.5}$$

Let $x_k(t) = \sum_{i=0}^{\infty} x_{ki}(t)$ in (6.5) and we get

$$x_{k0}(t) = c_k + \frac{1-\upsilon}{B(\upsilon)}\chi_k(t) + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1}\chi_k(s)ds, \tag{6.6}$$

$$x_{ki}(t) = -\frac{1-\upsilon}{B(\upsilon)}h_k(t)A_{k(i-1)} - \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1} h_k(s)A_{k(i-1)}ds, \quad i \geq 1. \tag{6.7}$$

The final solution will be

$$x_k(t) = \sum_{i=0}^{\infty} x_{ki}(t). \tag{6.8}$$

6.1. Analysis of convergence

6.1.1. Existence of a unique solution

Define the mapping $\Psi : \Omega \to \Omega$. Ω is the Banach space $(C^{(n)}(J), \|\cdot\|)$, where $C^{(n)}(J)$ is the class of continuous column vectors $Y = (\overline{y})'$ with norm $\|Y\| = \sum_{k=1}^{n} \max_{t\in J} |y_k(t)|$, and $(.)'$ denotes the matrix transpose.

Theorem 6.1. *The system (6.1)-(6.2) has a unique solution if* $0 < \phi < 1$, $\phi = \frac{LM[\Gamma(\upsilon)+T^{\upsilon}]}{B(\upsilon)\Gamma(\upsilon)}$, *where* $L = \sum_{m=1}^{n} L_m$, $M = \max\{M_1, M_2, \ldots, M_n\}$.

Proof. Equation (6.5) can be described as

$$\begin{aligned} Y(t) &= C + \frac{1-\upsilon}{B(\upsilon)}\chi(t) + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1}\chi(s)ds \\ &\quad -\frac{1-\upsilon}{B(\upsilon)}H(t)F(\overline{y}) - \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1} H(s)F(\overline{y})\,ds, \end{aligned}$$

where

$$\begin{aligned} C &= (c_1, c_2, \ldots, c_n)', \\ \chi(t) &= (\chi_1, \chi_2, \ldots, \chi_n)', \\ H(t) &= diag\{h_1, h_2, \ldots, h_n\}, \\ F(\overline{y}(t)) &= (f_1(\overline{y}), f_2(\overline{y}), \ldots, f_n(\overline{y}))'. \end{aligned}$$

The mapping $\Psi : \Omega \to \Omega$ is defined as

$$\Psi X(t) = C + \frac{1-\upsilon}{B(\upsilon)}\chi(t) + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t (t-s)^{\upsilon-1}\chi(s)ds$$

$$-\frac{1-\upsilon}{B(\upsilon)}H(t)F(\overline{y})-\frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}H(s)F(\overline{y})\,ds.$$

Let $X, Z \in \Omega$.

$$\begin{aligned}
\|\Psi Y(t)-\Psi Z(t)\| &= \left\|-\frac{1-\upsilon}{B(\upsilon)}H(t)(F(\overline{y})-F(\overline{z}))-\frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}H(s)(F(\overline{y})-F(\overline{z}))\,ds\right\| \\
&\leq \left\|\frac{1-\upsilon}{B(\upsilon)}H(t)(F(\overline{y})-F(\overline{z}))\right\|+\left\|\frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}H(s)(F(\overline{y})-F(\overline{z}))\,ds\right\| \\
&\leq \frac{1-\upsilon}{B(\upsilon)}\|H(t)\|\,\|F(\overline{x})-F(\overline{z})\|+\frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}\|H(s)\|\,\|F(\overline{x})-F(\overline{z})\|\,ds \\
&\leq \frac{(1-\upsilon)M}{B(\upsilon)}\left(\sum_{m=1}^n\max_{t\in J}|f_m(\overline{x})-f_m(\overline{z})|\right) \\
&\quad+\frac{\upsilon M}{B(\upsilon)\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}\left(\sum_{m=1}^n\max_{t\in J}|f_m(\overline{y})-f_m(\overline{z})|\right)ds \\
&\leq \frac{M}{B(\upsilon)}\left(\sum_{m=1}^n\max_{t\in J}|f_m(\overline{y})-f_m(\overline{z})|\right)\left[(1-\upsilon)+\frac{\upsilon}{\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}\,ds\right] \\
&\leq \frac{M}{B(\upsilon)}\left[1-\upsilon+\frac{\upsilon T^\upsilon}{\Gamma(\upsilon)}\right]\sum_{m=1}^n L_m\|Y-Z\| \\
&\leq \frac{M}{B(\upsilon)}\left[1+\frac{T^\upsilon}{\Gamma(\upsilon)}\right]\sum_{m=1}^n L_m\|Y-Z\| \\
&\leq \frac{M[\Gamma(\gamma)+T^\gamma]}{B(\gamma)\Gamma(\gamma)}\sum_{m=1}^n L_m\|Y-Z\| \\
&\leq \frac{M[\Gamma(\gamma)+T^\gamma]}{B(\gamma)\Gamma(\gamma)}\sum_{m=1}^n L_m\|Y-Z\| \\
&\leq \frac{LM[\Gamma(\upsilon)+T^\upsilon]}{B(\upsilon)\Gamma(\upsilon)}\|Y-Z\| \\
&\leq \phi\|Y-Z\|.
\end{aligned}$$

If $0<\phi<1$, the mapping Ψ will be a contraction, and then there exists a unique solution of the system (6.1)-(6.2). □

6.1.2. Convergence proof

Theorem 6.2. *The series solution (6.8) will converge if* $|y_{k1}|<\infty$ *and* $0<\phi<1$, $\phi=\frac{LM[\Gamma(\upsilon)+T^\upsilon]}{B(\upsilon)\Gamma(\upsilon)}$, *where* $L=\sum_{k=1}^n L_k$, $M=\max\{M_1,M_2,\ldots,M_n\}$.

Proof. Take a sequence $\{S_{kr}\}$ such that $S_{kr}=\sum_{i=0}^r y_{ki}(t)$ is a partial sums sequence of $\sum_{i=0}^\infty y_{ki}(t)$. We have

$$f(S_{kr})=\sum_{i=0}^p A_{ki}(y_{k0},y_{k1},\ldots,y_{kr}).$$

Let S_{kr} and S_{kw} be two partial sums where $r > w$. Our goal is to prove that $\{S_{kr}\}$ is a Cauchy sequence in this Banach space.

$$
\begin{aligned}
\|S_{kr} - S_{kw}\| &= \sum_{i=1}^{n} \max_{t\in J} |S_{ir} - S_{iw}| = \sum_{i=1}^{n} \max_{t\in J} \left| \sum_{j=w+1}^{r} x_{ij}(t) \right| \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left| \sum_{j=w+1}^{r} \frac{1-\upsilon}{B(\upsilon)} h_k(t) A_{k(i-1)} + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} h_k(s) A_{k(i-1)} ds \right| \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left| \frac{1-\upsilon}{B(\upsilon)} h_k(t) \sum_{j=w+1}^{r} A_{i(j-1)} + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\gamma-1} h_k(s) \sum_{j=w+1}^{r} A_{i(j-1)} ds \right| \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left| \frac{1-\upsilon}{B(\upsilon)} h_k(t) \sum_{j=w}^{r-1} A_{ij} + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} h_k(s) \sum_{j=w}^{r-1} A_{ij} ds \right| \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left| \frac{1-\upsilon}{B(\upsilon)} h_k(t) [f(S_{i(r-1)}) - f(S_{i(w-1)})] \right. \\
&\quad \left. + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} h_k(s) [f(S_{i(r-1)}) - f(S_{i(w-1)})] ds \right] \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left[\left| \frac{1-\upsilon}{B(\upsilon)} h_k(t) [f(S_{i(r-1)}) - f(S_{i(w-1)})] \right| \right. \\
&\quad \left. + \left| \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} h_k(s) [f(S_{i(r-1)}) - f(S_{i(w-1)})] ds \right| \right] \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left[\frac{1-\upsilon}{B(\upsilon)} |h_k(t)| \left| f(S_{i(r-1)}) - f(S_{i(w-1)}) \right| \right. \\
&\quad \left. + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} |h_k(s)| \left| f(S_{i(r-1)}) - f(S_{i(w-1)}) \right| ds \right] \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left[\frac{1-\upsilon}{B(\upsilon)} |h_k(t)| \left(L_i \sum_{j=1}^{n} \left| S_{j(r-1)} - S_{j(w-1)} \right| \right) \right. \\
&\quad \left. + \frac{\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} |h_k(s)| \left(L_i \sum_{j=1}^{n} \left| S_{j(r-1)} - S_{j(w-1)} \right| \right) ds \right] \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left(L_i \sum_{j=1}^{n} \left| S_{j(r-1)} - S_{j(w-1)} \right| \right) \left[\frac{1-\upsilon}{B(\upsilon)} M + \frac{M\upsilon}{B(\upsilon)\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} ds \right] \\
&\le \sum_{i=1}^{n} \max_{t\in J} \left(L_k \sum_{j=1}^{n} \left| S_{j(r-1)} - S_{j(w-1)} \right| \right) \left[\frac{1-\upsilon}{B(\upsilon)} M + \frac{MT^{\upsilon}}{B(\upsilon)\Gamma(\upsilon)} \right] \\
&\le \frac{LM[\Gamma(\upsilon) + T^{\upsilon}]}{B(\upsilon)\Gamma(\upsilon)} \left\| S_{k(r-1)} - S_{k(w-1)} \right\| \\
&\le \phi \left\| S_{k(r-1)} - S_{k(w-1)} \right\|.
\end{aligned}
$$

Let $r = w + 1$. Then,

$$\left\|S_{k(w+1)} - S_{kw}\right\| \leq \phi \left\|S_{kw} - S_{k(w-1)}\right\| \leq \phi^2 \left\|S_{k(w-1)} - S_{k(w-2)}\right\| \leq \cdots \leq \phi^w \|S_{k1} - S_{k0}\|.$$

From the triangle inequality,

$$\begin{aligned} \|S_{kr} - S_{kw}\| &\leq \left\|S_{k(w+1)} - S_{kw}\right\| + \left\|S_{k(w+2)} - S_{k(w+1)}\right\| + \cdots + \left\|S_{kr} - S_{k(r-1)}\right\| \\ &\leq \left[\phi^w + \phi^{w+1} + \cdots + \phi^{r-1}\right] \|S_{k1} - S_{k0}\| \\ &\leq \phi^w \left[1 + \phi + \cdots + \phi^{r-w-1}\right] \|S_{k1} - S_{k0}\| \\ &\leq \phi^w \left[\frac{1 - \phi^{r-w}}{1 - \phi}\right] \|y_{k1}(t)\|. \end{aligned}$$

Since $0 < \phi < 1$ and $r > w$, $(1 - \phi^{r-w}) \leq 1$. Consequently,

$$\|S_{kr} - S_{kw}\| \leq \frac{\phi^w}{1 - \phi} \|y_{k1}(t)\| \leq \frac{\phi^w}{1 - \phi} \max_{t \in J} |y_{k1}(t)|,$$

but $|y_{k1}(t)| < \infty$. As $w \to \infty$, then, $\|S_{kr} - S_{kw}\| \to 0$. Therefore, $\{S_{kr}\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{i=0}^{\infty} y_{ki}(t)$ will converge. □

6.1.3. Error estimation

Theorem 6.3. *The maximum absolute error of the series solution (6.8) is estimated as*

$$\max_{t \in J} \left| y_k(t) - \sum_{i=0}^{w} y_{ki}(t) \right| \leq \frac{\phi^w}{1 - \phi} \max_{t \in J} |y_{k1}(t)|.$$

Proof. From Theorem 6.2, we have

$$\|S_{kr} - S_{kw}\| \leq \frac{\phi^w}{1 - \phi} \max_{t \in J} |y_{k1}(t)|.$$

However, $S_{ir} = \sum_{i=0}^{r} y_{ki}(t)$ as $r \to \infty$. Then, $S_{kr} \to y_k(t)$. So

$$\|y_k(t) - S_{kw}\| \leq \frac{\phi^w}{1 - \phi} \max_{t \in J} |y_{k1}(t)|.$$

So, the maximum absolute error will be

$$\max_{t \in J} \left| y_k(t) - \sum_{i=0}^{w} y_{ki}(t) \right| \leq \frac{\phi^w}{1 - \phi} \max_{t \in J} |y_{k1}(t)|.$$

□

6.2. SIR epidemic model of arbitrary orders containing ABD

The SIR epidemic model of arbitrary orders involving the ABD is

$$\begin{aligned} {}^{AB}D_0^{\alpha}S(t) &= (1-p)\pi - \beta S I - \pi S, \\ {}^{AB}D_0^{\alpha}I(t) &= \beta S I - (\gamma+\pi) I, \\ {}^{AB}D_0^{\alpha}R(t) &= p\pi + \gamma I - \pi R, \end{aligned} \tag{6.9}$$

subject to

$$S(0) = N_1,\ I(0) = N_2,\ R(0) = N_3.$$

Applying the ADM to the system (6.9), we get the following solution algorithm:

$$S_0 = N_1 + {}^{AB}I^{\alpha}(1-p)\pi, \quad S_{j+1} = -\beta\, {}^{AB}I^{\alpha}(A_j) - \pi\, {}^{AB}I^{\alpha}\left(S_j\right), \tag{6.10}$$

$$I_0 = N_2, \quad I_{j+1} = \beta\, {}^{AB}I^{\alpha}\left(A_j\right) - {}^{AB}I^{\alpha}\left[(\gamma+\pi) I_j\right], \tag{6.11}$$

$$R_0 = N_3 + {}^{AB}I^{\alpha}(p\pi), \quad R_{j+1} = \gamma\, {}^{AB}I^{\alpha}\left(I_j\right) - \pi\, {}^{AB}I^{\alpha}\left(R_j\right). \tag{6.12}$$

Using the relations (6.10)–(6.12), the series solution of the system (6.9) will be

$$S(t) = \sum_{k=0}^{n} S_k(t), I(t) = \sum_{k=0}^{n} I_k(t), R(t) = \sum_{k=0}^{n} R_k(t).$$

Figures 9–11 show the ADM solution of the SIR system at different values of α ($\alpha = 1, 0.95, 0.85, 0.75$) and $n = 5$, respectively.

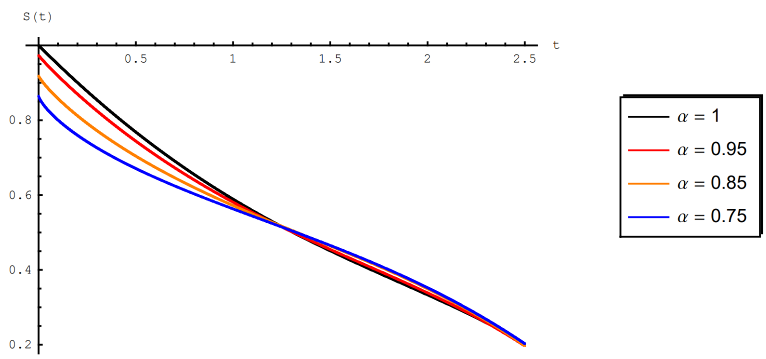

Figure 9. ADM solution of $S(t)$ in the Atangana-Baleanu sense at different values of α.

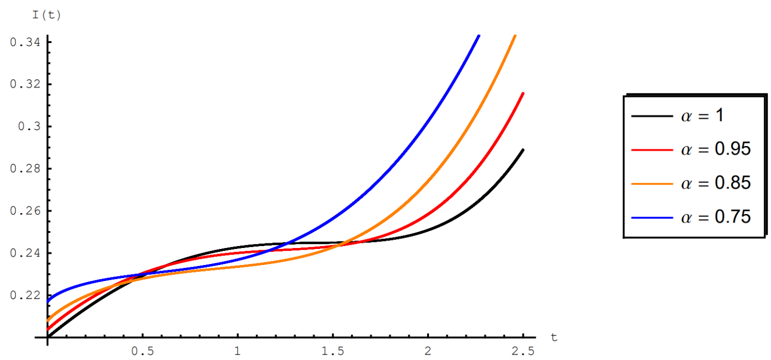

Figure 10. ADM solution of $I(t)$ in the Atangana-Baleanu sense at different values of α.

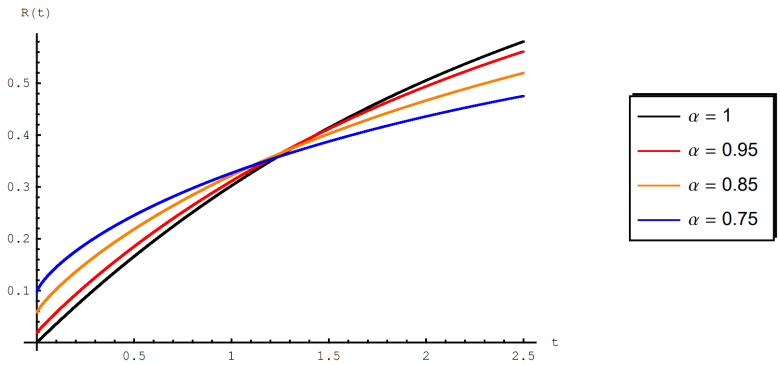

Figure 11. ADM solution of $R(t)$ in the Atangana-Baleanu sense at different values of α.

Tables 7–9 show ADs between the ADM and RK4 solutions of the SIR system at $\alpha = 1$, respectively.

Table 7. ADM and RK4 solutions of $S(t)$.

t	ADM solution	RK4 solution	AD
0	1	1	0
0.1	0.949	0.949	0.00004
0.2	0.901	0.901	4×10^{-6}
0.3	0.854	0.854	0.00001
0.4	0.810	0.810	0.00004
0.5	0.768	0.768	0.00003
0.6	0.729	0.729	0.00006
0.7	0.691	0.691	4×10^{-6}
0.8	0.655	0.655	0.00008
0.9	0.621	0.621	0.00027

Table 8. ADM and RK4 solutions of $I(t)$.

t	ADM solution	RK4 solution	AD
0	0.2	0.2	0
0.1	0.207	0.207	0.00001
0.2	0.214	0.214	0.00003
0.3	0.2195	0.2195	0.00002
0.4	0.225	0.225	0.00005
0.5	0.229	0.229	0.00003
0.6	0.233	0.233	0.00004
0.7	0.237	0.237	4×10^{-6}
0.8	0.239	0.244	0.005
0.9	0.241	0.239	0.002

Table 9. ADM and RK4 solutions of $R(t)$.

t	ADM solution	RK4 solution	ADs
0	0	0	0
0.1	0.036	0.0359	0.00001
0.2	0.070	0.0704	0.00001
0.3	0.104	0.1036	0.00004
0.4	0.135	0.1354	0.00004
0.5	0.166	0.1661	0.00002
0.6	0.196	0.1955	0.00004
0.7	0.224	0.2229	0.00096
0.8	0.251	0.2511	0.00003
0.9	0.277	0.2772	0.00002

From Tables 7–9, for $\alpha = 1$, ADM and RK4 are given enclosed values, as shown from the values of ADs between them.

Figure 12 shows the ADM solution of the SIR system at $\alpha = 0.5$ and $n = 5$.

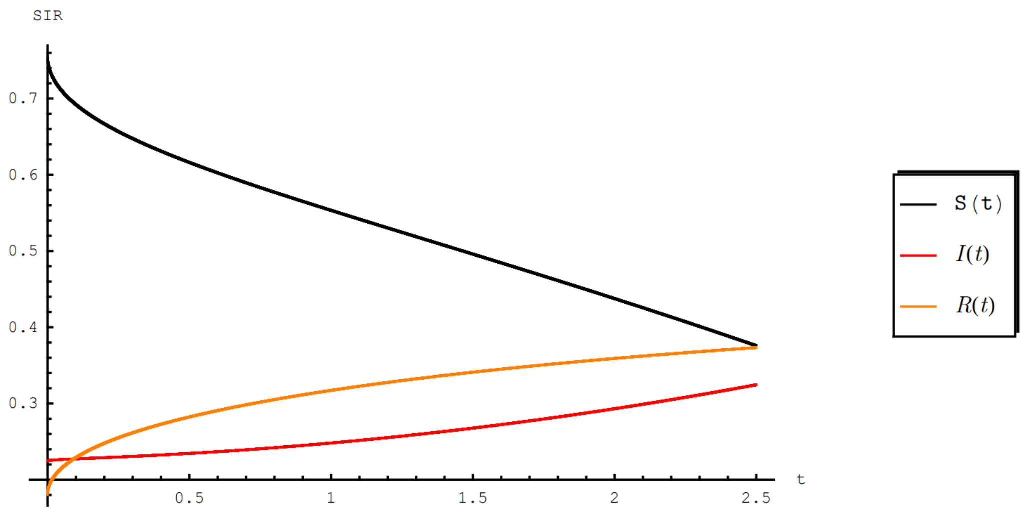

Figure 12. ADM solution of the SIR model in the Atangana-Baleanu sense at $\alpha = 0.5$.

From Figure 12, we see that the susceptible population reduces, whereas the infected population and the recovered population increase for a long time. In Figures 9–11, we see the effect of using different values of α on the SIR system.

7. Comparison between the three definitions

In this section, we aim to give a comparison between the previous three different FDs, as shown in the following figures. In Figures 13–23, we show the solution of the SIR system at multiple values of α ($\alpha = 1, 0.95, 0.85, 0.75$) and $n = 5$, as follows:

(1) The solution of $S(t)$ is given in Figures 13–15.

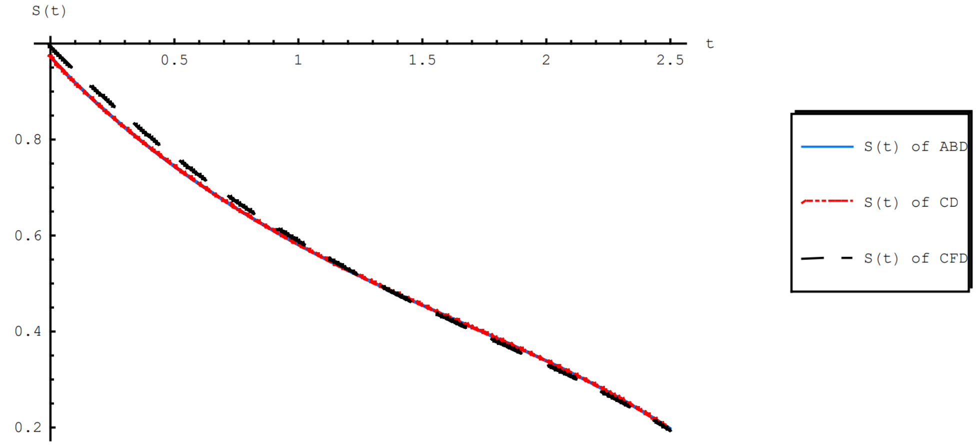

Figure 13. $S(t)$ solution of CFD, CD, ABD at $\alpha = 0.95$.

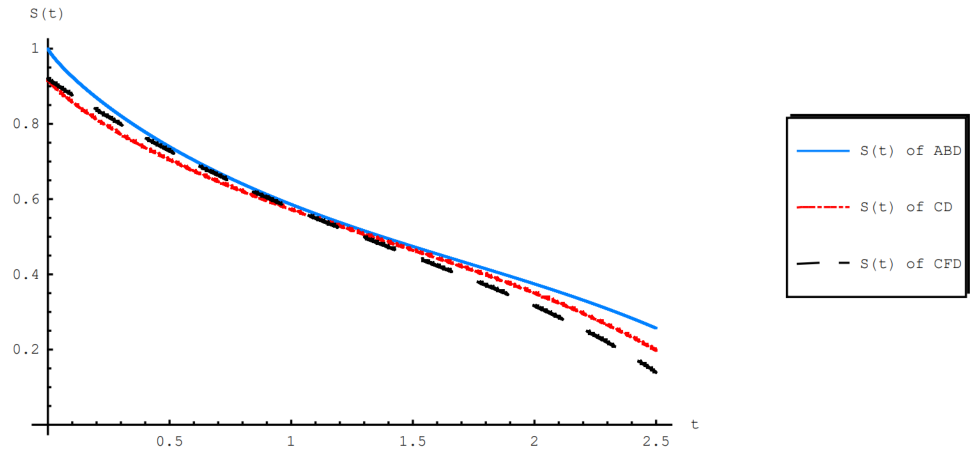

Figure 14. $S(t)$ solution of CFD, CD, ABD at $\alpha = 0.85$.

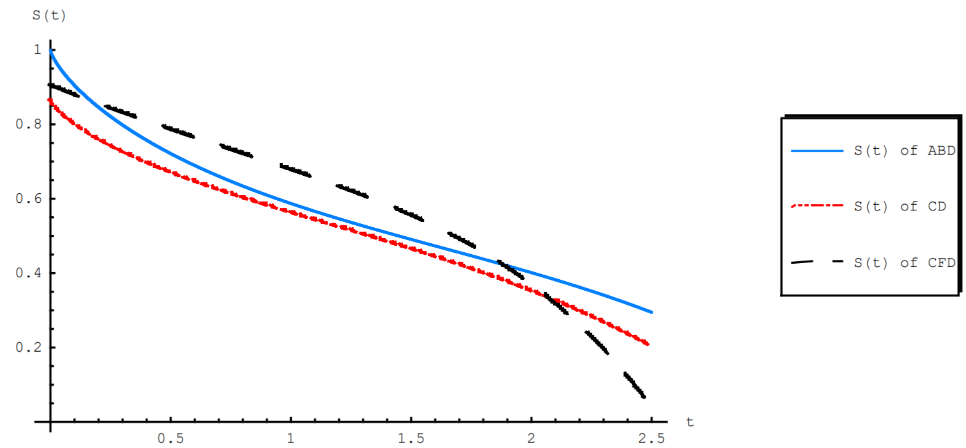

Figure 15. $S(t)$ solution of CFD, CD, ABD at $\alpha = 0.75$.

(2) The solution of $I(t)$ is given in Figures 16–19.

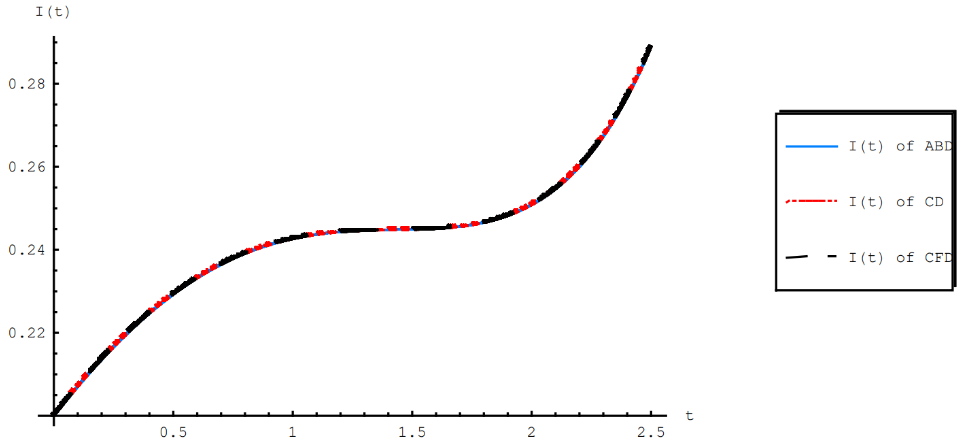

Figure 16. $I(t)$ solution of CFD, CD, ABD at $\alpha = 1$.

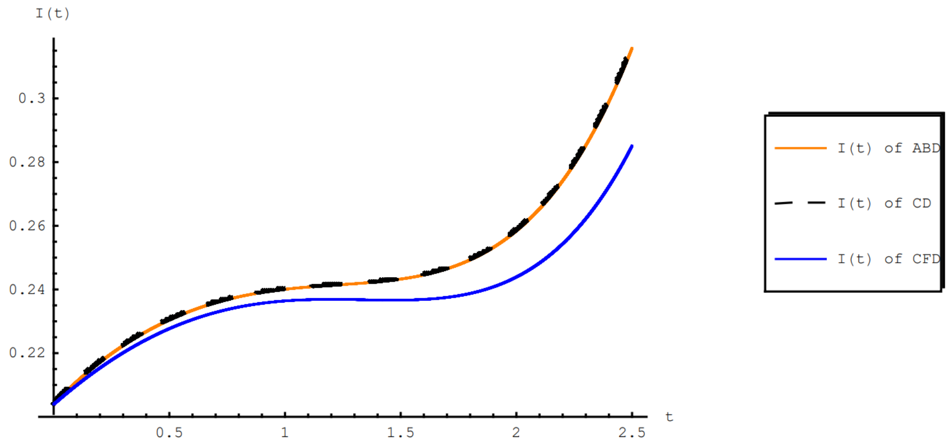

Figure 17. $I(t)$ solution of CFD, CD, ABD at $\alpha = 0.95$.

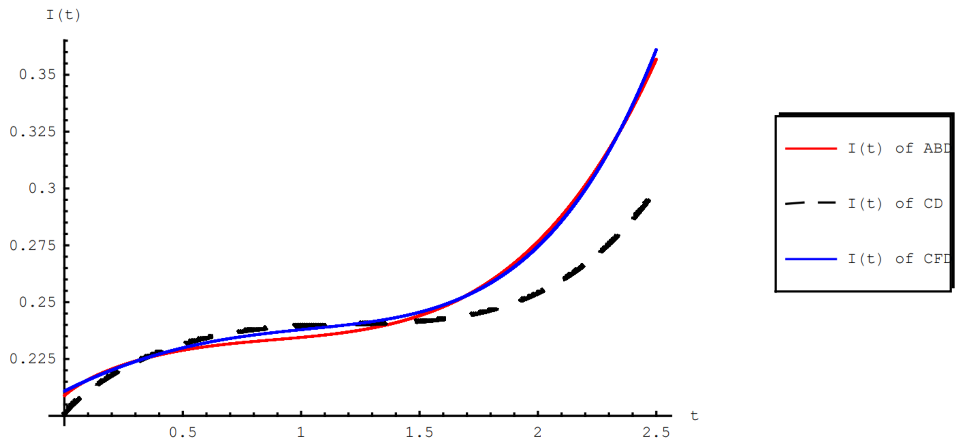

Figure 18. $I(t)$ solution of CFD, CD, ABD at $\alpha = 0.85$.

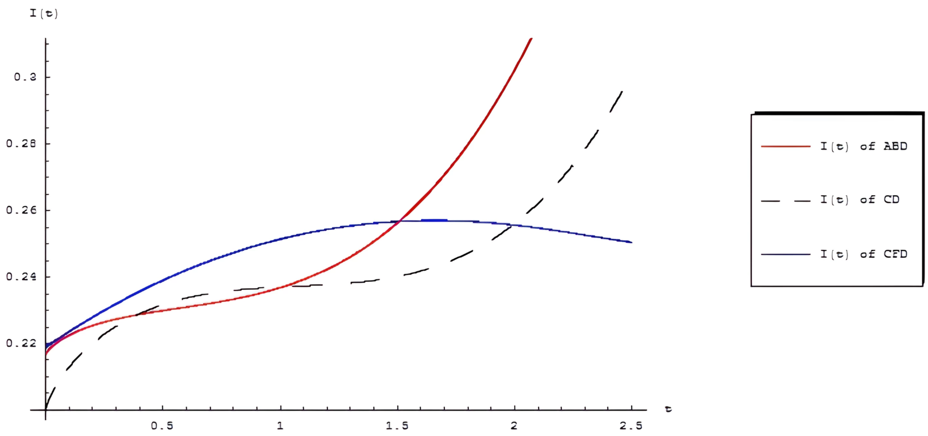

Figure 19. $I(t)$ solution of CFD, CD, ABD at $\alpha = 0.75$.

(3) The solution of $R(t)$ is given in Figures 20–23.

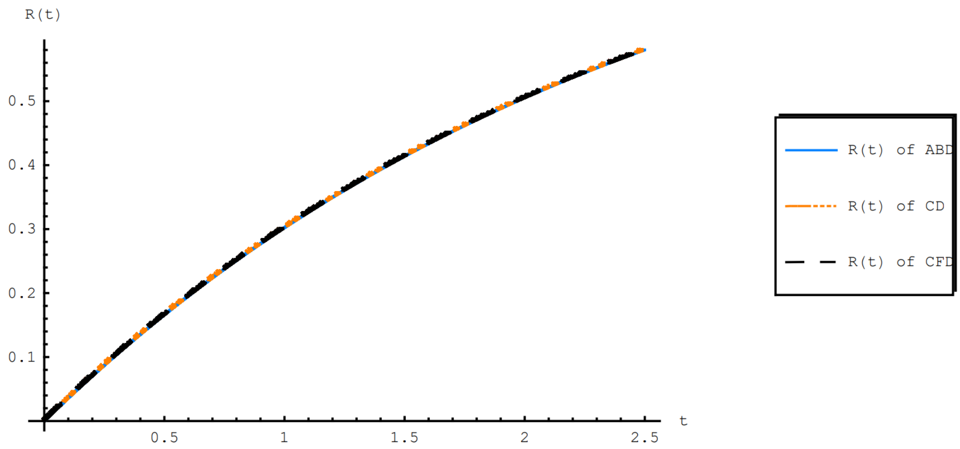

Figure 20. $R(t)$ solution of CFD, CD, ABD at $\alpha = 1$.

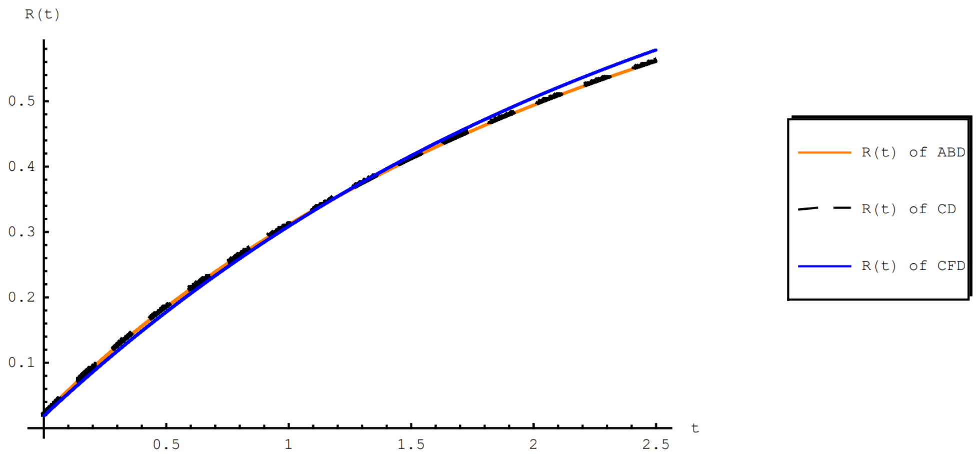

Figure 21. $R(t)$ solution of CFD, CD, ABD at $\alpha = 0.95$.

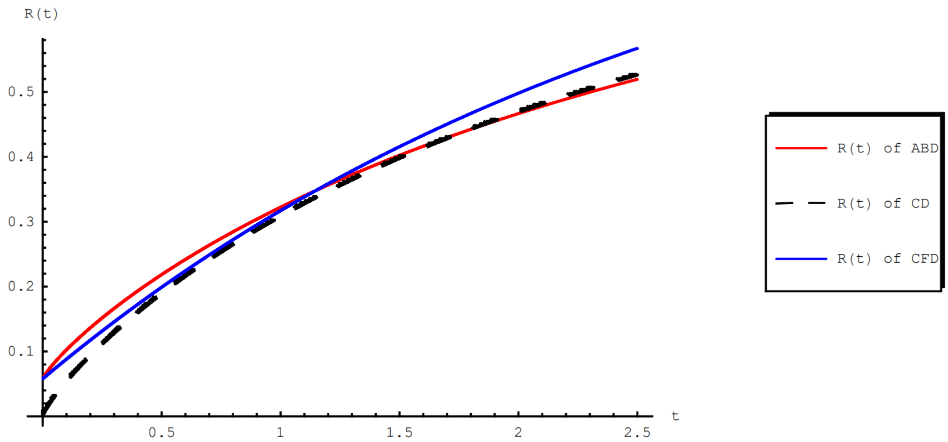

Figure 22. $R(t)$ solution of CFD, CD, ABD at $\alpha = 0.85$.

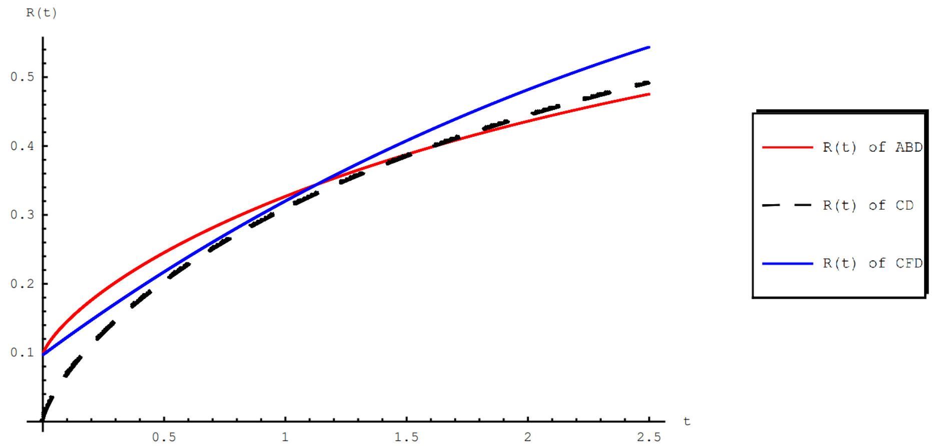

Figure 23. $R(t)$ solution of CFD, CD, ABD at $\alpha = 0.75$.

From Figures 13–23, we see the following:
i) In the integer case ($\alpha = 1$), the three FDs give the same solution (see Figures 13, 16 and 20).
ii) For fractional orders, we see that the ABD is closer to the CD than the CFD.

8. Conclusions

This research considered analytical and numerical solutions of an important fractional order model of epidemic childhood diseases (the SIR model) with three different definitions of fractional derivatives: CD, CFD, and ABD. The analytical solution was obtained using the ADM, while the numerical solution was obtained using the RK4 method. By calculating the absolute differences between these two methods, we see that the two solutions coincide (see Tables 1–9). A comparison is made between the solutions obtained with the three different definitions, and we see that, for integer order ($\alpha = 1$), the three different FDs give the same solution (see Figures 13, 16 and 20). Meanwhile, for fractional orders, we see that the ABD is closer to the CD than the CFD (see Figures 14–15, 17–19 and 21–23).

Author contributions

Eman A. A. Ziada and Osama Moaaz: Conceptualization, Methodology, Formal analysis; Salwa El-Morsy and Ahmad M. Alshamrani: Investigation, Writing-original draft; Sameh S. Askar and Monica Botros: Software, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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