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Research article

Duals of Gelfand-Shilov spaces of type $K\{M_p\}$ for the Hankel transformation

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Abstract: For $\mu \geq -\frac{1}{2}$, and under appropriate conditions on the sequence $\{M_p\}_{p=0}^{\infty}$ of weights, the elements, the (weakly, weakly*, strongly) bounded subsets, and the (weakly, weakly*, strongly) convergent sequences in the dual of a space $\mathcal{K}_\mu$ of type Hankel-$K\{M_p\}$ can be represented by distributional derivatives of functions and measures in terms of iterated adjoints of the differential operator $x^{-1}D_x$ and the Bessel operator $S_\mu = x^{-\mu-\frac{1}{2}}D_x x^{2\mu+1}D_x x^{-\mu-\frac{1}{2}}$. In this paper, such representations are compiled, and new ones involving adjoints of suitable iterations of the Zemanian differential operator $N_\mu = x^{\mu+\frac{1}{2}}D_x x^{-\mu-\frac{1}{2}}$ are proved. Prior to this, new descriptions of the topology of the space $\mathcal{K}_\mu$ are given in terms of the latter iterations.

Keywords: Bessel operator; boundedness; convergence; distribution; generalized function; $K\{M_p\}$ space; representation; test function; Zemanian operator; Zemanian space
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1. Introduction

As it is well known, Schwartz developed the theory of distributions in the late 1940's; a detailed exposition appears in his monograph [29]. Generalized functions of any kind, as well as their use to solve the Cauchy problem, were introduced by Gelfand and Shilov around 1953. In the period of 1956–58, these two authors published three volumes (in Russian) on the subject, which were translated into English during the 1960's [11–13]. Meanwhile, Friedman disseminated the ideas of Gelfand and Shilov in his book [10], enhancing them with more recent applications to differential equations, as well as a more complete treatment of the Cauchy problem.

Several test function spaces that were derived in the framework of the generalized Fourier transformation belong to the family of Gelfand-Shilov $K\{M_p\}$ spaces, whose theory was developed in [10, 12, 13] in connection with the Cauchy problem for various partial differential equations, boundary value problems for elliptic equations, and the problem of eigenfunction expansions for several differential operators. Among them are the spaces $\mathcal{D}_K$ and $\mathcal{S}$, introduced by Schwartz [29]; the space H (also denoted by $\mathcal{K}_1$), as developed by Sebastião e Silva [30] and later studied by Hasumi [16], Zieleźny [41], Sznajder and Zieleźny [34], and, somewhat more recently, by de Sousa Pinto [31]; the spaces denoted by $\mathcal{K}_p$ $(p > 1)$, as developed by Sznajder and Zieleźny [35]; and the spaces $\mathcal{S}_{\alpha,A}$ and $W_{M,a}$, developed by Gelfand and Shilov themselves [12, 13].

Most of the examples listed above have analogues in the Hankel transformation setting, such as those considered by Zemanian [38, 39], Betancor and Marrero [5], Betancor and Rodríguez-Mesa [6, 7], Durán [8], van Eijndhoven and Kerkhof [9], Lee [21], Pathak and Sahoo [26], and Pathak and Upadhyay [27]. In order to unify the underlying theory, Marrero introduced [24] and studied [22, 23, 25] the so-called Hankel-$K\{M_p\}$ spaces, which were intended to play the same role in the Hankel transformation setting as do the Gelfand-Shilov $K\{M_p\}$ spaces in the Fourier transformation setting. The study of Hankel-$K\{M_p\}$ spaces was continued by Arteaga and Marrero [1, 2].

In [10, p. 37], Friedman asserts the following:

> *One of the most interesting and important problems in the theory of generalized functions is the problem of finding the structure of generalized functions by expressing them in terms of differential operators acting on functions or on measures.*

Our aim in this paper is threefold: first, we want to briefly review the existing literature on the structure of distributions in spaces of type Hankel-$K\{M_p\}$; second, we want to obtain new structural results for these distributions in terms of the Zemanian differential operator N_μ [40, Section 5.3, Equation (3)]; and, third, we want to apply them in the characterization of the bounded subsets and the convergent sequences in the duals of spaces of type Hankel-$K\{M_p\}$.

The paper is organized as follows. In Section 2, the definition and topological properties, along with some examples of Hankel-$K\{M_p\}$ spaces, are recalled. Section 3 is devoted to reviewing the literature on the structural properties of the dual of a space of type Hankel-$K\{M_p\}$. The main results are established in Section 4, where a new description of the topology of a Hankel-$K\{M_p\}$ space is obtained; then, in Section 5, where such a description is applied to provide new results on the structure, boundedness, and convergence of distributions of type Hankel-$K\{M_p\}$.

Throughout the paper, the standard notation in distribution theory will be used. The letter I will stand for the interval $]0, \infty[$ and, unless otherwise stated, μ will be a fixed real parameter not less than $-\frac{1}{2}$, while C will represent a suitable positive constant which may vary from line to line.

2. The topology of Hankel-$K\{M_p\}$ spaces

Definition 2.1. ([24, Definition 2.1]) *Let $\{M_p\}_{p=0}^{\infty}$ be a sequence of continuous functions defined on $I =]0, \infty[$ such that*

$$1 = M_0(x) \le M_1(x) \le M_2(x) \le \dots \quad (x \in I).$$

We say that $\mathcal{K}_\mu$ is a space of type Hankel-$K\{M_p\}$, or just a Hankel-$K\{M_p\}$ space, provided that $\mathcal{K}_\mu$ consists of all of the complex-valued functions $\varphi \in C^\infty(I)$ such that

$$\|\varphi\|_{\mu,\infty,p} = \max_{0\le k\le p} \sup_{x\in I} |M_p(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)| < \infty \quad (p \in \mathbb{N}_0).$$

$\mathcal{K}_\mu$ is endowed with the locally convex topology generated by the sequence of norms $\left\{\|\cdot\|_{\mu,\infty,p}\right\}_{p=0}^{\infty}$. The dual space of $\mathcal{K}_\mu$ will be denoted by $\mathcal{K}'_\mu$.

What follows are examples of test function spaces of type Hankel-$K\{M_p\}$ arising in connection with the generalized Hankel transformation.

Example 2.2. *Let*

$$M_p(x) = \left(1 + x^2\right)^p \quad (x \in I,\ p \in \mathbb{N}_0).$$

The corresponding Hankel-$K\{M_p\}$ space is the Zemanian space $\mathcal{H}_\mu$ [38], [40, Chapter 5].

Example 2.3. *Fix $a > 0$. If the functions $\{M_p\}_{p=0}^{\infty}$ are allowed to take on the value ∞, then, with the convention that $0 \cdot \infty = 0$, the Zemanian space $\mathcal{B}_{\mu,a}$ [39] can be regarded as a Hankel-$K\{M_p\}$ space upon setting*

$$M_p(x) = \begin{cases} 1, & 0 < x < a \\ \infty, & x \geq a \end{cases} \quad (p \in \mathbb{N}_0).$$

Example 2.4. *Given $\alpha, A > 0$, define*

$$M_p(x) = \left(1 + x^2\right)^p \exp\left\{\frac{\alpha}{eA^{\frac{1}{\alpha}}}\left(1 - \frac{1}{p}\right)x^{\frac{1}{\alpha}}\right\} \quad (x \in I,\ p \in \mathbb{N}_0).$$

The resulting Hankel-$K\{M_p\}$ space is the space $\mathcal{H}_{\mu,\alpha,A}$, as introduced by Betancor and Marrero [5].

Example 2.5. *The space χ_μ, as defined by Betancor and Rodríguez-Mesa [6], is the Hankel-$K\{M_p\}$ space corresponding to the choice*

$$M_p(x) = \exp(px) \quad (x \in I,\ p \in \mathbb{N}_0).$$

Example 2.6. *The space $U_{\mu,M,a}^{\infty}$ developed by Pathak and Upadhyay [27] is also of type Hankel-$K\{M_p\}$, as can be seen upon setting*

$$M_p(x) = \exp\left\{M\left[a\left(1 - \frac{1}{p}\right)x\right]\right\} \quad (x \in I,\ p \in \mathbb{N}_0),$$

where $a > 0$,

$$M(x) = \int_0^x v(\xi)d\xi \quad (x \in I),$$

and the function $v = v(\xi)$ is continuous and increasing on $[0, \infty[$, with $v(0) = 0$ and $v(\infty) = \infty$.

In the previous examples, the sequence of weights $\{M_p\}_{p=0}^{\infty}$ satisfies at least one of the conditions in the following definition.

Definition 2.7. *The sequence $\{M_p\}_{p=0}^{\infty}$ is said to satisfy condition $(\cdot)$ for $\cdot$ = O, A, M, N, P, provided that the following hold:*

- (O) *The limit $\lim_{x\to 0^+} M_p(x)$ $(p \in \mathbb{N}_0)$ exists.*
- (A) *Given $r, p \in \mathbb{N}_0$, there exist $s \in \mathbb{N}_0$ and $b_{rp} > 0$ such that*

$$M_r(x)M_p(x) \leq b_{rp}M_s(x) \quad (x \in I).$$

(M) *Each M_p ($p \in \mathbb{N}_0$) is quasi-monotonic, that is, there exists $C_p > 0$ such that*

$$M_p(x) \leq C_p M_p(y) \quad (x, y \in I,\ x \leq y).$$

(N) *For every $p \in \mathbb{N}_0$, there exists $r \in \mathbb{N}_0$, $r > p$ such that the function*

$$m_{pr}(x) = \frac{M_p(x)}{M_r(x)} \quad (x \in I)$$

lies in $L^1(I)$ and satisfies

$$\lim_{x\to\infty} m_{pr}(x) = 0.$$

(P) *Given $p \in \mathbb{N}_0$, there exists $r \in \mathbb{N}_0$, $r > p$ for which*

$$\lim_{x\to\infty} \frac{M_p(x)}{M_r(x)} = 0.$$

By imposing appropriate combinations of the above conditions on the weights $\{M_p\}_{p=0}^{\infty}$, Hankel-$K\{M_p\}$ spaces can be endowed with interesting properties as topological vector spaces. It should be remarked that conditions similar to those in Definition 2.7 have been considered by several authors [12, 14, 15, 19, 20, 32, 33, 37] in order to develop a suitable theory of the Gelfand-Shilov spaces of type $K\{M_p\}$.

In fact, under adequate assumptions on the weights $\{M_p\}_{p=0}^{\infty}$, a space $\mathcal{K}_\mu$ of type Hankel-$K\{M_p\}$ can be nuclear [36, Definition III.50.1], Schwartz [17, Definition 3.15.1], Montel [17, Definition 3.9.1], and reflexive. We excerpt from [24] some of the results available along these lines, as well as those that reveal the relationship between $\mathcal{K}_\mu$ and the Zemanian spaces $\mathcal{H}_\mu$ and $\mathcal{B}_\mu$ (see Examples 2.2 and 2.3).

Proposition 2.8. ([24, Proposition 4.1]) *The space $\mathcal{K}_\mu$ is Fréchet. If the sequence $\{M_p\}_{p=0}^{\infty}$ satisfies (O) and (P), then $\mathcal{K}_\mu$ is Schwartz, Montel, and reflexive.*

Proposition 2.9. ([24, Proposition 4.3]) *Assume that $\{M_p\}_{p=0}^{\infty}$ satisfies conditions (O) or (M). Then, the injection $\mathcal{B}_\mu \hookrightarrow \mathcal{K}_\mu$ is continuous. If, additionally, $\{M_p\}_{p=0}^{\infty}$ satisfies (P), then $\mathcal{B}_\mu$ is dense in $\mathcal{K}_\mu$.*

Proposition 2.10. ([24, Proposition 4.4]) *Assume that $\{M_p\}_{p=0}^{\infty}$ satisfies (M) and (N).*

(i) *For every $p \in \mathbb{N}_0$, there exist $r \in \mathbb{N}_0$, $r > p$, and $C_{pr} > 0$ such that $x \leq C_{pr} M_r(x)$ ($x \in I$).*
(ii) *If $\{M_p\}_{p=0}^{\infty}$ satisfies (A) as well, then $\mathcal{K}_\mu \subset \mathcal{H}_\mu$ with a continuous embedding.*

Corollary 2.11. ([24, Corollary 4.5]) *If $\{M_p\}_{p=0}^{\infty}$ satisfies conditions (A), (M), and (N), then $\mathcal{B}_\mu \subset \mathcal{K}_\mu \subset \mathcal{H}_\mu$ with a continuous embedding. Moreover, $\mathcal{B}_\mu$ is dense in $\mathcal{K}_\mu$ and $\mathcal{K}_\mu$ is dense in $\mathcal{H}_\mu$.*

Proposition 2.12. ([24, Proposition 4.6]) *Under (A), (M), and (N), the topology of $\mathcal{K}_\mu$ is compatible with any one of the families of norms $\left\{\|\cdot\|_{\mu,q,p}\right\}_{p=0}^{\infty}$ ($q \in \mathbb{R}$, $1 \leq q < \infty$), where*

$$\|\varphi\|_{\mu,q,p} = \left\{\sum_{k=0}^{p} \int_0^\infty \left|M_p(x)\left(x^{-1}D\right)^k x^{-\mu-\frac{1}{2}}\varphi(x)\right|^q dx\right\}^{\frac{1}{q}} \quad (\varphi \in \mathcal{K}_\mu).$$

In this case, $\mathcal{K}_\mu$ is nuclear, Schwartz, Montel, and reflexive.

3. The dual space and its topology

Again, by imposing appropiate conditions on the weights $\{M_p\}_{p=0}^{\infty}$, the strong dual $\mathcal{K}'_{\mu,b}$ of a Hankel-$K\{M_p\}$ space $\mathcal{K}_\mu$ can be made nuclear, Schwartz, bornological [17, Definition 3.7.1], complete, Montel, and reflexive. Additionally, a wide range of structural results and characterizations of the bounded subsets and convergent sequences in $\mathcal{K}'_\mu$ are made available.

3.1. The topology of $\mathcal{K}'_\mu$

Proposition 3.1. ([24, Proposition 5.1]) *Let $\mathcal{K}_\mu$ be a Hankel-$K\{M_p\}$ space with a strong dual $\mathcal{K}'_{\mu,b}$.*

(*i*) *If the sequence $\{M_p\}_{p=0}^{\infty}$ satisfies (O) and (P), then $\mathcal{K}'_{\mu,b}$ is complete, bornological, Schwartz, Montel, and reflexive.*
(*ii*) *If $\{M_p\}_{p=0}^{\infty}$ satisfies $(A), (M)$, and (N), then $\mathcal{K}'_{\mu,b}$ is complete, bornological, Schwartz, Montel, reflexive, and nuclear.*

3.2. Structure, boundedness, and convergence in $\mathcal{K}'_\mu$

Next, we show that the functionals in the dual $\mathcal{K}'_\mu$ of a space $\mathcal{K}_\mu$ of type Hankel-$K\{M_p\}$ can be expressed as distributional derivatives of integrable functions and measures. The highest order of the differential operators that provide such representations is uniform over bounded subsets of $\mathcal{K}'_{\mu,b}$. Furthermore, the convergence to zero of a sequence in this space is determined by the convergence to zero, in their respective spaces, of the functions or measures representing the terms of the sequence.

In Propositions 3.2 and 3.3 below, the results on boundedness and convergence will be stated for the strong topology of $\mathcal{K}'_\mu$, but, it must be kept in mind that, under the same conditions, they are equally valid if the weak or weak* topologies are instead considered on that space. Indeed, by making appropriate assumptions on the weights, $\mathcal{K}_\mu$ becomes Montel, and, hence, reflexive (Propositions 2.8 and 2.12). Thus, the weak and weak* topologies of $\mathcal{K}'_\mu$ coincide, whereas the weak* and strong sequential convergence are equivalent on this space [36, Proposition II.34.6, Corollary 1]. Furthermore, given that $\mathcal{K}_\mu$ is Fréchet (Proposition 2.8), it is also barrelled [36, Definition II.33.1 and Proposition II.33.2, Corollary 1], and, in this class of spaces, the weak* and strong topologies share the same bounded sets [36, Theorem II.33.2].

At this point, we are in a position to state the first result on representation, boundedness, and convergence in the dual of $\mathcal{K}_\mu$. To this end, let $C(I)$ denote the space of all of the functions $f \in C[0, \infty[$ such that

$$\lim_{x\to\infty} f(x) = 0,$$

normed with

$$\|f\|_\infty = \sup_{x\in I} |f(x)|.$$

Its dual $C'(I)$ consists of all of the regular, complex Borel measures σ on $[0, \infty[$, with the total variation norm $|\sigma|$.

Proposition 3.2. *Assume that $\{M_p\}_{p=0}^{\infty}$ satisfies (O) and (P).*

(i) [24, Proposition 5.2] *A linear functional f belongs to $\mathcal{K}'_\mu$ if, and only if, there exist $p \in \mathbb{N}_0$ and $\sigma_k \in C'(I)$ ($k \in \mathbb{N}_0,\ 0 \le k \le p$) satisfying*

$$f = \sum_{k=0}^{p} x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k \left[M_p(x)\sigma_k\right].$$

(ii) [24, Proposition 5.4] *A set $B \subset \mathcal{K}'_{\mu,b}$ is bounded if, and only if, each $f \in B$ admits the representation*

$$f = \sum_{k=0}^{p} x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k \left[M_p(x)\sigma_{k,f}\right],$$

with $\sigma_{k,f} \in C'(I)$ ($k \in \mathbb{N}_0,\ 0 \le k \le p$) such that

$$\sum_{k=0}^{p} \int_0^\infty d\left|\sigma_{k,f}\right| \le C,$$

where $p \in \mathbb{N}_0$ and $C > 0$ do not depend on $f \in B$.

(iii) [24, Proposition 5.4] *A sequence $\left\{f_j\right\}_{j=0}^\infty$ converges to zero in $\mathcal{K}'_{\mu,b}$ if, and only if, each f_j admits the representation*

$$f_j = \sum_{k=0}^{p} x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k \left[M_p(x)\sigma_{k,j}\right] \quad (j \in \mathbb{N}_0),$$

with $\sigma_{k,j} \in C'(I)$ ($k \in \mathbb{N}_0,\ 0 \le k \le p$) such that $p \in \mathbb{N}_0$ does not depend on j and

$$\lim_{j\to\infty} \sum_{k=0}^{p} \int_0^\infty d\left|\sigma_{k,j}\right| = 0.$$

Proposition 3.3. *Assume that $\{M_p\}_{p=0}^\infty$ satisfies (A), (M), and (N).*

(i) [24, Proposition 5.3] *A linear functional f belongs to $\mathcal{K}'_\mu$ if, and only if, for every q, $1 < q \le \infty$, there exists $p \in \mathbb{N}_0$ such that f can be written as*

$$f = \sum_{k=0}^{p} x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k \left[M_p(x)g_k(x)\right],$$

with $g_k \in L^q(I)$ ($k \in \mathbb{N}_0,\ 0 \le k \le p$).

(ii) [24, Proposition 5.5] *A set $B \subset \mathcal{K}'_{\mu,b}$ is bounded if, and only if, given q, $1 < q \le \infty$, there exist $p \in \mathbb{N}_0$, $C > 0$, and, for each $f \in B$, functions $g_{k,f} \in L^q(I)$ ($k \in \mathbb{N}_0,\ 0 \le k \le p$) such that*

$$f = \sum_{k=0}^{p} x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k \left[M_p(x)g_{k,f}(x)\right],$$

with

$$\sum_{k=0}^{p} \left\|g_{k,f}\right\|_q \le C.$$

(*iii*) [24, Proposition 5.5] *A sequence* $\left\{f_j\right\}_{j=0}^{\infty}$ *converges to zero in* $\mathcal{K}'_{\mu,b}$ *if, and only if, for every* q, $1 < q \leq \infty$, *there exist* $p \in \mathbb{N}_0$ *and* $g_{k,j} \in L^q(I)$ $(k \in \mathbb{N}_0,\ 0 \leq k \leq p)$ *such that*

$$f_j = \sum_{k=0}^{p} x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k \left[M_p(x) g_{k,j}(x)\right] \quad (j \in \mathbb{N}_0),$$

with

$$\lim_{j\to\infty} \sum_{k=0}^{p} \left\|g_{k,j}\right\|_q = 0.$$

3.3. Representation through a single distributional derivative

Starting from Proposition 3.3, and adapting a technique of Kamiński [18], Marrero [25] obtained the results on structure, boundedness, and convergence in $\mathcal{K}'_\mu$, labeled below as Theorems 3.4, 3.5, and 3.6, respectively. Unlike Proposition 3.3, the aforementioned theorems have the advantage of allowing the elements of the dual to be expressed as the distributional derivative of a single continuous function under the same differential operator.

Theorem 3.4. ([25, Theorem 2.4]) *Assume that* $\{M_p\}_{p=0}^{\infty}$ *satisfies conditions* (A), (M), *and* (N). *Then, the following statements are equivalent:*

(*i*) *The functional* f *lies in* $\mathcal{K}'_\mu$.
(*ii*) *There exist* $k, p \in \mathbb{N}_0$ *and a function* F, *continuous on* I, *such that*

$$f = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k F(x) \tag{3.1}$$

and

$$M_p^{-1} F \in L^q(I) \tag{3.2}$$

for any q, $1 \leq q \leq \infty$.
(*iii*) *There exist* $k, p \in \mathbb{N}_0$ *and a function* F, *continuous on* I *and satisfying* (3.1), *such that* (3.2) *holds for some* q, $1 \leq q \leq \infty$.
(*iv*) *There exist* $k, p \in \mathbb{N}_0$ *and a function* F, *continuous on* I *and satisfying* (3.1), *such that* (3.2) *holds for* $q = \infty$.

Now, we shall state a characterization of boundedness in $\mathcal{K}'_\mu$.

Theorem 3.5. ([25, Theorem 2.5]) *Assume that* $\{M_p\}_{p=0}^{\infty}$ *satisfies conditions* (A), (M), *and* (N). *Then, the following four statements are equivalent:*

(*i*) *The set* $B \subset \mathcal{K}'_\mu$ *is (weakly, weakly*, strongly) bounded.*
(*ii*) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and, for each* $f \in B$, *a function* g_f, *continuous on* I, *such that*

$$f = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k g_f(x) \tag{3.3}$$

and

$$\left\|M_p^{-1} g_f\right\|_q \leq C \tag{3.4}$$

for any q, $1 \leq q \leq \infty$.

(iii) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and, for each* $f \in B$, *a function* g_f, *continuous on* I *and satisfying* (3.3), *such that* (3.4) *holds for some* q, $1 \leq q \leq \infty$.

(iv) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and, for each* $f \in B$, *a function* g_f, *continuous on* I *and satisfying* (3.3), *such that* (3.4) *holds for* $q = \infty$.

Finally, convergence in $\mathcal{K}'_\mu$ is described.

Theorem 3.6. ([25, Theorem 2.6]) *Assume that* $\{M_p\}_{p=0}^\infty$ *satisfies conditions* (A), (M), *and* (N). *Then, the following statements are equivalent:*

(i) *The sequence* $\left\{f_j\right\}_{j=0}^\infty$ *converges (weakly, weakly*, strongly) to zero in* $\mathcal{K}'_\mu$.

(ii) *There exist* $k, p \in \mathbb{N}_0$ *and* F_j, *continuous on* I, *such that*

$$f_j = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k F_j(x) \quad (j \in \mathbb{N}_0) \tag{3.5}$$

and

$$\lim_{j\to\infty} \left\|M_p^{-1} F_j\right\|_q = 0 \tag{3.6}$$

for any q, $1 \leq q \leq \infty$.

(iii) *There exist* $k, p \in \mathbb{N}_0$ *and* F_j, *continuous on* I *and satisfying* (3.5), *such that* (3.6) *holds for some* q, $1 \leq q \leq \infty$.

(iv) *There exist* $k, p \in \mathbb{N}_0$ *and* F_j, *continuous on* I *and satisfying* (3.5), *such that* (3.6) *holds for* $q = \infty$.

(v) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and functions* F_j, *continuous on* I *and satisfying* (3.5), *such that*

$$\left\|M_p^{-1} F_j\right\|_\infty \leq C \quad (j \in \mathbb{N}_0)$$

and $\lim_{j\to\infty} F_j(x) = 0$ *for almost all* $x \in I$.

3.4. Descriptions of $\mathcal{K}_\mu$ *and* $\mathcal{K}'_\mu$ *in terms of the Bessel operator*

Arteaga and Marrero [1] have shown that the topology of a Hankel-$K\{M_p\}$ space can be generated by families of norms of type L^q ($1 \leq q \leq \infty$) that involve the Bessel operator.

Definition 3.7. *For* $1 \leq q < \infty$, *consider the following families of norms on* $\mathcal{K}_\mu$:

$$\left.\begin{aligned} |\varphi|_{\mu,q,r} &= \sum_{k=0}^{r} \left\{ \int_0^\infty \left| M_r(x) x^{-\mu-\frac{1}{2}} S_\mu^k \varphi(x) \right|^q dx \right\}^{\frac{1}{q}} \\ |\varphi|_{\mu,\infty,r} &= \max_{0\leq k\leq r} \sup_{x\in I} |M_r(x) x^{-\mu-\frac{1}{2}} S_\mu^k \varphi(x)| \end{aligned}\right. \qquad (\varphi \in \mathcal{K}_\mu,\ r \in \mathbb{N}_0),$$

where

$$S_\mu = x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\mu-\frac{1}{2}} = D^2 - \frac{4\mu^2 - 1}{4x^2}$$

is the Bessel operator.

Theorem 3.8. ([1, Proposiciones 2.5 and 2.6]) *Under conditions* (A), (M), *and* (N) *on the weights* $\{M_p\}_{p=0}^\infty$, *any one of the families of norms* $\left\{|\cdot|_{\mu,q,r}\right\}_{r=0}^\infty$ $(1 \leq q \leq \infty)$ *generates the usual topology of* $\mathcal{K}_\mu$.

Remark 3.9. *Theorem 3.8 widens the class of spaces of type Hankel-K$\{M_p\}$ and shows how some of them, treated independently in the literature, actually coincide. In fact, for $a > 0$, let*

$$M_p(x) = \exp\left\{M\left(a\left[1-\frac{1}{p}\right]x\right)\right\} \quad (x \in I,\ p \in \mathbb{N}_0),$$

where $M \in C^2[0,\infty[$ satisfies that $M(0) = M'(0) = 0$, $M'(\infty) = \infty$, and $M''(x) > 0$ $(x \in I)$; also, let $\mathcal{K}_\mu$ be the corresponding Hankel-K$\{M_p\}$ space. Then, we get that $\mathcal{K}_\mu = x^{\mu+\frac{1}{2}} We_{M,a}$, where $We_{M,a}$ is the space introduced by van Eijndhoven and Kerkhof [9]. For the space $U^q_{\mu,M,a}$ $(1 \le q \le \infty)$ developed by Pathak and Upadhyay [27], the algebraic and topological identification

$$\mathcal{K}_\mu = U^q_{\mu,M,a} = U^\infty_{\mu,M,a} = x^{\mu+\frac{1}{2}} We_{M,a} \quad (1 \le q < \infty)$$

follows; see [7] as well.

More recently, on the basis of Theorem 3.8, and by combining techniques from [24, 25], Arteaga and Marrero [2] found new representations for the elements, the bounded subsets, and the convergent sequences in $\mathcal{K}'_\mu$, this time with the operator S_μ instead of the operator $x^{-1}D$ (see Sections 3.2 and 3.3). Their results are summarized below, beginning with the structure of the distributions in $\mathcal{K}'_\mu$.

Theorem 3.10. ([2, Theorem 4.1]) *Assume that the sequence of weights $\{M_p\}_{p=0}^\infty$ satisfies conditions (A), (M), and (N). The following statements are equivalent:*

(*i*) *The functional f lies in $\mathcal{K}'_\mu$.*
(*ii*) *For each q, $1 < q \le \infty$, there exist $r \in \mathbb{N}_0$ and $f_k \in L^q(I)$ $(k \in \mathbb{N}_0,\ 0 \le k \le r)$ such that*

$$f = \sum_{k=0}^{r} S_\mu^k \left[M_r(x) x^{-\mu-\frac{1}{2}} f_k(x)\right].$$

(*iii*) *There exist $k, p \in \mathbb{N}_0$ and $F \in C(I)$ such that $f = S_\mu^k x^{-\mu-\frac{1}{2}} F(x)$, with $M_p^{-1} F \in L^q(I)$ for all q, $1 \le q \le \infty$.*
(*iv*) *There exist $k, p \in \mathbb{N}_0$ and $F \in C(I)$ such that $f = S_\mu^k x^{-\mu-\frac{1}{2}} F(x)$, with $M_p^{-1} F \in L^q(I)$ for some q, $1 \le q \le \infty$.*
(*v*) *There exist $k, p \in \mathbb{N}_0$ and $F \in C(I)$ such that $f = S_\mu^k x^{-\mu-\frac{1}{2}} F(x)$, with $M_p^{-1} F \in L^\infty(I)$.*

Next, boundedness in $\mathcal{K}'_\mu$ is characterized.

Theorem 3.11. ([2, Theorem 5.1]) *Assume that the sequence of weights $\{M_p\}_{p=0}^\infty$ satisfies conditions (A), (M), and (N). The following five statements are equivalent:*

(*i*) *The set $B \subset \mathcal{K}'_\mu$ is (weakly, weakly*, strongly) bounded.*
(*ii*) *Given q, $1 < q \le \infty$, there exist $r \in \mathbb{N}_0$, $C > 0$, and, for each $f \in B$, functions $g_{f,i} \in L^q(I)$ $(i \in \mathbb{N}_0,\ 0 \le i \le r)$ such that*

$$f = \sum_{i=0}^{r} S_\mu^i \left[M_r(x) x^{-\mu-\frac{1}{2}} g_{f,i}(x)\right],$$

with $\sum_{i=0}^r \left\|g_{f,i}\right\|_q \le C$.

(*iii*) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and, for each* $f \in B$, *a function* $g_f \in C(I)$ *such that* $f = S_\mu^k x^{-\mu-\frac{1}{2}} g_f(x)$, *with* $\left\|M_p^{-1} g_f\right\|_q \leq C$ *for every* q, $1 \leq q \leq \infty$.
(*iv*) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and, for each* $f \in B$, *a function* $g_f \in C(I)$ *such that* $f = S_\mu^k x^{-\mu-\frac{1}{2}} g_f(x)$, *with* $\left\|M_p^{-1} g_f\right\|_q \leq C$ *for some* q, $1 \leq q \leq \infty$.
(*v*) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and, for each* $f \in B$, *a function* $g_f \in C(I)$ *such that* $f = S_\mu^k x^{-\mu-\frac{1}{2}} g_f(x)$, *with* $\left\|M_p^{-1} g_f\right\|_\infty \leq C$.

We close this section with the corresponding characterization of sequential convergence in $\mathcal{K}'_\mu$.

Theorem 3.12. ([2, Theorem 6.1]) *Under conditions* (*A*), (*M*), *and* (*N*) *on the sequence of weights* $\{M_p\}_{p=0}^\infty$, *the following statements are equivalent:*

(*i*) *The sequence* $\{f_n\}_{n=0}^\infty$ *converges (weakly, weakly*, strongly) to zero in* $\mathcal{K}'_\mu$.
(*ii*) *For each* q, $1 < q \leq \infty$, *there exist* $r \in \mathbb{N}_0$ *and* $f_{n,i} \in L^q(I)$ $(n, i \in \mathbb{N}_0,\ 0 \leq i \leq r)$ *such that*

$$f_n = \sum_{i=0}^{r} S_\mu^i \left[M_r(x) x^{-\mu-\frac{1}{2}} f_{n,i}(x) \right] \quad (n \in \mathbb{N}_0),$$

with $\lim_{n\to\infty} \sum_{i=0}^r \left\|f_{n,i}\right\|_q = 0$.
(*iii*) *There exist* $k, p \in \mathbb{N}_0$ *and* $F_n \in C(I)$ *such that*

$$f_n = S_\mu^k x^{-\mu-\frac{1}{2}} F_n(x) \quad (n \in \mathbb{N}_0),$$

with $\lim_{n\to\infty} \left\|M_p^{-1} F_n\right\|_q = 0$ *for all* q, $1 \leq q \leq \infty$.
(*iv*) *There exist* $k, p \in \mathbb{N}_0$ *and* $F_n \in C(I)$ *such that*

$$f_n = S_\mu^k x^{-\mu-\frac{1}{2}} F_n(x) \quad (n \in \mathbb{N}_0),$$

with $\lim_{n\to\infty} \left\|M_p^{-1} F_n\right\|_q = 0$ *for some* q, $1 \leq q \leq \infty$.
(*v*) *There exist* $k, p \in \mathbb{N}_0$ *and* $F_n \in C(I)$ *such that*

$$f_n = S_\mu^k x^{-\mu-\frac{1}{2}} F_n(x) \quad (n \in \mathbb{N}_0),$$

with $\lim_{n\to\infty} \left\|M_p^{-1} F_n\right\|_\infty = 0$.
(*vi*) *There exist* $k, p \in \mathbb{N}_0$, $C > 0$, *and* $F_n \in C(I)$ *such that*

$$f_n = S_\mu^k x^{-\mu-\frac{1}{2}} F_n(x),$$

with $\left\|M_p^{-1} F_n\right\|_\infty \leq C$ $(n \in \mathbb{N}_0)$ *and* $\lim_{n\to\infty} F_n(x) = 0$ *for almost all* $x \in I$.

4. A new description of the topology of Hankel-$K\{M_p\}$ spaces

First, we shall show that, for $\mu \geq -\frac{1}{2}$ and every p, $1 \leq p \leq \infty$, the families of norms $\left\{\rho_{p,r}^\mu\right\}_{r=0}^\infty$ generate the usual topology of a space $\mathcal{K}_\mu$ of type Hankel-$K\{M_p\}$, where, for $r \in \mathbb{N}_0$ and $\varphi \in \mathcal{K}_\mu$,

$$\begin{aligned} \rho_{\infty,r}^\mu(\varphi) &= \max_{0\leq k\leq r} \sup_{x\in I} |M_r(x) T_{\mu,k}\varphi(x)|, \\ \rho_{p,r}^\mu(\varphi) &= \max_{0\leq k\leq r} \left\{ \int_0^\infty \left|M_r(x) T_{\mu,k}\varphi(x)\right|^p dx \right\}^{\frac{1}{p}} \quad (1 \leq p < \infty). \end{aligned} \tag{4.1}$$

Here,

$$T_{\mu,k} = N_{\mu+k-1} \cdots N_\mu \quad (k \in \mathbb{N}), \tag{4.2}$$

where $N_\mu = x^{\mu+\frac{1}{2}} D x^{-\mu-\frac{1}{2}}$ denotes the Zemanian operator [40, pp. 135ff] and $T_{\mu,0}$ is the identity operator. The differential operators in (4.2) are interesting because of their symmetric behavior in the presence of the Hankel transformation; in fact, for an appropriate order of this transformation, it exchanges the order of the powers of the variable and the order of the differential operator defined in (4.2), an extremely useful operational rule that is very similar to its Fourier counterpart [40, Proof of Theorem 5.4-1].

This new description for the topology of $\mathcal{K}_\mu$ was motivated by [4, Theorem 3.3], where a similar result was established for the Zemanian space $\mathcal{H}_\mu$ (Example 2.2). The validity of this result requires assuming conditions (O), (A), (M), and (N) apply to the sequence of weights $\{M_p\}_{p=0}^\infty$ (Definition 2.7), which is a hypothesis that will be maintained throughout the entire section, although, for the sake of simplicity, it will not be made explicit on all occasions.

The proof of this first result imitates that of [4, Theorem 3.3]. Roughly speaking, it consists of introducing a new space of test functions, which we will denote by $\mathcal{S}_\mu$, and whose definition is formally analogous to that of $\mathcal{K}_\mu$, but with the operator $T_{\mu,k}$ in place of the operator $(x^{-1}D)^k x^{-\mu-\frac{1}{2}}$ $(k \in \mathbb{N}_0)$; this is followed by proving, with the aid of the open mapping theorem [28, Corollary 2.12], that, actually, $\mathcal{S}_\mu = \mathcal{K}_\mu$ (Theorem 4.4). Once this is done, it is not difficult to infer that the families of norms defined in (4.1) are equivalent over $\mathcal{S}_\mu$ (Proposition 4.6). At this point, we will apply techniques that are analogous to those used in the proof of the results in Section 3 in order to find representations of the elements, the (weakly, weakly*, strongly) bounded subsets, and the (weakly, weakly*, strongly) convergent sequences in the dual space $\mathcal{S}'_\mu = \mathcal{K}'_\mu$, this time as distributional derivatives induced by the adjoint of the operator $T_{\mu,k}$ $(k \in \mathbb{N}_0)$. Thus, Theorems 5.4–5.7 below generalize and improve, in a sense that will be specified in due course, their analogues for the space $\mathcal{H}'_\mu$ in [4].

We must emphasize that, although the ideas presented in this section are not entirely new, the results obtained in the general context of Hankel-$K\{M_p\}$ spaces have not appeared previously in the literature.

4.1. The space $\mathcal{S}_\mu$

Given $\mu \in \mathbb{R}$, denote by $\mathcal{S}_\mu$ the vector space of all smooth, complex-valued functions $\varphi = \varphi(x)$ defined on $I =]0, \infty[$ such that $\omega^\mu_{p,k}(\varphi) < \infty$, where

$$\omega^\mu_{p,k}(\varphi) = \sup_{x \in I} |M_p(x) T_{\mu,k} \varphi(x)| \quad (p, k \in \mathbb{N}_0)$$

and $T_{\mu,k}$ $(k \in \mathbb{N}_0)$ is as defined above.

A direct computation shows that, for $k \in \mathbb{N}_0$,

$$\begin{aligned} T_{\mu,k} &= x^{k+\mu+\frac{1}{2}} (x^{-1}D)^k x^{-\mu-\frac{1}{2}} \\ &= b^\mu_{k,0} x^{-k} + b^\mu_{k,1} x^{-k+1} D + \ldots + b^\mu_{k,k} D^k, \end{aligned} \tag{4.3}$$

where the coefficients $b^\mu_{k,j}$ $(0 \le j \le k)$ are appropriate constants, with $b^\mu_{k,k} = 1$.

The family $\Omega = \{\omega^\mu_{p,k}\}_{p,k=0}^\infty$ is a countable family of seminorms. This family is separating, because $\{\omega^\mu_{p,0}\}_{p=0}^\infty$ are norms. Consequently, Ω makes $\mathcal{S}_\mu$ into a countably multinormed space.

A family of norms $P = \{\rho^\mu_{\infty,r}\}_{r=0}^\infty$ that is equivalent to Ω, with the property that $\rho^\mu_{\infty,r} \leq \rho^\mu_{\infty,s}$ $(r, s \in \mathbb{N}_0)$, is obtained by setting

$$\rho^\mu_{\infty,r} = \max_{0\leq k\leq r} \omega^\mu_{r,k} \quad (r \in \mathbb{N}_0).$$

Proposition 4.1. *The space $\mathcal{S}_\mu$ $(\mu \in \mathbb{R})$ is Fréchet.*

Proof. Let $\{\varphi_n\}_{n=0}^\infty$ be a Cauchy sequence in $\mathcal{S}_\mu$; we want to prove that it converges in $\mathcal{S}_\mu$.

By (4.3), we have

$$D^k = T_{\mu,k} - (b^\mu_{k,0} x^{-k} + \ldots + b^\mu_{k,k-1} x^{-1} D^{k-1}). \tag{4.4}$$

Proceeding by induction on k, we find that, for each compact $K \subset I$, there exist constants $c^\mu_{k,j}$ $(0 \leq j \leq k)$ such that

$$\sup_{x\in K} |D^k\varphi(x)| \leq c^\mu_{k,0}\omega^\mu_{0,0}(\varphi) + c^\mu_{k,1}\omega^\mu_{0,1}(\varphi) + \ldots + c^\mu_{k,k}\omega^\mu_{0,k}(\varphi) \quad (\varphi \in \mathcal{S}_\mu).$$

Since $\omega^\mu_{0,k}(\varphi_n - \varphi_m) \xrightarrow[n,m\to\infty]{} 0$, the sequence $\{D^k\varphi_n\}_{n=0}^\infty$ is uniformly Cauchy on compact subsets of I for all $k \in \mathbb{N}_0$. Thus, there exists $\varphi \in C^\infty(I)$ such that

$$D^k\varphi_n(x) \xrightarrow[n\to\infty]{} D^k\varphi(x) \quad (k \in \mathbb{N}_0)$$

uniformly over compact subsets of I. This φ is the limit of $\{\varphi_n\}_{n=0}^\infty$ in $\mathcal{S}_\mu$. Indeed, for any $p, k \in \mathbb{N}_0$ and every $\varepsilon > 0$, there exists $N^\mu_{p,k} = N^\mu_{p,k}(\varepsilon) \in \mathbb{N}_0$ such that

$$|M_p(x)T_{\mu,k}(\varphi_n - \varphi_m)(x)| < \varepsilon \quad (x \in I,\ n, m \geq N^\mu_{p,k}).$$

Taking the limit as $n \to \infty$, it follows that

$$\omega^\mu_{p,k}(\varphi - \varphi_m) \leq \varepsilon \quad (m \geq N^\mu_{p,k}).$$

On the other hand, there exists $B^\mu_{p,k} > 0$, independent of m, such that

$$\omega^\mu_{p,k}(\varphi_m) \leq B^\mu_{p,k} \quad (m, p, k \in \mathbb{N}_0).$$

Thus,

$$\omega^\mu_{p,k}(\varphi) \leq B^\mu_{p,k} + \varepsilon \quad (p, k \in \mathbb{N}_0).$$

This shows that $\varphi \in \mathcal{S}_\mu$ and $\{\varphi_n\}_{n=0}^\infty$ converges to φ in $\mathcal{S}_\mu$, as asserted. □

Proposition 4.2. *For $\mu \geq -\frac{1}{2}$, the inclusion $\mathcal{K}_\mu \subset \mathcal{S}_\mu$ holds. Moreover, the embedding $\mathcal{K}_\mu \hookrightarrow \mathcal{S}_\mu$ is continuous.*

Proof. Let $p \in \mathbb{N}_0$. Proposition 2.10(i), along with condition (A) on the weights, yields $r, s \in \mathbb{N}_0$, $s > r > p + \mu + \frac{1}{2}$, and $C > 0$ such that

$$\begin{aligned}
&|M_p(x)T_{\mu,k}\varphi(x)| \\
&= |M_p(x)x^{k+\mu+\frac{1}{2}}(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)| \\
&\leq C|M_p(x)M_r(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)| \\
&\leq C|M_s(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)| \quad (x \in I,\ k \in \mathbb{N}_0,\ 0 \leq k \leq p)
\end{aligned}$$

whenever $\varphi \in \mathcal{K}_\mu$. According to Definition 2.1, this means that

$$\rho^\mu_{\infty,p}(\varphi) \leq C\|\varphi\|_{\mu,\infty,s} < \infty \quad (\varphi \in \mathcal{K}_\mu),$$

which completes the proof. □

Proposition 4.3. *Suppose that $\varphi \in \mathcal{S}_\mu$. Then, $\varphi \in \mathcal{K}_\mu$ if, and only if, the following limits exist:*

$$\lim_{x\to 0^+}(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x) \quad (k \in \mathbb{N}_0). \tag{4.5}$$

In other words,

$$\begin{aligned}\mathcal{K}_\mu &= \left\{\varphi \in \mathcal{S}_\mu : \text{ there exist } \lim_{x\to 0^+}(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)\ (k \in \mathbb{N}_0)\right\}\\ &= \left\{\varphi \in \mathcal{S}_\mu : (x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x) = O(1) \text{ as } x \to 0^+\ (k \in \mathbb{N}_0)\right\}.\end{aligned} \tag{4.6}$$

Proof. Note that the existence of a limit is a stronger condition than boundedness near the origin; thus, the first set on the right-hand side of (4.6) is contained in the second set.

Under (A), (M), and (N), we have that $\mathcal{K}_\mu \subset \mathcal{H}_\mu$ (Proposition 2.10(ii)). Hence, if $\varphi \in \mathcal{K}_\mu$, then $\varphi \in \mathcal{S}_\mu$ (by Proposition 4.2) and the limits given by (4.5) exist [40, Lemma 5.2-1].

Finally, suppose that $\varphi \in \mathcal{S}_\mu$ and $(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)$ is bounded near zero for all $k \in \mathbb{N}_0$. Since (O) holds, $M_p(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)$ is also bounded near zero for any $p, k \in \mathbb{N}_0$. On the other hand,

$$\begin{aligned}|M_p(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)| &= |M_p(x)x^{-k-\mu-\frac{1}{2}}T_{\mu,k}\varphi(x)|\\ &\leq |M_p(x)T_{\mu,k}\varphi(x)|\\ &\leq \omega^\mu_{p,k}(\varphi) < \infty \quad (1 \leq x < \infty,\ p, k \in \mathbb{N}_0).\end{aligned}$$

This proves that $\varphi \in \mathcal{K}_\mu$. □

Theorem 4.4. *For $\mu \geq -\frac{1}{2}$, the inclusion $\mathcal{S}_\mu \subset \mathcal{H}_\mu$ holds with a continuous embedding. Consequently, $\mathcal{S}_\mu = \mathcal{K}_\mu$ both algebraically and topologically.*

Proof. To show that $\mathcal{S}_\mu \subset \mathcal{H}_\mu$ and the inclusion map $\mathcal{S}_\mu \hookrightarrow \mathcal{H}_\mu$ is continuous, note that, given $p \in \mathbb{N}_0$, Proposition 2.10(i) and condition (A) on the weights yield $r \in \mathbb{N}_0$, $r > p$, and $C > 0$, for which $x^p \leq CM_r(x)$ $(x \in I)$. Therefore,

$$\begin{aligned}\sup_{x\in I}|x^p T_{\mu,k}\varphi(x)| &\leq C\sup_{x\in I}|M_r(x)T_{\mu,k}\varphi(x)|\\ &\leq C\rho^\mu_{\infty,r}(\varphi) < \infty \quad (\varphi \in \mathcal{S}_\mu,\ k \in \mathbb{N}_0,\ 0 \leq k \leq p),\end{aligned}$$

and it suffices to invoke [4, Theorem 3.3]. In particular, the limits given by (4.5) exist whenever $\varphi \in \mathcal{S}_\mu$ [40, Lemma 5.2-1], and, from Proposition 4.3, it follows that $\mathcal{S}_\mu = \mathcal{K}_\mu$.

This equality is also topological, as it can be deduced from Propositions 2.8, 4.1, and 4.2 by applying the open mapping theorem [28, Corollary 2.12]. □

Once it is proved that $\mathcal{S}_\mu \subset \mathcal{H}_\mu$, Proposition 4.5 is an immediate consequence of [4, Theorem 3.3 and Proposition 2.3]. Just for the sake of completeness, a direct proof is included.

Proposition 4.5. *Let $\mu \geq -\frac{1}{2}$. Every $\varphi \in \mathcal{S}_\mu$ is bounded. For any $k \in \mathbb{N}_0$, $D^k\varphi$ is rapidly decreasing at infinity; in particular, $\mathcal{S}_\mu$ is a dense proper subspace of $L^1(I)$.*

Proof. Every $\varphi \in \mathcal{S}_\mu$ is bounded because $\omega^k_{0,0}(\varphi) < \infty$.

To prove that $D^k\varphi$ ($k \in \mathbb{N}_0$) is rapidly decreasing at infinity, we proceed by induction on k, bearing in mind (4.4) and the fact that, given $m \in \mathbb{N}_0$, Proposition 2.10(i), along with condition (A), yields $r \in \mathbb{N}_0$, $r > m$, and $C > 0$ such that $x^m \leq CM_r(x)$ ($x \in I$). The inductive scheme is as follows.

$k = 0$: $x^m\varphi(x)$ ($m \in \mathbb{N}_0$) is bounded for $x \in I$ because

$$|x^m\varphi(x)| \leq C|M_r(x)\varphi(x)| = C\omega^\mu_{r,0}(\varphi) < \infty.$$

Therefore, $\varphi(x)$ is rapidly decreasing at infinity.

$k = 1$: $x^m D\varphi(x)$ ($m \in \mathbb{N}_0$) is bounded for $x \geq 1$ because, by (4.4), we have

$$x^m D\varphi(x) = \underbrace{x^m\, T_{\mu,1}\varphi(x)}_{\omega^\mu_{r,1}(\varphi)<\infty} - b^\mu_{1,0}\, \underbrace{x^{m-1}\varphi(x)}_{\substack{m\geq 1:\ \text{step } 0\\ m=0:\ \varphi \text{ bounded}}}\ .$$

Therefore, $D\varphi(x)$ is rapidly decreasing at infinity.

$k = 2$: $x^m D^2\varphi(x)$ ($m \in \mathbb{N}_0$) is bounded for $x \geq 1$ because, by (4.4), we have

$$x^m D^2\varphi(x) = \underbrace{x^m\, T_{\mu,2}\varphi(x)}_{\omega^\mu_{r,2}(\varphi)<\infty} - b^\mu_{2,0}\, \underbrace{x^{m-2}\varphi(x)}_{\substack{m\geq 2:\ \text{step } 0\\ m=0,1:\ \varphi \text{ bounded}}} - b_{2,1}\, \underbrace{x^{m-1}D\varphi(x)}_{\substack{m\geq 1:\ \text{step } 1\\ m=0:\ D\varphi \text{ bounded}}}\ .$$

Therefore, $D^2\varphi(x)$ is rapidly decreasing at infinity.

Assuming that the statement

$$x^m D^n\varphi(x)\ (m \in \mathbb{N}_0) \text{ is bounded for } x \geq 1$$

holds true for all $n \in \mathbb{N}_0$ with $0 \leq n \leq k$, we prove it for $k + 1$.

$k + 1$: $x^m D^{k+1}\varphi(x)$ ($m \in \mathbb{N}_0$) is bounded for $x \geq 1$ because, by (4.4), we have

$$\begin{aligned} x^m D^{k+1}\varphi(x) = {} & \underbrace{x^m\, T_{\mu,k+1}\varphi(x)}_{\omega^\mu_{r,k+1}(\varphi)<\infty} - b^\mu_{k+1,0}\, \underbrace{x^{m-k-1}\varphi(x)}_{\substack{m\geq k+1:\ \text{ind. hyp.}\\ 0\leq m\leq k:\ \varphi \text{ bounded}}} \\ & - b^\mu_{k+1,1}\, \underbrace{x^{m-k}D\varphi(x)}_{\substack{m\geq k:\ \text{ind. hyp.}\\ 0\leq m\leq k-1:\ D\varphi \text{ bounded}}} \\ & - \ldots \\ & - b^\mu_{k+1,k}\, \underbrace{x^{m-1}D^k\varphi(x)}_{\substack{m\geq 1:\ \text{ind. hyp.}\\ m=0:\ D^k\varphi \text{ bounded}}}\ . \end{aligned}$$

Therefore, $D^{k+1}\varphi(x)$ is rapidly decreasing at infinity and the induction is complete.

Finally, it is clear that

$$\mathcal{D}(I) \subset \mathcal{S}_\mu \subset L^1(I),$$

which implies that $\mathcal{S}_\mu$ is dense in $L^1(I)$. □

4.2. The topology of $\mathcal{S}_\mu$

Proposition 4.6. *For $\mu \geq -\frac{1}{2}$ and any p, $1 \leq p \leq \infty$, the families of norms $\{\rho^\mu_{p,r}\}_{r=0}^\infty$ generate the topology of $\mathcal{S}_\mu$, where, for $r \in \mathbb{N}_0$ and $\varphi \in \mathcal{S}_\mu$,*

$$\rho^\mu_{\infty,r}(\varphi) = \max_{0\leq k\leq r} \sup_{x\in I} |M_r(x) T_{\mu,k}\varphi(x)|,$$

$$\rho^\mu_{p,r}(\varphi) = \max_{0\leq k\leq r} \left\{ \int_0^\infty \left| M_r(x) T_{\mu,k}\varphi(x) \right|^p dx \right\}^{\frac{1}{p}} \quad (1 \leq p < \infty).$$

Proof. Fix $1 < p < \infty$ and $\varphi \in \mathcal{S}_\mu$. Given $r \in \mathbb{N}_0$, condition (N) yields $s \in \mathbb{N}_0$, $s > r$ such that

$$\int_0^\infty \frac{M_r(x)}{M_s(x)} dx < \infty. \tag{4.7}$$

Since

$$0 \leq \frac{M_r(x)}{M_s(x)} \leq 1 \quad (x \in I), \tag{4.8}$$

we may write the following:

$$\begin{aligned}
\rho^\mu_{p,r}(\varphi) &= \max_{0\leq k\leq r} \left\{ \int_0^\infty \left| M_r(x) T_{\mu,k}\varphi(x) \right|^p dx \right\}^{\frac{1}{p}} \\
&= \max_{0\leq k\leq r} \left\{ \int_0^\infty \left| M_s(x) T_{\mu,k}\varphi(x) \right|^p \left(\frac{M_r(x)}{M_s(x)} \right)^p dx \right\}^{\frac{1}{p}} \\
&\leq \max_{0\leq k\leq s} \sup_{x\in I} |M_s(x) T_{\mu,k}\varphi(x)| \left\{ \int_0^\infty \frac{M_r(x)}{M_s(x)} dx \right\}^{\frac{1}{p}} \\
&= \left\{ \int_0^\infty \frac{M_r(x)}{M_s(x)} dx \right\}^{\frac{1}{p}} \rho^\mu_{\infty,s}(\varphi).
\end{aligned}$$

A new application of (4.7) and (4.8), in combination with Hölder's inequality, leads to

$$\begin{aligned}
\rho^\mu_{1,r}(\varphi) &= \max_{0\leq k\leq r} \int_0^\infty \left| M_r(x) T_{\mu,k}\varphi(x) \right| dx \\
&= \max_{0\leq k\leq r} \int_0^\infty \left| M_s(x) T_{\mu,k}\varphi(x) \right| \frac{M_r(x)}{M_s(x)} dx \\
&\leq \max_{0\leq k\leq s} \left\{ \int_0^\infty \left| M_s(x) T_{\mu,k}\varphi(x) \right|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left(\frac{M_r(x)}{M_s(x)} \right)^q dx \right\}^{\frac{1}{q}} \\
&= \left\{ \int_0^\infty \frac{M_r(x)}{M_s(x)} dx \right\}^{\frac{1}{q}} \rho^\mu_{p,s}(\varphi).
\end{aligned}$$

Here, $q = p(p-1)^{-1}$ denotes the conjugate exponent of p.

Finally, given $k_0 \in \mathbb{N}_0$ and $\varphi \in \mathcal{S}_\mu$, the function $(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)$ is rapidly decreasing at infinity because $\mathcal{S}_\mu \subset \mathcal{H}_\mu$ (Theorem 4.4). Hence, for $\varphi \in \mathcal{S}_\mu$ and $r, k \in \mathbb{N}_0$ with $0 \leq k \leq r$, by using (M), we find that

$$|M_r(x) T_{\mu,k}\varphi(x)| = \left| M_r(x)\, x^{k+\mu+\frac{1}{2}} (x^{-1}D)^k x^{-\mu-\frac{1}{2}} \varphi(x) \right|$$

$$
\begin{aligned}
&= \left| M_r(x)\, x^{k+\mu+\frac{1}{2}} \int_x^\infty t(t^{-1}D)^{k+1} t^{-\mu-\frac{1}{2}} \varphi(t)\, dt \right| \\
&\le C \int_x^\infty \left| M_r(t)\, t^{(k+1)+\mu+\frac{1}{2}} (t^{-1}D)^{k+1} t^{-\mu-\frac{1}{2}} \varphi(t) \right| dt \\
&\le C \int_0^\infty \left| M_r(t) T_{\mu,k+1} \varphi(t) \right| dt \quad (x \in I).
\end{aligned}
$$

Consequently,

$$\rho^\mu_{\infty,r}(\varphi) \le C \rho^\mu_{1,r+1}(\varphi).$$

The proof is thus complete. □

5. Structure, boundedness, and convergence in $\mathcal{S}'_\mu$

Now that Proposition 4.6 has been proved, we intend to apply techniques similar to those employed in the proof of the results in Section 3 in order to characterize the elements, the (weakly, weakly*, strongly) bounded subsets, and the (weakly, weakly*, strongly) convergent sequences in the dual space $\mathcal{K}'_\mu = \mathcal{S}'_\mu$ ($\mu \ge -\frac{1}{2}$), this time as distributional derivatives induced by the adjoint of the operator $T_{\mu,k}$ ($k \in \mathbb{N}_0$). The main results correspond to Theorems 5.4–5.5, 5.6, and 5.7, respectively. Prior to deriving these results, three auxiliary lemmas must be established.

5.1. Auxiliary results

Lemma 5.1. *Let $\mu \ge -\frac{1}{2}$, and let $F \in C(I)$ be such that there exists $p \in \mathbb{N}_0$ with $M_p^{-1}F \in L^q(I)$ ($1 \le q \le \infty$). Then, for each $k \in \mathbb{N}_0$, there exist $p_k \in \mathbb{N}_0$, $p_k \ge p$, and a function $F_k \in C(I)$ satisfying*

$$x^{-\mu-\frac{1}{2}} (Dx^{-1})^k x^{k+\mu+\frac{1}{2}} F_k(x) = F(x) \quad (x \in I) \tag{5.1}$$

and

$$M_{p_k}^{-1} F_k \in L^q(I) \quad (1 \le q \le \infty). \tag{5.2}$$

Proof. We proceed by induction on k. The result is obvious if $k = 0$. Suppose that, given $k \in \mathbb{N}_0$, $k \ge 1$, there exist $p_k \in \mathbb{N}_0$, $p_k \ge p$, and a function $F_k \in C(I)$ satisfying (5.1) and (5.2). Use (N) to find $n, t \in \mathbb{N}_0$, $n > t > p_k$, such that

$$\int_0^\infty \frac{M_{p_k}(x)}{M_t(x)} dx < \infty, \tag{5.3}$$

$$\int_0^\infty \frac{M_t(x)}{M_n(x)} dx < \infty. \tag{5.4}$$

The inductive hypotheses, along with (M) and (5.3), allow us to write the following:

$$
\begin{aligned}
\frac{1}{M_t(x)} \int_0^x |F_k(\xi)|\, d\xi &\le C \int_0^\infty \left| \frac{F_k(\xi)}{M_t(\xi)} \right| d\xi \\
&= C \int_0^\infty \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right| \frac{M_{p_k}(\xi)}{M_t(\xi)} d\xi
\end{aligned}
$$

$$\leq C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right| \int_0^\infty \frac{M_{p_k}(\xi)}{M_t(\xi)} d\xi$$
$$= C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right|. \tag{5.5}$$

The function

$$\widetilde{F}_k(x) = x^{-k-\mu-\frac{1}{2}} \int_0^x F_k(\xi)\xi^{k+\mu+\frac{1}{2}} d\xi \quad (x \in I)$$

is continuous, and, by (5.1), we have

$$x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^{k+1} x^{(k+1)+\mu+\frac{1}{2}} \widetilde{F}_k(x)$$
$$= x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^{k} x^{k+\mu+\frac{1}{2}} F_k(x) = F(x) \quad (x \in I).$$

Furthermore, using (5.5), it follows that

$$\left| \frac{\widetilde{F}_k(x)}{M_t(x)} \right| = \frac{1}{M_t(x)} \left| x^{-k-\mu-\frac{1}{2}} \int_0^x F_k(\xi)\xi^{k+\mu+\frac{1}{2}} d\xi \right|$$
$$\leq \frac{1}{M_t(x)} \int_0^x |F_k(\xi)|\, d\xi$$
$$\leq C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right| \quad (x \in I).$$

Because $n > t$, given (5.2), it clearly follows that

$$\sup_{x \in I} \left| \frac{\widetilde{F}_k(x)}{M_n(x)} \right| \leq \sup_{x \in I} \left| \frac{\widetilde{F}_k(x)}{M_t(x)} \right|$$
$$\leq C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right| < \infty.$$

On the other hand, if $1 \leq q < \infty$, then, given (5.2), and with the aid of (5.4), we obtain

$$\int_0^\infty \left| \frac{\widetilde{F}_k(x)}{M_n(x)} \right|^q dx = \int_0^\infty \left| \frac{\widetilde{F}_k(x)}{M_t(x)} \right|^q \left(\frac{M_t(x)}{M_n(x)} \right)^q dx$$
$$\leq C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right|^q \int_0^\infty \frac{M_t(x)}{M_n(x)} dx$$
$$= C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right|^q < \infty.$$

To complete the induction, it suffices to take $p_{k+1} = n$ and $F_{k+1} = \widetilde{F}_k$. □

Lemma 5.2. *Let $\mu \geq -\frac{1}{2}$, and let $\mathcal{M}$ be a family of functions in $C(I)$ with the property that*

$$\sup_{F \in \mathcal{M}} \left\| M_p^{-1} F \right\|_q \leq A \quad (1 \leq q \leq \infty)$$

for certain $p \in \mathbb{N}_0$ and $A > 0$. Then, given $k \in \mathbb{N}_0$, there exist $p_k \in \mathbb{N}_0$, $p_k \geq p$, $C_k > 0$, and, for each $F \in \mathcal{M}$, a function $g_{k,F} \in C(I)$ such that

$$x^{-\mu-\frac{1}{2}}(Dx^{-1})^k x^{k+\mu+\frac{1}{2}} g_{k,F}(x) = F(x) \quad (x \in I) \tag{5.6}$$

and

$$\sup_{F\in\mathcal{M}} \left\| M_{p_k}^{-1} g_{k,F} \right\|_q \leq C_k \quad (1 \leq q \leq \infty). \tag{5.7}$$

Proof. The result holds trivially for $k = 0$. Proceeding by induction, fix $k \in \mathbb{N}_0$, $k \geq 1$. Let $p_k \in \mathbb{N}_0$, $p_k \geq p$, and $C_k > 0$, and, for each $F \in \mathcal{M}$, let $g_{k,F} \in C(I)$ satisfy (5.6) and (5.7). As in the proof of Lemma 5.1, for every $F \in \mathcal{M}$, we construct a function $\widetilde{g}_{k,F} \in C(I)$ such that

$$x^{-\mu-\frac{1}{2}}(Dx^{-1})^{k+1} x^{(k+1)+\mu+\frac{1}{2}} \widetilde{g}_{k,F}(x) = F(x) \quad (x \in I),$$

$$\sup_{x\in I} \left| \frac{\widetilde{g}_{k,F}(x)}{M_n(x)} \right| \leq C \sup_{\xi\in I} \left| \frac{g_{k,F}(\xi)}{M_{p_k}(\xi)} \right|,$$

and

$$\int_0^\infty \left| \frac{\widetilde{g}_{k,F}(x)}{M_n(x)} \right|^q dx \leq C \sup_{\xi\in I} \left| \frac{g_{k,F}(\xi)}{M_{p_k}(\xi)} \right|^q \quad (1 \leq q < \infty)$$

for some $n \in \mathbb{N}_0$, $n > p_k$, where the positive constant C does not depend on F. To complete the proof, it suffices to choose $p_{k+1} = n$ and $g_{k+1,F} = \widetilde{g}_{k,F}$, as well as to take into account the inductive hypotheses. □

The next result can be analogously established.

Lemma 5.3. *Let $\mu \geq -\frac{1}{2}$, and let $\left\{F_j\right\}_{j=0}^\infty$ be a sequence of functions in $C(I)$ for which there exists $p \in \mathbb{N}_0$ with*

$$\lim_{j\to\infty} \left\| M_p^{-1} F_j \right\|_q = 0 \quad (1 \leq q \leq \infty).$$

Then, for each $k \in \mathbb{N}_0$, there exist $p_k \in \mathbb{N}_0$, $p_k \geq p$, and $F_{k,j} \in C(I)$ ($j \in \mathbb{N}_0$) such that

$$x^{-\mu-\frac{1}{2}}(Dx^{-1})^k x^{k+\mu+\frac{1}{2}} F_{k,j}(x) = F_j(x) \quad (x \in I,\ j \in \mathbb{N}_0)$$

and

$$\lim_{j\to\infty} \left\| M_{p_k}^{-1} F_{k,j} \right\|_q = 0 \quad (1 \leq q \leq \infty).$$

5.2. Representation of the functionals in $\mathcal{S}'_\mu$

At this point, we are in a position to give several representations of the elements in the dual space of $\mathcal{S}_\mu$ in terms of the adjoint of the operator $T_{\mu,k}$ ($k \in \mathbb{N}_0$). This will be done in Theorems 5.4 and 5.5. The proof of Theorem 5.4 uses a fairly standard method, which can be traced back to [36] and was already used in [3, 4, 24], but we include it for the sake of completeness.

Theorem 5.4. *Let $\mu \geq -\frac{1}{2}$. A linear functional f belongs to $\mathcal{S}'_\mu$ if, and only if, for every q, $1 < q \leq \infty$, there exist $r \in \mathbb{N}_0$ and measurable functions g_k, with $M_r^{-1} g_k \in L^q(I)$ ($k \in \mathbb{N}_0$, $0 \leq k \leq r$), such that*

$$f = x^{-\mu-\frac{1}{2}} \sum_{k=0}^{r} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}}\, g_k(x). \tag{5.8}$$

Proof. Fix q, $1 < q \le \infty$, and let p, $1 \le p < \infty$, be the conjugate exponent of q.

If $f \in \mathcal{S}'_\mu$, then, by Proposition 4.6, there exist $r \in \mathbb{N}_0$ and $C > 0$ such that

$$|\langle f, \varphi\rangle| \le C\rho^\mu_{p,r}(\varphi) \quad (\varphi \in \mathcal{S}_\mu). \tag{5.9}$$

Let us denote by Γ_p the direct sum of $r + 1$ copies of $L^p(I)$, normed by

$$\|(f_k)_{0\le k\le r}\|_p = \max_{0\le k\le r} \|f_k\|_p, \tag{5.10}$$

and by Ξ_q the direct sum of $r + 1$ copies of $L^q(I)$, normed by

$$\|(f_k)_{0\le k\le r}\|_q = \sum_{k=0}^{r} \|f_k\|_q.$$

Consider the following injective map:

$$\begin{aligned} F : \mathcal{S}_\mu &\longrightarrow \Gamma_p \\ \varphi &\longmapsto F(\varphi) = \left(M_r(x)T_{\mu,k}\varphi(x)\right)_{0\le k\le r}, \end{aligned}$$

and define on $F(\mathcal{S}_\mu) \subset \Gamma_p$ the linear functional L by means of the formula

$$\langle L, F(\varphi)\rangle = \langle f, \varphi\rangle.$$

In view of (5.9) and (5.10), L is continuous, and its norm is, at most, C. We continue to denote by L a norm-preserving Hahn-Banach extension of this functional up to Γ_p. The Riesz representation $(f_k)_{0\le k\le r} \in \Xi_q$ of L over Γ_p satisfies

$$\|L\| = \sum_{k=0}^{r} \|f_k\|_q \le C \tag{5.11}$$

and

$$\begin{aligned} \langle f, \varphi\rangle &= \langle L, F(\varphi)\rangle \\ &= \sum_{k=0}^{r} \int_0^\infty f_k(x)M_r(x)x^{k+\mu+\frac{1}{2}} \left(x^{-1}D\right)^k x^{-\mu-\frac{1}{2}}\varphi(x)dx \\ &= \sum_{k=0}^{r} \left\langle f_k(x), M_r(x)x^{k+\mu+\frac{1}{2}} \left(x^{-1}D\right)^k x^{-\mu-\frac{1}{2}}\varphi(x)\right\rangle \\ &= \sum_{k=0}^{r} \left\langle x^{-\mu-\frac{1}{2}} \left(-Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} M_r(x)f_k(x), \varphi\right\rangle \\ &= \left\langle x^{-\mu-\frac{1}{2}} \sum_{k=0}^{r} \left(-Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} M_r(x)f_k(x), \varphi\right\rangle \quad \left(\varphi \in \mathcal{S}_\mu\right). \end{aligned}$$

Setting $g_k = (-1)^k f_k M_r$ ($k \in \mathbb{N}_0,\ 0 \le k \le r$), we obtain (5.8).

Conversely, if f is given by (5.8), then Hölder's inequality implies that

$$|\langle f, \varphi\rangle| \le \sum_{k=0}^{r} \int_0^\infty \left|M_r^{-1}(x)g_k(x)\right| \left|M_r(x)x^{k+\mu+\frac{1}{2}} \left(x^{-1}D\right)^k x^{-\mu-\frac{1}{2}}\varphi(x)\right|dx$$

$$\leq \rho_{p,r}^{\mu}(\varphi) \sum_{k=0}^{r} \left\| M_r^{-1} g_k \right\|_q \quad (\varphi \in \mathcal{S}_\mu),$$

which yields that $f \in \mathcal{S}'_\mu$. □

Theorem 5.5. *For $\mu \geq -\frac{1}{2}$, the following statements are equivalent:*

(i) *The functional f lies in $\mathcal{S}'_\mu$.*
(ii) *There exist $k, p \in \mathbb{N}_0$ and a function $F \in C(I)$ such that*

$$f = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} F(x), \tag{5.12}$$

with

$$M_p^{-1} F \in L^q(I) \tag{5.13}$$

for all q, $1 \leq q \leq \infty$.
(iii) *There exist $k, p \in \mathbb{N}_0$ and a function $F \in C(I)$ satisfying* (5.12) *such that* (5.13) *holds for some q, $1 \leq q \leq \infty$.*
(iv) *There exist $k, p \in \mathbb{N}_0$ and a function $F \in C(I)$ satisfying* (5.12) *such that* (5.13) *holds for $q = \infty$.*

Proof. (i) $\Rightarrow$ (ii) If $f \in \mathcal{S}'_\mu$, Theorem 5.4 yields that $p \in \mathbb{N}_0$ and measurable functions G_i ($i \in \mathbb{N}_0,\ 0 \leq i \leq p$) exist such that

$$f = x^{-\mu-\frac{1}{2}} \sum_{i=0}^{p} \left(Dx^{-1}\right)^i x^{i+\mu+\frac{1}{2}} G_i(x) \tag{5.14}$$

and

$$M_p^{-1} G_i \in L^\infty(I) \quad (i \in \mathbb{N}_0,\ 0 \leq i \leq p). \tag{5.15}$$

Apply (N) to get $n, t \in \mathbb{N}_0$, $n > t > p$, for which

$$\int_0^\infty \frac{M_p(x)}{M_t(x)} dx < \infty, \tag{5.16}$$

$$\int_0^\infty \frac{M_t(x)}{M_n(x)} dx < \infty. \tag{5.17}$$

Fix $i \in \mathbb{N}_0$, $0 \leq i \leq p$. By (M) and (5.16), we obtain

$$\begin{aligned} \frac{1}{M_t(x)} \int_0^x |G_i(\xi)|\, d\xi &\leq C \int_0^\infty \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \frac{M_p(\xi)}{M_t(\xi)} d\xi \\ &\leq C \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \int_0^\infty \frac{M_p(\xi)}{M_t(\xi)} d\xi \\ &= C \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \quad (x \in I). \end{aligned} \tag{5.18}$$

The function

$$\widetilde{G}_i(x) = x^{-i-\mu-\frac{1}{2}} \int_0^x G_i(\xi) \xi^{i+\mu+\frac{1}{2}} d\xi \quad (x \in I)$$

is continuous and satisfies

$$x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i+1}x^{(i+1)+\mu+\frac{1}{2}}\widetilde{G}_i(x)$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i}x^{i+\mu+\frac{1}{2}}G_i(x) \quad (x\in I).$$

Using (5.18), we may write

$$\begin{aligned}\frac{\left|\widetilde{G}_i(x)\right|}{M_t(x)} &= \frac{1}{M_t(x)}\left|x^{-i-\mu-\frac{1}{2}}\int_0^x G_i(\xi)\xi^{i+\mu+\frac{1}{2}}d\xi\right| \\ &\leq \frac{1}{M_t(x)}\int_0^x |G_i(\xi)|\,d\xi \\ &\leq C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right| \quad (x\in I). \end{aligned}\tag{5.19}$$

As $n>t$,

$$\begin{aligned}\sup_{x\in I}\left|\frac{\widetilde{G}_i(x)}{M_n(x)}\right| &\leq \sup_{x\in I}\left|\frac{\widetilde{G}_i(x)}{M_t(x)}\right| \\ &\leq C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right| < \infty. \end{aligned}$$

Furthermore, taking into account (5.19) and (5.17), we find that

$$\begin{aligned}\int_0^\infty\left|\frac{\widetilde{G}_i(x)}{M_n(x)}\right|^q dx &= \int_0^\infty\left|\frac{\widetilde{G}_i(x)}{M_t(x)}\right|^q\left(\frac{M_t(x)}{M_n(x)}\right)^q dx \\ &\leq C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right|^q\int_0^\infty\left(\frac{M_t(x)}{M_n(x)}\right)^q dx \\ &\leq C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right|^q\int_0^\infty\frac{M_t(x)}{M_n(x)}dx \\ &= C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right|^q < \infty \end{aligned}$$

whenever $1\leq q<\infty$. Now, applying Lemma 5.1 with $\mu+i+1\geq -\frac{1}{2}$ instead of $\mu\geq-\frac{1}{2}$, we obtain a function $F_i\in C(I)$ and a non-negative integer $s_i\geq n$ such that

$$x^{-(\mu+i+1)-\frac{1}{2}}\left(Dx^{-1}\right)^{p-i}x^{(p-i)+(\mu+i+1)+\frac{1}{2}}F_i(x) = \widetilde{G}_i(x) \quad (x\in I),$$

with

$$M_{s_i}^{-1}F_i\in L^q(I) \quad (1\leq q\leq\infty).$$

It follows that

$$x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i}x^{i+\mu+\frac{1}{2}}G_i(x)$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i+1}x^{(i+1)+\mu+\frac{1}{2}}\widetilde{G}_i(x)$$

$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i+1} x^{(\mu+i+1)+\frac{1}{2}}\left[x^{-(\mu+i+1)-\frac{1}{2}}\left(Dx^{-1}\right)^{p-i} x^{(p-i)+(\mu+i+1)+\frac{1}{2}}F_i(x)\right]$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i+1}\left(Dx^{-1}\right)^{p-i} x^{(p+1)+\mu+\frac{1}{2}}F_i(x)$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{p+1} x^{(p+1)+\mu+\frac{1}{2}}F_i(x) \quad (x \in I).$$

Set

$$F = \sum_{i=0}^{p} F_i, \quad m = \max_{0\le i\le p} s_i.$$

Then, $F \in C(I)$. Using (5.14) and (5.15), we have

$$f = x^{-\mu-\frac{1}{2}}\sum_{i=0}^{p}\left(Dx^{-1}\right)^{i} x^{i+\mu+\frac{1}{2}}G_i(x)$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{p+1} x^{(p+1)+\mu+\frac{1}{2}}\sum_{i=0}^{p}F_i(x)$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{p+1} x^{(p+1)+\mu+\frac{1}{2}}F(x)$$

and

$$M_m^{-1}F \in L^q(I) \quad (1 \le q \le \infty).$$

It is thus proved that (*i*) implies (*ii*).
(*ii*) $\Rightarrow$ (*iii*) This is obvious.
(*iii*) $\Rightarrow$ (*iv*) Suppose that there exist $k, p \in \mathbb{N}_0$ and a function $F \in C(I)$ satisfying (5.12) and (5.13) for some q, $1 \le q \le \infty$, and apply (N) to find $t \in \mathbb{N}_0$, $t > p$, such that

$$\int_0^\infty \frac{M_p(x)}{M_t(x)}dx < \infty. \tag{5.20}$$

The function

$$\widetilde{F}(x) = x^{-k-\mu-\frac{1}{2}}\int_0^x F(\xi)\xi^{k+\mu+\frac{1}{2}}d\xi \quad (x \in I)$$

is continuous, with

$$x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{k+1} x^{(k+1)+\mu+\frac{1}{2}}\widetilde{F}(x)$$
$$= x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{k} x^{k+\mu+\frac{1}{2}}F(x) = f.$$

By choosing $n > t$ and using (M), we obtain

$$\sup_{x\in I}\left|\frac{\widetilde{F}(x)}{M_n(x)}\right| \le C\sup_{x\in I}\left|x^{-k-\mu-\frac{1}{2}}\int_0^x \frac{F(\xi)}{M_p(\xi)}\frac{M_p(\xi)}{M_t(\xi)}\frac{M_t(\xi)}{M_n(\xi)}\xi^{k+\mu+\frac{1}{2}}d\xi\right|$$
$$\le C\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right|\frac{M_p(\xi)}{M_t(\xi)}d\xi.$$

If $q = 1$, (5.13) and the fact that $t > p$ imply that

$$\sup_{x\in I}\left|\frac{\widetilde{F}(x)}{M_n(x)}\right| \leq C\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right| d\xi < \infty.$$

If $q = \infty$, conditions (5.13) and (5.20) yield

$$\sup_{x\in I}\left|\frac{\widetilde{F}(x)}{M_n(x)}\right| \leq C\sup_{\xi\in I}\left|\frac{F(\xi)}{M_p(\xi)}\right| \int_0^\infty \frac{M_p(\xi)}{M_t(\xi)} d\xi < \infty.$$

Finally, if $1 < q < \infty$, then (5.13), Hölder's inequality, and (5.20) lead to

$$\begin{aligned}\sup_{x\in I}\left|\frac{\widetilde{F}(x)}{M_n(x)}\right| &\leq C\left\{\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right|^q d\xi\right\}^{\frac{1}{q}} \left\{\int_0^\infty \left(\frac{M_p(\xi)}{M_t(\xi)}\right)^{q'} d\xi\right\}^{\frac{1}{q'}} \\ &\leq C\left\{\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right|^q d\xi\right\}^{\frac{1}{q}} \left\{\int_0^\infty \frac{M_p(\xi)}{M_t(\xi)} d\xi\right\}^{\frac{1}{q'}} < \infty.\end{aligned}$$

Here, q' denotes the conjugate exponent of q. Therefore, (5.12) and (5.13) hold with $k + 1$ instead of k, $\widetilde{F}$ instead of F, and n instead of p, when $q = \infty$. This establishes (iv).
$(iv) \Rightarrow (i)$ To complete the proof, assume that there exist $k, p \in \mathbb{N}_0$ and $F \in C(I)$ such that

$$f = x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}}F(x),$$

with

$$M_p^{-1}F \in L^\infty(I).$$

This representation of f guarantees that $f \in \mathcal{S}'_\mu$, as it can be deduced from the following estimate:

$$\begin{aligned}|\langle f, \varphi\rangle| &= \left|(-1)^k \int_0^\infty F(x)x^{k+\mu+\frac{1}{2}}\left(x^{-1}D\right)^k x^{-\mu-\frac{1}{2}}\varphi(x)dx\right| \\ &\leq \sup_{x\in I}\left|\frac{F(x)}{M_p(x)}\right| \sup_{x\in I}\left|M_r(x)x^{k+\mu+\frac{1}{2}}\left(x^{-1}D\right)^k x^{-\mu-\frac{1}{2}}\varphi(x)\right| \int_0^\infty \frac{M_p(x)}{M_r(x)}dx \\ &\leq C\omega^\mu_{r,k}(\varphi) \quad (\varphi \in \mathcal{S}_\mu), \end{aligned} \tag{5.21}$$

where $r > p$ is chosen according to (N). Thus, (iv) implies (i), and we are done. □

5.3. Boundedness in $\mathcal{S}'_\mu$

Next, we shall characterize boundedness in $\mathcal{S}'_\mu$. Recall that, under (A), (M), and (N), $\mathcal{S}_\mu$ is reflexive (Theorem 4.4 and Proposition 2.12), which means that the weak and weak* topologies of $\mathcal{S}'_\mu$ coincide. Furthermore, because $\mathcal{S}_\mu$ is a Fréchet space (Proposition 4.1), it is barrelled [36, Definition II.33.1 and Proposition II.33.2, Corollary 1], and, in this class of spaces, the weak* and strong topologies share the same bounded sets [36, Theorem II.33.2].

Theorem 5.6. *For $\mu \geq -\frac{1}{2}$, the following five statements are equivalent:*

(i) *The set $B \subset \mathcal{S}'_\mu$ is (weakly, weakly*, strongly) bounded.*

(ii) *Given q, $1 < q \leq \infty$, there exist $r \in \mathbb{N}_0$, $C > 0$, and, for each $f \in B$, measurable functions $g_{k,f}$ ($k \in \mathbb{N}_0,\ 0 \leq k \leq r$) such that*

$$f = x^{-\mu-\frac{1}{2}} \sum_{k=0}^{r} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} g_{k,f}(x),$$

with

$$\sum_{k=0}^{r} \left\| M_r^{-1} g_{k,f} \right\|_q \leq C.$$

(iii) *There exist $k, p \in \mathbb{N}_0$, $C > 0$, and, for each $f \in B$, a function $g_f \in C(I)$ such that*

$$f = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} g_f(x), \tag{5.22}$$

with

$$\left\| M_p^{-1} g_f \right\|_q \leq C \tag{5.23}$$

for any q, $1 \leq q \leq \infty$.

(iv) *There exist $k, p \in \mathbb{N}_0$, $C > 0$, and, for each $f \in B$, a function $g_f \in C(I)$ satisfying* (5.22) *such that* (5.23) *holds for some q, $1 \leq q \leq \infty$.*

(v) *There exist $k, p \in \mathbb{N}_0$, $C > 0$, and, for each $f \in B$, a function $g_f \in C(I)$ satisfying* (5.22) *such that* (5.23) *holds for $q = \infty$.*

Proof. Since $\mathcal{S}_\mu$ is barrelled, a subset B of $\mathcal{S}'_\mu$ is (weakly, weakly*, strongly) bounded if, and only if, it is equicontinuous [36, Proposition II.33.1], which means that (5.9) holds for $r \in \mathbb{N}_0$ and $C > 0$ independent of $f \in B$. Now, (ii) can be obtained by using the same argument as in the proof of Theorem 5.4, taking into account condition (5.11) on the norms of the representing functions.

If (ii) holds, then, in particular, there exist $p \in \mathbb{N}_0$, $A > 0$, and, for each $f \in B$, measurable functions $G_{i,f}$ ($i \in \mathbb{N}_0,\ 0 \leq i \leq p$) such that

$$f = x^{-\mu-\frac{1}{2}} \sum_{i=0}^{p} \left(Dx^{-1}\right)^i x^{i+\mu+\frac{1}{2}} G_{i,f}(x),$$

with

$$\sum_{i=0}^{p} \left\| M_p^{-1} G_{i,f} \right\|_\infty \leq A.$$

Fix $i \in \mathbb{N}_0$, $0 \leq i \leq p$. The argument in the proof that (i) implies (ii) in Theorem 5.5 yields that $n \in \mathbb{N}_0$, $n > p$, $\widetilde{A} > 0$, and $\widetilde{G}_{i,f} \in C(I)$ exist such that

$$\begin{aligned} x^{-\mu-\frac{1}{2}} &\left(Dx^{-1}\right)^{i+1} x^{(i+1)+\mu+\frac{1}{2}} \widetilde{G}_{i,f}(x) \\ &= x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^i x^{i+\mu+\frac{1}{2}} G_{i,f}(x) \quad (x \in I), \end{aligned}$$

$$\sup_{x \in I} \left| \frac{\widetilde{G}_{i,f}(x)}{M_n(x)} \right| \leq C \sup_{\xi \in I} \left| \frac{G_{i,f}(\xi)}{M_p(\xi)} \right| \leq \widetilde{A},$$

and

$$\int_0^\infty \left|\frac{\widetilde{G}_{i,f}(x)}{M_n(x)}\right|^q dx \leq C \sup_{\xi\in I}\left|\frac{G_{i,f}(\xi)}{M_p(\xi)}\right|^q \leq \widetilde{A}^q \quad (1 \leq q < \infty),$$

where $\widetilde{A}$ is independent of $f \in B$. According to Lemma 5.2, applied with $\mu + i + 1 \geq -\frac{1}{2}$ instead of $\mu \geq -\frac{1}{2}$, there exist $s_i \in \mathbb{N}_0$, $s_i \geq n$, $C_i > 0$, and, for each $f \in B$, a function $F_{i,f} \in C(I)$ such that

$$x^{-(\mu+i+1)-\frac{1}{2}}\left(Dx^{-1}\right)^{p-i} x^{(p-i)+(\mu+i+1)+\frac{1}{2}} F_{i,f}(x) = \widetilde{G}_{i,f}(x) \quad (x \in I),$$

with

$$\left\|M_{s_i}^{-1} F_{i,f}\right\|_q \leq C_i \quad (1 \leq q \leq \infty).$$

Therefore,

$$\begin{aligned} &x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{i} x^{i+\mu+\frac{1}{2}} G_{i,f}(x) \\ &\qquad = x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{p+1} x^{(p+1)+\mu+\frac{1}{2}} F_{i,f}(x) \quad (x \in I). \end{aligned}$$

Setting

$$g_f = \sum_{i=0}^{p} F_{i,f}, \quad m = \max_{0\leq i\leq p} s_i, \quad C = \sum_{i=0}^{p} C_i,$$

we find that $g_f \in C(I)$,

$$f = x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{p+1} x^{(p+1)+\mu+\frac{1}{2}} g_f(x),$$

and

$$\left\|M_m^{-1} g_f\right\|_q \leq C \quad (1 \leq q \leq \infty),$$

where $m \in \mathbb{N}_0$ and $C > 0$ do not depend on $f \in B$. Thus, (*ii*) implies (*iii*).

It is apparent that (*iii*) implies (*iv*).

To prove that (*iv*) implies (*v*), assume that there exist $k, p \in \mathbb{N}_0$, $A > 0$, and, for each $f \in B$, a function $g_f \in C(I)$ satisfying

$$f = x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{k} x^{k+\mu+\frac{1}{2}} g_f(x)$$

with

$$\left\|M_p^{-1} g_f\right\|_q \leq A$$

for some q, $1 \leq q \leq \infty$. The argument in the proof that (*iii*) implies (*iv*) in Theorem 5.5 allows us to find $n \in \mathbb{N}_0$, $n > p$, $\widetilde{A} > 0$, and, for each $f \in B$, a function $\widetilde{g}_f \in C(I)$ such that

$$f = x^{-\mu-\frac{1}{2}}\left(Dx^{-1}\right)^{k+1} x^{(k+1)+\mu+\frac{1}{2}} \widetilde{g}_f(x),$$

with

$$\sup_{x\in I}\left|\frac{\widetilde{g}_f(x)}{M_n(x)}\right| \leq C\left\{\int_0^\infty \left|\frac{g_f(\xi)}{M_p(\xi)}\right|^q d\xi\right\}^{\frac{1}{q}} \leq \widetilde{A},$$

if $1 \leq q < \infty$, or

$$\sup_{x\in I}\left|\frac{\widetilde{g}_f(x)}{M_n(x)}\right| \leq C \sup_{\xi\in I}\left|\frac{g_f(\xi)}{M_p(\xi)}\right| \leq \widetilde{A},$$

if $q = \infty$. This establishes (*v*).

Finally, (*v*) and (5.21), with g_f ($f \in B$) instead of F, yield (*i*). □

5.4. Convergence in $\mathcal{S}'_\mu$

Next, we shall describe sequential convergence in the dual of $\mathcal{S}_\mu$. Under (A), (M), and (N), $\mathcal{S}_\mu$ is Montel and, hence, reflexive (Theorem 4.4 and Proposition 2.12). Thus, the weak and weak* topologies of $\mathcal{S}'_\mu$ coincide, and weak* and strong sequential convergence are equivalent in this space [36, Proposition II.34.6, Corollary 1].

Theorem 5.7. *For $\mu \geq -\frac{1}{2}$, the following statements are equivalent:*

(i) *The sequence $\left\{f_j\right\}_{j=0}^\infty$ converges (weakly, weakly*, strongly) to zero in $\mathcal{S}'_\mu$.*

(ii) *For each q, $1 < q \leq \infty$, there exist $p \in \mathbb{N}_0$ and measurable functions $g_{k,j}$ ($k \in \mathbb{N}_0,\ 0 \leq k \leq p$) such that*

$$f_j = x^{-\mu-\frac{1}{2}} \sum_{k=0}^{p} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} g_{k,j}(x) \quad (j \in \mathbb{N}_0),$$

with $\lim_{j\to\infty} \sum_{k=0}^{p} \left\|M_p^{-1} g_{k,j}\right\|_q = 0$.

(iii) *There exist $k, p \in \mathbb{N}_0$ and $F_j \in C(I)$ ($j \in \mathbb{N}_0$) such that*

$$f_j = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} F_j(x) \quad (j \in \mathbb{N}_0) \tag{5.24}$$

and

$$\lim_{j\to\infty} \left\|M_p^{-1} F_j\right\|_q = 0 \tag{5.25}$$

for any q, $1 \leq q \leq \infty$.

(iv) *There exist $k, p \in \mathbb{N}_0$ and $F_j \in C(I)$ ($j \in \mathbb{N}_0$) satisfying* (5.24) *and* (5.25) *for some q, $1 \leq q \leq \infty$.*

(v) *There exist $k, p \in \mathbb{N}_0$ and $F_j \in C(I)$ ($j \in \mathbb{N}_0$) satisfying* (5.24) *and* (5.25) *for $q = \infty$.*

(vi) *There exist $k, p \in \mathbb{N}_0$, $C > 0$, and $F_j \in C(I)$ ($j \in \mathbb{N}_0$) satisfying* (5.24)*, with*

$$\left\|M_p^{-1} F_j\right\|_\infty \leq C \quad (j \in \mathbb{N}_0)$$

and $\lim_{j\to\infty} F_j(x) = 0$ *for almost all $x \in I$.*

Proof. To show that (i) implies (ii), suppose that $\left\{f_j\right\}_{j=0}^\infty$ converges to zero in $\mathcal{S}'_\mu$. Then [24, Equation (5.7)], there exist $p \in \mathbb{N}_0$ and positive constants $\left\{C_j\right\}_{j=0}^\infty$ such that

$$\left|\langle f_j, \varphi\rangle\right| \leq C_j \|\varphi\|_{\mu,\infty,p} \quad \left(\varphi \in \mathcal{S}_\mu,\ j \in \mathbb{N}_0\right)$$

and $\lim_{j\to\infty} C_j = 0$. On the other hand (Proposition 4.6), given q, $1 < q \leq \infty$, there exist $C > 0$ and $r \in \mathbb{N}_0$ such that

$$\|\varphi\|_{\mu,\infty,p} \leq C\, \rho^\mu_{q,r}(\varphi) \quad (\varphi \in \mathcal{S}_\mu).$$

It follows that

$$\left|\langle f_j, \varphi\rangle\right| \leq C_j\, \rho^\mu_{q,r}(\varphi) \quad \left(\varphi \in \mathcal{S}_\mu,\ j \in \mathbb{N}_0\right), \tag{5.26}$$

where the redefined sequence $\left\{C_j\right\}_{j=0}^\infty$ still satisfies

$$\lim_{j\to\infty} C_j = 0. \tag{5.27}$$

Now, it suffices to provide the same argument as in the proof of Theorem 5.4, using (5.26) instead of (5.9) and keeping in mind (5.27) in relation to (5.11).

Parts (*iv*) and (*vi*) follow trivially from (*iii*) and (*v*), respectively.

The arguments in the proof of the corresponding results for Theorems 5.5 and 5.6, in combination with Lemma 5.3 instead of Lemmas 5.1 and 5.2, allow us to establish that (*ii*) implies (*iii*) and (*iv*) implies (*v*); we omit the details.

Finally, assuming that (*vi*) holds, to obtain (*i*), it suffices to choose $n \in \mathbb{N}_0$, $n > p$, according to (*N*) and apply Lebesgue's dominated convergence theorem to the following integrals:

$$\left\langle f_j, \varphi \right\rangle = (-1)^k \int_0^\infty \frac{F_j(x)}{M_p(x)} \frac{M_p(x)}{M_n(x)} M_n(x) T_{\mu,k}\varphi(x) dx \quad \left(\varphi \in \mathcal{S}_\mu,\ j \in \mathbb{N}_0\right).$$

This completes the proof. □

5.5. An example: $\mathcal{H}_\mu$

Theorems 5.4–5.7 characterize the structure, boundedness, and convergence in the dual of a wide range of spaces that arise in the context of the Hankel transformation (see Sections 2 and 3.4). For illustrative purposes, in Corollary 5.8, some of those results have been specialized for the Zemanian space $\mathcal{H}_\mu$, which, as already mentioned (Example 2.2), is none other than the Hankel-$K\{M_p\}$ space defined by the sequence of weights $\left\{\left(1 + x^2\right)^p\right\}_{p=0}^\infty$.

The application of Theorem 5.4 to $\mathcal{H}_\mu$ is as stated in [4, Theorem 5.5]. However, the representation of the elements in $\mathcal{H}'_\mu$ obtained as a consequence of Theorem 5.5 is an improvement upon the result of [4] in three ways. First, it involves the $T_{\mu,k}$-distributional derivative ($k \in \mathbb{N}_0$) of a single continuous function F. Second, the representing function F is not dependent on the exponent q; in fact, the products of F and the inverses of the weights belong to all classes $L^q(I)$, $1 \leq q \leq \infty$. Finally, as we have just highlighted, and unlike what occurs in Theorem 5.4, $q = 1$ is included in the range of exponents for which the representation is valid.

Remarks analogous to the preceding ones can be made regarding Theorem 5.7; particularly, the application of part (*ii*) to the space $\mathcal{H}_\mu$ is as stated in [4, Theorem 5.4], while parts (*iii*) to (*v*) improve upon this result along the lines described above.

Reference [4] contains no analogue of Theorem 5.6.

Corollary 5.8. *Let $\mathcal{H}'_\mu$ be the dual space of $\mathcal{H}_\mu$. Then, the following holds:*

(*i*) *A functional f belongs to $\mathcal{H}'_\mu$ if, and only if, there exist $k, p \in \mathbb{N}_0$ and a function $F \in C(I)$ such that*

$$f = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} F(x),$$

with

$$\left(1 + x^2\right)^{-p} F(x) \in L^q(I) \quad (1 \leq q \leq \infty).$$

(*ii*) *A set $B \subset \mathcal{H}'_\mu$ is (weakly, weakly*, strongly) bounded if, and only if, there exist $k, p \in \mathbb{N}_0$, $C > 0$, and, for every $f \in B$, a function $g_f \in C(I)$ such that*

$$f = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} g_f(x),$$

with

$$\left\| \left(1+x^2\right)^{-p} g_f(x) \right\|_q \le C \quad (1 \le q \le \infty).$$

(*iii*) *A sequence* $\left\{f_j\right\}_{j=0}^{\infty}$ *converges (weakly, weakly*, strongly) to zero in* $\mathcal{H}'_\mu$ *if, and only if, there exist* $k, p \in \mathbb{N}_0$ *and* $g_j \in C(I)$ *such that*

$$f_j = x^{-\mu-\frac{1}{2}} \left(Dx^{-1}\right)^k x^{k+\mu+\frac{1}{2}} g_j(x) \quad (j \in \mathbb{N}_0),$$

with

$$\lim_{j\to\infty} \left\| \left(1+x^2\right)^{-p} g_j(x) \right\|_q = 0 \quad (1 \le q \le \infty).$$

6. Conclusions

In Friedman's opinion [10, p. 37], one of the most interesting and important problems in the theory of generalized functions or distributions is to find their structure, that is, to express them in terms of differential operators acting on functions or on measures. This paper provides novel results in this direction for the spaces of type Hankel-$K\{M_p\}$. Namely, here it was shown that, for $\mu \ge -\frac{1}{2}$, and under certain conditions on the weights $\{M_p\}_{p=0}^{\infty}$, the elements, the (weakly, weakly*, strongly) bounded subsets, and the (weakly, weakly*, strongly) convergent sequences in the dual of a space $\mathcal{K}_\mu$ of type Hankel-$K\{M_p\}$ can be represented as distributional derivatives of functions and of measures that satisfy good properties, not only in terms of the differential operator $x^{-1}D$ and of the Bessel operator $S_\mu = x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\mu-\frac{1}{2}}$, but also in terms of suitable iterations $T_{\mu,k}$ ($k \in \mathbb{N}_0$) of the Zemanian differential operator $N_\mu = x^{\mu+\frac{1}{2}} D x^{-\mu-\frac{1}{2}}$. To this end, new descriptions for the usual topology of $\mathcal{K}_\mu$ in terms of the latter iterations were given.

The operators $T_{\mu,k}$ are defined by $T_{\mu,k} = N_{\mu+k-1} \cdots N_\mu$ for $k \in \mathbb{N}$, while $T_{\mu,0}$ is the identity operator. The interest of such operators lies in their symmetric behavior with respect to h_μ, the Hankel transformation of order μ. Indeed, given $\varphi \in L^1(I)$, we define

$$\left(h_\mu \varphi\right)(y) = \int_0^\infty \varphi(x)(xy)^{\frac{1}{2}} J_\mu(xy)dx \quad (y \in I),$$

where $I =]0, \infty[$ and J_μ is the Bessel function of the first kind and order μ. For a sufficiently smooth φ with a good boundary behavior, the identity

$$\begin{aligned}(-y)^m T_{\mu,k}(h_\mu \varphi)(y) &= \int_0^\infty (-x)^k T_{\mu,m} \varphi(x)(xy)^{\frac{1}{2}} J_{\mu+k+m}(xy)dx \\ &= h_{\mu+k+m}[(-x)^k T_{\mu,m} \varphi(x)](y) \quad (y \in I,\ m, k \in \mathbb{N}_0)\end{aligned}$$

holds [40, Proof of Theorem 5.4-1]. This rule closely resembles its Fourier counterpart and facilitates the operational calculus of the Hankel transformation, particularly when dealing with the Zemanian space $\mathcal{H}_\mu$ [38, 39], which can be regarded as a paradigm for the spaces of type Hankel-$K\{M_p\}$. In fact, the specialization to $\mathcal{H}_\mu$ of the results obtained in our work improves upon previous results from [4]. The findings of the present paper are thus expected to pave the way for a time-frequency analysis of the spaces $\mathcal{K}_\mu$.

Author contributions

Samuel García-Baquerín and Isabel Marrero: Conceptualization, Methodology, Investigation, Writing – Original Draft, Writing – Review & Editing. All authors of this article have contributed equally. Isabel Marrero: Supervision, Project Administration. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare they have no conflict of interest.

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