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Research article

Solution formula for generalized two-phase Stokes equations and its applications to maximal regularity: Model problems

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Abstract: In this paper, we give a solution formula for the two-phase Stokes equations with and without surface tension and gravity over the whole space with a flat interface. The solution formula has already been considered by Shibata and Shimizu. However, we have reconstructed the formula so that we are able to easily prove resolvent and maximal regularity estimates. The previous work required the assumption of additional conditions on normal components. Here, although we consider normal components, the assumption is weaker than before. The method is based on an H^∞-calculus which has already been applied for the Stokes problems with various boundary conditions in the half-space.

Keywords: solution formula; resolvent estimate; maximal regularity; two-phase Stokes equations
Mathematics Subject Classification: 35D35, 35B45, 35K51

1. Introduction

It is known that the motion of viscous incompressible fluids is governed by this Navier-Stokes equations. When we consider two fluids that are separated by a free surface, the analysis is a difficult problem. Mathematically, the free-boundary problem is formulated based on initial-boundary value problems, as follows. Let $\Omega_+(t)$ and $\Omega_-(t)$ be domains in $\mathbb{R}^n$ that contain different fluids, and let them have the same time-dependent boundary $\Gamma(t) = \partial\Omega_+(t)(= \partial\Omega_-(t))$ and $\mathbb{R}^n = \Omega_+(t) \cup \Omega_-(t) \cup \Gamma(t)$. The unknowns are the boundary $\Gamma(t)$, the velocity $v(x,t) = {}^t(v_1, \ldots, v_n)$, and pressure $\theta(x,t)$ defined on $\Omega(t) = \Omega_+(t) \cup \Omega_-(t)$. The equations are as follows:

$$\begin{cases} \rho(\partial_t v + (v\cdot\nabla)v) - \mathrm{Div}\, S(v,\theta) = 0 & \text{in } \Omega(t),\ t>0,\\ \mathrm{div}\, v = 0 & \text{in } \Omega(t),\ t>0,\\ [\![S(v,\theta)\nu_t]\!] = c_\sigma \mathcal{H}\nu_t + [\![\rho]\!] c_g x_n \nu_t & \text{on } \Gamma(t),\ t>0,\\ [\![v]\!] = 0 & \text{on } \Gamma(t),\ t>0,\\ V = v\cdot \nu_t & \text{on } \Gamma(t),\ t>0,\\ v|_{t=0} = v_0 & \text{in } \Omega(0). \end{cases} \tag{1.1}$$

Here, $S(v,\theta) = \mu D(v) - \theta I = (\mu(\partial_i v_j + \partial_j v_i) - \delta_{ij}\theta)_{ij}$ is an $n\times n$ symmetric stress tensor, V is the normal velocity of $\Gamma(t)$, ν_t is the unit outward normal vector pointing from $\Omega_+(t)$ to $\Omega_-(t)$, $\mathcal{H}$ is the mean curvature of $\Gamma(t)$, which is given by $\mathcal{H}\nu_t = \Delta_{\Gamma(t)}x$, where $\Delta_{\Gamma(t)}$ denotes the Laplace-Beltrami operator on $\Gamma(t)$. The letters ρ, μ, c_σ, and c_g denote the coefficients of density, viscosity, surface tension, and gravity, respectively. Here, ρ and μ are positive constants on each domain $\Omega_\pm(t)$. The symbol $[\![\cdot]\!]$ denotes a jump across the interface $\Gamma(t)$. For example, the quantity $[\![\rho]\!]$ means that $[\![\rho]\!] = \rho|_{\Omega_+(t)} - \rho|_{\Omega_-(t)}$ for the piecewise constant density ρ defined on $\Omega(t)$.

It is known that the Hanzawa transformation is a useful technique to solve free boundary problems. In this method, the unknown $\Gamma(t)$ is given by a height function defined on the boundary of a fixed domain. After applying this transformation, the equations become quasi-linear equations. Therefore, it is important to consider the linearized equations. In addition to the above discussion, maximal regularity for the linearized equations over the whole space with a flat interface is a necessary, as described below;

$$\begin{cases} \rho\partial_t U - \mathrm{Div}\, S(U,\Theta) = F & \text{in } \dot{\mathbb{R}}^n (:= \mathbb{R}^n_+ \cup \mathbb{R}^n_-), t>0,\\ \mathrm{div}\, U = F_d & \text{in } \dot{\mathbb{R}}^n, t>0,\\ \partial_t Y + U_n = D & \text{on } \mathbb{R}^n_0 (:= \partial\mathbb{R}^n_+), t>0,\\ [\![S(U,\Theta)\nu]\!] - ([\![\rho]\!]c_g + c_\sigma\Delta')Y\nu = [\![G]\!] & \text{on } \mathbb{R}^n_0, t>0,\\ [\![U]\!] = [\![H]\!] & \text{on } \mathbb{R}^n_0, t>0,\\ (U,Y)|_{t=0} = (0,0) & \text{in } \dot{\mathbb{R}}^n, \end{cases} \tag{1.2}$$

where F, F_d, D, G, and H are external forces and $\nu = (0,\dots,0,-1)$. Moreover, we consider the corresponding resolvent equations and the case that $c_\sigma = c_g = 0$:

$$\begin{cases} \rho\lambda u - \mathrm{Div}\, S(u,\theta) = f & \text{in } \dot{\mathbb{R}}^n,\\ \mathrm{div}\, u = f_d & \text{in } \dot{\mathbb{R}}^n,\\ \lambda\eta + u_n = d & \text{on } \mathbb{R}^n_0,\\ [\![S(u,\theta)\nu]\!] - ([\![\rho]\!]c_g + c_\sigma\Delta')\eta\nu = [\![g]\!] & \text{on } \mathbb{R}^n_0,\\ [\![u]\!] = [\![h]\!] & \text{on } \mathbb{R}^n_0, \end{cases} \tag{1.3}$$

$$\begin{cases} \rho\partial_t U - \mathrm{Div}\, S(U,\Theta) = F & \text{in } \dot{\mathbb{R}}^n, t>0,\\ \mathrm{div}\, U = F_d & \text{in } \dot{\mathbb{R}}^n, t>0,\\ [\![S(U,\Theta)\nu]\!] = [\![G]\!] & \text{on } \mathbb{R}^n_0, t>0,\\ [\![U]\!] = [\![H]\!] & \text{on } \mathbb{R}^n_0, t>0,\\ U|_{t=0} = 0 & \text{in } \dot{\mathbb{R}}^n, \end{cases} \tag{1.4}$$

$$
\begin{cases}
\rho\lambda u - \operatorname{Div} S(u,\theta) = f & \text{in } \dot{\mathbb{R}}^n,\\
\operatorname{div} u = f_d & \text{in } \dot{\mathbb{R}}^n,\\
[\![S(u,\theta)\nu]\!] = [\![g]\!] & \text{on } \mathbb{R}^n_0,\\
[\![u]\!] = [\![h]\!] & \text{on } \mathbb{R}^n_0.
\end{cases}
\tag{1.5}
$$

In this paper, we construct the solution formulas for these four problems. The approach is based on the standard method, which entails the use of partial Fourier transforms and Laplace transforms of the equations. When we solve ordinary differential equations, we need to consider the matrix. In the previous related works [30,33], the authors also derived the solution formulas by analyzing the ordinary differential equations and the matrix. However, our approach will be easier than before. We focus only on the determinant of the matrix and the order of growth of the cofactor matrix. We do not need to calculate the inverse of the 4×4 matrix. Thus, we are able to more effectively obtain the solution formulas. This is one of our main theorems. As an application, we are able to prove the resolvent estimate and maximal regularity estimate. When we obtain the solution formulas with a suitable form, we know that they have these estimates. This strategy has been shown in [17], which considered the Stokes equations with various boundary conditions in the half-space. We remark that the authors of [30, 33] had to assume additional conditions for h_n and H_n. They had to assume a regularity for the derivative of h_n in all directions. On the other hand, we can relax some conditions. We clarify that only $\partial_n h_n$ is the essential condition. This may be useful for future researchers to consider the external forces which act in the tangential direction. The computational complexity is also much less than before. Moreover, our result on maximal regularity for the problem with surface tension is easy to understand from the perspective of regularity theory. See Theorem 2.4. We expect that our method and analysis will be applied in future works.

There are several papers on two-phase free boundary problems. The problems can be divided into two cases: one is a compact free surface, and the other is a non-compact one. For simplicity, we only consider the first case. Tanaka [38] proved the global existence theorem in L^2 Sobolev-Slobodetskii space. Denisova proved the same results with $c_\sigma = 0$ in both Hölder space [5] and L^2-based Sobolev space [6]. Denisova and Solonnikov extended their results to be applicable to capillary fluids, i.e., $c_\sigma > 0$, in both the whole space [4] and bounded domain [7]. Shimizu [24] treated the case in which $c_\sigma = 0$ in L^p-L^q settings. Köhne et al. proved global well-posedness for the capillary fluids in L^p-settings as well as their asymptotic behavior, in [19]. Saito and Shibata [23] considered a comprehensive approach for two-phase problems. Moreover, there are some papers that focus on two-phase problems, e.g., varifold solutions [1] and viscosity solutions [15, 37]. For more results on resolvent estimates and maximal regularity, see also [17, 20, 22, 29–34].

This paper is organized as follows. First, we introduce some notation and state our main theorems in Section 2. The main objective of this work was to shorten the proofs of estimates and weaken the assumption on the normal components relative to that in the previous work [33]. In Section 3, we cite some theorems from [33], which is the standard way to consider solution formulas. This implies that it is enough to consider the cases that $f = f_d = 0$ and $F = F_d = 0$. The solution formula derived from the boundary data is the most important part. This is demonstrated in Section 4 for (1.4) and (1.5). Then, in Section 5, we prove the resolvent L_q estimate and maximal L_p-L_q estimate as based on Theorem 6.1 in [17]. Analysis of (1.2) and (1.3) is given in Section 6. The solution formulas and the estimates depend on the results for (1.4) and (1.5).

2. Main theorem

In this section, we provide and describe some notation and function spaces and give main theorems. Let $\mathbb{R}^n_+$, $\mathbb{R}^n_-$, $\mathbb{R}^n_0$ be the upper and lower half-spaces and the corresponding flat boundary, and let Q_+, Q_-, Q^n_0 be the corresponding time-space domains, as follows:

$$\begin{aligned}
&\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}, \quad \mathbb{R}^n_- := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n < 0\},\\
&\mathbb{R}^n_0 := \{x = (x', 0) = (x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n\},\\
&Q_+ := \mathbb{R}^n_+ \times (0, \infty), \quad Q_- := \mathbb{R}^n_- \times (0, \infty), \quad Q_0 := \mathbb{R}^n_0 \times (0, \infty).
\end{aligned}$$

Given a domain D, Lebesgue and Sobolev spaces are denoted by $L_q(D)$ and $W^m_q(D)$ with the norms $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W^m_q(D)}$. This is similar for the X-valued spaces $L_p(\mathbb{R}, X)$ and $W^m_p(\mathbb{R}, X)$. For a scalar function f and n-vector $\mathbf{f} = (f_1, \dots, f_n)$, we use the following symbols:

$$\begin{aligned}
\nabla f &= (\partial_1 f, \dots, \partial_n f), & \nabla^2 f &= (\partial_i\partial_j f \mid i, j = 1, \dots, n),\\
\nabla \mathbf{f} &= (\partial_i f_j \mid i, j = 1, \dots, n), & \nabla^2 \mathbf{f} &= (\partial_i\partial_j f_k \mid i, j, k = 1, \dots, n).
\end{aligned}$$

Even if $\mathbf{g} = (g_1, \dots, g_{\tilde{n}}) \in X^{\tilde{n}}$ for some $\tilde{n}$, we denote $\mathbf{g} \in X$ and $\|\mathbf{g}\|_X$ by $\sum_{j=1}^{\tilde{n}} \|g_j\|_X$ for simplicity. Set

$$\hat{W}^1_q(D) = \{\pi \in L_{q,\mathrm{loc}}(D) \mid \nabla\pi \in L_q(D)\}, \quad \hat{W}^1_{q,0}(D) = \{\pi \in \hat{W}^1_q(D) \mid \pi|_{\partial D} = 0\}$$

and let $\hat{W}^{-1}_q(D)$ denote the dual space of $\hat{W}^1_{q',0}(D)$, where $1/q + 1/q' = 1$. For $\pi \in \hat{W}^{-1}_q(D) \cap L_q(D)$, we have

$$\|\pi\|_{\hat{W}^{-1}_q(D)} = \sup\left\{\left|\int_D \pi\phi dx\right| \mid \phi \in \hat{W}^1_{q',0}(D), \|\nabla\phi\|_{L_{q'}(D)} = 1\right\}.$$

Although we usually consider the time interval $\mathbb{R}_+$ for initial-value problems, we consider the functions on $\mathbb{R}$ to enable use of the Fourier transform. Thus, and to consider Laplace transforms as Fourier transforms, we introduce some function spaces:

$$\begin{aligned}
L_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f : \mathbb{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\mathbb{R}, X),\ f(t) = 0 \text{ for } t < 0\},\\
W^m_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f \in L_{p,0,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma_0 t}\partial^j_t f(t) \in L_p(\mathbb{R}, X),\ j = 1, \dots, m\},\\
L_{p,0}(\mathbb{R}, X) &:= L_{p,0,0}(\mathbb{R}; X), \quad W^m_{p,0}(\mathbb{R}, X) := W^m_{p,0,0}(\mathbb{R}; X)
\end{aligned}$$

for some $\gamma_0 \geq 0$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, defined as follows:

$$\mathcal{F}[f](\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad \mathcal{F}^{-1}[g](x) = \mathcal{F}^{-1}_\xi[g](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi.$$

Similarly, let $\mathcal{L}$ and $\mathcal{L}^{-1}_\lambda$ denote the two-sided Laplace transform and its inverse, defined as follows:

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) dt, \quad \mathcal{L}^{-1}_\lambda[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\lambda) d\tau,$$

where $\lambda = \gamma + i\tau \in \mathbb{C}$. Given $s \geq 0$ and the X-valued function f, we use the following Bessel potential spaces to treat fractional orders:

$$H^s_{p,0,\gamma_0}(\mathbb{R}, X) := \{f : \mathbb{R} \to X \mid \Lambda^s_\gamma f := \mathcal{L}^{-1}_\lambda[|\lambda|^s \mathcal{L}[f](\lambda)](t) \in L_{p,0,\gamma}(\mathbb{R}, X) \text{ for any } \gamma \geq \gamma_0\},$$

$$H^s_{p,0}(\mathbb{R}, X) := H^s_{p,0,0}(\mathbb{R}, X).$$

Since we need to consider the n-th component of the velocity, we introduce the following function space:

$$E_q(\dot{\mathbb{R}}^n) := \{h_n \in W^2_q(\dot{\mathbb{R}}^n) \mid |\nabla'|^{-1}\partial_n h_n := \mathcal{F}^{-1}_{\xi'}|\xi'|^{-1}\mathcal{F}_{x'}(\partial_n h_n)(x', x_n) \in L_q(\dot{\mathbb{R}}^n)\}.$$

We remark that this condition is weaker than that described in [33] since they assumed that $|\nabla'|^{-1}h_n \in \hat{W}^1_q(\dot{\mathbb{R}}^n)$. It will be easier to apply this assumption to handle a difficult term. Let $\Sigma_{\varepsilon,\gamma} := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \varepsilon, |\lambda| \geq \gamma\}$ and $\Sigma_\varepsilon := \Sigma_{\varepsilon,0}$. Throughout this paper, let ρ, μ be positive constants on each domain $\mathbb{R}^n_\pm$, denoted by $\rho_\pm$ and $\mu_\pm$. Next, we shall state our main results.

Theorem 2.1. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then, for any $\lambda \in \Sigma_\varepsilon$ and*

$$f \in L_q(\dot{\mathbb{R}}^n), \quad f_d \in \hat{W}^{-1}_q(\mathbb{R}^n) \cap W^1_q(\dot{\mathbb{R}}^n), \quad g \in W^1_q(\dot{\mathbb{R}}^n), \quad h \in W^2_q(\dot{\mathbb{R}}^n), \quad h_n \in E_q(\dot{\mathbb{R}}^n)$$

problem (1.5) *admits a unique solution $(u, \theta) \in W^2_q(\dot{\mathbb{R}}^n) \times \hat{W}^1_q(\dot{\mathbb{R}}^n)$ with the following resolvent estimate:*

$$\begin{aligned}
&\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_q(\dot{\mathbb{R}}^n)} \\
\leq& C_{n,q,\varepsilon}\left\{\|(f, \lambda^{1/2}f_d, \nabla f_d, \lambda^{1/2}g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|f_d\|_{\hat{W}^{-1}_q(\mathbb{R}^n)}\right\}.
\end{aligned}$$

Theorem 2.2. *Let $1 < p, q < \infty$ and $\gamma_0 \geq 0$. Then, for any*

$$\begin{aligned}
&F \in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)), \quad F_d \in W^1_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^{-1}_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\dot{\mathbb{R}}^n)), \\
&G \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\dot{\mathbb{R}}^n)), \\
&H \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n)), \quad H_n \in W^1_{p,0,\gamma_0}(\mathbb{R}, E_q(\dot{\mathbb{R}}^n)),
\end{aligned}$$

problem (1.4) *admits a unique solution (U, Θ) such that*

$$\begin{aligned}
&U \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n)), \\
&\Theta \in L_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^1_q(\dot{\mathbb{R}}^n))
\end{aligned}$$

with the following maximal L_p-L_q regularity estimate:

$$\begin{aligned}
&\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda^{1/2}_\gamma \nabla U, \nabla^2 U, \nabla\Theta)\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\
\leq& C_{n,p,q,\gamma_0}\Big\{\|e^{-\gamma t}(F, \Lambda^{1/2}_\gamma F_d, \nabla F_d, \Lambda^{1/2}_\gamma G, \nabla G, \partial_t H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n)\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\
&\qquad + \|e^{-\gamma t}(\partial_t F_d, \gamma F_d)\|_{L_p(\mathbb{R}, \hat{W}^{-1}_q(\mathbb{R}^n))}\Big\},
\end{aligned}$$

for any $\gamma \geq \gamma_0$.

We can extend the above theorems to the problems (1.2) and (1.3). Let $c_\sigma > 0$ and $c_g > 0$.

Theorem 2.3. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then, there exists a constant $\gamma_0 \geq 1$ that depends on $\varepsilon > 0$ such that, for any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and*

$$f \in L_q(\dot{\mathbb{R}}^n), \quad f_d \in \hat{W}^{-1}_q(\mathbb{R}^n) \cap W^1_q(\dot{\mathbb{R}}^n), \quad g \in W^1_q(\dot{\mathbb{R}}^n),$$

$$h \in W_q^2(\dot{\mathbb{R}}^n), \quad h_n \in E_q(\dot{\mathbb{R}}^n), \quad d \in W_q^2(\dot{\mathbb{R}}^n)$$

problem (1.3) *admits a unique solution* $(u, \theta, \eta) \in W_q^2(\dot{\mathbb{R}}^n) \times \hat{W}_q^1(\dot{\mathbb{R}}^n) \times W_q^3(\dot{\mathbb{R}}^n)$ *with the following resolvent estimate:*

$$\begin{aligned}
&\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|\eta\|_{W_q^2(\dot{\mathbb{R}}^n)} + \|\eta\|_{W_q^3(\dot{\mathbb{R}}^n)} \\
\le& C_{n,q,\varepsilon,\gamma_0}\left\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|f_d\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|d\|_{W_q^2(\dot{\mathbb{R}}^n)}\right\}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
|\lambda|^{3/2}\|\eta\|_{W_q^1(\dot{\mathbb{R}}^n)} \le& C_{n,q,\varepsilon,\gamma_0}\left\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)}\right. \\
&\left. + |\lambda|\|f_d\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + |\lambda|^{1/2}\|d\|_{W_q^1(\dot{\mathbb{R}}^n)}\right\}
\end{aligned}$$

and

$$\begin{aligned}
|\lambda|^2\|\eta\|_{L_q(\dot{\mathbb{R}}^n)} \le& C_{n,q,\varepsilon,\gamma_0}\left\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)}\right. \\
&\left. + |\lambda|\|f_d\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + |\lambda|\|d\|_{L_q(\dot{\mathbb{R}}^n)}\right\}.
\end{aligned}$$

Theorem 2.4. *Let* $1 < p, q < \infty$. *Then, there exists a constant* $\gamma_0 \ge 1$ *such that, for any*

$$\begin{aligned}
F &\in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)), \quad F_d \in W^1_{p,0,\gamma_0}(\mathbb{R}, \hat{W}_q^{-1}(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n)), \\
G &\in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n)), \quad H \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n)), \\
H_n &\in W^1_{p,0,\gamma_0}(\mathbb{R}, E_q(\dot{\mathbb{R}}^n)), \quad D \in L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n)),
\end{aligned}$$

problem (1.2) *admits a unique solution* (U, Θ, Y) *such that*

$$\begin{aligned}
&U \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n)), \\
&\Theta \in L_{p,0,\gamma_0}(\mathbb{R}, \hat{W}_q^1(\dot{\mathbb{R}}^n)), \\
&Y \in L_{p,0,\gamma_0}(\mathbb{R}, W_q^3(\dot{\mathbb{R}}^n)) \cap W^1_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n))
\end{aligned}$$

with the following maximal L_p-L_q regularity estimate:

$$\begin{aligned}
&\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U, \nabla\Theta)\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\
&\quad + \|e^{-\gamma t}(\partial_t Y, \gamma Y)\|_{L_p(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t} Y\|_{L_p(\mathbb{R}, W_q^3(\dot{\mathbb{R}}^n))} \\
\le& C_{n,p,q,\gamma_0}\left\{\|e^{-\gamma t}(F, \Lambda_\gamma^{1/2} F_d, \nabla F_d, \Lambda_\gamma^{1/2} G, \nabla G, \partial_t H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))}\right. \\
&\left. + \|e^{-\gamma t}(\partial_t F_d, \gamma F_d)\|_{L_p(\mathbb{R}, \hat{W}_q^{-1}(\mathbb{R}^n))} + \|e^{-\gamma t} D\|_{L_p(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n))}\right\},
\end{aligned}$$

for any $\gamma \ge \gamma_0$. *Moreover, we see that, if* $D \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n))$, *then* $Y \in H^{3/2}_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n))$ *and*

$$\begin{aligned}
&\|e^{-\gamma t}\Lambda_\gamma^{3/2} Y\|_{L_p(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n))} \\
\le& C_{n,p,q,\gamma_0}\left\{\|e^{-\gamma t}(F, \Lambda_\gamma^{1/2} F_d, \nabla F_d, \Lambda_\gamma^{1/2} G, \nabla G, \partial_t H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))}\right. \\
&\left. + \|e^{-\gamma t}(\partial_t F_d, \gamma F_d)\|_{L_p(\mathbb{R}, \hat{W}_q^{-1}(\mathbb{R}^n))} + \|e^{-\gamma t} D\|_{L_p(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\Lambda_\gamma^{1/2} D\|_{L_p(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n))}\right\}
\end{aligned}$$

for any $\gamma \geq \gamma_0$. *Moreover, we see that, if* $D \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))$*, then* $Y \in W^2_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))$ *and*

$$\begin{aligned}&\|e^{-\gamma t}\partial_t^2 Y\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\ \leq & C_{n,p,q,\gamma_0}\Big\{\|e^{-\gamma t}(F, \Lambda_\gamma^{1/2}F_d, \nabla F_d, \Lambda_\gamma^{1/2}G, \nabla G, \partial_t H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n))\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\ &+\|e^{-\gamma t}(\partial_t F_d, \gamma F_d)\|_{L_p(\mathbb{R},\hat{W}_q^{-1}(\mathbb{R}^n))} + \|e^{-\gamma t}D\|_{L_p(\mathbb{R},W_q^2(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\partial_t D\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\Big\}\end{aligned}$$

for any $\gamma \geq \gamma_0$.

Remark 2.5. (i) *In Theorems 2.1 and 2.3, the uniqueness implies that, if* $f = f_d = [\![g]\!] = [\![h]\!] = 0$ *and* $[\![d]\!] = 0$*, then* $u = 0$, $\nabla\theta = 0$ *with* $[\![\theta]\!] = 0$*, and* $\eta|_{\mathbb{R}^n_0} = 0$*. In Theorems 2.2 and 2.4, the uniqueness has a similar implication.*
(ii) *By interpolation theory, we have*

$$\begin{aligned}W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n)) &\subset H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\dot{\mathbb{R}}^n)),\\ W^2_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap W^1_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n)) &\subset H^{3/2}_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\dot{\mathbb{R}}^n)).\end{aligned}$$

3. Reduction to the problem only with boundary data

In this section, we follow the method in [33]; thus, it is enough to consider the case that $f = f_d = 0$ and $F = F_d = 0$ by subtracting solutions of inhomogeneous data.

We start with whole-space problems.

Lemma 3.1. ([33, Lemma 2.1]) *Let* $1 < p, q < \infty$ *and* $\gamma_0 \geq 0$.
(1) *For any* $f_d \in \hat{W}^{-1}_q(\mathbb{R}^n) \cap W^1_q(\dot{\mathbb{R}}^n)$*, there exists a* $z \in W^2_q(\dot{\mathbb{R}}^n)$ *such that* $\operatorname{div} z = f_d$ *in* $\dot{\mathbb{R}}^n$, $[\![z]\!] = 0$ *on* $\mathbb{R}^n_0$*, and the following estimates hold:*

$$\begin{aligned}\|z\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C_{n,q}\|f_d\|_{\hat{W}^{-1}_q(\mathbb{R}^n)},\\ \|\nabla^{j+1}z\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C_{n,q}\|\nabla^j f_d\|_{L_q(\dot{\mathbb{R}}^n)}\ (j = 0, 1).\end{aligned}$$

(2) *For any* $F_d \in W^1_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^{-1}_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\dot{\mathbb{R}}^n))$*, there exists a*

$$Z \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n))$$

such that $\operatorname{div} Z = F_d$ *in* $\dot{\mathbb{R}}^n \times \mathbb{R}$, $[\![Z(t)]\!] = 0$ *on* $\mathbb{R}^n_0 \times \mathbb{R}$*, and the following estimates hold:*

$$\begin{aligned}\|e^{-\gamma t}(\partial_t Z, \gamma Z)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} &\leq C_{n,p,q}\|e^{-\gamma t}(\partial_t F_d, \gamma F_d)\|_{L_p(\mathbb{R},\hat{W}^{-1}_q(\mathbb{R}^n))},\\ \|e^{-\gamma t}\Lambda_\gamma^{1/2}\nabla Z\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} &\leq C_{n,p,q}\|e^{-\gamma t}\Lambda_\gamma^{1/2}F_d\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))},\\ \|e^{-\gamma t}\nabla^2 Z\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} &\leq C_{n,p,q}\|e^{-\gamma t}\nabla F_d\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\end{aligned}$$

for any $\gamma \geq \gamma_0$.

Setting $u = v + z$, $\tilde{f} = f - (\rho\lambda z - \Delta z)$ and $U = V + Z$, $\tilde{F} = F - (\rho\partial_t Z - \Delta Z)$, we would like to find $(v, \theta), (V, \Theta)$ such that

$$\begin{cases}\rho\lambda v - \operatorname{Div} S(v, \theta) = \tilde{f} & \text{in } \dot{\mathbb{R}}^n,\\ \operatorname{div} v = 0 & \text{in } \dot{\mathbb{R}}^n,\\ [\![S(v, \theta)\nu]\!] = [\![g - \mu D(z)\nu]\!] & \text{on } \mathbb{R}^n_0,\\ [\![v]\!] = [\![h]\!] & \text{on } \mathbb{R}^n_0.\end{cases} \tag{3.1}$$

and

$$\begin{cases} \rho\partial_t V - \operatorname{Div} S(V,\Theta) = \tilde{F} & \text{in } \dot{\mathbb{R}}^n, t > 0, \\ \operatorname{div} V = 0 & \text{in } \dot{\mathbb{R}}^n, t > 0, \\ [\![S(V,\Theta)\nu]\!] = [\![G - \mu D(Z)\nu]\!] & \text{on } \mathbb{R}^n_0, t > 0, \\ [\![V]\!] = [\![H]\!] & \text{on } \mathbb{R}^n_0, t > 0, \\ V|_{t=0} = 0 & \text{in } \mathbb{R}^{n-1}. \end{cases} \tag{3.2}$$

Let $\tilde{g} := g - \mu D(z)\nu$ and $\tilde{G} := G - \mu D(Z)\nu$. We see that

$$\begin{aligned} \|\tilde{f}\|_{L_q(\dot{\mathbb{R}}^n)} &\leq \|f\|_{L_q(\dot{\mathbb{R}}^n)} + C_{n,q}(|\lambda|\|f_d\|_{\hat{W}^{-1}_q(\mathbb{R}^n)} + \|\nabla f_d\|_{L_q(\dot{\mathbb{R}}^n)}), \\ \|e^{-\gamma t}\tilde{F}\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} &\leq \|e^{-\gamma t}F\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + C_{n,p,q}(\|e^{-\gamma t}\partial_t F_d\|_{L_p(\mathbb{R},\hat{W}^{-1}_q(\mathbb{R}^n))} + \|e^{-\gamma t}\nabla F_d\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}), \\ \|(\lambda^{1/2}\tilde{g}, \nabla\tilde{g})\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C\|(\lambda^{1/2}f_d, \nabla f_d, \lambda^{1/2}g, \nabla g)\|_{L_q(\dot{\mathbb{R}}^n)}, \\ \|e^{-\gamma t}(\Lambda^{1/2}_\gamma\tilde{G}, \nabla\tilde{G})\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} &\leq C\|e^{-\gamma t}(\Lambda^{1/2}_\gamma F_d, \nabla F_d, \Lambda^{1/2}_\gamma G, \nabla G)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}. \end{aligned}$$

Therefore, we can reduce the problem as follows: $f_d = 0, F_d = 0$.

Second, we would like to reduce the case of $f = 0$, $F = 0$. Let $P(\xi) = (P_{j,k})_{jk} = (\delta_{jk} - \xi_j\xi_k|\xi|^{-2})_{jk}$ be the Helmholtz decomposition. Then, the functions

$$\begin{aligned} \psi_\pm(x) &= \mathcal{F}^{-1}_\xi\left[\frac{P(\xi)\mathcal{F}_x f(\xi)}{\rho_\pm\lambda + \mu_\pm|\xi|^2}\right](x), \quad \phi_\pm(x) = -\mathcal{F}^{-1}_\xi\left[\frac{i\xi\cdot\mathcal{F}_x f(\xi)}{|\xi|^2}\right](x), \\ \Psi_\pm(x,t) &= \mathcal{L}_\lambda\mathcal{F}^{-1}_\xi\left[\frac{P(\xi)\mathcal{F}_x\mathcal{L}F(\xi,\lambda)}{\rho_\pm\lambda + \mu_\pm|\xi|^2}\right](x,t), \quad \Phi_\pm(x,t) = -\mathcal{L}_\lambda\mathcal{F}^{-1}_\xi\left[\frac{i\xi\cdot\mathcal{F}_x\mathcal{L}F(\xi,\lambda)}{|\xi|^2}\right](x,t) \end{aligned}$$

satisfy

$$\begin{aligned} &(\psi_\pm, \phi_\pm) \in W^2_q(\mathbb{R}^n) \times \hat{W}^1_q(\mathbb{R}^n), \\ &\rho_\pm\lambda\psi_\pm - \mu_\pm\Delta\psi_\pm + \nabla\phi_\pm = f, \quad \operatorname{div}\psi_\pm = 0 \quad \text{in } \mathbb{R}^n, \\ &\|(\lambda\psi_\pm, \lambda^{1/2}\nabla\psi_\pm, \nabla^2\psi_\pm, \nabla\phi_\pm)\|_{L_q(\mathbb{R}^n_\pm)} \leq C_{n,q,\varepsilon}\|f\|_{L_q(\dot{\mathbb{R}}^n)} \end{aligned}$$

and

$$\begin{aligned} &\Psi_\pm \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\mathbb{R}^n)), \quad \Phi_\pm \in L_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^1_q(\mathbb{R}^n)), \\ &\rho_\pm\partial_t\Psi_\pm - \mu_\pm\Delta\Psi_\pm + \nabla\Phi_\pm = F, \quad \operatorname{div}\Psi_\pm = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty), \quad \Psi_\pm|_{t=0} = 0, \\ &\|e^{-\gamma t}(\partial_t\Psi_\pm, \gamma\Psi_\pm, \Lambda^{1/2}_\gamma\nabla\Psi, \nabla^2\Psi_\pm, \nabla\Phi_\pm)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))} \leq C_{n,p,q,\gamma_0}\|e^{-\gamma t}F\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}, \end{aligned}$$

for any $1 < p, q < \infty$, $\gamma \geq \gamma_0 \geq 0$, $f \in L_q(\dot{\mathbb{R}}^n)$, $F \in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))$, and $\lambda \in \Sigma_\varepsilon$ with $0 < \varepsilon < \pi/2$, according to (3.19) in [34]. We define

$$(\psi, \phi, \Psi, \Phi) := \begin{cases} (\psi_+, \phi_+, \Psi_+, \Phi_+) \text{ for } x \in \mathbb{R}^n_+, \\ (\psi_-, \phi_-, \Psi_-, \Phi_-) \text{ for } x \in \mathbb{R}^n_-. \end{cases}$$

Then, we have that $[\![\phi]\!] = 0$ on $\mathbb{R}^n_0$ and $[\![\Phi(t)]\!] = 0$ on Q_0.

Setting $u := \psi + w$, $\theta := \phi + \kappa$ in (1.5) with $f_d = 0$ and $U := \Psi + W$, $\Theta = \Phi + \Xi$ in (1.4) with $F_d = 0$, respectively, we have

$$\begin{cases} \rho\lambda w - \mu\Delta w + \nabla\kappa = 0 & \text{in } \dot{\mathbb{R}}^n, \\ \operatorname{div} w = 0 & \text{in } \dot{\mathbb{R}}^n, \\ [\![S(w,\kappa)\nu]\!] = [\![g - \mu D(\psi)\nu]\!] & \text{on } \mathbb{R}^n_0, \\ [\![w]\!] = [\![h - \psi]\!] & \text{on } \mathbb{R}^n_0. \end{cases} \tag{3.3}$$

and

$$\begin{cases} \rho\partial_t W - \mu\Delta W + \nabla\Xi = 0 & \text{in } \dot{\mathbb{R}}^n, t > 0, \\ \operatorname{div} W = 0 & \text{in } \dot{\mathbb{R}}^n, t > 0, \\ [\![S(W,\Xi)\nu]\!] = [\![G - \mu D(\Psi)\nu]\!] & \text{on } \mathbb{R}^n_0, t > 0, \\ [\![W]\!] = [\![H - \Psi]\!] & \text{on } \mathbb{R}^n_0, t > 0, \\ W|_{t=0} = 0 & \text{in } \mathbb{R}^{n-1}. \end{cases} \tag{3.4}$$

Let

$$\tilde{g} := g - \mu D(\psi)\nu, \quad \tilde{h} := h - \psi,$$
$$\tilde{G} := G - \mu D(\Psi)\nu, \quad \tilde{H} := H - \Psi.$$

Since we have the estimates

$$\begin{aligned} &\|(\lambda^{1/2}\tilde{g}, \nabla\tilde{g}, \lambda\tilde{h}, \nabla^2\tilde{h}, \lambda|\nabla'|^{-1}\partial_n\tilde{h}_n)\|_{L_q(\dot{\mathbb{R}}^n)} \\ \le& C\|(f, \lambda^{1/2}g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)}, \\ &\|e^{-\gamma t}(\Lambda_\gamma^{1/2}\tilde{G}, \nabla\tilde{G}, \partial_t\tilde{H}, \nabla^2\tilde{H}, \partial_t(|\nabla'|^{-1}\partial_n\tilde{H}_n))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\ \le& C\|e^{-\gamma t}(F, \Lambda_\gamma^{1/2}G, \nabla G, \partial_t H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))}, \end{aligned}$$

we conclude that $f = f_d = 0$ and $F = F_d = 0$ are sufficient for Theorems 2.1 and 2.2, where we prove that

$$\|\lambda|\nabla'|^{-1}\partial_n\psi_n\|_{L_q(\dot{\mathbb{R}}^n)} \le C\|f\|_{L_q(\dot{\mathbb{R}}^n)}$$
$$\|e^{-\gamma t}\partial_t(|\nabla'|^{-1}\partial_n\Psi_n)\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \le C\|e^{-\gamma t}F\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))}$$

in the Appendix.

4. Solution formula for the problems without f, f_d and surface tension

We shall give a solution of the resolvent problem (1.5) with $f = f_d = 0$ and $\lambda \in \Sigma_\varepsilon$. We apply a partial Fourier transform with respect to the tangential direction $x' \in \mathbb{R}^{n-1}$. We apply notation

$$\hat{v}(\xi', x_n) := \mathcal{F}_{x'} v(\xi', x_n) := \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} v(x', x_n)\,dx',$$
$$\mathcal{F}^{-1}_{\xi'} w(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} w(\xi', x_n)\,d\xi'$$

for the functions $v, w : \mathbb{R}^n_\pm \to \mathbb{C}$. Let $u_\pm = {}^t(u_{\pm 1}, \dots, u_{\pm(n-1)}, u_{\pm n})$. Here and hereafter, the index j runs from 1 to $n-1$ unless stated otherwise.

We need to solve the following second-order ordinary differential equations:

$$\begin{cases} (\rho_\pm\lambda + \mu_\pm|\xi'|^2 - \mu_\pm\partial_n^2)\hat{u}_{\pm j} + i\xi_j\hat{\theta}_\pm = 0 & \text{in } x_n \neq 0, \\ (\rho_\pm\lambda + \mu_\pm|\xi'|^2 - \mu_\pm\partial_n^2)\hat{u}_{\pm n} + \partial_n\hat{\theta}_\pm = 0 & \text{in } x_n \neq 0, \\ \sum_{j=1}^{n-1} i\xi_j\hat{u}_{\pm j} + \partial_n\hat{u}_{\pm n} = 0 & \text{in } x_n \neq 0, \\ [\![\mu(i\xi_j\hat{u}_n + \partial_n\hat{u}_j)]\!] = -[\![\hat{g}_j]\!] & \text{on } x_n = 0, \\ [\![2\mu\partial_n\hat{u}_n - \hat{\theta}]\!] = -[\![\hat{g}_n]\!] & \text{on } x_n = 0, \\ [\![\hat{u}]\!] = [\![\hat{h}]\!] & \text{on } x_n = 0. \end{cases} \tag{4.1}$$

Set

$$A := \sqrt{\sum_{j=1}^{n-1}\xi_j^2}, \qquad B_\pm := \sqrt{\rho_\pm(\mu_\pm)^{-1}\lambda + A^2}$$

with positive real parts. Here, we consider ξ' to have complex values, as follows:

$$\xi_j \in \tilde{\Sigma}_\eta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \eta\} \cup \{z \in \mathbb{C} \setminus \{0\} \mid \pi - \eta < |\arg z|\}$$

for $\eta \in (0, \pi/4)$. The details are given in Lemma 5.2.

We find the solution of the form

$$\hat{u}_{\pm j}(\xi', x_n) = \alpha_{\pm j}(e^{\mp B_\pm x_n} - e^{\mp A x_n}) + \beta_{\pm j}e^{\mp B_\pm x_n} \ (j = 1, \dots, n), \qquad \hat{\theta}_\pm(\xi', x_n) = \gamma_\pm e^{\mp A x_n}.$$

Then, the equations become

$$\begin{cases} -\mu_\pm(B_\pm^2 - A^2)\alpha_{\pm j} + i\xi_j\gamma_\pm = 0, \\ -\mu_\pm(B_\pm^2 - A^2)\alpha_{\pm n} \mp A\gamma_\pm = 0, \\ -i\xi' \cdot \alpha'_\pm \pm A\alpha_{\pm n} = 0, \\ i\xi' \cdot (\alpha'_\pm + \beta'_\pm) \mp B_\pm(\alpha_{\pm n} + \beta_{\pm n}) = 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \mu_+(B_+^2 - A^2)\alpha_{+n} + \mu_+(B_+^2 + A^2)\beta_{+n} - \mu_-(B_-^2 - A^2)\alpha_{-n} - \mu_-(B_-^2 + A^2)\beta_{-n} = i\xi' \cdot [\![\hat{g}']\!], \\ \mu_+(B_+ - A)^2\alpha_{+n} - 2\mu_+AB_+\beta_{+n} + \mu_-(B_- - A)^2\alpha_{-n} - 2\mu_-AB_-\beta_{-n} = -A[\![\hat{g}'_n]\!], \\ (B_+ - A)\alpha_{+n} + B_+\beta_{+n} + (B_- - A)\alpha_{-n} + B_-\beta_{-n} = i\xi' \cdot [\![\hat{h}']\!], \\ \beta_{+n} - \beta_{-n} = [\![\hat{h}'_n]\!]. \end{cases}$$

This means that

$$\begin{bmatrix} \mu_+(B_+ + A) & \mu_+(B_+^2 + A^2) & -\mu_-(B_- + A) & -\mu_-(B_-^2 + A^2) \\ \mu_+(B_+ - A) & -2\mu_+AB_+ & \mu_-(B_- - A) & -2\mu_-AB_- \\ 1 & B_+ & 1 & B_- \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} (B_+ - A)\alpha_{+n} \\ \beta_{+n} \\ (B_- - A)\alpha_{-n} \\ \beta_{-n} \end{bmatrix}$$

$$= \begin{bmatrix} i\xi' \cdot [\![\hat{g}']\!] \\ -A[\![\hat{g}_n]\!] \\ i\xi' \cdot [\![\hat{h}']\!] \\ [\![\hat{h}_n']\!] \end{bmatrix} = \begin{bmatrix} i\xi'^T & 0 & 0 & 0 \\ 0 & -A & 0 & 0 \\ 0 & 0 & i\xi'^T & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} [\![\hat{g}_1]\!] \\ \vdots \\ [\![\hat{g}_n]\!] \\ [\![\hat{h}_1]\!] \\ \vdots \\ [\![\hat{h}_n]\!] \end{bmatrix},$$

where ξ'^T is the transpose of ξ'. We define

$$L := \begin{bmatrix} \mu_+(B_+ + A) & \mu_+(B_+^2 + A^2) & -\mu_-(B_- + A) & -\mu_-(B_-^2 + A^2) \\ \mu_+(B_+ - A) & -2\mu_+ AB_+ & \mu_-(B_- - A) & -2\mu_- AB_- \\ 1 & B_+ & 1 & B_- \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

$$R := (r_{ij}) := \begin{bmatrix} i\xi'^T & 0 & 0 & 0 \\ 0 & -A & 0 & 0 \\ 0 & 0 & i\xi'^T & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1 \le i \le 4, 1 \le j \le 2n).$$

We have

$$\begin{aligned} \det L = (\mu_+ - \mu_-)^2 A^3 - \{(3\mu_+ - \mu_-)\mu_+ B_+ + (3\mu_- - \mu_+)\mu_- B_-\} A^2 \\ - \{(\mu_+ B_+ + \mu_- B_-)^2 + \mu_+\mu_-(B_+ + B_-)^2\} A - (\mu_+ B_+ + \mu_- B_-)(\mu_+ B_+^2 + \mu_- B_-^2) \end{aligned}$$

and it is known, from [33, Lemma 5.5], that the determinant is not zero for $\lambda \in \Sigma_\varepsilon$, $\xi' \in \mathbb{R}^{n-1}$.

We introduce some new notation:

$$\begin{aligned} \mathcal{M}_+ &= \mathcal{M}_+(A, B_+, x_n) = \frac{e^{-B_+ x_n} - e^{-A x_n}}{B_+ - A}, \\ \mathcal{M}_- &= \mathcal{M}_-(A, B_-, x_n) = \frac{e^{B_- x_n} - e^{A x_n}}{B_- - A}, \\ (a_{i,j}) &:= (L^{-1} R)_{ij} = (\det L)^{-1} \Big(\sum_{s=1}^{4} L_{is} r_{sj}\Big)_{ij} \quad (1 \le i \le 4, 1 \le j \le 2n), \end{aligned}$$

where we use the cofactor matrix of L, denoted by $\mathrm{Cof}(L) = (L_{ij})$.

From these observations, we have

$$\begin{aligned} \hat{u}_{+n}(\xi', x_n) &= \sum_{k=1}^{n} \Big\{ (a_{1,k}\mathcal{M}_+ + a_{2,k} e^{-B_+ x_n})[\![\hat{g}_k]\!] + (a_{1,n+k}\mathcal{M}_+ + a_{2,n+k} e^{-B_+ x_n})[\![\hat{h}_k]\!] \Big\}, \\ \hat{u}_{-n}(\xi', x_n) &= \sum_{k=1}^{n} \Big\{ (a_{3,k}\mathcal{M}_- + a_{4,k} e^{B_- x_n})[\![\hat{g}_k]\!] + (a_{3,n+k}\mathcal{M}_- + a_{4,n+k} e^{B_- x_n})[\![\hat{h}_k]\!] \Big\}. \end{aligned}$$

To simplify, we define the following symbols for $k = 1, \ldots, n$:

$$\phi_{k,+n}(\lambda, \xi', x_n) = a_{1,k}\mathcal{M}_+ + a_{2,k} e^{-B_+ x_n},$$

$$\psi_{k,+n}(\lambda,\xi',x_n) = a_{1,n+k}\mathcal{M}_+ + a_{2,n+k}e^{-B_+x_n},$$
$$\phi_{k,-n}(\lambda,\xi',x_n) = a_{3,k}\mathcal{M}_- + a_{4,k}e^{B_-x_n},$$
$$\psi_{k,-n}(\lambda,\xi',x_n) = a_{3,n+k}\mathcal{M}_- + a_{4,n+k}e^{B_-x_n},$$

which yields the solution formulas for $u_{\pm n}$:

$$\hat{u}_{+n} = \hat{u}_{+n}(\xi',x_n) = \sum_{k=1}^{n}(\phi_{k,+n}[\![\hat{g}_k]\!] + \psi_{k,+n}[\![\hat{h}_k]\!]), \quad x_n > 0,$$
$$\hat{u}_{-n} = \hat{u}_{-n}(\xi',x_n) = \sum_{k=1}^{n}(\phi_{k,-n}[\![\hat{g}_k]\!] + \psi_{k,-n}[\![\hat{h}_k]\!]), \quad x_n < 0.$$

Since

$$\gamma_\pm = \mp\frac{\mu_\pm(B_\pm + A)}{A}(B_\pm - A)\alpha_{\pm n}$$

from the second equation of (4.2), by letting

$$\chi_{k,+}(\lambda,\xi',x_n) = -\frac{\mu_+(B_+ + A)}{A}a_{1,k}e^{-Ax_n},$$
$$\omega_{k,+}(\lambda,\xi',x_n) = -\frac{\mu_+(B_+ + A)}{A}a_{1,n+k}e^{-Ax_n},$$
$$\chi_{k,-}(\lambda,\xi',x_n) = \frac{\mu_-(B_- + A)}{A}a_{3,k}e^{Ax_n},$$
$$\omega_{k,-}(\lambda,\xi',x_n) = \frac{\mu_-(B_- + A)}{A}a_{3,n+k}e^{Ax_n},$$

we have

$$\hat{\theta}_\pm = \hat{\theta}_\pm(\xi',x_n) = \sum_{k=1}^{n}(\chi_{k,\pm}[\![\hat{g}_k]\!] + \omega_{k,\pm}[\![\hat{h}_k]\!]), \quad x_n \gtrless 0.$$

From the first equation of (4.2), $\alpha_{\pm j} = \mp(i\xi_j/A)\alpha_{\pm n}$. From the fourth and the sixth equations of (4.1), $\beta_{\pm j}$ satisfies

$$\begin{bmatrix}\mu_+B_+ & \mu_-B_-\\ 1 & -1\end{bmatrix}\begin{bmatrix}\beta_{+j}\\ \beta_{-j}\end{bmatrix} = \begin{bmatrix}[\![\hat{g}_j]\!]\\ [\![\hat{h}_j]\!]\end{bmatrix} + \begin{bmatrix}-\mu_+(B_+ - A)\alpha_{+j} + \mu_+i\xi_j\beta_{+n} - \mu_-(B_- - A)\alpha_{-j} - \mu_-i\xi_j\beta_{-n}\\ 0\end{bmatrix}$$
$$= \begin{bmatrix}[\![\hat{g}_j]\!]\\ [\![\hat{h}_j]\!]\end{bmatrix} + \frac{i\xi_j}{A}\begin{bmatrix}\mu_+ & \mu_+A & -\mu_- & -\mu_-A\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}(B_+ - A)\alpha_{+n}\\ \beta_{+n}\\ (B_- - A)\alpha_{-n}\\ \beta_{-n}\end{bmatrix},$$

$$\begin{bmatrix}\beta_{+j}\\ \beta_{-j}\end{bmatrix} = \frac{1}{\mu_+B_+ + \mu_-B_-}\left\{\begin{bmatrix}[\![\hat{g}_j]\!] + \mu_-B_-[\![\hat{h}_j]\!]\\ [\![\hat{g}_j]\!] - \mu_+B_+[\![\hat{h}_j]\!]\end{bmatrix} + \frac{i\xi_j}{A}\begin{bmatrix}\mu_+ & \mu_+A & -\mu_- & -\mu_-A\\ \mu_+ & \mu_+A & -\mu_- & -\mu_-A\end{bmatrix}\begin{bmatrix}(B_+ - A)\alpha_{+n}\\ \beta_{+n}\\ (B_- - A)\alpha_{-n}\\ \beta_{-n}\end{bmatrix}\right\}$$

$$= \frac{1}{\mu_+ B_+ + \mu_- B_-} \begin{bmatrix} [\![\hat g_j]\!] + \mu_- B_- [\![\hat h_j]\!] + \frac{i\xi_j}{A}\sum_{k=1}^n \{\mu_+ (a_{1,k} + Aa_{2,k})[\![\hat g_k]\!] - \mu_-(a_{3,n+k} + Aa_{4,n+k})[\![\hat h_k]\!]\} \\ [\![\hat g_j]\!] - \mu_+ B_+ [\![\hat h_j]\!] + \frac{i\xi_j}{A}\sum_{k=1}^n \{\mu_+ (a_{1,k} + Aa_{2,k})[\![\hat g_k]\!] - \mu_-(a_{3,n+k} + Aa_{4,n+k})[\![\hat h_k]\!]\} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \hat u_{\pm j} = \hat u_{\pm j}(\xi', x_n) &= (B_\pm - A)\alpha_{\pm j}\mathcal{M}_\pm + \beta_{\pm j} e^{\mp B_\pm x_n} \\ &=: \sum_{k=1}^n (\phi_{k,\pm j}[\![\hat g_k]\!] + \psi_{k,\pm j}[\![\hat h_k]\!]), \quad x_n \gtrless 0, \end{aligned}$$

where

$$\begin{aligned} \phi_{k,+j}(\lambda,\xi',x_n) &= -\frac{i\xi_j}{A} a_{1,k}\mathcal{M}_+ + \frac{1}{\mu_+B_+ + \mu_-B_-}(\delta_{k,j} + \frac{i\mu_+\xi_j}{A}(a_{1,k} + Aa_{2,k}))e^{-B_+x_n}, \\ \psi_{k,+j}(\lambda,\xi',x_n) &= -\frac{i\xi_j}{A} a_{1,n+k}\mathcal{M}_+ + \frac{1}{\mu_+B_+ + \mu_-B_-}(\mu_-B_-\delta_{k,j} - \frac{i\mu_-\xi_j}{A}(a_{3,n+k} + Aa_{4,n+k}))e^{-B_+x_n}, \\ \phi_{k,-j}(\lambda,\xi',x_n) &= \frac{i\xi_j}{A} a_{3,k}\mathcal{M}_- + \frac{1}{\mu_+B_+ + \mu_-B_-}(\delta_{k,j} + \frac{i\mu_+\xi_j}{A}(a_{1,k} + Aa_{2,k}))e^{B_-x_n}, \\ \psi_{k,-j}(\lambda,\xi',x_n) &= \frac{i\xi_j}{A} a_{3,n+k}\mathcal{M}_- - \frac{1}{\mu_+B_+ + \mu_-B_-}(\mu_+B_+\delta_{k,j} + \frac{i\mu_-\xi_j}{A}(a_{3,n+k} + Aa_{4,n+k}))e^{B_-x_n}. \end{aligned}$$

Here, we introduce a new notation, i.e., $[\![\![f]\!]\!](x', x_n) := f(x', x_n) - f(x', -x_n)$, for $f : \dot{\mathbb{R}}^n \to \mathbb{C}$. Then, we have that $[\![f]\!](x') = \lim_{x_n \to +0}[\![\![f]\!]\!](x', x_n)$ and $[\![a]\!](\xi') = \mp \int_{\mathbb{R}_\pm} [\![\![\partial_n a]\!]\!](\xi', y_n)dy_n$ for a function a with $a(\cdot, x_n) \to 0$ as $x_n \to \pm\infty$. So, we obtain the solution formulas of integral form;

$$\begin{aligned} u_{\pm j}(x) = \mp \sum_{k=1}^n \Big\{ &\int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\partial_n \phi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![g_k]\!]\!]\right](x,y_n)dy_n \\ &+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\phi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![\partial_n g_k]\!]\!]\right](x,y_n)dy_n \\ &+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\partial_n \psi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![h_k]\!]\!]\right](x,y_n)dy_n \\ &+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\psi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![\partial_n h_k]\!]\!]\right](x,y_n)dy_n \Big\}, \quad (j = 1,\ldots,n), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \theta_\pm(x) = \mp \sum_{k=1}^n \Big\{ &\int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\partial_n \chi_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![g_k]\!]\!]\right](x,y_n)dy_n \\ &+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\chi_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![\partial_n g_k]\!]\!]\right](x,y_n)dy_n \\ &+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\partial_n \omega_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![h_k]\!]\!]\right](x,y_n)dy_n \\ &+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\omega_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\![\partial_n h_k]\!]\!]\right](x,y_n)dy_n \Big\}. \end{aligned} \tag{4.4}$$

Since the Laplace-transformed non-stationary Stokes equations of (1.4) with $F = F_d = 0$ on $\mathbb{R}$ are equivalent to the resolvent problem (1.5) with $f = f_d = 0$, we have the following formula:

$$U_{\pm j}(x,t) = \mp \mathcal{L}^{-1}_\lambda \sum_{k=1}^n \Big\{ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'}\left[\partial_n \phi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}\mathcal{L}[\![\![G_k]\!]\!]\right](x,y_n)dy_n$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\phi_{k,\pm j}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![\partial_n G_k]\!]\right](x, y_n)dy_n \\
&+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\partial_n\psi_{k,\pm j}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![H_k]\!]\right](x, y_n)dy_n \\
&+ \left.\int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\psi_{k,\pm j}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![\partial_n H_k]\!]\right](x, y_n)dy_n\right\}, \quad (j = 1, \ldots, n), \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
\Theta_\pm(x, t) = \mp\mathcal{L}^{-1}_\lambda \sum_{k=1}^{n} \bigg\{ &\int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\partial_n\chi_{k,\pm}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![G_k]\!]\right](x, y_n)dy_n \\
&+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\chi_{k,\pm}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![\partial_n G_k]\!]\right](x, y_n)dy_n \\
&+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\partial_n\omega_{k,\pm}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![H_k]\!]\right](x, y_n)dy_n \\
&+ \int_{\mathbb{R}_\pm} \mathcal{F}^{-1}_{\xi'} \left[\omega_{k,\pm}(\lambda, \xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}[\![\partial_n H_k]\!]\right](x, y_n)dy_n \bigg\}. \qquad (4.6)
\end{aligned}
$$

5. Proofs of estimates for the problem without surface tension and gravity

We decompose the solutions (4.3) and (4.4) to obtain the independent variables on the right-hand side of the resolvent estimates. Analysis of the solutions (4.5) and (4.6) are based on the same analysis for the resolvent problems.

We shall provide a theorem to prove the main theorems. Let us respectively define the operators T and $\tilde{T}_\gamma$ by

$$
\begin{aligned}
T[m]f(x) &= \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m(\xi', x_n + y_n)\mathcal{F}_{x'} f](x, y_n)dy_n, \\
\tilde{T}_\gamma[m_\lambda]g(x, t) &= \mathcal{L}^{-1}_\lambda \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m_\lambda(\xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}g](x, y_n, \lambda)dy_n, \\
&= [e^{\gamma t}\mathcal{F}^{-1}_{\tau\to t} T[m_\lambda]\mathcal{F}_{t\to\tau}(e^{-\gamma t} g)](x, t),
\end{aligned}
$$

where $\lambda = \gamma + i\tau \in \Sigma_\varepsilon$, m, m_λ are $\mathbb{C}$-valued functions, $f : \mathbb{R}^n_+ \to \mathbb{C}$, and $g : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{C}$. The following theorem was taken from [17]. See also [18], where $\mathcal{R}$-boundedness and the difference from previous works are written.

Theorem 5.1. ([17, Theorem 6.1]) (i) *Let m satisfy the following two conditions:*
(a) *There exists $\eta \in (0, \pi/2)$ such that $\{m(\cdot, x_n), x_n > 0\} \subset H^\infty(\tilde{\Sigma}^{n-1}_\eta)$.*
(b) *There exist $\eta \in (0, \pi/2)$ and $C > 0$ such that* $\sup_{\xi' \in \tilde{\Sigma}^{n-1}_\eta} |m(\xi', x_n)| \le C x_n^{-1}$ *for all $x_n > 0$.*
Then, $T[m]$ is a bounded linear operator on $L_q(\mathbb{R}^n_+)$ for every $1 < q < \infty$.
(ii) *Let $\gamma_0 \ge 0$ and m_λ satisfy the following two conditions:*
(c) *There exists $\eta \in (0, \pi/2 - \varepsilon)$ such that, for each $x_n > 0$ and $\gamma \ge \gamma_0$,*

$$
\tilde{\Sigma}^n_\eta \ni (\tau, \xi') \mapsto m_\lambda(\xi', x_n) \in \mathbb{C}
$$

is bounded and holomorphic.
(d) *There exist $\eta \in (0, \pi/2 - \varepsilon)$ and $C > 0$ such that* $\sup\{|m_\lambda(\xi', x_n)| \mid (\tau, \xi') \in \tilde{\Sigma}^n_\eta\} \le C x_n^{-1}$ *for all $\gamma \ge \gamma_0$*

and $x_n > 0$.
Then, $\tilde{T}_\gamma[m_\lambda]$ *satisfies*

$$\|e^{-\gamma t}\tilde{T}_\gamma[m_\lambda]g\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))} \le C\|e^{-\gamma t}g\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))}$$

for every $\gamma \ge \gamma_0$ *and* $1 < p, q < \infty$.

By using the identities given by

$$B_\pm^2 = \rho_\pm(\mu_\pm)^{-1}\lambda + \sum_{m=1}^{n-1}\xi_m^2, \qquad 1 = \frac{B_\pm^2}{B_\pm^2} = \frac{\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}}{B_\pm^2}\lambda^{1/2} - \sum_{m=1}^{n-1}\frac{i\xi_m}{B_\pm^2}(i\xi_m),$$

we have

$$\begin{aligned}
u_{\pm j}(x) = \mp\sum_{k=1}^{n}\Bigg\{&\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\partial_n\phi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\lambda^{1/2}g_k]\!]\Big](x,y_n)dy_n\\
&-\sum_{m=1}^{n-1}\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[i\xi_m B_\pm^{-2}\partial_n\phi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\partial_m g_k]\!]\Big](x,y_n)dy_n\\
&+\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[\phi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\partial_n g_k]\!]\Big](x,y_n)dy_n\\
&+\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[B_\pm^{-2}\partial_n\psi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![(\rho_\pm(\mu_\pm)^{-1}\lambda-\Delta')h_k]\!]\Big](x,y_n)dy_n\\
&-\sum_{m=1}^{n-1}\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[i\xi_m B_\pm^{-2}\psi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\partial_m\partial_n h_k]\!]\Big](x,y_n)dy_n\Bigg\}\\
&\mp\sum_{k=1}^{n-1}\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\psi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\lambda^{1/2}\partial_n h_k]\!]\Big](x,y_n)dy_n\\
&+\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[\rho_\pm(\mu_\pm)^{-1}AB_\pm^{-2}\psi_{k,\pm j}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\lambda|\nabla'|^{-1}\partial_n h_n]\!]\Big](x,y_n)dy_n
\end{aligned}$$

$$\begin{aligned}
\theta_\pm(x) = \mp\sum_{k=1}^{n}\Bigg\{&\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\partial_n\chi_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\lambda^{1/2}g_k]\!]\Big](x,y_n)dy_n\\
&-\sum_{m=1}^{n-1}\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[i\xi_m B_\pm^{-2}\partial_n\chi_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\partial_m g_k]\!]\Big](x,y_n)dy_n\\
&+\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\left[\chi_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\partial_n g_k]\!]\right](x,y_n)dy_n\\
&+\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[B_\pm^{-2}\partial_n\omega_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![(\rho_\pm(\mu_\pm)^{-1}\lambda-\Delta')h_k]\!]\Big](x,y_n)dy_n\\
&-\sum_{m=1}^{n-1}\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[i\xi_m B_\pm^{-2}\omega_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\partial_m\partial_n h_k]\!]\Big](x,y_n)dy_n\Bigg\}\\
&\mp\sum_{k=1}^{n-1}\int_{\mathbb{R}_\pm}\mathcal{F}_{\xi'}^{-1}\Big[\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\omega_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\lambda^{1/2}\partial_n h_k]\!]\Big](x,y_n)dy_n
\end{aligned}$$

$$+\int_{\mathbb{R}_\pm}\mathcal{F}^{-1}_{\xi'}\left[\rho_\pm(\mu_\pm)^{-1}AB_\pm^{-2}\omega_{k,\pm}(\lambda,\xi',x_n+y_n)\mathcal{F}_{x'}[\![\lambda|\nabla'|^{-1}\partial_n h_n]\!]\right](x,y_n)dy_n$$

for $j=1,\dots,n$.

Let $S^{u_j}_\pm(\lambda,\xi',x_n)$ and $S^\theta_\pm(\lambda,\xi',x_n)$ be considered as follows;

$$S^{u_j}_\pm(\lambda,\xi',x_n):=\begin{cases}\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\partial_n\phi_{k,\pm j}(\lambda,\xi',x_n) & k\in\{1,\dots,n\},\\ i\xi_m B_\pm^{-2}\partial_n\phi_{k,\pm j}(\lambda,\xi',x_n) & k\in\{1,\dots,n\}, m\in\{1,\dots,n-1\},\\ \phi_{k,\pm j}(\lambda,\xi',x_n) & k\in\{1,\dots,n\},\\ B_\pm^{-2}\partial_n\psi_{k,\pm j}(\lambda,\xi',x_n) & k\in\{1,\dots,n\},\\ i\xi_m B_\pm^{-2}\psi_{k,\pm j}(\lambda,\xi',x_n) & k\in\{1,\dots,n\}, m\in\{1,\dots,n-1\},\\ \rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\psi_{k,\pm j}(\lambda,\xi',x_n) & k\in\{1,\dots,n-1\},\\ \rho_\pm(\mu_\pm)^{-1}AB_\pm^{-2}\psi_{n,\pm j}(\lambda,\xi',x_n), & \end{cases}$$

$$S^\theta_\pm(\lambda,\xi',x_n):=\begin{cases}\rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\partial_n\chi_{k,\pm}(\lambda,\xi',x_n) & k\in\{1,\dots,n\},\\ i\xi_m B_\pm^{-2}\partial_n\chi_{k,\pm}(\lambda,\xi',x_n) & k\in\{1,\dots,n\}, m\in\{1,\dots,n-1\},\\ \chi_{k,\pm}(\lambda,\xi',x_n) & k\in\{1,\dots,n\},\\ B_\pm^{-2}\partial_n\omega_{k,\pm}(\lambda,\xi',x_n) & k\in\{1,\dots,n\},\\ i\xi_m B_\pm^{-2}\omega_{k,\pm}(\lambda,\xi',x_n) & k\in\{1,\dots,n\}, m\in\{1,\dots,n-1\},\\ \rho_\pm(\mu_\pm)^{-1}\lambda^{1/2}B_\pm^{-2}\omega_{k,\pm}(\lambda,\xi',x_n) & k\in\{1,\dots,n-1\},\\ \rho_\pm(\mu_\pm)^{-1}AB_\pm^{-2}\omega_{n,\pm}(\lambda,\xi',x_n). & \end{cases}$$

We shall confirm that all of the symbols are bounded in the sense that

$$\begin{aligned}&\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde{\Sigma}^{n-1}_\eta\\ \ell,\ell'=1,\dots,n-1}}\left\{(|\lambda|+|\lambda|^{1/2}|\xi_\ell|+|\xi_\ell||\xi_{\ell'}|)|S^{u_j}_\pm|+(|\lambda|^{1/2}+|\xi_\ell|)|\partial_n S^{u_j}_\pm|+|\partial_n^2 S^{u_j}_\pm|+|\xi_\ell||S^\theta_\pm|+|\partial_n S^\theta_\pm|\right\}\\ &< C(\pm x_n)^{-1}\end{aligned}\tag{5.1}$$

for suitable ε,η; thus, we can prove Theorem 2.1 with $f=f_d=0$.

Following the method in [17], we have some identities:

$$\begin{aligned}\partial_n\mathcal{M}_\pm(A,B_\pm,x_n)&=\mp e^{\mp B_\pm x_n}\mp A\mathcal{M}_\pm(A,B_\pm,x_n),\\ \partial_n^2\mathcal{M}_\pm(A,B_\pm,x_n)&=(A+B_\pm)e^{\mp B_\pm x_n}+A^2\mathcal{M}_\pm(A,B_\pm,x_n),\\ \partial_n^3\mathcal{M}_\pm(A,B_\pm,x_n)&=\mp(A^2+AB_\pm+B_\pm^2)e^{\mp B_\pm x_n}\mp A^3\mathcal{M}_\pm(A,B_\pm,x_n)\end{aligned}$$

as well as the following useful lemma, where we let $\tilde{A}:=\sqrt{\sum_{j=1}^{n-1}|\xi_j|^2}$.

Lemma 5.2. ([17, Lemma 6.3]) *Let $0<\varepsilon<\pi/2$ and $0<\eta<\varepsilon/2$. Then, for any $(\lambda,\xi',x_n)\in\Sigma_\varepsilon\times\tilde{\Sigma}^{n-1}_\eta\times\mathbb{R}_\pm$, we have*

$$\begin{aligned}c\tilde{A}\le\operatorname{Re}A\le|A|\le\tilde{A},\\ c(|\lambda|^{1/2}+\tilde{A})\le\operatorname{Re}B_\pm\le|B_\pm|\le C(|\lambda|^{1/2}+\tilde{A}),\end{aligned}$$

$$\begin{aligned}
|\partial_n^m e^{\mp A x_n}| &\le C\tilde{A}^{-1+m}(\pm x_n)^{-1}, \text{ for } x_n \gtrless 0,\\
|\partial_n^m e^{\mp B_\pm x_n}| &\le C(|\lambda|^{1/2}+\tilde{A})^{-1+m}(\pm x_n)^{-1}, \text{ for } x_n \gtrless 0,\\
|\mathcal{M}_\pm(A,B_\pm,x_n)| &\le C(|\lambda|^{1/2}+\tilde{A})^{-1}\tilde{A}^{-1}(\pm x_n)^{-1}, \text{ for } x_n \gtrless 0,\\
|\partial_n^m \mathcal{M}_\pm(A,B_\pm,x_n)| &\le C(|\lambda|^{1/2}+\tilde{A})^{-2+m}(\pm x_n)^{-1}, \text{ for } x_n \gtrless 0, m\neq 0,\\
c(|\lambda|^{1/2}+\tilde{A}) &\le |\mu_+B_+ + \mu_-B_-|
\end{aligned}$$

for $m = 0,1,2,3$, *with positive constants* c *and* C, *which are independent of* λ, ξ', x_n.

We recall that

$$\begin{aligned}
L^{-1} &= \begin{bmatrix} \mu_+(B_+ + A) & \mu_+(B_+^2 + A^2) & -\mu_-(B_- + A) & -\mu_-(B_-^2+A^2)\\ \mu_+(B_+ - A) & -2\mu_+AB_+ & \mu_-(B_- - A) & -2\mu_-AB_-\\ 1 & B_+ & 1 & B_-\\ 0 & 1 & 0 & -1 \end{bmatrix}^{-1}\\
&= (\det L)^{-1}\mathrm{Cof}(L) = (\det L)^{-1}(L_{ij})_{ij}.
\end{aligned}$$

From the cofactor expansion, we have

$$|L_{is}| \le \begin{cases} C(|\lambda|^{1/2}+\tilde{A}) & \text{for } (i,s) = (2,1),(2,2),(4,1),(4,2),\\ C(|\lambda|^{1/2}+\tilde{A})^2 & \text{for } (i,s) = (1,1),(1,2),(2,3),(3,1),(3,2),(4,3),\\ C(|\lambda|^{1/2}+\tilde{A})^3 & \text{for } (i,s) = (1,3),(2,4),(3,3),(4,4),\\ C(|\lambda|^{1/2}+\tilde{A})^4 & \text{for } (i,s) = (1,4),(3,4). \end{cases}$$

Then,

$$|\sum_{s=1}^4 L_{is}r_{sj}| \le \begin{cases} C\tilde{A}(|\lambda|^{1/2}+\tilde{A}) & \text{for } (i,j) = (2,1),\ldots,(2,n),(4,1),\ldots,(4,n),\\ C\tilde{A}(|\lambda|^{1/2}+\tilde{A})^2 & \text{for } (i,j) = \begin{cases}(1,1),\ldots,(1,n),(2,n+1),\ldots,(2,2n-1),\\ (3,1),\ldots,(3,n),(4,n+1),\ldots,(4,2n-1),\end{cases}\\ C\tilde{A}(|\lambda|^{1/2}+\tilde{A})^3 & \text{for } (i,j) = (1,n+1),\ldots,(1,2n-1),(3,n+1),\ldots,(3,2n-1),\\ C(|\lambda|^{1/2}+\tilde{A})^3 & \text{for } (i,j) = (2,2n),(4,2n),\\ C(|\lambda|^{1/2}+\tilde{A})^4 & \text{for } (i,j) = (1,2n),(3,2n). \end{cases}$$

We need to derive the boundedness for $\det L$.

Lemma 5.3. *Let* $0 < \varepsilon < \pi/2$ *and* $0 < \eta < \varepsilon/2$. *Then, there exist positive constants* c *and* C *such that*

$$c(|\lambda|^{1/2}+\tilde{A})^3 \le |\det L| \le C(|\lambda|^{1/2}+\tilde{A})^3 \qquad (\lambda\in\Sigma_\varepsilon, \xi'\in\tilde{\Sigma}_\eta^{n-1}).$$

Proof. Let the angle of A be $\theta\in(0,\eta)$, i.e., $A = |A|e^{i\theta}$. Since the function of $\det L = \det L(A,B_+,B_-)$ is homogeneous function of $A, B_\pm$, it follows that

$$|\det L(A,B_+,B_-)| = |\det L(|A|, B_+e^{-i\theta}, B_-e^{-i\theta})| \ge c(|\lambda e^{-2i\theta}| + |A|^2)^{3/2} \ge c(|\lambda|^{1/2}+\tilde{A})^3$$

from the previous results on real values in [30, Lemma 5.5], where we have chosen small η such that $0 < 2\eta < \varepsilon$. It is easy to check the estimate from above. □

We calculate the estimates of $\phi_{k,\pm j}$ and $\psi_{k,\pm j}$ by combining all of the estimates above, as follows:

$$\begin{aligned}
|\partial_n^m \phi_{k,\pm n}| &\le |a_{1,k}||\partial_n^m \mathcal{M}_\pm| + |a_{2,k}||\partial_n^m e^{\mp B_\pm x_n}| \\
&\le |\det L|^{-1}\Big(\Big|\sum_{s=1}^4 L_{1s} r_{sk}\Big||\partial_n^m \mathcal{M}_\pm| + \Big|\sum_{s=1}^4 L_{2s} r_{sk}\Big||\partial_n^m e^{\mp B_\pm x_n}|\Big) \\
&\le C(|\lambda|^{1/2} + \tilde{A})^{-2+m}(\pm x_n)^{-1}, \\
|\partial_n^m \phi_{k,\pm j}| &\le C(|a_{1,k}||\partial_n^m \mathcal{M}_\pm| + |\mu_+ B_+ + \mu_- B_-|^{-1}(1 + |a_{1,k}| + \tilde{A}|a_{2,k}|)|\partial_n^m e^{\mp B_\pm x_n}|) \\
&\le C(|\lambda|^{1/2} + \tilde{A})^{-2+m}(\pm x_n)^{-1},
\end{aligned}$$

for $m = 0, 1, 2, 3$ and $j, k = 1, \ldots, n$; similarly,

$$\begin{aligned}
|\partial_n^m \psi_{k,\pm j}| &\le \begin{cases} C(|\lambda|^{1/2} + \tilde{A})^{-1+m}(\pm x_n)^{-1} & \text{if } k \in \{1, \ldots, n-1\} \\ C(|\lambda|^{1/2} + \tilde{A})^{-1}\tilde{A}^m(\pm x_n)^{-1} & \text{if } k = n, \end{cases} \\
|\partial_n^m \chi_{k,\pm}| &\le C\tilde{A}^{-1+m}(\pm x_n)^{-1},\ k \in \{1, \ldots, n\}, \\
|\partial_n^m \omega_{k,\pm}| &\le \begin{cases} C(|\lambda|^{1/2} + \tilde{A})\tilde{A}^{-1+m}(\pm x_n)^{-1} & \text{if } k \in \{1, \ldots, n-1\} \\ C(|\lambda|^{1/2} + \tilde{A})^2\tilde{A}^{-2+m}(\pm x_n)^{-1} & \text{if } k = n, \end{cases}
\end{aligned}$$

for $m = 0, 1, 2, 3$ and $j = 1, \ldots, n$.

We remark that $\psi_{n,\pm j}$ and $\omega_{n,\pm}$ are the coefficients of h_n. They are different from $\psi_{k,\pm j}$ and $\omega_{k,\pm}$. Therefore, we need to assume additional assumptions on normal components.

These estimates lead to the inequality (5.1), which encompasses the estimates λu, $\lambda^{1/2}\partial_\ell u$, $\partial_\ell\partial_{\ell'} u$, $\lambda^{1/2}\partial_n u$, $\partial_\ell\partial_n u$, $\partial_n^2 u$, $\partial_\ell\theta$, and $\partial_n\theta$.

We also see that the new symbols $S_\pm^{u_j}$ and $S_\pm^\theta$, multiplied by λ, ξ_ℓ, and ∂_n, are holomorphic in $(\tau, \xi') \in \tilde{\Sigma}_\eta^n$. Therefore, we are able to apply Theorem 5.1, where we employ a change of variables from x_n to $-x_n$, also, note that $\|[\![f]\!]\|_{L_q(\mathbb{R}_+^n)} \le \|f\|_{L_q(\dot{\mathbb{R}}^n)}$.

Theorem 5.4. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then, for any $\lambda \in \Sigma_\varepsilon$, $g \in W_q^1(\dot{\mathbb{R}}^n)$, $h \in W_q^2(\dot{\mathbb{R}}^n)$, and $h_n \in E_q(\dot{\mathbb{R}}^n)$, the problem (1.5) with $f = f_d = 0$ admits a solution $(u, \theta) \in W_q^2(\dot{\mathbb{R}}^n) \times \hat{W}_q^1(\dot{\mathbb{R}}^n)$ with the following resolvent estimate:*

$$\begin{aligned}
\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_q(\dot{\mathbb{R}}^n)} &\le C\|(\lambda^{1/2} g, \nabla g, \lambda h, \lambda^{1/2}\nabla h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} \\
&\le C\|(\lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)}
\end{aligned}$$

for some positive constant C.

This theorem and the estimates in Section 3 can be applied to derive the existence in Theorem 2.1. The uniqueness has been derived in [30,33], where they considered the homogeneous equation and the dual problem.

For the non-stationary Stokes equations, we have the following theorem according to Theorem 5.1:

Theorem 5.5. *Let $1 < p, q < \infty$ and $\gamma_0 \ge 0$. Then, for any*

$$\begin{aligned}
H &\in W_{p,0,\gamma_0}^1(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n)), \\
H_n &\in W_{p,0,\gamma_0}^1(\mathbb{R}, E_q(\dot{\mathbb{R}}^n)),
\end{aligned}$$

the problem (1.4) *with* $F = F_d = 0$ *admits a solution* (U, Π) *such that*

$$U \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\dot{\mathbb{R}}^n)),$$
$$\Pi \in L_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^1_q(\dot{\mathbb{R}}^n))$$

with the following maximal L_p-L_q *regularity:*

$$\begin{aligned}
&\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U, \nabla\Pi)\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\
&\leq C\|e^{-\gamma t}(\partial_t H, \Lambda_\gamma^{1/2}\nabla H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\
&\leq C\|e^{-\gamma t}(\partial_t H, \nabla^2 H, \partial_t(|\nabla'|^{-1}\partial_n H_n))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))}
\end{aligned}$$

for any $\gamma \geq \gamma_0$ *with some positive constant* $C = C_{n,p,q,\gamma_0}$*, depending only on* $n, p, q,$ *and* γ_0.

6. On the problems with surface tension and gravity

In this section, we consider the problems (1.2) and (1.3). We shall prove Theorems 2.3 and 2.4. Let (v, τ) and (V, Υ) be solutions to the following problems:

$$\begin{cases}
\rho\lambda v - \operatorname{Div} S(v, \tau) = f & \text{in } \dot{\mathbb{R}}^n, \\
\operatorname{div} v = f_d & \text{in } \dot{\mathbb{R}}^n, \\
[\![S(v, \tau)\nu]\!] = [\![g]\!] & \text{on } \mathbb{R}^n_0, \\
[\![v]\!] = [\![h]\!] & \text{on } \mathbb{R}^n_0,
\end{cases} \tag{6.1}$$

$$\begin{cases}
\rho\partial_t V - \operatorname{Div} S(V, \Upsilon) = F & \text{in } \dot{\mathbb{R}}^n, t > 0, \\
\operatorname{div} V = F_d & \text{in } \dot{\mathbb{R}}^n, t > 0, \\
[\![S(V, \Upsilon)\nu]\!] = [\![G]\!] & \text{on } \mathbb{R}^n_0, t > 0, \\
[\![V]\!] = [\![H]\!] & \text{on } \mathbb{R}^n_0, t > 0, \\
V|_{t=0} = 0 & \text{in } \dot{\mathbb{R}}^n.
\end{cases} \tag{6.2}$$

We shall find the solutions (w, κ, η) and (W, Ξ, Y) satisfying

$$\begin{cases}
\rho\lambda w - \operatorname{Div} S(w, \kappa) = 0 & \text{in } \dot{\mathbb{R}}^n, \\
\operatorname{div} w = 0 & \text{in } \dot{\mathbb{R}}^n, \\
\lambda\eta + w_n = d - v_n =: \tilde{d} & \text{on } \mathbb{R}^n_0, \\
[\![S(w, \kappa)\nu]\!] - ([\![\rho]\!]c_g + c_\sigma\Delta')\eta\nu = 0 & \text{on } \mathbb{R}^n_0, \\
[\![w]\!] = 0 & \text{on } \mathbb{R}^n_0,
\end{cases} \tag{6.3}$$

$$\begin{cases}
\rho\partial_t W - \operatorname{Div} S(W, \Xi) = 0 & \text{in } \dot{\mathbb{R}}^n, t > 0, \\
\operatorname{div} W = 0 & \text{in } \dot{\mathbb{R}}^n, t > 0, \\
\partial_t Y + W_n = D - V_n =: \tilde{D} & \text{on } \mathbb{R}^n_0, t > 0, \\
[\![S(V, \Xi)\nu]\!] - ([\![\rho]\!]c_g + c_\sigma\Delta')Y\nu = 0 & \text{on } \mathbb{R}^n_0, t > 0, \\
[\![V]\!] = 0 & \text{on } \mathbb{R}^n_0, t > 0, \\
(V, Y)|_{t=0} = (0, 0) & \text{in } \dot{\mathbb{R}}^n.
\end{cases} \tag{6.4}$$

Then $(u, \theta, \eta) = (v + w, \tau + \kappa, \eta)$ and $(U, \Theta, Y) = (V + W, \Upsilon + \Xi, Y)$ are the solutions of (1.3) and (1.2). To solve the equations in (6.3), it is enough to consider that

$$([\![\hat{h}]\!], [\![\hat{g}']\!], [\![\hat{g}_n]\!]) = (0, 0, -([\![\rho]\!]c_g - c_\sigma A^2)\hat{\eta})$$

in (4.1), and that

$$\begin{cases} \lambda\hat{\eta} + \hat{w}_n = \hat{\tilde{d}} & \text{on } \mathbb{R}^n_0, \\ \hat{w}_{\pm j} = \phi_{n,\pm j}[\![\hat{g}_n]\!] & \text{in } \mathbb{R}^n_\pm \ (j = 1, \cdots, n). \end{cases} \tag{6.5}$$

Note that $\phi_{n,+n}(\lambda, \xi', 0) = \phi_{n,-n}(\lambda, \xi', 0) = a_{2,n} = a_{4,n} = (\det L)^{-1}A\{\mu_+(B_+ + A) + \mu_-(B_- + A)\}$. Therefore we have the following solution formulas:

$$\begin{aligned} \hat{\eta}(\lambda, \xi') &= \frac{\det L}{\lambda \det L - A\{\mu_+(B_+ + A) + \mu_-(B_- + A)\}([\![\rho]\!]c_g - c_\sigma A^2)}\hat{\tilde{d}}, \\ \hat{w}_{\pm j}(\lambda, \xi', x_n) &= -\phi_{n,\pm j}([\![\rho]\!]c_g - c_\sigma A^2)\hat{\eta} \qquad (j = 1, \cdots, n), \\ \hat{\kappa}_\pm(\lambda, \xi', x_n) &= -\chi_{n,\pm}([\![\rho]\!]c_g - c_\sigma A^2)\hat{\eta} \end{aligned}$$

with the following estimate:

$$\begin{aligned} &\mathcal{L}(\lambda, \xi') := \lambda \det L - A\{\mu_+(B_+ + A) + \mu_-(B_- + A)\}([\![\rho]\!]c_g - c_\sigma A^2) \\ &|\mathcal{L}(\lambda, \xi')| \geq c(|\lambda| + \tilde{A})(|\lambda|^{1/2} + \tilde{A})^3 \end{aligned}$$

for $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times \tilde{\Sigma}^{n-1}_\eta$ with $0 < \varepsilon < \pi/2$, $0 < \eta < \varepsilon/2$, and $\gamma_0 \geq 1$. The proof for $\xi' \in \mathbb{R}^{n-1}$ is in [33, Lemma 6.1]. However, the proof for complex values is almost the same.

Let η extend suitably from $\mathbb{R}^{n-1}$ to $\dot{\mathbb{R}}^n$. Since we have the estimate

$$\sup_{\substack{(\lambda,\xi') \in \Sigma_{\varepsilon,\gamma_0} \times \tilde{\Sigma}^{n-1}_\eta \\ \ell = 1,\ldots,n-1}} \left\{ (|\lambda| + |\xi_\ell|)\frac{\det L}{\mathcal{L}} \right\} < C$$

and holomorphy, we are able to prove, by applying Fourier multiplier theory as in [22, Proposition 4.3.10, Theorem 4.3.3], that

$$\begin{aligned} \|(\lambda\eta, \nabla\eta)\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C\|\tilde{d}\|_{L_q(\dot{\mathbb{R}}^n)}, \\ \|(\lambda\nabla\eta, \nabla^2\eta)\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C\|\nabla\tilde{d}\|_{L_q(\dot{\mathbb{R}}^n)}, \\ \|(\lambda\nabla^2\eta, \nabla^3\eta)\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C\|\nabla^2\tilde{d}\|_{L_q(\dot{\mathbb{R}}^n)}, \end{aligned}$$

for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$. Then, from the results in the previous section, it follows that

$$\begin{aligned} \|(\lambda w, \lambda^{1/2}\nabla w, \nabla^2 w, \nabla\kappa)\|_{L_q(\dot{\mathbb{R}}^n)} &\leq C\|(\lambda^{1/2}g_n, \nabla g_n)\|_{L_q(\dot{\mathbb{R}}^n)} \\ &\leq C\|(\lambda^{1/2}\eta, \lambda^{1/2}\nabla^2\eta, \nabla\eta, \nabla^3\eta)\|_{L_q(\dot{\mathbb{R}}^n)} \\ &\leq C\|\tilde{d}\|_{W^2_q(\dot{\mathbb{R}}^n)}, \end{aligned}$$

where we have used $|\lambda|^{1/2} \leq |\lambda|$ when $\lambda \in \Sigma_{\varepsilon,\gamma_0}$;also, C is dependent on γ_0, as well as the constants $[\![\rho]\!]$, c_g, and c_σ. This yields that

$$\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|\eta\|_{W^2_q(\dot{\mathbb{R}}^n)} + \|\eta\|_{W^3_q(\dot{\mathbb{R}}^n)}$$

$$
\begin{aligned}
&\le \|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v, \lambda w, \lambda^{1/2}\nabla w, \nabla^2 w, \nabla\tau, \nabla\kappa)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|\eta\|_{W_q^2(\dot{\mathbb{R}}^n)} + \|\eta\|_{W_q^3(\dot{\mathbb{R}}^n)} \\
&\le C_{n,q,\varepsilon,\gamma_0}\left\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|f_d\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|\tilde{d}\|_{W_q^2(\dot{\mathbb{R}}^n)}\right\} \\
&\le C_{n,q,\varepsilon,\gamma_0}\left\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|f_d\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|d\|_{W_q^2(\dot{\mathbb{R}}^n)}\right\}
\end{aligned}
$$

since

$$
\begin{aligned}
\|\tilde{d}\|_{W_q^2(\dot{\mathbb{R}}^n)} &\le C_{n,q,\varepsilon}(\|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + \|v\|_{W_q^2(\dot{\mathbb{R}}^n)}) \\
&\le C_{n,q,\varepsilon,\gamma_0}(\|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + |\lambda|\|v\|_{L_q(\dot{\mathbb{R}}^n)} + \|\nabla^2 v\|_{L_q(\dot{\mathbb{R}}^n)}) \\
&\le C_{n,q,\varepsilon,\gamma_0}(\|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + \|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|f_d\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)}).
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
|\lambda|^{3/2}\|\eta\|_{W_q^1(\dot{\mathbb{R}}^n)} &\le |\lambda|^{1/2}\|\tilde{d}\|_{W_q^1(\dot{\mathbb{R}}^n)} \\
&\le |\lambda|^{1/2}\|d\|_{W_q^1(\dot{\mathbb{R}}^n)} + |\lambda|^{1/2}\|v\|_{W_q^1(\dot{\mathbb{R}}^n)} \\
&\le C_{n,q,\varepsilon,\gamma_0}\Big\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} \\
&\qquad + |\lambda|\|g\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + |\lambda|^{1/2}\|d\|_{W_q^1(\dot{\mathbb{R}}^n)}\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
|\lambda|^2\|\eta\|_{L_q(\dot{\mathbb{R}}^n)} &\le |\lambda|\|\tilde{d}\|_{L_q(\dot{\mathbb{R}}^n)} \\
&\le |\lambda|\|d\|_{L_q(\dot{\mathbb{R}}^n)} + |\lambda|\|v\|_{L_q(\dot{\mathbb{R}}^n)} \\
&\le C_{n,q,\varepsilon,\gamma_0}\Big\{\|(f, \lambda^{1/2} f_d, \nabla f_d, \lambda^{1/2} g, \nabla g, \lambda h, \nabla^2 h, \lambda|\nabla'|^{-1}\partial_n h_n)\|_{L_q(\dot{\mathbb{R}}^n)} \\
&\qquad + |\lambda|\|g\|_{\hat{W}_q^{-1}(\dot{\mathbb{R}}^n)} + \|d\|_{W_q^2(\dot{\mathbb{R}}^n)} + |\lambda|\|d\|_{L_q(\dot{\mathbb{R}}^n)}\Big\}.
\end{aligned}
$$

The proof of Theorem 2.4 is the same as above.

7. Conclusions

In this paper, the solution formulas for generalized two-phase Stokes equations have been derived. By using the solution formulas, we have proved the existence of resolvent L_q estimates and maximal L_p-L_q estimates. The method is based on H^∞-calculus, whereas the previous works were based on $\mathcal{R}$-boundedness. The complexity of the calculation is comparatively less, and the conditions on the normal component is relaxed. Our method does not require the estimates of derivatives of the Fourier symbols. Although we were able to obtain an explicit form of the solution, we only applied the order of the coefficient of boundary source terms. Thus, we have not only considered the standard free-boundary condition, we have also considered the problem with surface tension and gravity. This strategy will be useful for future works when we consider other terms.

Appendix

Proof of the estimate for normal components

Proof of the estimate $\|\lambda|\nabla'|^{-1}\partial_n\psi_n\|_{L_q(\dot{\mathbb{R}}^n)} \leq C\|f\|_{L_q(\dot{\mathbb{R}}^n)}$. We see that

$$\begin{aligned}&\lambda|\nabla'|^{-1}\partial_n\psi_{\pm n}\\ &=\sum_{k=1}^{n-1}\mathcal{F}_\xi^{-1}\left(\lambda\frac{i\xi_n}{|\xi'|}\frac{1}{\rho_\pm\lambda+\mu_\pm|\xi|^2}(\frac{-\xi_n\xi_k}{|\xi|^2})\right)\mathcal{F}_xf_k+\mathcal{F}_\xi^{-1}\left(\lambda\frac{i\xi_n}{|\xi'|}\frac{1}{\rho_\pm\lambda+\mu_\pm|\xi|^2}(1-\frac{\xi_n^2}{|\xi|^2})\right)\mathcal{F}_xf_n.\end{aligned}$$

All symbols denoted by

$$\lambda\frac{i\xi_n}{|\xi'|}\frac{1}{\rho_\pm\lambda+\mu_\pm|\xi|^2}\frac{-\xi_n\xi_k}{|\xi|^2},\quad \lambda\frac{i\xi_n}{|\xi'|}\frac{1}{\rho_\pm\lambda+\mu_\pm|\xi|^2}(1-\frac{\xi_n^2}{|\xi|^2})=\lambda\frac{i\xi_n}{|\xi'|}\frac{1}{\rho_\pm\lambda+\mu_\pm|\xi|^2}\frac{|\xi'|^2}{|\xi|^2}$$

are bounded and holomorphic in $\lambda \in \Sigma_\varepsilon$, $\xi \in \tilde{\Sigma}_\eta^n$ for small ε, η, where we regard $|\xi'| = \sqrt{\sum_{j=1}^{n-1}\xi_j^2} = A$ and $|\xi|^2 = A^2 + \xi_n^2$ as complex functions. Therefore, by Fourier multiplier theory, we have

$$\|\lambda|\nabla'|^{-1}\partial_n\psi_n\|_{L_q(\dot{\mathbb{R}}^n)} \leq \sum_\pm \|\lambda|\nabla'|^{-1}\partial_n\psi_{\pm n}\|_{L_q(\mathbb{R}^n)} \leq C\|f\|_{L_q(\dot{\mathbb{R}}^n)}.$$

The other estimate follows similarly. □

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no competing interests.

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