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Research article

Convex contractions on extended b-metric spaces

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Abstract: This research investigated different types of convex contractions in the setting of extended b-metric spaces from the point of view of the existence and uniqueness of their fixed points. The assumptions imposed on involved mappings refer to convexity of order 2, two-sided convexity or Ćirić-type convexity, which also fulfill a continuity type condition. An example was provided to emphasize the usability of the results.

Keywords: extended b-metric space; generalized metric space; convex contraction; generalized contraction
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1. Introduction

Fixed point theory is one of the most important and significant research fields in nonlinear and functional analysis since it provides some of the most useful tools to solve many problems in applied sciences and engineering, such as dynamical systems, game theory, optimization theory, the existence of solutions to integrals, and differential and matrix equations. One of the most useful results in fixed point theory is the Banach-Picard-Caccioppoli contraction principle [1, 2]. It states that a mapping T on a complete metric space X onto itself has a unique fixed point provided that T is a contractive mapping, i.e., there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. Due to its importance, the result has since been studied and generalized.

Broadly, there are two manners in the generalization of fixed point theorems. One is by weakening the conditions of the contraction and the other way is by generalizing the underlying space (which is usually done by weakening the triangle inequality).

In 1962, Rakotch [3] generalized the contraction condition by defining a family of functions such that the contraction principle was still valid when the constant k was replaced by a function with suitable properties. After that, in 1969, Boyd and Wong [4] improved the result of Rakotch by introducing in the right hand member of the contraction an upper semicontinuous function $\varphi(\cdot)$ such that $\varphi(t) < t$, for any $t > 0$ and, in the same year, Meir and Keeler [5] generalized the latter result by introducing a condition of weakly uniformly strict contraction. Inspired by the contraction principle, in 1968, the concept of the Kannan contraction [6] was introduced. Note that this type of contraction may not always be continuous, thus the independence of the Kannan fixed point theorem of the contraction principle. Also, note that a Kannan contraction characterizes the completeness of a metric space, as was shown in [7]. In 1972, the Chatterjea's fixed point theorem [8] was stated, while in 1971, Reich [9] generalized Banach's fixed point theorem and Kannan's fixed point theorem by putting together the terms from the right-hand side of both contractions. In the same manner, Hardy and Rogers [10] generalized, in 1973, the results of Reich and Chatterjea and one year later, in 1974, Ćirić [11] improved the Hardy-Rogers contraction. Also, one can consider different types of rational contractions, for example, in [12], researchers used an ICS function and some generalized weak contractions of Boyd-Wong-type to generalize the results from [13]. For a more perspicuous and comprehensive view of the contractive mappings that admit a unique fixed point, we address the reader to Rhoades [14] and the references therein. In 1982, Istrăţescu [15] inaugurated the class of convex contractive mappings by introducing several convexity conditions, thus bringing out new generalizations for the contraction principle. The convex contractions, too, were later subjected to generalization. For example, in [16], Miandaragh et al. defined the notions of generalized convex contractions and generalized convex contractions of order 2 and proved fixed point theorems regarding these type of mappings.

In recent years, several generalizations of classical metric spaces have been given. For instance, Bakhtin [17] and Czerwik [18] introduced the concept of b-metric spaces. With the emergence of this space, a myriad of novel results has concurrently surfaced. For example, in [19], Ali et al. defined on b-metric spaces the notions of Hardy-Rogers-type (F-α)-contractions and Hardy-Rogers-type (F-α^*)-contractions and then established fixed point theorems for these contractions. In [20], Kamran et al. introduced on the same ambiental space the concepts of Feng-Liu-type (F-α)-contractions and Feng-Liu-type (F-α^*)-contractions and proved fixed point results regarding these contractions. In [21], Shatanawi et al. used a contraction condition by means of a comparison function to prove a result regarding a unique common fixed point of two mappings.

In 2014, Kirk and Shahzad [22] defined the notion of strong b-metric spaces. In 2000, Branciari [23] defined the framework of generalized metric spaces (also known as rectangular metric spaces) and generalized metric spaces of order v, and in the same year, Hitzler and Seda [24] introduced the notion of dislocated metric spaces. In 2014, Khojasteh et al. [25] defined the concept of θ-metric spaces. For related further results, including the metrization of such spaces, see [26]. In 2015 and 2018, Jleli and Samet introduced the generalized metric spaces [27] and F-metric spaces [28], respectively. In 2017, Kamran et al. [29] defined the concept of extended b-metric spaces, in this way generalizing the b-metric spaces. For fixed point results in the setting of this space, the reader can consult, for example, some of the following articles and references therein. In [30], Samreen et al. came up with a generalization of some of the main results from [31–33]. In [34], Alqahtani et al. proved fixed point theorems for two mappings that form an (α,β)-orbital-cyclic-admissible pair and obtained corollaries for (α, β)-orbital-cyclic-admissible mappings and α-orbital-admissible mappings. In [35], Abdeljawad et al. defined the concepts of Θ_e-contractions and Hardy-Rogers-type Θ-contractions and proved fixed point theorems for each one of them in the setting of extended b-metric spaces. Also, in [36], Shatanawi et al. introduced the notion of $\alpha - \psi$-contractive mappings and proved a fixed point result for such functions. In [37], Alqahtani et al. proved some fixed point theorems for an orbitally continuous self-map T on a T-orbitally complete extended b-metric space. In [38], Mitrović et al. proved the fixed point theorems of Reich [9] and Nadler [39] in the setting of extended b-metric spaces. Huang et al. [40] and Alqahtani et al. [41] determined fixed point results for some rational type contractions and in [42], Kiran et al. generalized the Hardy-Rogers fixed point theorem [10] in the setting of extended b-metric spaces and also proved some theorems for multi-valued mappings.

This paper is organized as follows: In Section 2, preliminary concepts and notions are recalled, such as b-metric spaces, extended b-metric spaces, and the convex contractive mappings used in the main results. Also, a new type of convex contractive mapping is presented—the Ćirić-convex contraction. In Section 3, some fixed point theorems are formulated and proven in the setting of extended b-metric spaces for the contractive mappings defined in the previous section.

2. Preliminary definitions and auxiliary results

First, recall the definition of a b-metric space.

Definition 2.1. [17, 18] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric provided that, for all $x, y, z \in X$,

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space of constant s.

Second, recall the definition of an extended b-metric space.

Definition 2.2. [29] Let X be a nonempty set and θ: $X \times X \to [1, \infty)$. A function d_θ: $X \times X \to [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$, it satisfies:

- $d_\theta(x, y) = 0$ if and only if $x = y$,
- $d_\theta(x, y) = d_\theta(y, x)$,
- $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

A pair (X, d_θ) is called an extended b-metric space.

It is obvious that a b-metric space is a particular case of an extended b-metric space by taking $\theta(x, y) = s$. The following example shows that the class of extended b-metric spaces is larger than the class of b-metric spaces, in the sense of inclusion.

Example 2.1. [43] Let $X = [-1, 1]$ and θ: $X \times X \to [1, \infty)$ be defined by

$$\theta(x, y) = \frac{1 + x^2 + y^2}{x^2 + y^2},$$

if $x^2 + y^2 > 0$ and $\theta(0,0) = 1$. Define d_θ: $X \times X \to [0, \infty)$,

$$d_\theta(x,y) = \begin{cases} 0, & \text{if and only if } x = y, \\ \frac{1}{x^2}, & \text{if } xy = 0 \text{ and } x^2 + y^2 \neq 0, \\ \frac{1}{x^2y^2}, & \text{if } 0 \neq x \neq y \neq 0. \end{cases}$$

Thus, d_θ defines an extended b-metric on X, therefore (X, d_θ) is an extended b-metric space. Note that (X, d_θ) is not a b-metric space. To prove this, consider $x, y \in [-1, 1] \setminus \{0\}$ such that $x \neq y$. We have that

$$\frac{d_\theta(x,y)}{d_\theta(x,0) + d_\theta(0,y)} = \frac{\frac{1}{x^2y^2}}{\frac{1}{x^2} + \frac{1}{y^2}} = \frac{1}{x^2 + y^2}.$$

Note that

$$\sup\{\frac{1}{x^2 + y^2} : x, y \in [-1, 1] \setminus \{0\}, x \neq y\} = +\infty.$$

Therefore, it is impossible to find $s \geq 1$ such that

$$d_\theta(x,y) \leq s[d_\theta(x,0) + d_\theta(0,y)].$$

Onwards, we recollect the concepts of convergence, Cauchy sequence, and completeness in an extended b-metric space.

Definition 2.3. [29] Let (X, d_θ) be an extended b-metric space. Then a sequence $\{x_n\}_n$ in X is said to be:

- Convergent if and only if there exists $x \in X$ such that $d_\theta(x_n, x) \to 0$ as $n \to \infty$, and we write $\lim_{n\to\infty} x_n = x$,
- Cauchy if and only if $d_\theta(x_n, x_m) \to 0$ as $n, m \to \infty$.

The extended b-metric space (X, d_θ) is complete if every Cauchy sequence converges in X. We note that the extended b-metric d_θ is not a continuous function in general.

A useful result in proving fixed point theorems in the setting of extended b-metric spaces is presented here.

Lemma 2.1. [30] Let (X, d_θ) be an extended b-metric space. Then every convergent sequence has a unique limit.

Proof. Consider a convergent sequence $\{x_n\}_n$ of X and presume there exist $u, v \in X$ such that

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} x_n = v.$$

Then,

$$d_\theta(u,v) \leq \theta(u,v)[d_\theta(u,x_n) + d_\theta(x_n,v)].$$

As $\theta(\cdot,\cdot)$ is finite, by taking the limit when $n \to \infty$ in the previous inequality, we obtain that $d_\theta(u,v) = 0$, thus $u = v$. Therefore, the limit of $\{x_n\}_n$ is unique. □

Remark 2.1. If we define $\theta(\cdot,\cdot)$ as θ: $X \times X \to [1, \infty]$, then the uniqueness of a convergent sequence would not yield from Definition 2.2.

Henceforth, consider X to be a nonempty set and (X, d) to be an extended b-metric space.

A usual property of functions used in proving fixed point results for convex contractive mappings is stated next.

Definition 2.4. [44] A mapping $T\colon X \to X$ is called orbitally continuous if

$$\lim_{i\to\infty} T^{n_i}x = z$$

implies

$$\lim_{i\to\infty} T(T^{n_i}x) = Tz,$$

where T^n denotes the n-fold composition of T with itself.

In the following, we present the types of convex contractive mappings that will be used in the main results.

Definition 2.5. [15] A mapping $T\colon X \to X$ is said to be a convex contraction of order 2 if there exist $a, b \in [0, 1)$ with $a + b < 1$ such that, for all $x, y \in X$, the following inequality holds:

$$d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y).$$

Note that this class of mappings contains the class of contractive mappings in the sense of [1, 2].

Definition 2.6. [15] A mapping $T\colon X \to X$ is said to be a two-sided convex contraction if there exist $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ such that, for all $x, y \in X$, the following inequality holds:

$$d(T^2x, T^2y) \leq a_1d(x, Tx) + a_2d(Tx, T^2x) + b_1d(y, Ty) + b_2d(Ty, T^2y).$$

Definition 2.7. [15] A mapping $T\colon X \to X$ is said to be a convex contraction of type 2 if there exist constants $c_0, c_1, a_1, a_2, b_1, b_2 \in [0, 1)$ with $c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$ such that, for all $x, y \in X$, the following inequality holds:

$$\begin{aligned} d(T^2x, T^2y) \leq & c_0d(x, y) + c_1d(Tx, Ty) + a_1d(x, Tx) \\ & + a_2d(Tx, T^2x) + b_1d(y, Ty) + b_2d(Ty, T^2y). \end{aligned}$$

Definition 2.8. [15] A mapping $T\colon X \to X$ is said to be a convex contraction of order $k \geq 2$ if there exist $a_0, a_1, \ldots, a_{k-1} \in [0, 1)$ with $a_0 + a_1 + \ldots + a_{k-1} < 1$ such that, for all $x, y \in X$, the following inequality holds:

$$d(T^kx, T^ky) \leq a_0d(x, y) + a_1d(Tx, Ty) + \ldots + a_{k-1}d(T^{k-1}x, T^{k-1}y).$$

Another type of convex contractive mapping is presented next.

Definition 2.9. A mapping $T\colon X \to X$ is said to be a Ćirić-convex contraction if there exists $h \in [0, 1)$ such that, for all $x, y \in X$, the following inequality holds:

$$d(T^2x, T^2y) \leq h\max\{d(x, y), d(Tx, Ty), d(x, Tx), d(Tx, T^2x), d(y, Ty), d(Ty, T^2y)\}.$$

3. Main results

A valuable lemma that will be used throughout the proofs of the theorems in this section is given below.

Lemma 3.1. Let (X, d) be an extended b-metric space, let $T\colon X \to X$ be a mapping and the sequence $\{x_n\}_n$ of Picard iterations based on an initial point $x_0 \in X$, i.e., $x_n = T^n x_0$, for all $n \geq 0$. If there exists $\lambda \in [0, 1)$ such that

$$d(x_n, x_{n+1}) \leq \sqrt{\lambda^{n-1}} \max\{d(x_0, x_1), d(x_1, x_2)\}$$

for all $n \geq 0$, and there exist

$$\beta < \frac{1}{\sqrt{\lambda}}$$

and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, then $\{x_n\}_n$ is a Cauchy sequence.

Proof. Set $d_n = d(x_n, x_{n+1})$, for all $n \geq 0$ and $M = \max\{d_0, d_1\}$. Consider n, p, $j \in \mathbb{N}$ with $n \geq n_0$, $p \geq 1$ and $j \in \{n, n+1, \ldots, n+p-1\}$. Obviously, $\beta\sqrt{\lambda} < 1$.

The following estimations will justify that $\{x_n\}_n$ is a Cauchy sequence:

$$\begin{aligned}
d(x_n, x_{n+p}) \leq& \theta(x_n, x_{n+p}) d(x_n, x_{n+1}) + \theta(x_n, x_{n+p}) d(x_{n+1}, x_{n+p}) \\
\leq& \theta(x_n, x_{n+p}) d_n + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) d(x_{n+1}, x_{n+2}) \\
&+ \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) d(x_{n+2}, x_{n+p}) \\
\leq& \ldots \\
\leq& \theta(x_n, x_{n+p}) d_n + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) d_{n+1} \\
&+ \ldots + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \cdots \theta(x_{n+p-1}, x_{n+p}) d_{n+p-1} \\
\leq& \beta \sqrt{\lambda^{n-1}} M + \beta^2 \sqrt{\lambda^n} M + \ldots + \beta^p \sqrt{\lambda^{n+p-2}} M \\
\leq& \beta \sqrt{\lambda^{n-1}} M \Big[1 + \beta\sqrt{\lambda} + \ldots + \beta^{p-1} \sqrt{\lambda^{p-1}}\Big] \\
=& \beta \sqrt{\lambda^{n-1}} M \frac{1 - (\beta\sqrt{\lambda})^p}{1 - \beta\sqrt{\lambda}} \\
\leq& \beta \sqrt{\lambda^{n-1}} M \frac{1}{1 - \beta\sqrt{\lambda}} \to 0,
\end{aligned}$$

when $n \to \infty$. □

The bounding condition imposed on the function $\theta(\cdot, \cdot)$ was dictated by the necessity to ensure that the Picard sequence under consideration meets the Cauchy sequence definition. Note that, in the previous lemma, if we replace the condition

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

with the condition

$$\lim_{n,m\to\infty} \theta(x_n, x_m) \le \beta \quad \text{or} \quad \lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{\sqrt{h}},$$

the conclusion would still be valid.

Another lemma, which will prove to be useful, is presented as follows. For the forthcoming result, which is self-evident and straightforward, we shall opt to omit the proof.

Lemma 3.2. Let (X, d) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous mapping. If the Picard iterations sequence based on an initial point $x_0 \in X$, i.e., $x_n = T^n x_0$, for all $n \ge 0$, is a Cauchy sequence, then T has a fixed point.

From now on, we present fixed point theorems for the previously introduced convex contractive mappings.

Theorem 3.1. *Let (X, d) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous convex contraction of type 2. Suppose there exist $x_0 \in X$,*

$$\beta < \sqrt{\frac{1 - b_2}{c_0 + c_1 + a_1 + a_2 + b_1}}$$

and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \le \beta^{j-n+1}$$

holds for all $n \ge n_0$, $p \ge 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \ge 0$.

Then T has a unique fixed point.

Proof. $\{x_n\}_n$ is the Picard iteration sequence based on the initial point x_0.

Set $d_n = d(x_n, x_{n+1})$, for all $n \ge 0$,

$$\lambda = \frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2} \quad \text{and} \quad M = \max\{d_0, d_1\}.$$

Affirmation 1. $d_n \le \sqrt{\lambda^{n-1}} M$, for all $n \ge 0$.

The statement will be proven by complete induction on n. For $n = 0$ and $n = 1$, the inequality is obvious. For $n = 2$, it follows that

$$\begin{aligned} d_2 =& d(x_2, x_3) = d(T^2 x_0, T^2 x_1) \\ &\le c_0 d(x_0, x_1) + c_1 d(x_1, x_2) + a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + b_1 d(x_1, x_2) + b_2 d(x_2, x_3), \end{aligned}$$

then

$$\begin{aligned} d_2(1 - b_2) &\le (c_0 + a_1) d_0 + (c_1 + a_2 + b_1) d_1 \\ &\le (c_0 + a_1) M + (c_1 + a_2 + b_1) M \\ &= (c_0 + c_1 + a_1 + a_2 + b_1) M, \end{aligned}$$

thus

$$d_2 \leq \frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2} M = \lambda M < \sqrt{\lambda} M = \sqrt{\lambda^{2-1}} M.$$

Therefore, the base step is verified. Henceforth, consider $k \geq 1$ such that

$$d_k \leq \sqrt{\lambda^{k-1}} M \quad \text{and} \quad d_{k-1} \leq \sqrt{\lambda^{k-2}} M.$$

Considering these inequalities, we present the inductive step:

$$\begin{aligned} d_{k+1} &= d(x_{k+1}, x_{k+2}) \\ &= d(T^2 x_{k-1}, T^2 x_k) \\ &\leq c_0 d_{k-1} + c_1 d_k + a_1 d_{k-1} + a_2 d_k + b_1 d_k + b_2 d_{k-1}, \end{aligned}$$

then

$$\begin{aligned} d_{k+1}(1 - b_2) &\leq (c_0 + a_1) d_{k-1} + (c_1 + a_2 + b_1) d_k \\ &\leq (c_0 + a_1) \sqrt{\lambda^{k-2}} M + (c_1 + a_2 + b_1) \sqrt{\lambda^{k-1}} M \\ &\leq (c_0 + a_1) \sqrt{\lambda^{k-2}} M + (c_1 + a_2 + b_1) \sqrt{\lambda^{k-2}} M \\ &= \sqrt{\lambda^{k-2}} M (c_0 + a_1 + c_1 + a_2 + b_1), \end{aligned}$$

thus

$$\begin{aligned} d_{k+1} &\leq \sqrt{\lambda^{k-2}} M \frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2} \\ &= \sqrt{\lambda^{k-2}} M \lambda \\ &= \sqrt{\lambda^{(k+1)-1}} M. \end{aligned}$$

Therefore, by complete induction on n, we conclude that $d_n \leq \sqrt{\lambda^{n-1}} M$, for all $n \geq 0$.

Affirmation 2. T has a fixed point.

By using Lemma 3.1, we conclude that the sequence $\{x_n\}_n$ is Cauchy. Also, by making use of Lemma 3.2, we get that T has a fixed point $u \in X$.

Affirmation 3. T has a unique fixed point.

Assume that there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then,

$$\begin{aligned} d(u, v) &= d(T^2 u, T^2 v) \\ &\leq c_0 d(u, v) + c_1 d(Tu, Tv) + a_1 d(u, Tu) + a_2 d(Tu, T^2 u) + b_1 d(v, Tv) + b_2 d(Tv, T^2 v) \\ &= (c_0 + c_1) d(u, v) \\ &< d(u, v) \end{aligned}$$

is a contradiction.

Consequently, u is the only fixed point of T. □

By setting $a_1 = a_2 = b_1 = b_2 = 0$ and $c_0 = c_1 = 0$ in Theorem 3.1, we obtain the corresponding results for convex contractions of order 2 and two-sided convex contractions, respectively.

Corollary 3.1. Let (X, d) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous convex contraction of order 2. If there exist $x_0 \in X$,

$$\beta < \frac{1}{\sqrt{a+b}},$$

$n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$, then T has a unique fixed point.

Corollary 3.2. Let (X, d) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous two-sided convex contraction. Suppose there exist $x_0 \in X$,

$$\beta < \sqrt{\frac{1-b_2}{a_1+a_2+b_1}}$$

and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$.

Then T has a fixed point that is unique.

Theorem 3.2. *Let (X, d) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous Ćirić-convex contraction. Presume there exist $x_0 \in X$,*

$$\beta < \frac{1}{\sqrt{h}}$$

and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$.

Then T has a unique fixed point.

Proof. Consider $\{x_n\}_n$ as the Picard iteration sequence based on the initial point x_0.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. Without loss of generality, we may assume that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N}$. Set $d_n = d(x_n, x_{n+1})$, for all $n \geq 0$ and $M = \max\{d_0, d_1\}$.

Affirmation 1. $d_n \leq \sqrt{h^{n-1}}M$, for all $n \geq 0$.

The statement will be proven by complete induction on n. For $n = 0$ and $n = 1$, the inequality is obvious. For $n = 2$, it follows that:

$$\begin{aligned} d_2 = d(x_2, x_3) &= d(T^2x_0, T^2x_1) \\ &\leq h\max\{d_0, d_1, d_0, d_1, d_1, d_2\} \\ &= h\max\{d_0, d_1, d_2\} \\ &= h\max\{M, d_2\}. \end{aligned}$$

If $\max\{M, d_2\} = d_2$, then

$$d_2 \leq hd_2 < d_2,$$

is a contradiction.

Thus, $\max\{M, d_2\} = M$. Eventually,

$$d_2 \leq hM \leq \sqrt{h}M = \sqrt{h^{2-1}}M.$$

Therefore, the base step is verified. Henceforth, consider $k \geq 1$ such that

$$d_k \leq \sqrt{h^{k-1}}M \quad \text{and} \quad d_{k-1} \leq \sqrt{h^{k-2}}M.$$

Considering these inequalities, we present the inductive step:

$$\begin{aligned} d_{k+1} &= d(x_{k+1}, x_{k+2}) = d(T^2x_{k-1}, T^2x_k) \\ &\leq \max\{d_{k-1}, d_k, d_{k+1}\}. \end{aligned}$$

If $\max\{d_{k-1}, d_k, d_{k+1}\} = d_{k+1}$, then

$$d_{k+1} \leq hd_{k+1} < d_{k+1},$$

which is a contradiction.

Therefore,

$$\max\{d_{k-1}, d_k, d_{k+1}\} = \max\{d_{k-1}, d_k\}.$$

Then,

$$\begin{aligned} d_{k+1} &\leq h\max\{d_{k-1}, d_k\} \\ &\leq h\max\{\sqrt{h^{k-2}}M, \sqrt{h^{k-1}}M\} \\ &= h\sqrt{h^{k-2}}M \\ &= \sqrt{h^{(k+1)-1}}M. \end{aligned}$$

Thus, by complete induction on n, we conclude that $d_n \leq \sqrt{h^{n-1}}M$, for all $n \geq 0$.

Affirmation 2. T has a fixed point.

By using Lemma 3.1 with $\lambda = h$, $\{x_n\}_n$ is a Cauchy sequence. Now, employing Lemma 3.2, T has a fixed point $u \in X$.

Affirmation 3. T has a unique fixed point.

Assume that there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then,

$$\begin{aligned} d(u, v) &= d(T^2 u, T^2 v) \\ &\leq h \max\{d(u, v), d(Tu, Tv), d(u, Tu), d(Tu, T^2 u), d(v, Tv), d(Tv, T^2 v)\} \\ &= hd(u, v) < d(u, v), \end{aligned}$$

which is a contradiction.

Consequently, u is the only fixed point of T. □

In order to prove the next theorem, we first need to prove the following result.

Lemma 3.3. Let (X, d) be an extended b-metric space and let $T: X \to X$ be a mapping and the sequence $\{x_n\}_n$ of Picard iterations based on an initial point $x_0 \in X$, i.e., $x_n = T^n x_0$, for all $n \geq 0$. If there exist $\lambda \in [0, 1)$ and an integer $k \geq 2$ such that

$$d(x_n, x_{n+1}) \leq \sqrt[k]{\lambda^{n-k}} M$$

for all $n \geq 0$, where

$$M = \max\{d(x_0, x_1), d(x_1, x_2), \ldots, d(x_{k-1}, x_k)\}$$

and there exist $\beta < \frac{1}{\sqrt[k]{\lambda}}$ and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, then $\{x_n\}_n$ is a Cauchy sequence.

Proof. Set $d_n = d(x_n, x_{n+1})$, for all $n \geq 0$ and $M = \max\{d_0, d_1\}$. Consider n, p, $j \in \mathbb{N}$ with $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$. Obviously, $\beta \sqrt[k]{\lambda} < 1$.

The following estimations will justify that $\{x_n\}_n$ is a Cauchy sequence.

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \theta(x_n, x_{n+p}) d(x_n, x_{n+1}) + \theta(x_n, x_{n+p}) d(x_{n+1}, x_{n+p}) \\ &\leq \theta(x_n, x_{n+p}) d_n + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) d(x_{n+1}, x_{n+2}) \\ &\quad + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) d(x_{n+2}, x_{n+p}) \\ &\leq \ldots \\ &\leq \theta(x_n, x_{n+p}) d_n + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) d_{n+1} \\ &\quad + \ldots + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \cdots \theta(x_{n+p-1}, x_{n+p}) d_{n+p-1} \\ &\leq \beta \sqrt[k]{\lambda^{n-k}} M + \beta^2 \sqrt[k]{\lambda^{n-k+1}} M + \ldots + \beta^p \sqrt[k]{\lambda^{n-k+p-1}} M \\ &\leq \beta \sqrt[k]{\lambda^{n-k}} M \Big[1 + \beta \sqrt[k]{\lambda} + \ldots + \beta^{p-1} \sqrt[k]{\lambda^{p-1}} \Big] \\ &= \beta \sqrt[k]{\lambda^{n-k}} M \frac{1 - (\beta \sqrt[k]{\lambda})^p}{1 - \beta \sqrt[k]{\lambda}} \\ &\leq \beta \sqrt[k]{\lambda^{n-k}} M \frac{1}{1 - \beta \sqrt[k]{\lambda}} \to 0, \end{aligned}$$

when $n \to \infty$. □

The following result is an extension of Corollary 3.1.

Theorem 3.3. *Let (X, d) be a complete extended b-metric space, let k be an integer such that $k \geq 2$, and let $T\colon X \to X$ be an orbitally continuous convex contraction of order k. Suppose there exist $x_0 \in X$,*

$$\beta < \frac{1}{\sqrt[k]{a_0 + a_1 + \ldots + a_{k-1}}}$$

and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$.

Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Construct the sequence $\{x_n\}_n$ of Picard iterations based on the initial point x_0. Set $d_n = d(x_n, x_{n+1})$, for all $n \geq 0$. Set

$$\lambda = a_0 + a_1 + \ldots + a_{k-1} \quad \text{and} \quad M = \max\{d_0, d_1, \ldots, d_{k-1}\}.$$

Affirmation 1. $d_n \leq \sqrt[k]{\lambda^{n-k}} M$, for all $n \geq 0$.

The statement will be proven by complete induction on n.

For $n \in \{0, 1, \ldots, k-1\}$, the inequality is obvious. For $n = k$, it follows that

$$\begin{aligned}
d_k &= d(x_k, x_{k+1}) = d(T^k x_0, T^k x_1)\\
&\leq a_0 d(x_0, x_1) + a_1 d(x_1, x_2) + \ldots + a_{k-1} d(x_{k-1}, x_k)\\
&= a_0 d_0 + a_1 d_1 + \ldots + a_{k-1} d_{k-1}\\
&\leq a_0 M + a_1 M + \ldots + a_{k-1} M\\
&= \lambda M \leq M = \lambda^0 M\\
&= \sqrt[k]{\lambda^{k-k}} M.
\end{aligned}$$

Thus, the base step is verified. Henceforth, consider $s \geq k$ such that $d_j \leq \sqrt[k]{\lambda^{j-k}} M$, for all $j \leq s$. Considering these inequalities, we present the inductive step:

$$\begin{aligned}
d_{s+1} &= d(x_{s+1}, x_{s+2}) = d(T^k x_{s-k+1}, T^k x_{s-k+2})\\
&\leq a_0 d_{s-k+1} + a_1 d_{s-k+2} + \ldots + a_{k-1} d_s\\
&\leq a_0 \sqrt[k]{\lambda^{s-2k+1}} M + a_1 \sqrt[k]{\lambda^{s-2k+2}} M + \ldots + a_{k-1} \sqrt[k]{\lambda^{s-k}} M\\
&\leq a_0 \sqrt[k]{\lambda^{s-2k+1}} M + a_1 \sqrt[k]{\lambda^{s-2k+1}} M + \ldots + a_{k-1} \sqrt[k]{\lambda^{s-2k+1}} M\\
&= M \sqrt[k]{\lambda^{s-2k+1}} [a_0 + a_1 + \ldots a_{k-1}]\\
&= M \sqrt[k]{\lambda^{s-2k+1}} \lambda\\
&= \sqrt[k]{\lambda^{(s+1)-k}} M.
\end{aligned}$$

Affirmation 2. T has a fixed point.

By using Lemma 3.3, $\{x_n\}_n$ is a Cauchy sequence. Now, employing Lemma 3.2, T has a fixed point $u \in X$.

Affirmation 3. T has a unique fixed point.

Assume that there exists $v \in X$ such that $v \neq u$ and $Tv = v$. Then,

$$\begin{aligned} d(u, v) &= d(T^k u, T^k v) \\ &\leq a_0 d(u, v) + a_1 d(Tu, Tv) + \ldots + a_{k-1} d(T^{k-1}u, T^{k-1}v) \\ &= a_0 d(u, v) + a_1 d(u, v) + \ldots + a_{k-1} d(u, v) \\ &= (a_0 + a_1 + \ldots + a_{k-1}) d(u, v) \\ &< d(u, v), \end{aligned}$$

which is a contradiction.

Consequently, u is the only fixed point of T. □

By taking $c_0 = a_1 = b_1 = 0$ and $a_2 = b_2$ in Theorem 3.1, we get the following power contraction version of the Reich fixed point theorem for extended b-metric spaces.

Corollary 3.3. Let (X, d_θ) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous mapping. Suppose there exist $a, b \in [0, 1)$ with $a + 2b < 1$ such that the inequality

$$d_\theta(T^2x, T^2y) \leq a d_\theta(Tx, Ty) + b[d_\theta(Tx, T^2x) + d_\theta(Ty, T^2y)]$$

holds for any $x, y \in X$. Also, presume there exist $x_0 \in X, \beta < \sqrt{\frac{1-b}{a+b}}$, and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$. Then T has a unique fixed point.

By setting $c_0 = a_1 = a_2 = b_1 = b_2 = 0$ in Theorem 3.1, we obtain the following extension of the contraction principle in the framework of extended b-metric spaces.

Corollary 3.4. Let (X, d_θ) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous mapping. If there exists $k \in [0, 1)$ such that the inequality

$$d_\theta(T^2x, T^2y) \leq k d_\theta(Tx, Ty)$$

holds for any $x, y \in X$, and if there exist $x_0 \in X, \beta < \frac{1}{\sqrt{k}}$, and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$, then T has a unique fixed point.

If we pick $c_1 = a_1 = a_2 = b_1 = b_2 = 0$ in Theorem 3.1, we get a power contraction version of the contraction principle on extended b-metric spaces.

Corollary 3.5. Let (X, d_θ) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous mapping. Presume there exists $k \in [0, 1)$ such that the inequality

$$d_\theta(T^2x, T^2y) \leq kd_\theta(x, y)$$

holds for any $x, y \in X$, and suppose there exist $x_0 \in X, \beta < \frac{1}{\sqrt{k}}$, and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$. Then, T has a fixed point, which is unique.

By choosing $c_0 = c_1 = a_1 = b_1 = 0$ and $a_2 = b_2$ in Theorem 3.1, we obtain a power contraction version of the Kannan fixed point theorem in the setting of extended b-metric spaces.

Corollary 3.6. Let (X, d_θ) be a complete extended b-metric space and let $T\colon X \to X$ be an orbitally continuous mapping. If there exists $k \in [0, \frac{1}{2})$ such that the inequality

$$d_\theta(T^2x, T^2y) \leq k[d_\theta(Tx, T^2x) + d_\theta(Ty, T^2y)]$$

holds for any $x, y \in X$, and if there exist $x_0 \in X, \beta < \sqrt{\frac{1-k}{k}}$, and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, $n \geq 0$, then T has a unique fixed point.

By taking $\theta(\cdot, \cdot) \equiv s \geq 1$, we obtain the following analogous result for Theorem 3.1 in the setting of b-metric spaces.

Corollary 3.7. Let (X, d) be a complete b-metric space and let $T\colon X \to X$ be an orbitally continuous convex contraction of type 2. If

$$s < \sqrt{\frac{1 - b_2}{c_0 + c_1 + a_1 + a_2 + b_1}},$$

then T has a unique fixed point.

Also, we obtain the corresponding results for Corollaries 3.1 and 3.2 for b-metric spaces.

Corollary 3.8. Let (X, d) be a complete b-metric space and let $T\colon X \to X$ be an orbitally continuous two-sided convex contraction. If

$$s < \sqrt{\frac{1 - b_2}{a_1 + a_2 + b_1}},$$

then T has a fixed point that is unique.

Corollary 3.9. Let (X, d) be a complete b-metric space and let $T\colon X \to X$ be an orbitally continuous convex contraction of order 2. Suppose that

$$s < \frac{1}{\sqrt{a+b}}.$$

Then T has a unique fixed point.

Consequently, we also acquire the analogous results for Theorems 3.2 and 3.3 in the setting of b-metric spaces.

Corollary 3.10. Let (X, d) be a complete b-metric space and let $T\colon X \to X$ be an orbitally continuous Ćirić-convex contraction. Presume

$$s < \frac{1}{\sqrt{h}}.$$

Then T has a unique fixed point.

Corollary 3.11. Let (X, d) be a complete extended b-metric space, let k be an integer such that $k \geq 2$, and let $T\colon X \to X$ be an orbitally continuous convex contraction of order k. Suppose that

$$s < \frac{1}{\sqrt[k]{a_0 + a_1 + \ldots + a_{k-1}}}.$$

Then T has a unique fixed point.

By taking $s = 1$, we obtain the main results from [15] and the analogue of Theorem 3.2 in the setting of metric spaces, stated next.

Corollary 3.12. Let (X, d) be a complete metric space and let $T\colon X \to X$ be a Ćirić-convex contraction. If T is orbitally continuous, then T has a unique fixed point.

Now, we provide an application of our proven results by presenting an example of a mapping which fulfills the conditions of a convex contractive mapping of order 2 so that it has a unique fixed point, following the work in [34]. Similar examples can be found for all the other types of convex contractive mappings presented in this paper.

Example 3.1. Let $X = [0, 1]$. Define $\theta\colon X \times X \to [1, \infty)$,

$$\theta(x, y) = \begin{cases} \frac{x+y+1}{x+y}, & \text{if } (x, y) \neq (0, 0), \\ 1.25, & \text{if } (x, y) = (0, 0). \end{cases}$$

Define $d_\theta\colon X \times X \to [0, \infty)$,

$$d_\theta(x, y) = \begin{cases} 0, & \text{if and only if } x = y, \\ \frac{1}{x}, & \text{if } xy = 0, \text{and } x^2 + y^2 \neq 0, \\ \frac{1}{xy}, & \text{if } 0 \neq x \neq y \neq 0. \end{cases}$$

Define $T\colon X \to X$,

$$Tx = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{4}], \\ 0, & \text{if } x \in (\frac{1}{4}, 1]. \end{cases}$$

We shall prove that the mapping T satisfies the conditions stated in Theorem 3.1, thus T has a unique fixed point.

Affirmation 1. (X, d_θ) is an extended b-metric space and is not a b-metric space.

Justification 1. As it was shown in [34], d_θ is an extended b-metric, thus (X, d_θ) is an extended b-metric space.

Due to an argument similar to the one used in Example 2.1, we conclude that (X, d_θ) is not a b-metric space.

Affirmation 2. T is a convex contraction of order 2.

Justification 2. Note that

$$T^2x = \begin{cases} 4x, & \text{if } x \in [0, \frac{1}{8}], \\ 0, & \text{if } x \in (\frac{1}{8}, 1]. \end{cases}$$

Observe, by induction, that

$$T^kx = \begin{cases} 2^kx, & \text{if } x \in \left[0, \frac{1}{2^{k+1}}\right], \\ 0, & \text{if } x \in (\frac{1}{2^{k+1}}, 1]. \end{cases}$$

To prove that T is a convex contraction of order 2, we have to find constants a,b $\in$ $[0, 1)$ with $a + b < 1$ such that the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

holds for any $x, y \in [0, 1]$. We will prove that $a = \frac{1}{2}$ and $b = \frac{1}{8}$ satisfy the contractive inequality.

Let $x, y \in X$. We have to consider different cases.

Case 1. If $x = y$, note that $Tx = Ty$ and $T^2x = T^2y$, thus

$$d_\theta(x, y) = d_\theta(Tx, Ty) = d_\theta(T^2x, T^2y) = 0,$$

and then, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes $0 \leq 0$, thus the contractive inequality holds for any a and b.

Next, we have to consider the cases where $x = 0$ or $y = 0$. Without loss of generality, due to the symmetry of d_θ, we can assume that $y = 0$.

Case 2. If $x \in (0, \frac{1}{8}]$ and $y = 0$, note that $Tx = 2x$, $T^2x = 4x$, $Ty = 0$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$\frac{1}{4x} \leq \frac{a}{2x} + \frac{b}{x},$$

or, equivalently,

$$1 \leq 2a + 4b,$$

which holds for $a = \frac{1}{2}$ and $b = \frac{1}{8}$.

Case 3. If $x \in (\frac{1}{8}, \frac{1}{4}]$ and $y = 0$, note that $Tx = 2x$, $T^2x = 0$, $Ty = 0$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$0 \leq \frac{a}{2x} + \frac{b}{x},$$

which holds for any a and $b \in [0, 1)$.

Case 4. If $x \in (\frac{1}{4}, 1]$ and $y = 0$, note that $Tx = 0$, $T^2x = 0$, $Ty = 0$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes $0 \leq 0$, thus the contractive inequality holds for any a and b.

Henceforth, assume that $0 \neq x \neq y \neq 0$.

Case 5. If $x, y \in (0, \frac{1}{8}]$, note that $Tx = 2x$, $T^2x = 4x$, $Ty = 2y$, and $T^2y = 4y$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$\frac{1}{16xy} \leq \frac{a}{4xy} + \frac{b}{xy},$$

or, equivalently,

$$1 \leq 4a + 16b,$$

which holds for $a = \frac{1}{2}$ and $b = \frac{1}{8}$.

Case 6. If $x, y \in (\frac{1}{8}, \frac{1}{4}]$, note that $Tx = 2x$, $T^2x = 0$, $Ty = 2y$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$0 \leq \frac{a}{4xy} + \frac{b}{xy},$$

thus the contractive inequality holds for any a and b.

Case 7. If $x, y \in (\frac{1}{4}, 1]$, note that $Tx = 0$, $T^2x = 0$, $Ty = 0$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$0 \leq 0 + \frac{b}{xy},$$

which holds for any $b \in [0, 1)$.

Due to the symmetry of d_θ, it is enough to consider only three more cases.

Case 8. If $x \in (0, \frac{1}{8}]$ and $y \in (\frac{1}{8}, \frac{1}{4}]$, note that $Tx = 2x$, $T^2x = 4x$, $Ty = 2y$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$\frac{1}{4x} \leq \frac{a}{4xy} + \frac{b}{xy},$$

or, equivalently,

$$y \leq a + 4b.$$

Note that with $a = \frac{1}{2}$ and $b = \frac{1}{8}$, we get that

$$y \leq 1 = a + 4b.$$

Case 9. If $x \in (0, \frac{1}{8}]$ and $y \in (\frac{1}{4}, 1]$, note that $Tx = 2x$, $T^2x = 4x$, $Ty = 0$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$\frac{1}{4x} \leq \frac{a}{2x} + \frac{b}{xy},$$

or, equivalently,

$$y \leq 2ay + 4b.$$

Considering $a = \frac{1}{2}$ and $b = \frac{1}{8}$, we get that

$$y \leq y + \frac{1}{2} = 2ay + 4b.$$

Case 10. If $x \in (\frac{1}{8}, \frac{1}{4}]$ and $y \in (\frac{1}{4}, 1]$, note that $Tx = 2x$, $T^2x = 0$, $Ty = 0$, and $T^2y = 0$. Thus, the inequality

$$d_\theta(T^2x, T^2y) \leq ad_\theta(Tx, Ty) + bd_\theta(x, y)$$

becomes

$$0 \leq \frac{a}{2x} + \frac{b}{xy},$$

which holds for any $a, b \in [0, 1)$.

Consequently, considering the cases presented above, we can conclude that T is a convex contraction of order 2 with $a = \frac{1}{2}$ and $b = \frac{1}{8}$.

Affirmation 3. There exist

$$\beta < \frac{1}{\sqrt{a+b}}$$

and $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq \beta^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, where $x_n = T^n x_0$, for all $n \geq 0$.

Justification 3. Let $p \geq 1$.

$$\frac{1}{\sqrt{a+b}} = \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{8}}} = \frac{2\sqrt{10}}{5}.$$

Let

$$\beta = 1.26 < \frac{2\sqrt{10}}{5}.$$

To prove that there exists $n_0 \in \mathbb{N}$ such that the inequality

$$\prod_{i=n}^{j} \theta(x_i, x_{n+p}) \leq 1.26^{j-n+1}$$

holds for all $n \geq n_0$, $p \geq 1$, and $j \in \{n, n+1, \ldots, n+p-1\}$, it is enough to prove that there exists $n_0 \in \mathbb{N}$ such that $\theta(x_k, x_{n+p}) < 1.26$ holds for all $k \in \{n, n+1, \ldots, n+p-1\}$, $k \geq n_0$.

Let $k \in \{n, n+1, \ldots, n+p-1\}$. If $x_0 = 0$, $T^n x_0 = 0$, for all $n \in \mathbb{N}$. Therefore,

$$\theta(x_k, x_{n+p}) = \theta(0, 0) = 1.25 \leq 1.26.$$

If $x_0 \in (0, 1]$, let $q \in \mathbb{N}$ be the largest number such that $x_0 \in (0, \frac{1}{2^q}]$. Set $n_0 = q + 1$. Therefore, $x_k = T^k x_0 = 0$, for any $k \geq n_0$. As $k \leq n + p$, obviously, $x_{n+p} = 0$ also. Therefore,

$$\theta(x_k, x_{n+p}) = \theta(0, 0) = 1.25 \leq 1.26$$

for any $k \geq n_0$.

Therefore, all the conditions from Theorem 3.1 are satisfied.

Remark that 0 is a fixed point of T. We can conclude that 0 is the only fixed point of T.

4. Conclusions

This research article delved into the exploration of convex contractions within the realm of extended b-metric spaces, with a particular emphasis on the existence and uniqueness of fixed points under various convexity conditions. The introduction of the Ćirić-convex contraction enriches the fixed point theory by broadening the understanding of contractive mappings.

The findings underscore the versatility and potency of convex contractions in generalized metric spaces, offering novel insights and extending the boundaries of traditional fixed point theorems. This work not only contributes to the existing literature by providing generalizations and new perspectives on convex contractions, but also lays the groundwork for future research in the setting of generalized extended b-metric spaces, such as new extended b-metric spaces [45], controlled metric-type spaces [46] and double controlled metric-type spaces [47], etc.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no competing interests.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/FM-3-1-133-181
2. R. Caccioppoli, Un teorema generale sull' esistenza di elementi uniti in una transformazione funzionale, *Rend. Accad. Lincei*, **11** (1930), 794–799.
3. E. Rakotch, A note on contractive mappings, *Proc. Amer. Math. Soc.*, **13** (1962), 459–465. https://doi.org/10.1090/S0002-9939-1962-0148046-1
4. D. Boyd, J. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.*, **20** (1969), 458–464. https://doi.org/10.1090/S0002-9939-1969-0239559-9
5. A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, **28** (1969), 326–329. https://doi.org/10.1016/0022-247X(69)90031-6
6. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.
7. V. Subrahmanyam, Completeness and fixed points, *Monatsh. Math.*, **80** (1975), 325–330. https://doi.org/10.1007/BF01472580
8. S. K. Chatterjea, Fixed-point theorems, *C. R. Acad. Bulg. Sci.*, **25** (1972), 727–730.

9. S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.*, **14** (1971), 121–124. https://doi.org/10.4153/CMB-1971-024-9
10. G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.*, **16** (1973), 201–206. https://doi.org/10.4153/CMB-1973-036-0
11. L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267–273.
12. H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa, N. Tahat, Theorems for Boyd-Wong-type contractions in ordered metric spaces, *Abstr. Appl. Anal.*, **2012** (2012), 359054. https://doi.org/10.1155/2012/359054
13. N. V. Luong, N. X. Thuan, Fixed point theorem for generalized weak contractions satisfying rational expressions in ordered metric spaces, *Fixed Point Theory Appl.*, **2011** (2011), 46. https://doi.org/10.1186/1687-1812-2011-46
14. B. E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, **226** (1977), 257–290. https://doi.org/10.1090/S0002-9947-1977-0433430-4
15. V. I. Istrăţescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters. I, *Ann. Mat. Pura Appl.*, **130** (1982), 89–104. https://doi.org/10.1007/BF01761490
16. M. A. Miandaragh, M. Postolache, S. Rezapour, Approximate fixed points of generalized convex contractions, *Fixed Point Theory Appl.*, **2013** (2013), 255. https://doi.org/10.1186/1687-1812-2013-255
17. I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal.*, **30** (1989), 26–37.
18. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inf. Univ. Ostraviensis*, **1** (1993), 5–11.
19. M. U. Ali, T. Kamran, M. Postolache, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, *Nonlinear Anal. Modell. Control*, **22** (2017), 17–30. https://doi.org/10.15388/NA.2017.1.2
20. T. Kamran, M. Postolache, M. U. Ali, Q. Kiran, Feng and Liu type F-contraction in b-metric spaces with application to integral equations, *J. Math. Anal.*, **7** (2016), 18–27.
21. W. Shatanawi, A. Pitea, R. Lazović, Contraction conditions using comparison functions on b-metric spaces, *Fixed Point Theory Appl.*, **2014** (2014), 135. https://doi.org/10.1186/1687-1812-2014-135
22. W. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Springer, 2014. https://doi.org/10.1007/978-3-319-10927-5
23. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. https://doi.org/10.5486/pmd.2000.2133
24. P. Hitzler, A. K. Seda, Dislocated topologies, *J. Electr. Eng.*, **51** (2000), 3–7.
25. F. Khojasteh, E. Karapınar, S. Radenovic, θ-metric space: a generalization, *Math. Probl. Eng.*, **2013** (2013), 504609. https://doi.org/10.1155/2013/504609
26. N. V. Dung, Further results on θ-metric spaces, *Fixed Point Theory*, **25** (2024), 99–110.

27. M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, *Fixed Point Theory Appl.*, **2015** (2015), 61. https://doi.org/10.1186/s13663-015-0312-7
28. M. Jleli, B. Samet, On a new generalization of metric spaces, *Fixed Point Theory Appl.*, **20** (2018), 128. https://doi.org/10.1007/s11784-018-0606-6
29. T. Kamran, M. Samreen, Q. U. Ain, A generalization of b-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 19. https://doi.org/10.3390/math5020019
30. M. Samreen, T. Kamran, M. Postolache, Extened b-metric space, extended b-comparison function and nonlinear contractions, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **80** (2018), 21–28.
31. M. Samreen, Q. Kiran, T. Kamran, Fixed point theorems for φ-contractions, *J. Inequal. Appl.*, **2014** (2014), 266. https://doi.org/10.1186/1029-242X-2014-266
32. P. D. Proinov, A generalization of the Banach contraction principle with high order of convergence of succesive approximations, *Nonlinear Anal.*, **67** (2007), 2361–2369. https://doi.org/10.1016/j.na.2006.09.008
33. T. L. Hicks, B. E. Rhoades, A Banach type fixed point theorem, *Math. Japonica*, **24** (1979), 327–330.
34. B. Alqahtani, A. Fulga, E. Karapınar, Common fixed point results on an extended b-metric space, *J. Inequal. Appl.*, **2018** (2018), 158. https://doi.org/10.1186/s13660-018-1745-4
35. T. Abdeljawad, R. P. Agarwal, E. Karapınar, S. K. Panda, Solution of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, *Symmetry*, **11** (2019), 686. https://doi.org/10.3390/sym11050686
36. W. Shatanawi, K. Abodayeh, A. Mukheimer, Some fixed point theorems in extended b-metric spaces, *UPB Sci. Bull.*, **80** (2018), 71–78.
37. B. Alqahtani, A. Fulga, E. Karapınar, Non-unique fixed point results in extended b-metric space, *Mathematics*, **6** (2018), 68. https://doi.org/10.3390/math6050068
38. Z. D. Mitrović, H. Işik, S. Radenović, The new results in extended b-metric spaces and applications, *Int. J. Nonlinear Anal. Appl.*, **11** (2020), 473–482. https://doi.org/10.22075/IJNAA.2019.18239.1998
39. S. B. Nadler, Multivalued contraction mappings, *Pacific J. Math.*, **30** (1969), 475–488. https://doi.org/10.2140/PJM.1969.30.475
40. H. Huang, Y. M. Singh, M. S. Khan, S. Radenović, Rational type contractions in extended b-metric spaces, *Symmetry*, **13** (2021), 614. https://doi.org/10.3390/sym13040614
41. B. Alqahtani, A. Fulga, E. Karapınar, V. Rakočević, Contractions with rational inequalities in the extended b-metric space, *J. Inequal. Appl.*, **2019** (2019), 220. https://doi.org/10.1186/s13660-019-2176-6
42. Q. Kiran, N. Alamgir, N. Mlaiki, H. Aydi, On some new fixed-point results in complete extended b-metric spaces, *Mathematics*, **7** (2019), 476. https://doi.org/10.3390/math7050476
43. N. Haokip, N. Goswami, B. C. Tripathy, Common fixed point results for (α-β)-orbital-cyclic admissible triplet in extended b-metric spaces, *Thai J. Math.*, **20** (2022), 563–576.

44. L. B. Ćirić, On contraction type mappings, *Math. Balkanica*, **1** (1971), 52–57.

45. H. Aydi, A. Felhi, T. Kamran, E. Karapınar, M. U. Ali, On nonlinear contractions in new extended *b*-metric spaces, *Appl. Appl. Math.*, **14** (2019), 537–547.

46. N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6** (2018), 194. https://doi.org/10.3390/math6100194

47. T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, *Mathematics*, **6** (2018), 320. https://doi.org//10.3390/math6120320