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Research article

Subordinations and superordinations studies using q-difference operator

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Abstract: The results of this work belong to the field of geometric function theory, being based on differential subordination methods. Using the idea of the $\mathfrak{q}$-calculus operators, we define the $\mathfrak{q}$-analogue of the multiplier- Ruscheweyh operator of a specific family of linear operators, $I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)$. Our major goal is to build and investigate some analytic function subclasses using $I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)$. Also, some differential subordination and superordination results are obtained. Moreover, based on the new theoretical results, several examples are constructed. For every differential superordination under investigation, the best subordinant is provided.

Keywords: analytic function; differential subordination; superordination; $\mathfrak{q}$-difference operator; $\mathfrak{q}$-analogue Catas operator; $\mathfrak{q}$-analogue of Ruscheweyh operator
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1. Introduction

Some of the topics in geometric function theory are based on $\mathfrak{q}$-calculus operator and differential subordinations. Ismail et al. defined the class of $\mathfrak{q}$-starlike functions in 1990 [1], presenting the first uses of $\mathfrak{q}$-calculus in geometric function theory. Several authors focused on the $\mathfrak{q}$-analogue of the Ruscheweyh differential operators established in [2] and the $\mathfrak{q}$-analogue of the Sălăgean differential operators defined in [3]. Examples include the investigation of differential subordinations using a specific $\mathfrak{q}$-Ruscheweyh type derivative operator in [4].

In what follows, we recall the main concepts used in this research.

We denote by H the class of analytic functions in the open unit disc $\mathbf{U} := \{\xi \in \mathbb{C} : |\xi| < 1\}$. Also, $H[a, n]$ denotes the subclass of H, containing the functions $f \in H$ given by

$$\mathfrak{f}(\xi) = a + a_n\xi^n + a_{n+1}\xi^{n+1} + ..., \qquad \xi \in \mathbf{U}.$$

Another well-known subclass of H is class $A(n)$, which consists of $\mathfrak{f} \in H$ and is given by

$$\mathfrak{f}(\xi) = \xi + \sum_{\kappa=n+1}^{\infty} a_\kappa \xi^\kappa, \ \xi \in \mathbf{U}, \tag{1.1}$$

with $n \in \mathbb{N} = \{1, 2, ...\}$, and A=$A(1)$.

The subclass K is defined by

$$K = \left\{ f \in \mathbf{A} : \operatorname{Re}\left(\frac{\xi \mathfrak{f}''(\xi)}{\mathfrak{f}'(\xi)} + 1 \right) > 0, \ \mathfrak{f}(0) = 0, \ \mathfrak{f}'(0) = 1, \ \xi \in \mathbf{U} \right\},$$

means the class of convex functions in the unit disk $\mathbf{U}$.

For two functions $\mathfrak{f}, \mathcal{L}$ (belong) to $A(n)$, $\mathfrak{f}$ given by (1.1), and $\mathcal{L}$ is given by the next form

$$\mathcal{L}(\xi) = \xi + \sum_{\kappa=n+1}^{\infty} b_\kappa \xi^\kappa, \ \xi \in \mathbf{U},$$

the well-known *convolution product* was defined as: $^* : A \to A$

$$(\mathfrak{f} * \mathcal{L})(\xi) := \xi + \sum_{\kappa=n+1}^{\infty} a_\kappa b_\kappa \xi^\kappa, \ \xi \in \mathbf{U}.$$

In particular [5,6], several applications of Jackson's $\mathfrak{q}$-difference operator $\mathfrak{d}_\mathfrak{q} : A \to A$ are defined by

$$\mathfrak{d}_\mathfrak{q}\mathfrak{f}(\xi) := \begin{cases} \frac{\mathfrak{f}(\xi) - \mathfrak{f}(\mathfrak{q}\xi)}{(1-\mathfrak{q})\xi} & (\xi \neq 0; \ 0 < \mathfrak{q} < 1), \\ \\ \mathfrak{f}'(0) & (\xi = 0). \end{cases} \tag{1.2}$$

Maybe we can put just $\kappa \in \mathbb{N} = \{1, 2, 3, ..\}$. It is written once previously

$$\mathfrak{d}_\mathfrak{q} \left\{ \sum_{\kappa=1}^{\infty} a_\kappa \xi^\kappa \right\} = \sum_{\kappa=1}^{\infty} [\kappa]_\mathfrak{q} \, a_\kappa \xi^{\kappa-1}, \tag{1.3}$$

where

$$[\kappa]_\mathfrak{q} = \frac{1 - \mathfrak{q}^\kappa}{1 - \mathfrak{q}} = 1 + \sum_{n=1}^{\kappa-1} \mathfrak{q}^n, \ \lim_{q \to 1^-} [\kappa]_q = \kappa.$$

$$[\kappa]_\mathfrak{q}! = \begin{cases} \prod_{n=1}^{\kappa} [n]_\mathfrak{q}, & \kappa \in \mathbb{N}, \\ 1 & \kappa = 0. \end{cases} \tag{1.4}$$

In [7], Aouf and Madian investigate the $\mathfrak{q}$-analogue Cătas operator $I_{\mathfrak{q}}^{s}(\lambda,\ell): A \to A$ $(s \in \mathbb{N}_0, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1)$ as follows:

$$I_{\mathfrak{q}}^{s}(\lambda,\ell)\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty}\left(\frac{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}{[1+\ell]_{\mathfrak{q}}}\right)^{s} a_{\kappa}\xi^{\kappa}, \quad (s \in \mathbb{N}_0, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1).$$

Also, the $\mathfrak{q}$-Ruscheweyh operator $\mathfrak{R}_{\mathfrak{q}}^{\mu}\mathfrak{f}(\xi)$ was investigated in 2014 by Aldweby and Darus [8]

$$\mathfrak{R}_{\mathfrak{q}}^{\mu}\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty}\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa}, \ (\mu \geq 0, 0 < \mathfrak{q} < 1),$$

where $[a]_{\mathfrak{q}}$ and $[a]_{\mathfrak{q}}!$ are defined in (1.4).

Let be

$$\mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s}(\xi) = \xi + \sum_{\kappa=2}^{\infty}\left(\frac{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}{[1+\ell]_{\mathfrak{q}}}\right)^{s} \xi^{\kappa}.$$

Now we define a new function $\mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s,\mu}(\xi)$ in terms of the Hadamard product (or convolution) such that:

$$\mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s}(\xi) * \mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s,\mu}(\xi) = \xi + \sum_{\kappa=2}^{\infty}\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}\xi^{\kappa}.$$

Next, driven primarily by the $\mathfrak{q}$-Ruscheweyh operator and the $\mathfrak{q}$-Cătas operator, we now introduce the operator $I_{\mathfrak{q},\mu}^{s}(\lambda,\ell): A \to A$ is defined by

$$I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi) = \mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s,\mu}(\xi) * \mathfrak{f}(\xi), \tag{1.5}$$

where $s \in \mathbb{N}_0, \ell, \lambda, \mu \geq 0, 0 < \mathfrak{q} < 1$. For $\mathfrak{f} \in \mathbf{A}$ and (1.5), it is obvious

$$I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty}\psi_{\mathfrak{q}}^{*s}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa}, \tag{1.6}$$

where

$$\psi_{\mathfrak{q}}^{*s}(\kappa,\lambda,\ell) = \left(\frac{[1+\ell]_{\mathfrak{q}}}{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}\right)^{s}.$$

We observe that:

(*i*) If $s = 0$ and $\mathfrak{q} \to 1^{-}$, we get $\mathfrak{R}^{\mu}\mathfrak{f}(\xi)$ is a Russcheweyh differential operator [9] investigated by numerous authors [10–12].

(*ii*) If we set $\mathfrak{q} \to 1^{-}$, we obtain $I_{\lambda,\ell,\mu}^{m}\mathfrak{f}(\xi)$ which was presented by Aouf and El-Ashwah [13].

(*iii*) If we set $\mu = 0$ and $\mathfrak{q} \to 1^{-}$, we obtain $J_{\mathfrak{p}}^{m}(\lambda,\ell)\mathfrak{f}(\xi)$, presented by El-Ashwah and Aouf (with $p = 1$) [14].

(*iv*) If $\mu = 0$, $\ell = \lambda = 1$, and $\mathfrak{q} \to 1^{-}$, we get $\wp^{\alpha}\mathfrak{f}(\xi)$, investigated by Jung et al. [15].

(*v*) If $\mu = 0$, $\lambda = 1$, $\ell = 0$, and $\mathfrak{q} \to 1^{-}$, we obtain $I^{s}\mathfrak{f}(\xi)$, presented by Sălăgean [16].

(*vi*) If we set $\mu = 0$ and $\lambda = 1$, we obtain $I_{\mathfrak{q},s}^{\ell}\mathfrak{f}(\xi)$, presented by Shah and Noor [17].

(vii) If we set $\mu = 0$, $\lambda = 1$, and q$\rightarrow 1^{-}$, we obtain $J^{s}_{\mathfrak{q},\ell}$ Srivastava–Attiya operator: see [18, 19].
($viii$) $I^{1}_{\mathfrak{q},0}(1,0) = \int_0^{\xi} \frac{\mathfrak{f}(t)}{t} \mathfrak{d}_{\mathfrak{q}} t$. ($\mathfrak{q}$-Alexander operator [17]).
(ix) $I^{1}_{\mathfrak{q},0}(1,\ell) = \frac{[1+\rho]_{\mathfrak{q}}}{\xi^{\rho}} \int_0^{\xi} t^{\rho-1} \mathfrak{f}(t) \mathfrak{d}_{\mathfrak{q}} t$ ($\mathfrak{q}$-Bernardi operator [20]).
(x) $I^{1}_{\mathfrak{q},0}(1,1) = \frac{[2]_{\mathfrak{q}}}{\xi} \int_0^{\xi} \mathfrak{f}(t) \mathfrak{d}_{\mathfrak{q}} t$ ($\mathfrak{q}$-Libera operator [20]).
Moreover, we have
(i) $I^{s}_{\mathfrak{q},\mu}(1,0)\mathfrak{f}(\xi) = I^{s}_{\mathfrak{q},\mu}\mathfrak{f}(\xi)$

$$\mathfrak{f}(\xi) \in \mathbf{A} : I^{s}_{\mathfrak{q},\mu}\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \left(\frac{1}{[\kappa]_{\mathfrak{q}}} \right)^{s} \frac{[\kappa + \mu - 1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}! [\kappa - 1]_{\mathfrak{q}}!} a_{\kappa} \xi^{\kappa}, \ (s \in \mathbb{N}_0, \mu \geq 0, 0 < \mathfrak{q} < 1, \xi \in \mathbf{U}).$$

(ii) $I^{s}_{\mathfrak{q},\mu}(1,\ell)\mathfrak{f}(\xi) = I^{s,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(\xi)$

$$\begin{aligned} \mathfrak{f}(\xi) \quad &\in \quad \mathbf{A} : I^{s,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \left(\frac{[1+\ell]_{\mathfrak{q}}}{[\kappa+\ell]_{\mathfrak{q}}} \right)^{s} \frac{[\kappa + \mu - 1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}! [\kappa - 1]_{\mathfrak{q}}!} a_{\kappa} \xi^{\kappa}, \\ (s \quad &\in \quad \mathbb{N}_0, \ell > 0, \mu \geq 0, 0 < \mathfrak{q} < 1, \xi \in \mathbf{U}). \end{aligned}$$

(iii) $I^{s}_{\mathfrak{q},\mu}(\lambda,0)\mathfrak{f}(\xi) = I^{s,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(\xi)$

$$\begin{aligned} \mathfrak{f}(\xi) \quad &\in \quad \mathbf{A} : I^{s,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \left(\frac{1}{1 + \lambda([\kappa]_{\mathfrak{q}} - 1)} \right)^{s} \frac{[\kappa + \mu - 1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}! [\kappa - 1]_{\mathfrak{q}}!} a_{\kappa} \xi^{\kappa}, \\ (s \quad &\in \quad \mathbb{N}_0, \lambda > 0, \mu \geq 0, 0 < \mathfrak{q} < 1, \xi \in \mathbf{U}). \end{aligned}$$

Since the investigation of $\mathfrak{q}$-difference equations using function theory tools explores various properties, this direction has been considered in many works. Thus, several authors used the $\mathfrak{q}$-calculus based linear extended operators recently defined for investigating theories of differential subordination and subordination (see [21–32]). Applicable problems involving $\mathfrak{q}$-difference equations and $\mathfrak{q}$-analogues of mathematical physical problems are studied extensively for: Dynamical systems, $\mathfrak{q}$-oscillator, $\mathfrak{q}$-classical, and quantum models; $\mathfrak{q}$-analogues of mathematical-physical problems, including heat and wave equations; and sampling theory of signal analysis [33, 34].

We denote by Φ the class of analytic univalent functions $\varphi(\xi)$, which are convex functions with $\varphi(0) = 1$ and $\mathrm{Re}\varphi(\xi) > 0$ in $\mathbf{U}$.

The differential subordination theory, studied by Miller and Mocanu [35], is based on the following definitions:

$\mathfrak{f}$ is subordinate to $\mathcal{L}$ in $\mathbf{U}$, denote it as $\mathfrak{f} \prec \mathcal{L}$ if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$ for all $\xi \in \mathbf{U}$, such that $\mathfrak{f}(\xi) = \mathcal{L}(\varpi(\xi))$. Moreover, if $\mathcal{L}$ is univalent in $\mathbf{U}$, we have:

$$\mathfrak{f}(\xi) \prec \mathcal{L}(\xi) \Leftrightarrow \mathfrak{f}(0) = \mathcal{L}(0) \quad \text{and} \quad \mathfrak{f}(\mathbf{U}) \subset \mathcal{L}(\mathbf{U}).$$

Let $\Phi(r, s, t; \xi) : \mathbb{C}^3 \times \mathbf{U} \to \mathbb{C}$ and let $\mathfrak{h}$ in $\mathbf{U}$ be a univalent function. An analytic function λ in $\mathbf{U}$, which validates the differential subordination, is a solution of the differential subordination

$$\Phi(\lambda(\xi), \xi\lambda'(\xi), \xi^2\lambda''(\xi); \xi) \prec \mathfrak{h}(\xi). \tag{1.7}$$

We call $\mathfrak{V}$ a dominant of the solutions of the differential subordination in (1.7) if $\lambda(\xi) \prec \mathfrak{V}(\xi)$ for all λ satisfying (1.7). A dominant $\widetilde{\varkappa}$ is called the best dominant of (1.7) if $\widetilde{\mathfrak{V}}(\xi) \prec \mathfrak{V}(\xi)$ for all the dominants $\mathfrak{V}$.

The following definitions characterize both of the theories of differential superordination that Miller and Mocanu introduced in 2003 [36]:

$\mathfrak{f}$ is superordinate to $\mathcal{L}$, denotes as $\mathcal{L} \prec \mathfrak{f}$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$ for all $\xi \in \mathbf{U}$, such that $\mathcal{L}(\xi) = \mathfrak{f}(\varpi(\xi))$. For the univalent function $\mathfrak{f}$, we have

$$\mathcal{L}(\xi) \prec \mathfrak{f}(\xi) \Leftrightarrow \mathfrak{f}(0) = \mathcal{L}(0) \quad \text{and} \quad \mathcal{L}(\mathbf{U}) \subset \mathfrak{f}(\mathbf{U}).$$

Let $\Phi(r, s; \xi) : \mathbb{C}^2 \times \mathbf{U} \to \mathbb{C}$ and let $\mathfrak{h}$ in $\mathbf{U}$ be an analytic function. A solution of the differential superordination is the univalent function λ such that $\Phi(\lambda(\xi), \xi\lambda^{'}(\xi); \xi)$ is univalent in $\mathbf{U}$ satisfy the differential superordination

$$\mathfrak{h}(\xi) \prec \Phi(\lambda(\xi), \xi\lambda^{'}(\xi); \xi), \tag{1.8}$$

then λ is called to be a solution of the differential superordination in (1.8). We call the function $\mathfrak{V}$ a subordinant of the solutions of the differential superordination in (1.8) if $\mathfrak{V}(\xi) \prec \lambda(\xi)$ for all λ satisfying (1.8). A subordinant $\widetilde{\mathfrak{V}}$ is called the best subordinant of (1.8) if $\mathfrak{V}(\xi) \prec \widetilde{\mathfrak{V}}(\xi)$ for all the subordinants $\mathfrak{V}$.

Let $\wp$ say the collection of injective and analytic functions on $\overline{\mathbf{U}} \setminus E(\chi)$, with $\chi^{'}(\xi) \neq 0$ for $\xi \in \partial\mathbf{U} \setminus E(\chi)$ and

$$E(\chi) = \{\varsigma : \varsigma \in \partial\mathbf{U} \ : \ \lim_{\xi \to \varsigma} \chi(\xi) = \infty\}.$$

Also, $\wp(a)$ is the subclass of $\wp$ with $\chi(0) = a$.

The proofs of our main results and findings in the upcoming sections can benefit from the usage of the following lemmas:

Lemma 1.1. *(Miller and Mocanu [35]). Suppose* $\mathfrak{g}$ *is convex in* $\mathbf{U}$, *and*

$$\mathfrak{h}(\xi) = n\gamma\xi\mathfrak{g}^{'}(\xi) + \mathfrak{g}(\xi),$$

with $\xi \in \mathbf{U}$, *n is +ve integer and* $\gamma > 0$. *When*

$$\mathfrak{g}(0) + \mathfrak{p}_n\xi^n + \mathfrak{p}_{n+1}\xi^{n+1} + ... = \mathfrak{p}(\xi), \quad \xi \in \mathbf{U},$$

is analytic in $\mathbf{U}$, *and*

$$\gamma\xi\mathfrak{p}^{'}(\xi) + \mathfrak{p}(\xi) \prec \mathfrak{h}(\xi), \quad \xi \in \mathbf{U},$$

holds, then

$$\mathfrak{p}(\xi) \prec \mathfrak{g}(\xi),$$

holds as well.

Lemma 1.2. *(Hallenbeck and Ruscheweyh [37], see also (Miller and Mocanu [38], Th. 3.1.b, p.71)). Let* $\mathfrak{h}$ *be a convex with* $\mathfrak{h}(0) = a$, *and let* $\gamma \in \mathbb{C}^*$ *with* $Re(\gamma) \geq 0$. *When* $\mathfrak{p} \in H[a, n]$ *and*

$$\mathfrak{p}(\xi) + \frac{\xi\mathfrak{p}^{'}(\xi)}{\gamma} \prec \mathfrak{h}(\xi), \qquad \xi \in \mathbf{U},$$

holds, then

$$\mathfrak{p}(\xi) \prec \mathfrak{g}(\xi) \prec \mathfrak{h}(\xi), \qquad \xi \in \mathbf{U},$$

holds for

$$\mathfrak{g}(\xi) = \frac{\gamma}{n\xi^{(\gamma/n)}} \int_0^{\xi} \mathfrak{h}(t)t^{(\gamma/n)-1}dt, \qquad \xi \in \mathbf{U}.$$

Lemma 1.3. *(Miller and Mocanu [35]) Let* $\mathfrak{h}$ *be a convex with* $\mathfrak{h}(0) = a$*, and let* $\gamma \in \mathbb{C}^*$*, with* $Re(\gamma) \geq 0$*. When* $\mathfrak{p} \in Q \cap H[a,n]$*,* $\mathfrak{p}(\xi) + \frac{\xi \mathfrak{p}'(\xi)}{\gamma}$ *is a univalent in* $\mathbf{U}$ *and*

$$\mathfrak{h}(\xi) \prec \mathfrak{p}(\xi) + \frac{\xi \mathfrak{p}'(\xi)}{\gamma}, \qquad \xi \in \mathbf{U},$$

holds, then

$$\mathfrak{g}(\xi) \prec \mathfrak{p}(\xi), \qquad \xi \in \mathbf{U},$$

holds as well, for $\mathfrak{g}(\xi) = \frac{\gamma}{n\xi^{(\gamma/n)}} \int_0^{\xi} \mathfrak{h}(t)t^{(\gamma/n)-1}dt$*,* $\xi \in \mathbf{U}$ *the best subordinant.*

Lemma 1.4. *(Miller and Mocanu [35]) Let a convex* $\mathfrak{g}$ *be in* $\mathbf{U}$*, and*

$$\mathfrak{h}(\xi) = \mathfrak{g}(\xi) + \frac{\xi \mathfrak{g}'(\xi)}{\gamma}, \qquad \xi \in \mathbf{U},$$

with $\gamma \in \mathbb{C}^*$*,* $Re(\gamma) \geq 0$*. If* $\mathfrak{p} \in Q \cap H[a,n]$*,* $\mathfrak{p}(\xi) + \frac{\xi \mathfrak{p}'(\xi)}{\gamma}$ *is a univalent in* $\mathbf{U}$ *and*

$$\mathfrak{g}(\xi) + \frac{\xi \mathfrak{g}'(\xi)}{\gamma} \prec \mathfrak{p}(\xi) + \frac{\xi \mathfrak{p}'(\xi)}{\gamma}, \qquad \xi \in \mathbf{U},$$

holds, then

$$\mathfrak{g}(\xi) \prec \mathfrak{p}(\xi), \qquad \xi \in \mathbf{U},$$

holds as well, for $\mathfrak{g}(\xi) = \frac{\gamma}{n\xi^{(\gamma/n)}} \int_0^{\xi} \mathfrak{h}(t)t^{(\gamma/n)-1}dt$*,* $\xi \in \mathbf{U}$ *the best subordinant.*

For $\acute{a}, \varrho, \acute{c}$ and ć(ć $\notin \mathbb{Z}_0^-$) let consider the following Gaussian hypergeometric function is

$${}_2F_1(\acute{a}, \varrho; \acute{c}; \xi) = 1 + \frac{\acute{a}\varrho}{\acute{c}} . \frac{\xi}{1!} + \frac{\acute{a}(\acute{a}+1)\varrho(\varrho+1)}{\acute{c}(\acute{c}+1)} . \frac{\xi^2}{2!} + \dots .$$

For $\xi \in \mathbf{U}$, the above series completely converges to an analytic function in $\mathbf{U}$, (see, for details, [[39], Chapter 14]).

Lemma 1.5. *[39] For* $\acute{a}, \varrho$ *and* $\acute{c}$ *(*$\acute{c} \notin \mathbb{Z}_0^-$*), complex parameters*

$$\int_0^1 t^{\varrho-1}(1-t)^{\acute{c}-\varrho-1}(1-\xi t)^{-\acute{a}}dt = \frac{\Gamma(\varrho)\Gamma(\acute{c}-\varrho)}{\Gamma(\acute{c})} {}_2F_1(\acute{a}, \varrho; \acute{c};\xi) \quad (\mathrm{Re}(\acute{c}) > \mathrm{Re}(\varrho) > 0);$$

$$
\begin{aligned}
{}_2F_1(\acute{a},\varrho;\acute{c};\xi) &= {}_2F_1(\varrho,\acute{a};\acute{c};\xi);\\
{}_2F_1(\acute{a},\varrho;\acute{c};\xi) &= (1-\xi)^{-\acute{a}}\,{}_2F_1(\acute{a},\acute{c}-\varrho;\acute{c};\frac{\xi}{\xi-1});\\
{}_2F_1(1,1;2;\frac{\acute{a}\xi}{\acute{a}\xi+1}) &= \frac{(1+\acute{a}\xi)\ln(1+\acute{a}\xi)}{\acute{a}\xi};\\
{}_2F_1(1,1;3;\frac{\acute{a}\xi}{\acute{a}\xi+1}) &= \frac{2(1+\acute{a}\xi)}{\acute{a}\xi}\left(1-\frac{\ln(1+\acute{a}\xi)}{\acute{a}\xi}\right).
\end{aligned}
$$

A $\mathfrak{q}$-multiplier-Ruscheweyh operator is considered in the study reported in this paper to create a novel convex subclass of normalized analytic functions in the open unit disc $\mathbf{U}$. Then, employing the techniques of differential subordination and superordination theory, this subclass is examined in more detail.

2. Differential subordination results

$I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)$ given in (1.6) is a $\mathfrak{q}$-multiplier-Ruscheweyh operator that is applied to define the new class of normalized analytic functions in the open unit disc $\mathbf{U}$.

Definition 2.1. Let $\alpha \in [0,1)$. The class $\mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$ involves of the function $\mathfrak{f} \in \mathbf{A}$ with

$$
\mathrm{Re}\left(I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)' > \alpha, \qquad \xi \in \mathbf{U}. \tag{2.1}
$$

We use the following denotations:
(*i*) $\mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell;0) = \mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell)$.
(*ii*) $\mathfrak{S}^0_{\mathfrak{q},0}(\lambda,\ell;\alpha) = \mathfrak{S}(\alpha)$ $(\mathrm{Re}\mathfrak{f}(\xi)' > \alpha)$, see Ding et al. [40].
(*iii*) $\mathfrak{S}^0_{\mathfrak{q},0}(\lambda,\ell;0) = \mathfrak{S}$ $(\mathrm{Re}\mathfrak{f}(\xi)' > 0)$, see MacGregor [41].
The first result concerning the class $\mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$ establishes its convexity.

Theorem 2.1. *The class $\mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$ is closed under convex combination.*

Proof. Consider

$$
\mathfrak{f}_j(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{j\kappa}\xi^\kappa,\ \xi \in \mathbf{U},\ \ j = 1,2,
$$

being in the class $\mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$. It suffices to demonstrate that

$$
\mathfrak{f}(\xi) = \eta\mathfrak{f}_1(\xi) + (1-\eta)\mathfrak{f}_2(\xi),
$$

belongs to the class $\mathfrak{S}^s_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$, with η a positive real number.

$\mathfrak{f}$ is given by:

$$
\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty}(\eta a_{1\kappa} + (1-\eta)a_{2\kappa})\xi^\kappa,\ \xi \in \mathbf{U},
$$

and

$$
I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty}\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}(\eta a_{1\kappa} + (1-\eta)a_{2\kappa})\xi^\kappa. \tag{2.2}
$$

Differentiating (2.2), we have

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)' = 1 + \sum_{\kappa=2}^{\infty} \psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}(\eta a_{1\kappa} + (1-\eta)a_{2\kappa})\kappa\xi^{\kappa-1}.$$

Hence

$$\begin{aligned}\operatorname{Re}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)' &= 1 + \operatorname{Re}\left(\eta \sum_{\kappa=2}^{\infty} \kappa\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{1\kappa}\xi^{\kappa-1}\right) \\ &\quad + \operatorname{Re}\left((1-\eta) \sum_{\kappa=2}^{\infty} \kappa\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{2\kappa}\xi^{\kappa-1}\right). \end{aligned} \tag{2.3}$$

Taking into account that $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$, we can write

$$\operatorname{Re}\left(\eta \sum_{\kappa=2}^{\infty} \kappa\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{j\kappa}\xi^{\kappa-1}\right) > \eta(\alpha-1). \tag{2.4}$$

Using relation (2.4), we get from relation (2.3):

$$\operatorname{Re}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)' > 1 + \eta(\alpha-1) + (1-\eta)(\alpha-1) = \alpha.$$

It demonstrated that the set $\mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$ is convex. □

Next, we study a class of differential subordinations $\mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$ and a $\mathfrak{q}$-multiplier-Ruscheweyh operator $I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)$ involving convex functions.

Theorem 2.2. *For $\mathfrak{g}$ to be convex, we define*

$$\mathfrak{h}(\xi) = \mathfrak{g}(\xi) + \frac{\xi\mathfrak{g}'(\xi)}{\mathfrak{a}+2}, \qquad \mathfrak{a} > 0,\ \xi \in \mathbf{U}. \tag{2.5}$$

For $\mathfrak{f} \in \mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)$, consider

$$F(\xi) = \frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+1}} \int_0^{\xi} t^{\mathfrak{a}}\mathfrak{f}(t)dt,\ \xi \in \mathbf{U}, \tag{2.6}$$

then the differential subordination

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)' \prec \mathfrak{h}(\xi), \tag{2.7}$$

implies the differential subordination

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)' \prec \mathfrak{g}(\xi),$$

for the best dominant.

Proof. We can write (2.6) as:

$$\xi^{a+1}F(\xi) = (\mathfrak{a}+2)\int_0^{\xi} t^{a}\mathfrak{f}(t)dt, \quad \xi \in \mathbf{U},$$

and differentiating it, we get

$$\xi F^{'}(\xi) + (\mathfrak{a}+1)F(\xi) = (\mathfrak{a}+2)\mathfrak{f}(\xi)$$

and

$$\xi\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'} + (\mathfrak{a}+1)I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi) = (\mathfrak{a}+2)I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi), \quad \xi \in \mathbf{U}.$$

Differentiating the last relation, we obtain

$$\frac{\xi\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{''}}{\mathfrak{a}+2} + \left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'} = \left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}, \quad \xi \in \mathbf{U},$$

and (2.7) can be written as

$$\frac{\xi\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{''}}{\mathfrak{a}+2} + \left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'} \prec \frac{\xi\mathfrak{g}^{'}(\xi)}{\mathfrak{a}+2} + \mathfrak{g}(\xi). \tag{2.8}$$

Denoting

$$\mathfrak{p}(\xi) = \left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'} \in H[1,1], \tag{2.9}$$

differential subordination (2.8) has the next type:

$$\frac{\xi\mathfrak{p}^{'}(\xi)}{\mathfrak{a}+2} + \mathfrak{p}(\xi) \prec \frac{\xi\mathfrak{g}^{'}(\xi)}{\mathfrak{a}+2} + \mathfrak{g}(\xi).$$

Through Lemma 1.1, we find

$$\mathfrak{p}(\xi) \prec \mathfrak{g}(\xi),$$

then

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'} \prec \mathfrak{g}(\xi),$$

where $\mathfrak{g}$ is the best dominant. □

Theorem 2.3. *Denoting*

$$I_{\mathfrak{a}}(\mathfrak{f})(\xi) = \frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+1}}\int_0^{\xi} t^{\mathfrak{a}}\mathfrak{f}(t)dt, \;\; \mathfrak{a} > 0, \tag{2.10}$$

then,

$$I_{\mathfrak{a}}\left[\mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha)\right] \subset \mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha^{*}), \tag{2.11}$$

where

$$\alpha^{*} = (2\alpha-1) - (\alpha-1)_{2}F_{1}(1,1,\mathfrak{a}+3;\frac{1}{2}). \tag{2.12}$$

Proof. Using Theorem 2.2 for $\mathfrak{h}(\xi) = \frac{1-(2\alpha-1)\xi}{1-\xi}$, and using the identical procedures as Theorem 2.2, proof then

$$\frac{\xi \mathfrak{p}'(\xi)}{\mathfrak{a}+2} + \mathfrak{p}(\xi) \prec \mathfrak{h}(\xi),$$

holds, with $\mathfrak{p}$ defined by (2.9).

Through Lemma 1.2, we find

$$\mathfrak{p}(\xi) \prec \mathfrak{g}(\xi) \prec \mathfrak{h}(\xi),$$

similar to

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)' \prec \mathfrak{g}(\xi) \prec \mathfrak{h}(\xi),$$

where

$$\begin{aligned}\mathfrak{g}(\xi) &= \frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+2}} \int_0^{\xi} t^{\mathfrak{a}+1} \frac{1-(2\alpha-1)t}{1-t} dt \\ &= (2\alpha-1) - \frac{2(\mathfrak{a}+2)(\alpha-1)}{\xi^{\mathfrak{a}+2}} \int_0^{\xi} \frac{t^{\mathfrak{a}+1}}{1-t} dt.\end{aligned}$$

By using Lemma 1.5, we get

$$\mathfrak{g}(\xi) = (2\alpha-1) - 2(\alpha-1)(1-\xi)^{-1}{}_2F_1(1,1,\mathfrak{a}+3;\frac{\xi}{\xi-1}).$$

Since $\mathfrak{g}$ is a convex function and $\mathfrak{g}(\mathbf{U})$ is symmetric around the real axis, we have

$$\begin{aligned}\operatorname{Re}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)' &\geq \min_{|\xi|=1} \operatorname{Re}\mathfrak{g}(\xi) = \operatorname{Re}\mathfrak{g}(-1) = \alpha^* \\ &= (2\alpha-1) - (\alpha-1){}_2F_1(1,1,\mathfrak{a}+3;\frac{1}{2}).\end{aligned}$$

□

If we put $\alpha = 0$, in Theorem 2.3, we obtain

Corollary 2.1. *Let*

$$I_{\mathfrak{a}}(\mathfrak{f})(\xi) = \frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+1}} \int_0^{\xi} t^{\mathfrak{a}} \mathfrak{f}(t) dt, \ \ \mathfrak{a} > 0,$$

then,

$$I_{\mathfrak{a}}\left[\mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\right] \subset \mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha^*),$$

where

$$\alpha^* = -1 + {}_2F_1(1,1,\mathfrak{a}+3;\frac{1}{2}).$$

Example 2.1. If $\mathfrak{a} = 0$ in Corollary 2.1, we get:

$$I_0(\mathfrak{f})(\xi) = \frac{2}{\xi} \int_0^{\xi} \mathfrak{f}(t) dt,$$

then,

$$I_0\left[\mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\right] \subset \mathfrak{S}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\alpha^*),$$

where

$$\alpha^* = -1 + {}_2F_1(1,1,3;\frac{1}{2}) = 3 - 4\ln 2.$$

Theorem 2.4. *Let* $\mathfrak{g}$ *be the convex with* $\mathfrak{g}(0) = 1$*, we define*

$$\mathfrak{h}(\xi) = \xi\mathfrak{g}^{'}(\xi) + \mathfrak{g}(\xi),\ \xi \in \mathbf{U}.$$

If $\mathfrak{f} \in \mathbf{A}$ *verifies*

$$\left(I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'} \prec \mathfrak{h}(\xi),\ \xi \in \mathbf{U}, \tag{2.13}$$

then the sharp differential subordination

$$\frac{I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi} \prec \mathfrak{g}(\xi), \quad \xi \in \mathbf{U}, \tag{2.14}$$

holds.

Proof. Considering

$$\mathfrak{p}(\xi) = \frac{I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi} = \frac{\xi + \sum_{\kappa=2}^{\infty}\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_\kappa\xi^\kappa}{\xi} = 1 + \mathfrak{p}_1\xi + \mathfrak{p}_2\xi^2 + ..., \qquad \xi \in \mathbf{U},$$

clearly $\mathfrak{p} \in H[1,1]$, this we can write

$$\xi\mathfrak{p}(\xi) = I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi),$$

and differentiating it, we obtain

$$\left(I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'} = \xi\mathfrak{p}^{'}(\xi) + \mathfrak{p}(\xi).$$

Subordination (2.13) takes the form

$$\xi\mathfrak{p}^{'}(\xi) + \mathfrak{p}(\xi) \prec \mathfrak{h}(\xi) = \xi\mathfrak{g}^{'}(\xi) + \mathfrak{g}(\xi), \tag{2.15}$$

Lemma 1.1, allows us to have $\mathfrak{p}(\xi) \prec \mathfrak{g}(\xi)$, then (2.14) holds. □

Theorem 2.5. *Let* $\mathfrak{h}$ *be the convex and* $\mathfrak{h}(0) = 1$*, if* $\mathfrak{f} \in \mathbf{A}$ *verifies*

$$\left(I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'} \prec \mathfrak{h}(\xi),\ \xi \in \mathbf{U}, \tag{2.16}$$

then we obtain the subordination

$$\frac{I^s_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi} \prec \mathfrak{g}(\xi), \quad \xi \in \mathbf{U},$$

for the convex function $\mathfrak{g}(\xi) = (2\alpha - 1) + \frac{2(\alpha-1)}{\xi}\ln(1-\xi)$*, being the best dominant.*

Proof. Let

$$\mathfrak{p}(\xi)=\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}=1+\sum_{\kappa=2}^{\infty}\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa-1}\in H[1,1],\qquad \xi\in\mathbf{U}.$$

By differentiating it, we get

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}=\xi\mathfrak{p}^{'}(\xi)+\mathfrak{p}(\xi),$$

and differential subordination (2.16) becomes

$$\xi\mathfrak{p}^{'}(\xi)+\mathfrak{p}(\xi)\prec\mathfrak{h}(\xi),$$

Lemma 1.2 allows us to have

$$\mathfrak{p}(\xi)\prec\mathfrak{g}(\xi)=\frac{1}{\xi}\int_{0}^{\xi}\mathfrak{h}(t)dt,$$

then

$$\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}\prec\mathfrak{g}(\xi)=(2\alpha-1)+\frac{2(\alpha-1)}{\xi}\ln(1-\xi),$$

for $\mathfrak{g}$ is the best dominant. □

If we put $\alpha=0$ in Theorem 2.5, we have

Corollary 2.2. *Considering the convex* $\mathfrak{h}$ *with* $\mathfrak{h}(0)=1$*, if* $\mathfrak{f}\in\mathbf{A}$ *verifies*

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}\prec\mathfrak{h}(\xi),\ \xi\in\mathbf{U},$$

then we obtain the subordination

$$\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}\prec\mathfrak{g}(\xi)=-1-\frac{2}{\xi}\ln(1-\xi),\quad \xi\in\mathbf{U},$$

for the convex function $\mathfrak{g}(\xi)$*, which is the best dominant.*

Example 2.2. From Corollary 2.2, if

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}\prec\mathfrak{h}(\xi),\ \xi\in\mathbf{U},$$

we obtain

$$\operatorname{Re}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}\geq\min_{|\xi|=1}\operatorname{Re}\mathfrak{g}(\xi)=\operatorname{Re}\mathfrak{g}(-1)=-1+2\ln 2,$$

Theorem 2.6. *Let* $\mathfrak{g}$ *be a convex function with* $\mathfrak{g}(0)=1$*. We define* $\mathfrak{h}(\xi)=\xi\mathfrak{g}^{'}(\xi)+\mathfrak{g}(\xi)$, $\xi\in\mathbf{U}$*. If* $\mathfrak{f}\in\mathbf{A}$ *verifies*

$$\left(\frac{\xi I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\right)^{'}\prec\mathfrak{h}(\xi),\ \xi\in\mathbf{U},\tag{2.17}$$

then

$$\frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\prec\mathfrak{g}(\xi),\quad \xi\in\mathbf{U},\tag{2.18}$$

holds.

Proof. For

$$\mathfrak{p}(\xi)=\frac{I_{\mathfrak{q},\mu}^{s+1}(\lambda,\ell)\mathfrak{f}(\xi)}{I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)}=\frac{\xi+\sum_{\kappa=2}^{\infty}\psi_{\mathfrak{q}}^{*s+1}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa}}{\xi+\sum_{\kappa=2}^{\infty}\psi_{\mathfrak{q}}^{*s}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa}}.$$

By differentiating it, we get

$$\mathfrak{p}'(\xi)=\frac{\left(I_{\mathfrak{q},\mu}^{s+1}(\lambda,\ell)\mathfrak{f}(\xi)\right)'}{I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)}-\mathfrak{p}(\xi)\frac{\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)\right)'}{I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)}.$$

then

$$\xi\mathfrak{p}'(\xi)+\mathfrak{p}(\xi)=\left(\frac{\xi I_{\mathfrak{q},\mu}^{s+1}(\lambda,\ell)\mathfrak{f}(\xi)}{I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)}\right)'.$$

Differential subordination (2.17), then we obtain (2.15), and Lemma 1.1 allows us to have $\mathfrak{p}(\xi)\prec\mathfrak{g}(\xi)$, then (2.18) holds. □

3. Differential superordination results

This section examines differential superordinations with respect to a first-order derivative of a $\mathfrak{q}$-multiplier-Ruscheweyh operator $I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)$. For every differential superordination under investigation, we provide the best subordinant.

Theorem 3.1. *Considering $\mathfrak{f}\in\mathbf{A}$, a convex $\mathfrak{h}$ in $\mathbf{U}$ such that $\mathfrak{h}(0)=1$, and $F(\xi)$ defined in (2.6). We assume that $\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)\right)'$ is a univalent in $\mathbf{U}$, $\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)\right)'\in Q\cap H[1,1]$. If*

$$\mathfrak{h}(\xi)\prec\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)\right)',\quad\xi\in\mathbf{U},\tag{3.1}$$

holds, then

$$\mathfrak{g}(\xi)\prec\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)\right)',\quad\xi\in\mathbf{U},$$

with $\mathfrak{g}(\xi)=\frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+2}}\int_0^{\xi}t^{\mathfrak{a}+1}\mathfrak{h}(t)dt$ the best subordinant.

Proof. Differentiating (2.6), then $\xi F'(\xi)+(\mathfrak{a}+1)F(\xi)=(\mathfrak{a}+2)\mathfrak{f}(\xi)$ can be expressed as

$$\xi\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)\right)'+(\mathfrak{a}+1)I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)=(\mathfrak{a}+2)I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi),$$

which, after differentiating it again, has the form

$$\frac{\xi\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)\right)''}{(\mathfrak{a}+2)}+\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)\right)'=\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)\right)'.$$

Using the final relation, (3.1) can be expressed

$$\mathfrak{h}(\xi)\prec\frac{\xi\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)\right)''}{(\mathfrak{a}+2)}+\left(I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)F(\xi)\right)'.\tag{3.2}$$

Define

$$\mathfrak{p}(\xi)=\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'},\ \ \xi\in\mathbf{U}, \tag{3.3}$$

and putting (3.3) in (3.2), we obtain $\mathfrak{h}(\xi)\prec\frac{\xi\mathfrak{p}^{'}(\xi)}{(\mathfrak{a}+2)}+\mathfrak{p}(\xi)$, $\xi\in\mathbf{U}$. Using Lemma 1.3, given $n=1$, and $\alpha=\mathfrak{a}+2$, it results in $\mathfrak{g}(\xi)\prec\mathfrak{p}(\xi)$, similar $\mathfrak{g}(\xi)\prec\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'}$, with the best subordinant $\mathfrak{g}(\xi)=\frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+2}}\int_0^{\xi}t^{\mathfrak{a}+1}\mathfrak{h}(t)dt$ convex function. □

Theorem 3.2. *Let* $\mathfrak{f}\in\mathbf{A}$, $F(\xi)=\frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+1}}\int_0^{\xi}t^{\mathfrak{a}}\mathfrak{f}(t)dt$, *and* $\mathfrak{h}(\xi)=\frac{1-(2\alpha-1)\xi}{1-\xi}$ *where* $Re\mathfrak{a}>-2$, $\alpha\in[0,1)$. *Suppose that* $\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}$ *is a univalent in* $\mathbf{U}$, $\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'}\in Q\cap H[1,1]$ *and*

$$\mathfrak{h}(\xi)\prec\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'},\ \ \xi\in\mathbf{U}, \tag{3.4}$$

then

$$\mathfrak{g}(\xi)\prec\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'},\ \ \xi\in\mathbf{U},$$

is satisfied for the convex function $\mathfrak{g}(\xi)=(2\alpha-1)-2(\alpha-1)(1-\xi)^{-1}{}_2F_1(1,1,\mathfrak{a}+3;\frac{\xi}{\xi-1})$ *as the best subordinant.*

Proof. Let $\mathfrak{p}(\xi)=\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'}$. We can express (3.4) as follows when Theorem 3.1 is proved:

$$\mathfrak{h}(\xi)=\frac{1-(2\alpha-1)\xi}{1-\xi}\prec\frac{\xi\mathfrak{p}^{'}(\xi)}{\mathfrak{a}+2}+\mathfrak{p}(\xi).$$

By using Lemma 1.4, we obtain $\mathfrak{g}(\xi)\prec\mathfrak{p}(\xi)$, with

$$\begin{aligned}\mathfrak{g}(\xi)&=\frac{\mathfrak{a}+2}{\xi^{\mathfrak{a}+2}}\int_0^{\xi}\frac{1-(2\alpha-1)t}{1-t}t^{\mathfrak{a}+1}dt\\&=(2\alpha-1)-2(\alpha-1)(1-\xi)^{-1}{}_2F_1(1,1,\mathfrak{a}+3;\frac{\xi}{\xi-1})\prec\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)F(\xi)\right)^{'},\end{aligned}$$

$\mathfrak{g}$ is convex and the best subordinant. □

Theorem 3.3. *Let* $\mathfrak{f}\in\mathbf{A}$ *and* $\mathfrak{h}$ *be a convex function with* $\mathfrak{h}(0)=1$. *Assuming that* $\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}$ *is a univalent and* $\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}\in Q\cap H[1,1]$, *if*

$$\mathfrak{h}(\xi)\prec\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'},\ \ \xi\in\mathbf{U}, \tag{3.5}$$

holds, then

$$\mathfrak{g}(\xi)\prec\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi},\ \ \xi\in\mathbf{U},$$

is satisfied for the convex function $\mathfrak{g}(\xi)=\frac{1}{\xi}\int_0^{\xi}\mathfrak{h}(t)dt$, *the best subordinant.*

Proof. Denoting

$$\mathfrak{p}(\xi)=\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}=\frac{\xi+\sum_{\kappa=2}^{\infty}\psi^{*s}_{\mathfrak{q}}(\kappa,\lambda,\ell)\frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa}}{\xi}\in H[1,1],$$

we can write $I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)=\xi\mathfrak{p}(\xi)$ and differentiating it, we have

$$\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}=\xi\mathfrak{p}^{'}(\xi)+\mathfrak{p}(\xi).$$

With this notation, differential superordination (3.5) becomes

$$\mathfrak{h}(\xi)\prec\xi\mathfrak{p}^{'}(\xi)+\mathfrak{p}(\xi).$$

Using Lemma 1.3, we obtain

$$\mathfrak{g}(\xi)\prec\mathfrak{p}(\xi)=\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}\quad for\ \mathfrak{g}(\xi)=\frac{1}{\xi}\int_{0}^{\xi}\mathfrak{h}(t)dt,$$

convex and the best subordinant. □

Theorem 3.4. *Suppose that* $\mathfrak{h}(\xi)=\frac{1-(2\alpha-1)\xi}{1-\xi}$ *with* $\alpha\in[0,1)$. *For* $\mathfrak{f}\in\mathbf{A}$, *assume that* $\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'}$ *is a univalent and* $\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}\in Q\cap H[1,1]$. *If*

$$\mathfrak{h}(\xi)\prec\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)^{'},\quad\xi\in\mathbf{U},\tag{3.6}$$

holds, then

$$\mathfrak{g}(\xi)\prec\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi},\quad\xi\in\mathbf{U},$$

where

$$\mathfrak{g}(\xi)=(2\alpha-1)+\frac{2(\alpha-1)}{\xi}\ln(1-\xi).$$

Proof. After presenting Theorem 3.3's proof for $\mathfrak{p}(\xi)=\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi}$, superordination (3.6) takes the form

$$\mathfrak{h}(\xi)=\frac{1-(2\alpha-1)\xi}{1-\xi}\prec\xi\mathfrak{p}^{'}(\xi)+\mathfrak{p}(\xi).$$

By using Lemma 1.3, we obtain $\mathfrak{g}(\xi)\prec\mathfrak{p}(\xi)$, with

$$\begin{aligned}\mathfrak{g}(\xi)&=\frac{1}{\xi}\int_{0}^{\xi}\frac{1-(2\alpha-1)t}{1-t}dt\\&=(2\alpha-1)+\frac{2(\alpha-1)}{\xi}\ln(1-\xi)\prec\frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{\xi},\end{aligned}$$

$\mathfrak{g}$ is convex and the best subordinant. □

Theorem 3.5. *Let* $\mathfrak{h}$ *be a convex function, with* $\mathfrak{h}(0) = 1$. *For* $\mathfrak{f} \in \mathbf{A}$, *let* $\left(\frac{\xi I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\right)'$ *is univalent in* $\mathbf{U}$ *and* $\frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)} \in Q \cap H[1,1]$. *If*

$$\mathfrak{h}(\xi) \prec \left(\frac{\xi I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\right)', \quad \xi \in \mathbf{U}, \tag{3.7}$$

holds, then

$$\mathfrak{g}(\xi) \prec \frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}, \quad \xi \in \mathbf{U},$$

where the convex $\mathfrak{g}(\xi) = \frac{1}{\xi}\int_0^{\xi} \mathfrak{h}(t)dt$ *is the best subordinant .*

Proof. Let

$$\mathfrak{p}(\xi) = \frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)},$$

after differentiating it, we can write

$$\mathfrak{p}'(\xi) = \frac{\left(I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)'}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)} - \mathfrak{p}(\xi)\frac{\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)\right)'}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)},$$

in the form $\xi\mathfrak{p}'(\xi) + \mathfrak{p}(\xi) = \left(\frac{\xi I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\right)'$.

Differential superordination (3.7) becomes $\mathfrak{h}(\xi) \prec \xi\mathfrak{p}'(\xi) + \mathfrak{p}(\xi)$. Applying Lemma 1.3, we obtain $\mathfrak{g}(\xi) \prec \mathfrak{p}(\xi) = \frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}$, with the convex $\mathfrak{g}(\xi) = \frac{1}{\xi}\int_0^{\xi} \mathfrak{h}(t)dt$, the best subordinant. □

Theorem 3.6. *Assume that* $\mathfrak{h}(\xi) = \frac{1-(2\alpha-1)\xi}{1-\xi}$ *with* $\alpha \in [0,1)$. *For* $\mathfrak{f} \in \mathbf{A}$, *suppose that* $\left(\frac{\xi I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\right)'$ *is univalent and* $\frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)} \in Q \cap H[1,1]$. *If*

$$\mathfrak{h}(\xi) \prec \left(\frac{\xi I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}\right)', \quad \xi \in \mathbf{U}, \tag{3.8}$$

holds, then

$$\mathfrak{g}(\xi) \prec \frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}, \quad \xi \in \mathbf{U},$$

where

$$\mathfrak{g}(\xi) = (2\alpha - 1) + \frac{2(\alpha-1)}{\xi}\ln(1-\xi).$$

Proof. By using $\mathfrak{p}(\xi) = \frac{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\xi)}$, differential superordination (3.8) takes the form

$$\mathfrak{h}(\xi) = \frac{1-(2\alpha-1)\xi}{1-\xi} \prec \xi\mathfrak{p}'(\xi) + \mathfrak{p}(\xi).$$

By using Lemma 1.3, we get $\mathfrak{g}(\xi) \prec \mathfrak{p}(\xi)$, with

$$\begin{aligned}\mathfrak{g}(\xi) &= \frac{1}{\xi}\int_0^{\xi} \frac{1-(2\alpha-1)t}{1-t}dt \\ &= (2\alpha-1) + \frac{2(\alpha-1)}{\xi}\ln(1-\xi) \prec \frac{I_{\mathfrak{q},\mu}^{s+1}(\lambda,\ell)\mathfrak{f}(\xi)}{I_{\mathfrak{q},\mu}^{s}(\lambda,\ell)\mathfrak{f}(\xi)},\end{aligned}$$

$\mathfrak{g}$ is convex and the best subordinant. □

4. Conclusions

A new class of analytical normalized functions $\mathfrak{S}_{\mathfrak{q},\mu}^{s}(\lambda, \ell; \alpha)$, given in Definition 2.1, is related to the novel findings proven in this study given in Definition 2.1. To introduce some subclasses of univalent functions, we develop the $\mathfrak{q}$-analogue multiplier-Ruscheweyh operator $I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)$ using the notion of a $\mathfrak{q}$-difference operator. The $\mathfrak{q}$-Ruscheweyh operator and the $\mathfrak{q}$-Cătas operator are also used to introduce and study distinct subclasses. In Section 2, these subclasses are subsequently examined in more detail utilizing differential subordination theory methods. Regarding the $\mathfrak{q}$-analogue multiplier-Ruscheweyh operator$I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)$ and its derivatives of first and second order, we derive differential superordinations in Section 3. For every differential superordination under investigation, the best subordinant is provided.

Author contributions

The authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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