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Research article

On soft covering spaces in soft topological spaces

Mohammed Abu Saleem*

Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan

* **Correspondence:** Email: m_abusaleem@bau.edu.jo.

Abstract: In this paper, we present the concept of a soft covering map on a soft topological space. We also introduce the notion of a soft local homeomorphism and establish the relationship between soft local homeomorphism and soft open mapping. Additionally, we demonstrate that a soft local homeomorphism does not necessarily imply a soft covering map. We provide several characterizations and restriction theorems. Moreover, we deduce the necessary and sufficient conditions for a soft continuous map to be a soft covering map.

Keywords: soft covering map; soft covering space; soft local homeomorphism
Mathematics Subject Classification: 7M10, 57M12, 54-XX, 54A40

1. Introduction

The precise and accurate attributes of classical mathematical tools arise from their utilization in modeling, reasoning, and computation. This accuracy is a result of the application of two-valued logic in classical mathematics, imbuing a sense of certainty. In contrast, intricate challenges in fields like economics, physics, engineering, biology, sociology, and medicine entail elements of uncertainty and incomplete data. In these cases, conventional mathematical methods are insufficient to resolve such complexities. To address this limitation, there are disciplines like fuzzy set theory [1]. Homomorphism problems in fuzzy information systems, fuzzy covering decision systems, fuzzy-β-covering-based multi-granulation rough sets, and neutrosophic theory were studied in [2–7]. In this approach, probability theory and rough set theory are commonly employed, and each of these comes with its limitations. This led to the inception of a new theory, known as soft set theory [8]. Many researchers have examined and explored the fundamental concepts underlying soft sets [9–13]. Soft sets have been applied in diverse fields by researchers [14–17] including but not limited to fields such as operations research, game theory, function smoothness, measurement theory, and probability. Multiple researchers have utilized soft set theory to investigate diverse mathematical structures [18]. In [19], the scholars presents soft topology as an exceptional extension of classical topology. Within the area of topology, a multitude of fundamental notions, such as soft separation axioms, soft connected spaces, and soft locally path-connected spaces, etc. [20–30], have been expanded and enhanced through the utilization of soft sets. Nevertheless, there is still considerable scope for substantial contributions in this field. A cornerstone within the field of algebraic topology is the study of covering spaces, an essential concept that delves into the intricate properties of topological spaces. These properties maintain their integrity even among continuous transformations like stretching and bending. The significance of covering spaces becomes particularly pronounced when unraveling the essence of the fundamental group, homotopy, and other paramount topological invariants that characterize spaces [31–33]. The primary objective of this paper is to undertake a comprehensive theoretical exploration of the realm of soft-covering space theory.

2. Preliminaries

To establish the subsequent results, we require a review of fundamental concepts and characteristics concerning soft sets and soft topological spaces. First, let X denote an inital universe with a cardinality of at least 2, S represents a set of parameters, and 2^X stands for the power set of X. For our convenience, we will introduce the required concepts for the same set of parameters S while also providing analogous definitions for other sets of parameters, A and B, both subsets of S.

Definition 2.1. *[8] Let us consider $G : S \longrightarrow 2^X$ as mappping in the universe X. We call the set $(G,\ S) = \{(s,\ G(s)) : s \in S\}$ to be a soft set. A specific pair $(s, G(s))$, is known as a soft element of $(G,\ S)$. To keep it brief, we adopt the representation G_S rather than $(G,\ S)$.*

Definition 2.2. *[19, 34] Let G_S be a soft set G_S defined over the set X and $x \in X$. Then, $x \in G_S$ whenever $x \in G(s)$ for every $s \in S$, and $x \notin G_S$ whenever $x \notin G(s)$ for at least one $s \in S$. Also, $x \Subset G_S$ whenever $x \in G(s)$ or at least one $s \in S$, and $x \not\Subset G_S$ whenever $x \notin G(s)$ for each $s \in S$. The symbols $\in$ and $\Subset$ are referred to as natural belong and partial belong, respectively.*

Definition 2.3. *[17] A soft set F_A is said to be a soft subset of the soft set G_B if $A \subseteq B$ and $F(a) \subseteq G(a)$, $\forall\ a \in A$. In this situation, we write $F_A \subseteq G_B$. Also, $F_A = G_B$ when $F_A \subseteq G_B$ and $G_B \subseteq F_A$. The collection of soft sets defined over a parameter set A and with respect to a universe X is represented by the symbol $CS(X_A)$.*

Definition 2.4. *[9, 16] The intersection of soft sets F_S and G_S over X is the soft set H_S which is obtained by combining the soft sets F_S and G_S through the intersection operation, in which $H(s) = F(s) \cap G(s)$, for all $s \in S$ and represented by $F_S \cap G_S$. The union of the soft sets F_S and G_S over X is the soft set H_S obtained by combining the soft sets F_S and G_S through the union operation, in which $H(s) = F(s) \cup G(s)$, for all $s \in S$ and represented by $F_S \cup G_S$.*

Definition 2.5. *[9] The soft set F_S is said to be a null soft set if $F(s) = \phi$ for all $s \in S$. The null soft set will be represented by the symbol Φ_S. The soft set F_S is said to be an absolute soft set if $F(s) = X$ for all $s \in S$. The absolute set is represented by X_S.*

Definition 2.6. *[19] A soft topology on a set X is defined as a collection τ of soft sets over X, determined by the fixed parameters set S and satisfies:*

(i) X_S and Φ_S belongs τ.

(ii) If $F_S, G_S \in \tau$, then $F_S \cap G_S \in \tau$.
(iii) If $W_{\gamma S} \in \tau$ for every γ in some index set Λ, then $\cup_{\gamma\in\Lambda} W_{\gamma S} \in \tau$.

The triple (X, τ, S) is referred to as a soft topological space. Each element of τ is referred to as a soft open set, while its relative complement is termed a soft closed set.

Definition 2.7. *[19] Let A be a nonempty soft subset of (X, τ, S). Then, $\tau_A = \{A \cap G_S : G_S \in \tau\}$ is said to be a soft relative topology on A. Additionally, we refer to the triple (A, τ_A, S) as the soft subspace of X.*

Definition 2.8. *[30] Let (X, τ, S) be a soft topological space. Then,*
(i) the soft interior of soft sets U_S over X, denoted by $int(U_S)$, is the union of all soft sets that are contained in U_S.
(ii) The soft set U_S is called a soft (nhood) of $x \in X$, if there is a soft open set V_S in which $x \in V_S \subseteq U_S$.
(iii) The soft set F_S is called a soft closure of F_S, denoted by $cl(F_S)$, is the smallest soft closed set over X that contains F_S.

Definition 2.9. *[10] Let $\Gamma_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ be a soft mapping defined as a pair (Γ, φ), where Γ and φ represent mappings $\Gamma : X \longrightarrow Y$, $\varphi : A \longrightarrow B$. Consider the soft subsets G_P and H_Q of $CS(X_A)$ and $CS(Y_B)$, respectively. Then, the image of G_P and the pre-image of H_Q are given by:*
(i) $\Gamma_\varphi(G_P) = \left(\Gamma_\varphi(G)\right)_B$ can be considered as a soft subset of $CS(Y_B)$ in which

$$\Gamma_\varphi(G)(b) = \begin{cases} \bigcup_{a\in\varphi^{-1}(b)\cap P} \Gamma(G(a)), & \varphi^{-1}(b) \cap P \neq \phi, \\ \phi, & \textit{otherwise}. \end{cases} \quad \forall b \in B.$$

(ii) $\Gamma_\varphi^{-1}(H_Q) = \left(\Gamma_\varphi^{-1}(H)\right)_A$ can be considered as a soft subset of $CS(X_A)$ in which

$$\Gamma_\varphi^{-1}(H)(a) = \begin{cases} \Gamma^{-1}(H(\varphi(a))), & \varphi(a) \in Q, \\ \phi, & \textit{otherwise}. \end{cases} \quad \forall a \in A.$$

Definition 2.10. *[35] A soft mapping $\Gamma_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ is characterized as follows:*
(i) Γ_φ is injective whenever both Γ and φ are injective.
(ii) Γ_φ is onto whenever both Γ and φ are onto.
(iii) Γ_φ is bijective whenever both Γ and φ are bijective.

Definition 2.11. *[35] A soft map $\Gamma_\varphi : (X, \tau, A) \to (Y, \delta, B)$ is defined as follows:*
(i) It is soft continuous if the pre-image of every soft open subset in (Y, δ, B) is itself a soft open subset in (X, τ, A).
(ii) It is considered soft open if the image of all soft open subsets in soft open in (X, τ, A) becomes a soft open subset in (Y, δ, B).
(iii) It qualifies as a soft homeomorphism if it is bijective, soft continuous, and soft open.

Definition 2.12. *[36] A collection $\left\{V_{\gamma_S} : \gamma \in \Lambda\right\}$ of soft open sets is referred to as a soft open cover of the soft topological space (X, τ, S) if $X_S = \cup_{\gamma\in\Lambda} V_{\gamma S}$.*

Definition 2.13. *[22] Suppose $F_A \in CS(X_A)$ and $G_B \in CS(Y_B)$. The Cartesian product of F_A and G_B is a soft set $L_{A\times B}$, where $L : A \times B \longrightarrow \mathcal{P}(X \times Y)$ is defined as $L(s,t) = F(s) \times G(t) = \{(x_1, x_2) : x_1 \in F(s), x_2 \in G(t)\}$.*

Definition 2.14. *[23] A soft topological space (X, T, S) is regarded as soft locally connected at a soft element $x \in X$ if, for any soft open set F_S, $x \in F_S$, there exists a soft connected soft open set G_S that contains x and is also a subset of F_S.*

Definition 2.15. *[24] The largest soft connected soft subspace of a soft topological space (X, T, S) is called a soft component.*

3. Main results

In this section, we provide the concept of a soft covering-map and its corresponding soft-covering space. We delve into defining these fundamental notions while also delving into the characterization of their inherent properties.

Definition 3.1. *Let (X, τ_X, A), (Y, τ_Y, B) be two soft topological spaces, and suppose that $p_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ is soft continuous, onto map. Then, the soft open set U_B of $CS(Y_B)$ is said to be soft-evenly covered by p_φ if $p_\varphi^{-1}(U_B)$ can be represented as the union of disjoint open sets $V_{\alpha A}$ in $CS(X_A)$ for every $\alpha \in \Lambda$. The restriction $p_\varphi|V_{\alpha A}$ is a soft homeomorphism of $V_{\alpha A}$ onto U_B. The family $\{V_{\alpha A}\}$ will be called a soft partition of $p_\varphi^{-1}(U_B)$ into soft slices. Furthermore, if each soft point y of Y_B has a soft (nhood) U_B that is soft-evenly covered by p_φ, then p_φ is called a soft covering-map, and $CS(X_A)$ is said to be a soft-covering space of $CS(Y_B)$.*

From now on, we will consider the parametric map $\varphi : A \to B$ as a parametric onto map.

Example 3.2. *Let (X, τ_X, A) be a soft topological space, and let $i_\varphi : CS(X_A) \longrightarrow CS(X_A)$ be the soft identity map. Then, i_φ is a soft covering-map. More generally, consider the space $\hat{X} = X \times \{1, 2, \ldots, n\}$ (n-disjoint copies of X). The soft map $p_\varphi : CS\left(\hat{X}_A\right) \longrightarrow CS(X_A)$ given by $p_\varphi((x, j), A) = (x, A)$ for all j is a soft covering-map. It should be noted that the inverse image of all soft open sets in $CS(X_A)$ has exactly n-disjoint soft pre-images in $CS\left(\hat{X}_A\right)$, corresponding to each soft-evenly covered open sets in $CS(X_A)$.*

Theorem 3.3. *Let $\varphi : A \to B$ be a parametric map, and let us consider the map $p : \mathbb{R} \to S^1$ given by $p(s) = e^{2\pi i s} = (\cos(2\pi s), \sin(2\pi s))$. Then, the soft map $p_\varphi : CS(\mathbb{R}_A) \longrightarrow CS\left(S^1_B\right)$ is a soft covering-map.*

Proof. First note that the domain of p has the standard topology, and S^1 is considered as a subspace of the usual plane. Then, p is onto because p wraps the line around S^1 infinitely many times. Now, based on the definition (2.10), we can conclude that φ is onto and p_φ is a soft onto map. Also, p_φ is soft continuous, since for any open subset H_L of $CS\left(S^1_B\right)$, $(L \subseteq B)$ and $p_\varphi^{-1}(H_L) = \left(p_\varphi^{-1}(H)\right)_A$, $\forall a \in A$, $\varphi(a) \in L \implies p_\varphi^{-1}(H)(a) = p^{-1}(H(\varphi(a)))$ is a soft open set and if $\varphi(a) \notin L \implies p_\varphi^{-1}(H)(a) = \phi_S$ (soft open). Moreover, for every point $b \in S^1_B$, there is a soft open set V_S of b in $CS\left(S^1_B\right)$ that is soft-evenly covered by p_φ and is an infinite soft-sheeted cover. Hence, $p_\varphi : CS(\mathbb{R}_A) \longrightarrow CS\left(S^1_B\right)$ is a soft covering-map. □

Theorem 3.4. *Let $p_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ be a soft covering-map. If Y_{0B} is a soft subspace of Y_B, and if $X_{0A} = p_\varphi^{-1}(Y_{0B})$, then the map $p_{0\varphi} : CS(X_{0A}) \longrightarrow CS(Y_{0B})$ obtained by restricting p_φ is a covering-map.*

Proof. Given $y_0 \in Y_{0B}$, let U_B be a soft open set in $CS\ (Y_B)$ containing y_0 that is soft-evenly covered by p_φ, and let $\{V_{\alpha A}\}$ be a soft partition of $p_\varphi^{-1}(U_B)$ into soft slices. Then, $U_B \cap Y_{0B}$ is a soft (nhood) of y_0 in $CS\ (Y_{0B})$ and the soft sets $V_{\alpha A} \cap X_{0A}$ are disjoint soft open sets in $CS(X_{0A})$ whose union is $p_\varphi^{-1}\ (U_B \cap Y_{0B})$ and each is mapped soft homeomorphically onto $U_B \cap Y_{0B}$ by p_φ. □

Theorem 3.5. *If* $p_\varphi : CS\ (X_A) \longrightarrow CS\ (Y_B)$ *and* $\acute{p}_{\acute{\varphi}} : CS(\acute{X}_{\acute{A}}) \longrightarrow CS\left(\acute{Y}_{\acute{B}}\right)$ *are soft covering-maps, then* $p_\varphi \times \acute{p}_{\acute{\varphi}} : CS\ (X_A) \times CS(\acute{X}_{\acute{A}}) \longrightarrow CS\ (Y_B) \times CS(\acute{Y}_{\acute{B}})$ *is a soft covering-map.*

Proof. Given $y \in Y$, $\acute{y} \in \acute{Y}$, and consider U_B and $\acute{U}_{\acute{B}}$ are (nbds) of y and $\acute{y}$, respectively, which are soft-evenly covered by p_φ and $\acute{p}_{\acute{\varphi}}$. Let $\{V_{\alpha A}\}$ and $\{\acute{V}_{\beta\acute{A}}\}$ be soft partitions of $p_\varphi^{-1}(U_B)$ and $\acute{p}_{\acute{\varphi}}^{-1}(\acute{U}_{\acute{B}})$, respectively, into soft slices. Then, the inverse image under $p_\varphi \times \acute{p}_{\acute{\varphi}}$ of the soft open set $U_B \times \acute{U}_{\acute{B}}$ is the union of all the sets $V_{\alpha A} \times \acute{V}_{\beta\acute{A}}$. These are disjoint soft open sets of $CS\ (X_A) \times CS(\acute{X}_{\acute{A}})$, and each is mapped soft homeomorphically onto $U_B \times \acute{U}_{\acute{B}}$ by $p_\varphi \times \acute{p}_{\acute{\varphi}}$. □

The next example points out that the product of soft covering-maps is regarded as a soft covering-map.

Example 3.6. *If* $\mathbb{R}^n_A = S^1_B \times \cdots \times S^1_B$ *is the soft n-dimensional torus (product of soft n-circles), then the soft map* $p_\varphi : CS\left(\mathbb{R}^n_A\right) \longrightarrow CS\left(T^n_A\right)$*, in which* $p(s_1, \ldots, s_n) = (e^{2\pi i s_1}, \ldots, e^{2\pi i s_n})$ *is a soft covering-map.*

Definition 3.7. *A soft continuous map* $\Psi_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ *is called a soft local homeomorphism if, for every soft point* $x \in X$ *that has soft open (nhood)* V_A*, in which* $\Psi_\varphi\ (V_A)$ *is soft open in* $CS(X_A)$ *with the restriction mapping* $\Psi_\varphi|V_A$ *is a soft homeomorphism of* V_A *onto* $\Psi_\varphi(V_A)$*.*

Theorem 3.8. *Every soft local homeomorphism is a soft open mapping.*

Proof. Suppose that $\Psi_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ is a soft local homeomorphism and V_A is a soft open in $CS(X_A)$. If $w \in \Psi_\varphi(V_A)$, then there is $z \in V_A$ such that $\Psi_\varphi\ (z) = w$. By assumption, there is a soft open (nhood) U_B of w in $CS(Y_B)$, and a soft open (nhood) W_A of z in $CS(X_A)$ in which Ψ_φ maps W_A homeomorphically onto U_B. Since $V_A \cap W_A$ is soft open in W_A, and U_B is soft open in $CS(Y_B)$, $\Psi_\varphi(V_A \cap W_A)$ is soft open in $CS(Y_B)$. Obviously, $w \in \Psi_\varphi(V_A \cap W_A) \subset \Psi_\varphi(V_A)$. Thus, $\Psi_\varphi(V_A)$ can be considered as a soft (nhood) of w. Therefore, Ψ_φ is soft open mapping. □

The converse of Theorem (3.8) is not true, as can be demonstrated by the following example:

Example 3.9. *Let* $\varphi : A \to A$ *be a parametric map, and consider the map* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by* $f(x, y) = x$*. Then, the soft map* $f_\varphi : CS\left(\mathbb{R}^2_A\right) \longrightarrow CS\ (\mathbb{R}_A)$ *is clearly soft open mappping and not soft local homeomorphism since no soft (nbd) of any soft point in* $CS\left(\mathbb{R}^2_A\right)$ *is homeomorphic to a soft set in* $CS\ (\mathbb{R}_A)$*.*

However, it is important to note that a soft local homeomorphism does not necessarily imply a soft covering-map, as seen in the subsequent example.

Example 3.10. *Let* J *represent an open interval* $(0,\ m)$ *with the standard topology where* $m > 1$ *is an integer, and suppose that* $\Psi_\varphi : CS(J_A) \longrightarrow CS(S^1_B)$ *is a soft map in which* $\Psi(s) = e^{2\pi i s}$*. As a soft local homeomorphism is restricted to a soft open subset, we obtain* Ψ_φ *is a soft local homeomorphism. Meanwhile,* Ψ_φ *as a result is soft onto it doesn't qualify as a soft covering-map. The reason behind this is that the element* $1_B \in CS(S^1_B)$ *doesn't have a soft (nhood) that can be evenly covered. The space* $CS(J_A)$ *can be seen as a soft, open, and finite spiral over* $CS(S^1_B)$*.*

In soft covering theory, the study of soft covering-maps can be simplified by focusing on soft covering-maps with a base space that is soft (path) connected.

Theorem 3.11. *Let (Y, τ_Y, B) be a soft locally path-connected and $p_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ be a soft covering-map. Then, every soft point in $CS(Y_B)$ has a soft path-connected open (nhood) U_B in which every soft path-component of $p_\varphi^{-1}(V_B)$ is soft mapped homeomorphically onto V_B by p_φ.*

Proof. Consider a soft point $y \in Y$, and let U_B be a soft (nhood) of y in $CS(Y_B)$. Assume $p_\varphi^{-1}(U_B) = \cup_\gamma G_{\gamma A}$, in which every $G_{\gamma A}$ is soft open in $CS(X_A)$, $p_\varphi|G_{\gamma A}$ is a soft homeomorphism between $G_{\gamma A}$ and U_B, and $G_{\gamma A} \cap G_{\delta A} = \phi$ for $\gamma \neq \delta$. It follows from (Y, τ_Y, B) is a soft locally path-connected that U_B contains a soft path-connected (nhood) V_B of y. Let $W_{\gamma A} = G_{\gamma A} \cap p_\varphi^{-1}(V_B)$, $\forall \gamma$. Then every $W_{\gamma A}$ are soft open in $CS(X_A)$ and $p_\varphi^{-1}(V_B) = \cup_\gamma W_{\gamma A}$. Also, $p_\varphi|(W_{\gamma A})$ is a homeomorphism between $W_{\gamma A}$ and V_B. Because V_B is a soft path-connected, the same holds true for $W_{\gamma A}$. As $W_{\gamma A} \cap W_{\delta A} = \phi$ for $\gamma \neq \delta$, every $W_{\gamma A}$ is soft path-component of $p_\varphi^{-1}(V_B)$. □

Theorem 3.12. *Let (Y, τ_Y, B) be a soft locally path-connected, then a soft continuous map $p_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ is a soft covering-map iff, for every soft path-component M_B in $CS(Y_B)$, $p_\varphi|p_\varphi^{-1}(M_B) : p_\varphi^{-1}(M_B) \longrightarrow M_B$ is a soft covering-map.*

Proof. Assume $p_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ is a soft covering-map and $y \in M_B$. If U_B is a soft open (nbhd) of y in $CS(Y_B)$, and V_B is a soft path-component of U_B containing y, we have $V_B \subset M_B$, for M_B is a soft path-component in $CS(Y_B)$. Since (Y, τ_Y, B) is soft locally path-connected, V_B is soft open in $CS(Y_B)$ and so soft open in M_B. Obviously, V_B is soft-evenly covered by $q_\varphi = p_\varphi|p_\varphi^{-1}(M_B)$, and q_φ is a soft covering-map.

On the other hand, suppose that $q_\varphi : p_\varphi^{-1}(M_B) \longrightarrow M_B$, $y \longmapsto p(y)$ is a soft covering-map for all soft path-component M_B in $CS(Y_B)$, $y \in Y$, and let M_B be the soft path-component in $CS(Y_B)$, $y \in M_B$. Using the assumption, there exists a soft open (nhood) U_B of y in M_B, which is soft-evenly covered by q_φ. It follows from (Y, τ_Y, B) is soft locally path-connected, that the soft path-component M_B is soft open in $CS(Y_B)$. This implies that U_B is soft open in $CS(Y_B)$. Moreover, all soft open subsets of $p_\varphi^{-1}(M_B)$ are soft open in $CS(X_A)$. It has become evident that U_B is soft-evenly covered by p_φ. Hence, p_φ is a soft covering-map. □

Theorem 3.13. *Let $p_\varphi : CS(X_A) \longrightarrow CS(Y_B)$ be a soft covering-map. If (Y, τ_Y, B) is a soft locally path-connected with a soft path-component M_A in $CS(X_A)$, then $p_\varphi(M_A)$ is a soft path-component in $CS(Y_B)$ and $p_\varphi|M_A : CS(M_A) \longrightarrow CS\left(p_\varphi(M_A)\right)$ is a soft covering-map.*

Proof. Let (Y, τ_Y, B) be a soft locally path-connected, and suppose M_A is a soft path-component in $CS(X_A)$. To show that $p_\varphi(M_A)$ is a soft path-component in $CS(Y_B)$, it is enough to prove that it is a soft component, due to the indistinguishable nature of the soft components and soft path components in $CS(Y_B)$; it becomes clear that $p_\varphi(M_A)$ is soft connected. Now, to show $p_\varphi(M_A)$ is soft closed and soft open in $CS(Y_B)$, let $y \in cl(p_\varphi(M_A))$. Since (Y, τ_Y, B) is soft locally path conected, there is a soft path-connected open (nhood) U_B of y. Accordingly, each soft sheet $\hat{U}_B$ over U_B is soft path-connected. We have $M_A \cap p_\varphi^{-1}(U_B) \neq \phi$, for $U_B \cap p_\varphi(M_A) \neq \phi$. So there is a soft sheet $\hat{U}_B$ over U_B in which $\hat{U}_B \cap M_A \neq \phi$. It follows from M_A is a soft path-component in $CS(X_A)$, that $\hat{U}_B \subseteq M_A$. So, $U_B = p_\varphi(\hat{U}_B) \subseteq p_\varphi(M_A)$ and $y \in int(p_\varphi(M_A))$. This implies that $cl(p_\varphi(M_A)) \subseteq int(p_\varphi(M_A))$, and, therefore, $p_\varphi(M_A)$ is both soft closed and soft open. It follows that $p_\varphi(M_A)$ is a soft path-component in $CS\ (Y_B)$. Now, we show that $q_\varphi = p_\varphi|M_A : CS\ (M_A) \longrightarrow CS\left(p_\varphi(M_A)\right)$ is a soft covering-map. Let $y \in p_\varphi\ (M_A)$ and U_B be a soft path-connected soft (nbd) of y in $CS\ (Y_B)$. Thus, $U_B \subseteq p_\varphi(M_A)$. If $\hat{U}_B$ is a soft sheet over U_B and $\hat{U}_B \cap M_A \neq \phi$, then $\hat{U}_B \subseteq M_A$. Consequently, we can deduce that $q_\varphi^{-1}(U_B)$ is the disjoint union of those soft sheets $\hat{U}_B$ over U_B, each of which has an intersection with M_A. This implies that U_B is soft-evenly covered by q_φ and q_φ is a soft covering-map. □

4. Conclusions

This paper highlights the importance of soft covering-maps and spaces in soft topology theory. By introducing the notions of soft covering-maps and spaces, we have unearthed their pivotal role in connecting traditional topological concepts with the nuanced world of vague and imprecise information. Additionally, one can view soft covering space as a generalization or extension of covering space in geometric topology. Through a meticulous exploration of their properties, we have established a foundation for understanding the intricate interplay between soft covering maps and soft local homeomorphisms.

Use of AI tools declaration

The author declare he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares have no conflict of interest.

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