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Research article

Generalized warped product submanifolds of Lorentzian concircular structure manifolds

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Abstract: We began by considering invariant, anti-invariant, proper slant, and pointwise slant submanifolds of a Lorentzian concircular structure manifold. Subsequently, we explored two distinct categories of warped product submanifolds. The first category encompassed the fiber submanifold as an anti-invariant submanifold, while the second category included the fiber submanifold as a pointwise slant submanifold. We established several fundamental results concerning these submanifold classes. Additionally, we demonstrated the existence of such submanifold classes through specific examples. Moreover, we derived inequalities for the squared norm of the second fundamental form.

Keywords: $(LCS)_n$-manifold; skew CR-submanifold; warped product
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1. Introduction

In the year 2003, Shaikh [18] introduced a fascinating mathematical concept known as $(LCS)_n$-manifold, which stands for Lorentzian concircular structure manifold. This concept has profound implications in the field of general relativity. It was subsequently discovered that $(LCS)_n$-spacetimes are intricately connected to generalized Robertson Walker spacetimes [10], a well-established framework in cosmology.

The $(LCS)_n$-structure has garnered considerable attention due to its wide-ranging applications in the general theory of relativity. Researchers, as evidenced by studies such as [19, 20], have explored the various implications and consequences of this structure within the framework of Einstein's theory.

One intriguing property of the $(LCS)_n$-structure is its invariance under conformal transformations. This means that the structure remains unaltered when subjected to a conformal transformation, a mathematical operation that preserves angles but alters distances.

The concept of slant submanifolds was first introduced in the seminal work by Chen [3]. Building upon this notion, the idea of slant immersions of Riemannian manifolds into almost contact metric manifolds was further developed by Lotta [9]. Pointwise slant submanifolds, another variant of this concept, were introduced and investigated by Etayo [7]. For more comprehensive information on these topics, one may read [13, 17].

To explore additional classes of submanifolds within this manifold framework, researchers are suggested to go through Atehui [1] and Hui et al. [8].

The notion of warped product manifolds, on the other hand, originated from the pioneering work of Bishop and O'Neill [2] and has since been extensively studied in the literature, see [4–6, 8, 14, 22, 24]. The existence or non-existence of such product manifolds holds great significance, as it contributes to our understanding of the geometric structures and properties of these manifolds.

2. Preliminaries

Let $(\bar{\Sigma}, g)$ be an n-dimensional Lorentzian manifold with Lorentzian metric g and $\bar{\nabla}$ be the Levi-Civita connection for g. The $(LCS)_n$-manifold is defined as an n-dimensional Lorentzian manifold equipped with

- ξ, a unit timelike concircular vector field,
- η, ξ's associated 1-form,
- an $(1, 1)$ tensor field ϕ,

such that

$$\bar{\nabla}_P \xi = \alpha \phi P, \tag{2.1}$$

for some non-zero scalar function α which satisfies

$$\bar{\nabla}_P \alpha = P\alpha = d\alpha(P) = \rho\eta(P), \tag{2.2}$$

where $\rho = -(\xi\alpha)$ is also a scalar and $P \in \Gamma(T\bar{\Sigma})$. A $(LCS)_n$-manifold becomes a LP-Sasakian manifold when $\alpha = 1$ [11, 12].

From [18], we get some basic relations in a $(LCS)_n$-manifold $(n > 2)$ $\bar{\Sigma}$:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi P) = 0, \quad g(\phi P, \phi Q) = g(P, Q) + \eta(P)\eta(Q), \tag{2.3}$$

$$\phi^2 P = P + \eta(P)\xi, \tag{2.4}$$

$$(\bar{\nabla}_P \phi)Q = \alpha\{g(P, Q)\xi + 2\eta(P)\eta(Q)\xi + \eta(Q)P\}, \tag{2.5}$$

for all $P,\ Q,\ Z \in \Gamma(T\bar{\Sigma})$. Throughout the paper, we denote a $(LCS)_n$-manifold by $\bar{\Sigma}$.

We consider a submanifold $\Sigma \hookrightarrow \bar{\Sigma}$ with induced metric g and suppose that ∇, $\nabla^{\perp}$ denotes the induced connections on $T\Sigma$ and $T^{\perp}\Sigma$ of Σ, respectively. In this regard, the Gauss and Weingarten formulae are

$$\bar{\nabla}_P Q = \nabla_P Q + \zeta(P, Q), \tag{2.6}$$

and

$$\bar{\nabla}_P V = -A_V P + \nabla^{\perp}_P V, \tag{2.7}$$

for all $P, Q \in \Gamma(T\Sigma)$ and $V \in \Gamma(T^{\perp}\Sigma)$, where the second fundamental form is denoted by ζ and A_V denotes the shape operator (corresponding to V) for the immersion $\Sigma \hookrightarrow \bar{\Sigma}$ such that $g(\zeta(P, Q), V) = g(A_V P, Q)$.

For $P \in \Gamma(T\Sigma)$, the gradient $\boldsymbol{\nabla} i$ of a differentiable function i on Σ is defined by

$$g(\boldsymbol{\nabla} i, P) = Pi. \tag{2.8}$$

We also have

$$\text{(a) } \phi P = hP + kP, \ \text{ (b) } \phi V = lV + fV, \tag{2.9}$$

for any $P \in \Gamma(T\Sigma)$ and $V \in \Gamma(T^{\perp}\Sigma)$, where hP, lV are the tangential components and kP, fV are the normal components.

A submanifold $\Sigma \hookrightarrow \bar{\Sigma}$ is said to be invariant if $\phi(T_p\Sigma) \subseteq T_p\Sigma$ and anti-invariant if $\phi(T_p\Sigma) \subseteq T^{\perp}_p\Sigma$ for every $p \in \Sigma$.

A submanifold $\Sigma \hookrightarrow \bar{\Sigma}$ is said to be slant if for each non-zero vector $P \in T_p\Sigma$, the angle $\beta (0 \leq \beta \leq \frac{\pi}{2})$ between ϕP and $T_p\Sigma$ is a constant, i.e., it is independent of the choice of $p \in \Sigma$. Again Σ is said to be pointwise slant of $\bar{\Sigma}$ if β depends on P.

From [21], we find that a submanifold $\Sigma \hookrightarrow \bar{\Sigma}$ with $\xi \in \Gamma(T\Sigma)$ is pointwise slant if and only if

$$h^2 = \cos^2\beta (I + \eta \otimes \xi), \tag{2.10}$$

for some real valued function β defined on $T\Sigma$. Also if $\mathcal{D}^{\beta}$ is a pointwise slant distribution on pointwise slant submanifold Σ with $\xi \in \Gamma(T\Sigma)$, then

$$g(hZ, hW) = \cos^2\beta \{g(Z, W) + \eta(Z)\eta(W)\}, \tag{2.11}$$
$$g(kZ, kW) = \sin^2\beta \{g(Z, W) + \eta(Z)\eta(W)\}, \tag{2.12}$$

for any $Z, \ W \in \Gamma(\mathcal{D}^{\beta})$.

Let (N_1, g_1) and (N_2, g_2) be two semi-Riemannian manifolds and i be a positive smooth function on N_1. The warped product of (N_1, g_1) and (N_2, g_2) is denoted by $N_1 \times_i N_2 := (N_1 \times N_2, g)$, where

$$g = g_1 + i^2 g_2, \tag{2.13}$$

and i is the warping function. From [16], we have

$$\nabla_U P = \nabla_P U = (P \ln i)U, \forall P \in \Gamma(TN_1) \text{ and } U \in \Gamma(TN_2). \tag{2.14}$$

We consider Σ_I, $\Sigma_{\perp}$, Σ_{β}, and Σ_{ψ} as invariant, anti-invariant, proper slant, and proper pointwise slant submanifolds of $\bar{\Sigma}$. In this paper, we study the following two different classes of warped product submanifolds of $\bar{\Sigma}$.

First Class: $\Sigma = \Sigma_1 \times_i \Sigma_{\perp}$ with ξ tangent to Σ_1, where $\Sigma_1 = \Sigma_I \times \Sigma_{\beta}$. This class of submanifolds are known as warped product skew-CR submanifolds [15].

Second Class: $\Sigma = \Sigma_2 \times_i \Sigma_{\psi}$ with ξ tangent to Σ_2, where $\Sigma_2 = \Sigma_I \times \Sigma_{\perp}$. This class of submanifolds are known as warped product CR-slant submanifolds [23,25].

Throughout this paper, we consider the tangent spaces of Σ_I, $\Sigma_{\perp}$, Σ_{β}, and Σ_{ψ} as $\mathcal{D}^I$, $\mathcal{D}^{\perp}$, $\mathcal{D}^{\beta}$, and $\mathcal{D}^{\psi}$, respectively.

3. Submanifolds of the First Class

First, we construct an example of a submanifold of the First Class.

Example 1. *Consider the Euclidean space* $\mathbb{R}^{13}$ *with the cartesian coordinates* $(u_1, v_1, \cdots, u_6, v_6, t)$ *and para contact structure*

$$\phi\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial v_i}, \quad \phi\left(\frac{\partial}{\partial v_j}\right) = \frac{\partial}{\partial u_j}, \ \phi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i, j \leq 6.$$

It is clear that $\mathbb{R}^{13}$ *is a Lorentzian manifold with usual semi-Euclidean metric tensor. For any non-zero* λ, τ, *and* $\beta \in [0, \frac{\pi}{2}]$, *let* Σ *be a submanifold of* $\mathbb{R}^{13}$ *defined by the immersion map* $\chi : \mathbb{R}^6 \to \mathbb{R}^{13}$ *as*

$$\begin{aligned}\chi(\lambda, \tau, \beta, \mu, \varrho, t) = (&\lambda\cos\beta, \lambda\sin\beta, \tau\cos\beta, \tau\sin\beta, 4\lambda + 3\tau, 3\lambda + 4\tau,\\ &-\tau\cos\beta, \tau\sin\beta, -\lambda\cos\beta, \lambda\sin\beta, \mu, \varrho, t).\end{aligned}$$

Then the tangent space of Σ *is spanned by the following vectors*

$$\begin{aligned}
J_1 &= \cos\beta\frac{\partial}{\partial u_1} + \sin\beta\frac{\partial}{\partial v_1} + 4\frac{\partial}{\partial u_3} + 3\frac{\partial}{\partial v_3} - \cos\beta\frac{\partial}{\partial u_5} + \sin\beta\frac{\partial}{\partial v_5},\\
J_2 &= \cos\beta\frac{\partial}{\partial u_2} + \sin\beta\frac{\partial}{\partial v_2} + 3\frac{\partial}{\partial u_3} + 4\frac{\partial}{\partial v_3} - \cos\beta\frac{\partial}{\partial u_4} + \sin\beta\frac{\partial}{\partial v_4},\\
J_3 &= -\lambda\sin\beta\frac{\partial}{\partial u_1} + \lambda\cos\beta\frac{\partial}{\partial v_1} - \tau\sin\beta\frac{\partial}{\partial u_2} + \tau\cos\beta\frac{\partial}{\partial v_2} + \tau\sin\beta\frac{\partial}{\partial u_4}\\
&+ \tau\cos\beta\frac{\partial}{\partial v_4} + \lambda\sin\beta\frac{\partial}{\partial u_5} + \lambda\cos\beta\frac{\partial}{\partial v_5},\\
J_4 &= \frac{\partial}{\partial u_6},\ J_5 = \frac{\partial}{\partial v_6},\ \text{and } J_6 = \frac{\partial}{\partial t}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\phi J_1 &= \cos\beta\frac{\partial}{\partial v_1} + \sin\beta\frac{\partial}{\partial u_1} + 4\frac{\partial}{\partial v_3} + 3\frac{\partial}{\partial u_3} - \cos\beta\frac{\partial}{\partial v_5} + \sin\beta\frac{\partial}{\partial u_5},\\
\phi J_2 &= \cos\beta\frac{\partial}{\partial v_2} + \sin\beta\frac{\partial}{\partial u_2} + 3\frac{\partial}{\partial v_3} + 4\frac{\partial}{\partial u_3} - \cos\beta\frac{\partial}{\partial v_4} + \sin\beta\frac{\partial}{\partial u_4},
\end{aligned}$$

$$\begin{aligned}
\phi J_3 &= -\lambda\sin\beta\frac{\partial}{\partial v_1} + \lambda\cos\beta\frac{\partial}{\partial u_1} - \tau\sin\beta\frac{\partial}{\partial v_2} + \tau\cos\beta\frac{\partial}{\partial u_2} + \tau\sin\beta\frac{\partial}{\partial v_4}\\
&+ \tau\cos\beta\frac{\partial}{\partial u_4} + \lambda\sin\beta\frac{\partial}{\partial v_5} + \lambda\cos\beta\frac{\partial}{\partial u_5},\\
\phi J_4 &= \frac{\partial}{\partial v_6},\ \phi J_5 = \frac{\partial}{\partial u_6},\ \text{and } \phi J_6 = 0.
\end{aligned}$$

Therefore, it is clear that $\mathcal{D}^I = span\{J_4, J_5, J_6\}$ *is an invariant distribution,* $\mathcal{D}^\beta = span\{J_1, J_2\}$ *is a slant distribution with slant angle* $\cos^{-1}(\frac{25}{27})$, *and* $\mathcal{D}^\perp = span\{J_3\}$ *is an anti-invaiant distribution.*

Hence Σ is a skew CR-submanifold. Denote the integral manifolds of $\mathcal{D}^I$, $\mathcal{D}^\perp$, and $\mathcal{D}^\beta$ by Σ_I, $\Sigma_\perp$, and Σ_β, respectively. Then the product metric g of Σ is given by

$$g = -dt^2 + 27(d\lambda^2 + d\tau^2) + (d\mu^2 + d\varrho^2) + 2(\lambda^2 + \tau^2)d\beta^2.$$

Consequently Σ is a warped product skew CR-submanifold of type $\Sigma_1 \times_i \Sigma_\perp$ of $\mathbb{R}^{13}$, where $\Sigma_1 = \Sigma_I \times \Sigma_\beta$ with warping function $i = \sqrt{2(\lambda^2 + \tau^2)}$.

We take $dim\Sigma_I = 2a + 1$, $dim\Sigma_\perp = b$, $dim\Sigma_\beta = 2c$ and their corresponding tangent spaces are $\mathcal{D}^I \oplus \{\xi\}$, $\mathcal{D}^\perp$, and $\mathcal{D}^\beta$, respectively.

Assume that $\{x_1, x_2, \cdots, x_a, x_{a+1} = \phi x_1, \cdots, x_{2a} = \phi x_a, x_{2a+1} = \xi\}$, $\{x_{2a+2} = x_1^*, \cdots, x_{2a+b+1} = x_b^*\}$, and $\{x_{2a+b+2} = \hat{x}_1, x_{2a+b+3} = \hat{x}_2, \cdots, x_{2a+b+c+1} = \hat{x}_c, x_{2a+b+c+2} = \hat{x}_{c+1} = \sec\beta h\hat{x}_1, \cdots, x_{2a+b+2c+1}(= x_m) = \hat{x}_{2c} = \sec\beta h\hat{x}_c\}$ are local orthonormal frames of $\mathcal{D}^I \oplus \{\xi\}$, $\mathcal{D}^\perp$, and $\mathcal{D}^\beta$, respectively.

Then the local orthonormal frames for $\phi D^\perp$ and kD^β are $\{x_{m+1} = \tilde{x}_1 = \phi x_1^*, \cdots, x_{m+b} = \tilde{x}_b = \phi x_b^*\}$ and $\{x_{m+b+1} = \tilde{x}_{b+1} = \csc\beta k\hat{x}_1, \cdots, x_{m+b+c} = \tilde{x}_{b+c} = \csc\beta k\hat{x}_c, x_{m+b+c+1} = \tilde{x}_{b+c+1} = \csc\beta\sec\beta kh\hat{x}_1, \cdots, x_{m+b+2c} = \tilde{x}_{b+2c} = \csc\beta\sec\beta kh\hat{x}_c\}$, respectively. Also $\{x_{m+b+2c+1}, \cdots, x_n\}$ is a normal subbundle. We denote it by ν. Clearly ν is ϕ invariant and $dim\ \nu = (n - m - b - 2c)$.

First, we prove the following lemmas:

Lemma 1. *Let $\Sigma = \Sigma_1 \times_i \Sigma_\perp$ be a warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to $\Sigma_1 = \Sigma_I \times \Sigma_\beta$. Then we have*

$$g(\zeta(P, Q), \phi Z) = g(\zeta(P, Z), kU) = g(\zeta(P, U), \phi Z) = 0, \tag{3.1}$$

and

$$g(\zeta(U, Z), kV) + g(\zeta(U, V), \phi Z) = 0, \tag{3.2}$$

for every P, $Q \in \Gamma(\Sigma_I)$, $Z \in \Gamma(\Sigma_\perp)$, and $U, V \in \Gamma(\Sigma_\beta)$.

Proof. For P, $Q \in \Gamma(\Sigma_I)$, $Z \in \Gamma(\Sigma_\perp)$, and $U, V \in \Gamma(\Sigma_\beta)$, we find

$$g(\zeta(P, Q), \phi Z) = g(\nabla_Q \phi P, Z) - g((\bar{\nabla}_Q \phi)P, Z), \tag{3.3}$$

$$g(\zeta(P, Z), kU) = -g(\nabla_Z \phi P, U) - g((\bar{\nabla}_Z \phi)P, U) + g(P, \nabla_Z hU), \tag{3.4}$$

and

$$g(\zeta(P, U), \phi Z) = g(\nabla_U \phi P, Z) - g((\bar{\nabla}_U \phi)P, Z). \tag{3.5}$$

Using (2.5) and (2.14) in (3.3)–(3.5), we get (3.1).
Also we have

$$g(\zeta(U, V), \phi Z) = -g(hV, \nabla_U Z) - g((\bar{\nabla}_U \phi)V, Z) + g(\bar{\nabla}_U kV, Z). \tag{3.6}$$

Using (2.5) and (2.14) in (3.6), we get (3.2). □

Lemma 2. *Let $\Sigma = \Sigma_1 \times_i \Sigma_\perp$ be a warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to $\Sigma_1 = \Sigma_I \times \Sigma_\beta$. Then we have*

$$g(\zeta(P, Z), \phi W) = \{(\phi P \ln i) - \alpha\eta(P)\}g(Z, W), \tag{3.7}$$

$$g(\zeta(\phi P, Z), \phi W) = \{(P \ln i) + \alpha\eta(P)\}g(Z, W), \tag{3.8}$$

and

$$g(\zeta(Z, U), \phi W) + g(\zeta(Z, W), kU) = \{(hU \ln i) - \alpha\eta(U)\}g(Z, W), \tag{3.9}$$

for every $P \in \Gamma(\Sigma_I)$, Z, $W \in \Gamma(\Sigma_\perp)$, and $U \in \Gamma(\Sigma_\beta)$.

Proof. For $P \in \Gamma(\Sigma_I)$, $Z,\ W \in \Gamma(\Sigma_\perp)$, and $U \in \Gamma(\Sigma_\beta)$, we find

$$g(\zeta(P, Z), \phi W) = -g(\bar{\nabla}_Z \phi P, W) - g((\bar{\nabla}_Z \phi)P, W). \tag{3.10}$$

Using (2.5) and (2.14) in (3.10), we get (3.7). Replacing P by ϕP and applying $(\xi \ln i) = \alpha$ in (3.7), we get (3.8).

Also we have

$$g(\zeta(Z, U), \phi W) = -g(\bar{\nabla}_Z hU, W) + g(\bar{\nabla}_Z kU, W) - g((\bar{\nabla}_Z \phi)U, W). \tag{3.11}$$

Using (2.5) and (2.14) in (3.11), we get (3.9). □

Corollary 1. *Let $\Sigma = \Sigma_1 \times_i \Sigma_\perp$ be a $\mathcal{D}^\perp - \mathcal{D}^\theta$ warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to $\Sigma_1 = \Sigma_I \times \Sigma_\beta$, then we have*

$$g(\zeta(Z, W), kU) = \{(hU \ln i) - \alpha\eta(U)\}g(Z, W), \tag{3.12}$$

and

$$g(\zeta(Z, W), khU) = \cos^2\theta[(U \ln i) - \alpha\eta(U)]g(Z, W), \tag{3.13}$$

for every $Z,\ W \in \Gamma(\Sigma_\perp)$, and $U \in \Gamma(\Sigma_\beta)$.

Now we establish an inequality on a submanifold Σ of the First Class of $\bar{\Sigma}$.

Theorem 1. *Let $\Sigma = \Sigma_1 \times_i \Sigma_\perp$ be a $\mathcal{D}^\perp - \mathcal{D}^\beta$ mixed geodesic warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to Σ_I, where $\Sigma_1 = \Sigma_I \times \Sigma_\beta$. Then the squared norm of the second fundamental form satisfies*

$$||\zeta||^2 \geq b[\{2(||\ \nabla^I \ln i\ ||^2)\} + \cot^2\beta\ ||\ \nabla^\beta \ln i\ ||^2], \tag{3.14}$$

where $\nabla^I \ln i$ and $\nabla^\beta \ln i$ are the gradient of $\ln i$ along Σ_I and Σ_β, respectively, and for the case of equality, Σ_1 becomes totally geodesic and $\Sigma_\perp$ becomes totally umbilical in $\bar{\Sigma}$.

Proof. From (2.8), we have

$$||\zeta||^2 = \sum_{p,q=1}^{m} g(\zeta(x_p, x_q), \zeta(x_p, x_q)) = \sum_{r=m+1}^{n} g(\zeta(x_p, x_q), x_r)^2.$$

Decomposing the above relation for our constructed frames, we get

$$\begin{aligned} ||\zeta||^2 = & \sum_{r=m+1}^{n} \sum_{p,q=1}^{2a+1} g(\zeta(x_p, x_q), x_r)^2 + \sum_{r=m+1}^{n} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), x_r)^2 \\ & + \sum_{r=m+1}^{n} \sum_{i,j=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), x_r)^2 + 2 \sum_{r=m+1}^{n} \sum_{p=1}^{b} \sum_{q=1}^{2c} g(\zeta(x_p^*, \hat{x}_q), x_r)^2 \\ & +2 \sum_{r=m+1}^{n} \sum_{p=1}^{b} \sum_{q=1}^{2a+1} g(\zeta(x_p^*, x_q), x_r)^2 + 2 \sum_{r=m+1}^{n} \sum_{p=1}^{2c} \sum_{q=1}^{2a+1} g(\zeta(\hat{x}_p, x_q), x_r)^2. \end{aligned} \tag{3.15}$$

Now, again decomposing (3.15) along the normal subbundles $\phi\mathcal{D}^{\perp}$, $k\mathcal{D}^{\beta}$, and ν, we get

$$
\begin{aligned}
\|\zeta\|^2 = & \sum_{r=m+1}^{m+b}\sum_{p,q=1}^{2a+1} g(\zeta(x_p,x_q),x_r)^2 + \sum_{r=m+b+1}^{m+b+2c}\sum_{p,q=1}^{2a+1} g(\zeta(x_p,x_q),x_r)^2 \\
& + \sum_{r=m+b+2c+1}^{n}\sum_{p,q=1}^{2a+1} g(\zeta(x_p,x_q),x_r)^2 + \sum_{r=m+1}^{m+b}\sum_{p,q=1}^{b} g(\zeta(x_p^*,x_q^*),x_r)^2 \\
& + \sum_{r=m+b+1}^{m+b+2c}\sum_{p,q=1}^{b} g(\zeta(x_p^*,x_q^*),x_r)^2 + \sum_{r=m+b+2c+1}^{n}\sum_{p,q=1}^{b} g(\zeta(x_p^*,x_q^*),x_r)^2 \\
& + \sum_{r=m+1}^{m+b}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p,\hat{x}_q),x_r)^2 + \sum_{r=m+b+1}^{m+b+2c}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p,\hat{x}_q),x_r)^2 \\
& + \sum_{r=m+b+2c+1}^{n}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p,\hat{x}_q),x_r)^2 + 2\sum_{r=m+1}^{m+b}\sum_{p=1}^{b}\sum_{q=1}^{2c} g(\zeta(x_p^*,\hat{x}_q),x_r)^2 \\
& +2\sum_{r=m+b+1}^{m+b+2c}\sum_{p=1}^{b}\sum_{q=1}^{2c} g(\zeta(x_p^*,\hat{x}_q),x_r)^2 + 2\sum_{r=m+b+2c+1}^{n}\sum_{p=1}^{b}\sum_{q=1}^{2c} g(\zeta(x_p^*,\hat{x}_q),x_r)^2 \\
& +2\sum_{r=m+1}^{m+b}\sum_{p=1}^{2a+1}\sum_{q=1}^{b} g(\zeta(x_p,\overset{*}{x}_q),x_r)^2 + 2\sum_{r=m+b+1}^{m+b+2c}\sum_{p=1}^{2a+1}\sum_{q=1}^{b} g(\zeta(x_p,\overset{*}{x}_q),x_r)^2 \\
& +2\sum_{r=m+b+2c+1}^{n}\sum_{p=1}^{2a+1}\sum_{q=1}^{b} g(\zeta(x_p,\overset{*}{x}_q),x_r)^2 + 2\sum_{r=m+1}^{m+b}\sum_{p=1}^{2a+1}\sum_{q=1}^{2c} g(\zeta(x_p,\hat{x}_q),x_r)^2 \\
& +2\sum_{r=m+b+1}^{m+b+2c}\sum_{p=1}^{2a+1}\sum_{q=1}^{2c} g(\zeta(x_p,\hat{x}_q),x_r)^2 + 2\sum_{r=m+b+2c+1}^{n}\sum_{p=1}^{2a+1}\sum_{q=1}^{2c} g(\zeta(x_p,\hat{x}_q),x_r)^2.
\end{aligned}
\tag{3.16}
$$

Now, we can not find any relation for a warped product in the form $g(\zeta(E,F),\nu)$ for any $E, F \in \Gamma(T\Sigma)$. So, we leave the positive third, sixth, ninth, twelfth, fifteenth, and eighteenth terms of (3.16). Also, using Lemma 3.1 and the $\mathcal{D}^{\perp} - \mathcal{D}^{\beta}$ mixed geodesic property of Σ in (3.16), we get

$$
\begin{aligned}
|\zeta\|^2 \geq & \sum_{r=1}^{b}\sum_{p,q=2a+1}^{b} g(\zeta(x_p,x_q),k\hat{x}_r)^2 + \sum_{r=1}^{b}\sum_{p,q=1}^{b} g(\zeta(x_p^*,x_q^*),\phi x_r^*)^2 \\
& + \sum_{r=1}^{2c}\sum_{p,q=1}^{b} g(\zeta(x_p^*,x_q^*),k\hat{x}_r)^2 + \sum_{r=1}^{2c}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p,\hat{x}_q),k\hat{x}_r)^2 \\
& + 2\sum_{r=1}^{b}\sum_{p=1}^{2a+1}\sum_{q=1}^{b} g(\zeta(x_p,x_q^*),\phi x_r^*)^2 + 2\sum_{r=1}^{2c}\sum_{p=1}^{2a+1}\sum_{q=1}^{2c} g(\zeta(x_p,\hat{x}_q),k\hat{x}_r)^2.
\end{aligned}
\tag{3.17}
$$

Also, we have no relation for a warped product of the forms $g(\zeta(Z,W),\phi\mathcal{D}^{\perp})$, $g(\zeta(P,Q),k\mathcal{D}^{\beta})$, $g(\zeta(P,U),k\mathcal{D}^{\beta})$, and $g(\zeta(U,V),k\mathcal{D}^{\beta})$ for any $P,\ Q \in \Gamma(\mathcal{D}^{I})$ $Z,\ W \in \Gamma(\mathcal{D}^{\perp})$, $U,\ V \in \Gamma(\mathcal{D}^{\beta} \oplus \{\xi\})$.

So, we leave these terms from (3.17) and obtain

$$\|\zeta\|^2 \geq \sum_{r=1}^{2c} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), k\hat{x}_r)^2 + 2\sum_{r=1}^{b} \sum_{p=1}^{2a+1} \sum_{q=1}^{b} g(\zeta(x_p, x_q^*), \phi x_r^*)^2. \quad (3.18)$$

Now

$$\begin{aligned}\sum_{r=1}^{2c} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), k\hat{x}_r)^2 &= \csc^2\beta \sum_{r=1}^{c} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), k\hat{x}_r)^2 \\ &+ \csc^2\beta \sec^2\beta \sum_{r=1}^{c} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), kh\hat{x}_r)^2.\end{aligned}$$

Using Corollary 3.1, the above relation reduces to

$$\begin{aligned}\sum_{r=1}^{2c} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), k\hat{x}_r)^2 &= b\csc^2\beta \sum_{r=1}^{2c} \left[h(\hat{x}_r \ln i) - \eta(\hat{x}_r)\right]^2 \\ &+ b\cot^2\beta[\sum_{r=1}^{2c} \left[(\hat{x}_r \ln i) + \alpha\eta(\hat{x}_r)\right]^2.\end{aligned} \quad (3.19)$$

Now, since $\eta(\hat{x}_r = 0)$, for every $r = 1, 2, \cdots 2c$. So (3.19) turns into

$$\sum_{r=1}^{2c} \sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), k\hat{x}_r)^2 = b\cot^2\beta \|\nabla^{\beta} \ln i\|^2. \quad (3.20)$$

On the other hand

$$\begin{aligned}\sum_{r=1}^{b} \sum_{p=1}^{2a+1} \sum_{q=1}^{b} g(\zeta(x_p, x_q^*), \phi x_r^*)^2 &= \sum_{r=1}^{b} \sum_{p=1}^{a} \sum_{q=1}^{b} g(\zeta(x_p, x_q^*), \phi x_r^*)^2 \\ &+ \sum_{r=1}^{b} \sum_{p=1}^{a} \sum_{q=1}^{b} g(\zeta(\phi x_p, x_q^*), \phi x_r^*)^2 + \sum_{r=1}^{b} g(\zeta(\xi, x_q^*), \phi x_r^*)^2.\end{aligned}$$

Using Lemma 3.2 in the above relation, we obtain

$$\begin{aligned}\sum_{r=1}^{b} \sum_{p=1}^{2a+1} \sum_{q=1}^{b} g(\zeta(x_p, x_q^*), \phi x_r^*)^2 &= b\sum_{p=1}^{a} \left[(\phi x_p \ln i) - \eta(x_p)\right]^2 \\ &+ b\sum_{p=1}^{a} \left[(x_p \ln i) + \alpha\eta(x_p)\right]^2 + b\alpha.\end{aligned}$$

Since $\eta(x_p) = 0$ for every $p = 1, 2, \cdots, a$, using the relation $\xi(\ln i) = \alpha$, the above equation reduces to

$$\sum_{r=1}^{b} \sum_{p=1}^{2a+1} \sum_{q=1}^{b} g(\zeta(x_p, x_q^*), \phi x_r^*)^2 = b\|\nabla^{I} \ln i\|^2. \quad (3.21)$$

Using (3.20) and (3.21) in (3.18), we get the inequality (3.14).

If the equality of (3.14) holds, then after omitting ν component terms of (3.16), we get $\zeta(\mathcal{D}^I,\mathcal{D}^I)\perp\nu$, $\zeta(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\nu$, $\zeta(\mathcal{D}^\beta,\mathcal{D}^\beta)\perp\nu$, $\zeta(\mathcal{D}^\perp,\mathcal{D}^\beta)\perp\nu$, $\zeta(\mathcal{D}^I,\mathcal{D}^\perp)\perp\nu$, and $\zeta(\mathcal{D}^I,\mathcal{D}^\beta)\perp\nu$. Also, for the neglected terms of (3.17), we get $\zeta(\mathcal{D}^I,\mathcal{D}^I)\perp k\mathcal{D}^\beta$, $\zeta(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\phi\mathcal{D}^\perp$, $\zeta(\mathcal{D}^\beta,\mathcal{D}^\beta)\perp k\mathcal{D}^\beta$, $\zeta(\mathcal{D}^I,\mathcal{D}^\beta)\perp k\mathcal{D}^\beta$. Next, for $\mathcal{D}^\beta-\mathcal{D}^\perp$ mixed geodesicness and Lemma 3.1, we get $\zeta(\mathcal{D}^I,\mathcal{D}^I)\perp\phi\mathcal{D}^\perp$ and $\zeta(\mathcal{D}^\beta,\mathcal{D}^\beta)\perp\phi\mathcal{D}^\perp$.

Thus, we get $\zeta(\mathcal{D}^I,\mathcal{D}^I)=0$, $\zeta(\mathcal{D}^\beta,\mathcal{D}^\beta)=0$, $\zeta(\mathcal{D}^I,\mathcal{D}^\beta)=0$ and $\zeta(\mathcal{D}^\perp,\mathcal{D}^\perp)\subset k\mathcal{D}^\beta$.

Therefore Σ_1 is totally geodesic in Σ and hence in $\bar{\Sigma}$ [2]. Again, since $\Sigma_\perp$ is totally umbilical in Σ [2], with the fact that $\zeta(\mathcal{D}^\perp,\mathcal{D}^\perp)\subset k\mathcal{D}^\beta$, we conclude that $\Sigma_\perp$ is totally umbilical in $\bar{\Sigma}$. □

Theorem 2. *Let $\Sigma=\Sigma_1\times_i\Sigma_\perp$ be a $\mathcal{D}^\perp-\mathcal{D}^\beta$ mixed geodesic warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to $\Sigma_\perp$, where $\Sigma_1=\Sigma_\perp\times\Sigma_\beta$. Then the squared norm of the second fundamental form satisfies*

$$||\zeta||^2\geq b[2(||\ \nabla^I\ln i\ ||^2)+\cot^2\beta\{||\ \nabla^\beta\ln i\ ||^2-\alpha^2\}], \tag{3.22}$$

where $\nabla^I\ln i$ and $\nabla^\beta\ln i$ are the gradient of $\ln i$ along Σ_I and Σ_β, respectively, and for the case of equality, Σ_1 becomes totally geodesic and $\Sigma_\perp$ becomes totally umbilical in $\bar{\Sigma}$.

4. Submanifolds of the Second Class

First, we construct an example of a submanifold of the Second Class.

Example 2. *Consider the semi-Euclidean space $\mathbb{R}^{21}$ with the cartesian coordinates $(u_1,v_1,u_2,v_2,\cdots,u_{10},v_{10},t)$ and para contact structure*

$$\phi\left(\frac{\partial}{\partial u_i}\right)=\frac{\partial}{\partial v_i},\quad \phi\left(\frac{\partial}{\partial v_j}\right)=\frac{\partial}{\partial u_j},\ \phi\left(\frac{\partial}{\partial t}\right)=0,\quad 1\leq i,j\leq 10.$$

It is clear that $\mathbb{R}^{21}$ is a Lorentzian manifold with usual semi-Euclidean metric tensor. For any non-zero λ, τ, and $\beta,\psi\in[0,\frac{\pi}{2}]$, let Σ be a submanifold of $\mathbb{R}^{21}$ defined by the immersion map $\chi:\mathbb{R}^7\to\mathbb{R}^{21}$ as

$$\begin{aligned}\chi(\lambda,\ \tau,\ \beta,\ \psi,\ \mu,\ \varrho,\ t)=(&\lambda\cos\beta,\ \lambda\sin\beta,\ \tau\cos\beta,\ \tau\sin\beta,\ \lambda\cos\psi,\ \lambda\sin\psi,\ \tau\cos\psi,\\ &\tau\sin\psi,\ 4\beta+3\psi,\ 3\beta+4\psi,\ -\tau\cos\beta,\ \tau\sin\beta,\ -\lambda\cos\beta,\ \lambda\sin\beta,\ -\tau\cos\psi,\ \tau\sin\psi\\ &-\lambda\cos\psi,\ \lambda\sin\psi,\ \mu,\ \varrho,t).\end{aligned}$$

Then the tangent space of Σ is spanned by the following vectors

$$\begin{aligned}J_1 &= \cos\beta\frac{\partial}{\partial u_1}+\sin\beta\frac{\partial}{\partial v_1}+\cos\psi\frac{\partial}{\partial u_3}+\sin\psi\frac{\partial}{\partial v_3}\\ &- \cos\beta\frac{\partial}{\partial u_7}+\sin\beta\frac{\partial}{\partial v_7}-\cos\psi\frac{\partial}{\partial u_9}+\sin\psi\frac{\partial}{\partial v_9},\\ J_2 &= \cos\beta\frac{\partial}{\partial u_2}+\sin\beta\frac{\partial}{\partial v_2}+\cos\psi\frac{\partial}{\partial u_4}+\sin\psi\frac{\partial}{\partial v_4}\\ &- \cos\beta\frac{\partial}{\partial u_6}+\sin\beta\frac{\partial}{\partial v_6}-\cos\psi\frac{\partial}{\partial u_8}+\sin\psi\frac{\partial}{\partial v_8},\\ J_3 &= -\lambda\sin\beta\frac{\partial}{\partial u_1}+\lambda\cos\beta\frac{\partial}{\partial v_1}-\tau\sin\beta\frac{\partial}{\partial u_2}+\tau\cos\beta\frac{\partial}{\partial v_2}+4\frac{\partial}{\partial u_5}\end{aligned}$$

$$
\begin{aligned}
&+ \quad 3\frac{\partial}{\partial v_5} + \tau\sin\beta\frac{\partial}{\partial u_6} + \tau\cos\beta\frac{\partial}{\partial v_6} + \lambda\sin\beta\frac{\partial}{\partial u_7} + \lambda\cos\beta\frac{\partial}{\partial v_7},\\
J_4 &= -\lambda\sin\psi\frac{\partial}{\partial u_3} + \lambda\cos\psi\frac{\partial}{\partial v_3} - \tau\sin\psi\frac{\partial}{\partial u_4} + \tau\cos\psi\frac{\partial}{\partial v_4} + 3\frac{\partial}{\partial u_5}\\
&+ \quad 4\frac{\partial}{\partial v_5} + \tau\sin\psi\frac{\partial}{\partial u_8} + \tau\cos\psi\frac{\partial}{\partial v_8} + \lambda\sin\psi\frac{\partial}{\partial u_9} + \lambda\cos\psi\frac{\partial}{\partial v_9},\\
J_5 &= \frac{\partial}{\partial u_{10}},\ J_6 = \frac{\partial}{\partial v_{10}},\ \text{and}\ J_7 = \frac{\partial}{\partial t}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\phi J_1 &= \cos\beta\frac{\partial}{\partial v_1} + \sin\beta\frac{\partial}{\partial u_1} + \cos\psi\frac{\partial}{\partial v_3} + \sin\psi\frac{\partial}{\partial u_3}\\
&- \cos\beta\frac{\partial}{\partial v_7} + \sin\beta\frac{\partial}{\partial u_7} - \cos\psi\frac{\partial}{\partial v_9} + \sin\psi\frac{\partial}{\partial u_9},\\
\phi J_2 &= \cos\beta\frac{\partial}{\partial v_2} + \sin\beta\frac{\partial}{\partial u_2} + \cos\psi\frac{\partial}{\partial v_4} + \sin\psi\frac{\partial}{\partial u_4}\\
&- \cos\beta\frac{\partial}{\partial v_6} + \sin\beta\frac{\partial}{\partial u_6} - \cos\psi\frac{\partial}{\partial v_8} + \sin\psi\frac{\partial}{\partial u_8},
\end{aligned}
$$

$$
\begin{aligned}
\phi J_3 &= -\lambda\sin\beta\frac{\partial}{\partial v_1} + \lambda\cos\beta\frac{\partial}{\partial u_1} - \tau\sin\beta\frac{\partial}{\partial v_2} + \tau\cos\beta\frac{\partial}{\partial u_2} + 4\frac{\partial}{\partial v_5}\\
&+ \quad 3\frac{\partial}{\partial u_5} + \tau\sin\beta\frac{\partial}{\partial v_6} + \tau\cos\beta\frac{\partial}{\partial u_6} + \lambda\sin\beta\frac{\partial}{\partial v_7} + \lambda\cos\beta\frac{\partial}{\partial u_7},\\
\phi J_4 &= -\lambda\sin\psi\frac{\partial}{\partial v_3} + \lambda\cos\psi\frac{\partial}{\partial u_3} - \tau\sin\psi\frac{\partial}{\partial v_4} + \tau\cos\psi\frac{\partial}{\partial u_4} + 3\frac{\partial}{\partial v_5}\\
&+ \quad 4\frac{\partial}{\partial u_5} + \tau\sin\psi\frac{\partial}{\partial v_8} + \tau\cos\psi\frac{\partial}{\partial u_8} + \lambda\sin\psi\frac{\partial}{\partial v_9} + \lambda\cos\psi\frac{\partial}{\partial u_9},\\
\phi J_5 &= \frac{\partial}{\partial v_{10}},\ \phi J_6 = \frac{\partial}{\partial u_{10}},\ \text{and}\ \phi J_7 = 0.
\end{aligned}
$$

Therefore, it is clear that $\mathcal{D}^I = span\{J_5,\ J_6,\ J_7\}$ *is an invariant distribution,* $\mathcal{D}^\psi = span\{J_3,\ J_4\}$ *is a pointwise slant distribution with pointwise slant function* $\cos^{-1}(\frac{25}{2\lambda^2+2\tau^2+25})$*, and* $\mathcal{D}^\perp = span\{J_3, J_4\}$ *is an anti-invaiant distribution. Hence* Σ *is a CR-slant submanifold. Denote the integral manifolds of* $\mathcal{D}^I$*,* $\mathcal{D}^\perp$*, and* $\mathcal{D}^\psi$ *by* Σ_I*,* $\Sigma_\perp$*, and* Σ_β*, respectively. Then the product metric g of* Σ *is given by*

$$g = -dt^2 + 4(d\lambda^2 + d\tau^2) + (d\mu^2 + d\varrho^2) + (4\lambda^2 + 4\tau^2 + 25)(d\beta^2 + d\psi^2).$$

Consequently Σ *is a warped product CR-slant submanifold of type* $\Sigma_2 \times_i \Sigma_\beta$ *of* $\mathbb{R}^{21}$*, where* $\Sigma_2 = \Sigma_I \times \Sigma_\psi$ *with warping function* $i = \sqrt{4\lambda^2 + 4\tau^2 + 25}$*.*

Now we prove the following lemmas:

Lemma 3. *Let* $\Sigma = \Sigma_2 \times_i \Sigma_\psi$ *be a warped product submanifold of* $\bar{\Sigma}$ *such that* ξ *is tangent to* $\Sigma_2 = \Sigma_I \times \Sigma_\perp$*. Then we have*

$$g(\zeta(P,Q), kU) = g(\zeta(P,U), \phi Z) = g(\zeta(P,Z), kU) = 0, \tag{4.1}$$

and

$$g(\zeta(Z, PU), \phi W) + g(\zeta(Z, W), khU) = 0, \tag{4.2}$$

for every $P,\ Q \in \Gamma(\Sigma_I)$, $Z,\ W \in \Gamma(\Sigma_\perp)$, *and* $U \in \Gamma(\Sigma_\psi)$.

Proof. For $P,\ Q \in \Gamma(\Sigma_I)$, $Z,\ W \in \Gamma(\Sigma_\perp)$, and $U \in \Gamma(\Sigma_\psi)$, we find

$$g(\zeta(P, Q), kU) = -g(\phi P, \nabla_Q U) - g((\bar{\nabla}_Q \phi)P, U), \tag{4.3}$$

$$g(\zeta(P, U), \phi Z) = -g(\bar{\nabla}_U \phi P, Z) - g((\bar{\nabla}_U \phi)P, Z) + g(P, \bar{\nabla}_U, Z), \tag{4.4}$$

and

$$g(\zeta(P, Z), kU) = g(\phi P, \bar{\nabla}_Z U) - g((\bar{\nabla}_Z \phi)P, U). \tag{4.5}$$

Using (2.5) and (2.14) in (4.3)–(4.5), we get (4.1).

Also,

$$g(\zeta(U, V), \phi Z) = -g(hV, \bar{\nabla}_U Z) - g((\bar{\nabla}_U \phi)V, Z) + g(\bar{\nabla}_U kV, Z). \tag{4.6}$$

Using (2.5) and (2.14) in (3.6), we get (3.2). □

Lemma 4. *Let* $\Sigma = \Sigma_2 \times_i \Sigma_\beta$ *be a warped product CR-slant submanifold of* $\bar{\Sigma}$ *such that* ξ *is tangent to* $\Sigma_2 = \Sigma_I \times \Sigma_\perp$. *Then we have*

$$g(\zeta(P, U), kV) = \{(\phi P \ln i) - \alpha\eta(P)\}g(U, V) - (P \ln i)g(U, hV), \tag{4.7}$$

$$g(\zeta(\phi P, U), kV) = \{(P \ln i) + \alpha\eta(P)\}g(U, V) - (\phi P \ln i)g(U, hV), \tag{4.8}$$

and

$$g(\zeta(U, hV), \phi Z) + g(\zeta(U, Z), khV) = -\cos^2\psi(Z \ln i)g(U, V) - \eta(Z)g(U, hV), \tag{4.9}$$

for every $P \in \Gamma(\Sigma_I)$, $Z \in \Gamma(\Sigma_\perp)$, *and* $U,\ V \in \Gamma(\Sigma_\psi)$.

Proof. For $P \in \Gamma(\Sigma_I)$, $Z \in \Gamma(\Sigma_\perp)$, and $U,\ V \in \Gamma(\Sigma_\psi)$, we find

$$g(\zeta(P, U), kV) = -g(\bar{\nabla}_U \phi P, V) - g((\bar{\nabla}_U \phi)P, V). \tag{4.10}$$

Using (2.5) and (2.14) in (4.10), we get (4.7) and replacing P by ϕP in (4.7), we get (4.8).
Also we have

$$g(\zeta(U, hV), \phi Z) = -g(\bar{\nabla}_U Z, hV) + g(\bar{\nabla}_U khV, Z) - g((\bar{\nabla}_U \phi)hV, Z). \tag{4.11}$$

Using (2.5) and (2.14) in (4.11), we get (4.9). □

Corollary 2. *Let* $\Sigma = \Sigma_2 \times_i \Sigma_\psi$ *be a* $\mathcal{D}^\perp - \mathcal{D}^\psi$ *mixed geodesic warped product submanifold of* $\bar{\Sigma}$ *such that* ξ *is tangent to* $\Sigma_2 = \Sigma_I \times \Sigma_\perp$, *then we have*

$$g(\zeta(U, hV), \phi Z) = -\cos^2\psi(Z \ln i)g(U, V) - \alpha\eta(Z)g(U, hV), \tag{4.12}$$

and

$$g(\zeta(U, V), \phi Z) = -(Z \ln i)g(U, hV) - \alpha\eta(Z)g(U, V). \tag{4.13}$$

Now we establish the following inequality on a warped product submanifold Σ of $\bar{\Sigma}$ of the Second Class.

Theorem 3. *Let $\Sigma = \Sigma_2 \times_i \Sigma_\psi$ be a $\mathcal{D}^\perp - \mathcal{D}^\psi$ mixed geodesic warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to Σ_I, where $\Sigma_2 = \Sigma_I \times \Sigma_\perp$. Then the squared norm of the second fundamental form satisfies*

$$||\zeta||^2 \geq 2c[\{(\csc^2\beta + \cot^2\beta) \parallel \nabla^I \ln i \parallel^2\} + \cos^2\psi \parallel \nabla^\perp \ln i \parallel^2], \tag{4.14}$$

where $\nabla^I \ln i$ and $\nabla^\perp \ln i$ are the gradient of $\ln i$ along Σ_I and $\Sigma_\perp$, respectively, and for the case of equality, Σ_2 becomes totally geodesic and Σ_ψ becomes totally umbilical in $\bar{\Sigma}$.

Proof. For our constructed frame field, the second fundamental form ζ satisfies the relation (3.16). Now, similar to Theorem 3.1, we leave the positive third, sixth, ninth, twelfth, fifteenth, and eighteenth terms of (3.16).

Also, using Lemma 4.1 and the $\mathcal{D}^\perp - \mathcal{D}^\beta$ mixed geodesic property of Σ, from (3.16), we get

$$\begin{aligned} |\zeta||^2 \geq & \sum_{r=1}^{b}\sum_{p,q=1}^{2a+1} g(\zeta(x_p, x_q), k\hat{x}_r)^2 + \sum_{r=1}^{b}\sum_{p,q=1}^{b} g(\zeta(x_p^*, x_q^*), \phi x_r^*)^2 \\ + & \sum_{r=1}^{b}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), \phi x_r^*)^2 + \sum_{r=1}^{2c}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), k\hat{x}_r)^2 \\ + & 2\sum_{r=1}^{b}\sum_{p=1}^{2a+1}\sum_{q=1}^{b} g(\zeta(x_p, x_q^*), \phi x_r^*)^2 + 2\sum_{r=1}^{2c}\sum_{p=1}^{2a+1}\sum_{q=1}^{2c} g(\zeta(x_p, \hat{x}_q), k\hat{x}_r)^2. \end{aligned} \tag{4.15}$$

Also, we have no relation for a warped product of the forms $g(\zeta(Z, W), \phi\mathcal{D}^\perp)$, $g(\zeta(P, Q), k\mathcal{D}^\psi)$, $g(\zeta(P, Z), \phi\mathcal{D}^\perp)$, and $g(\zeta(U, V), k\mathcal{D}^\psi)$ for any $P,\ Q \in \Gamma(\mathcal{D}^I \oplus \{\xi\})$, $Z,\ W \in \Gamma(\mathcal{D}^\perp)$, $U,\ V \in \Gamma(\mathcal{D}^\psi)$. So, we leave these terms from (4.15) and obtain

$$||\zeta||^2 \geq \sum_{r=1}^{b}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), \phi x_r^*)^2 + 2\sum_{r=1}^{2c}\sum_{p=1}^{2a+1}\sum_{q=1}^{2c} g(\zeta(x_p, \hat{x}_q), k\hat{x}_r)^2. \tag{4.16}$$

Now

$$\sum_{r=1}^{b}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), \phi x_r^*)^2 = \sum_{r=1}^{b}\sum_{p,q=1}^{c} g(\zeta(\hat{x}_p, \hat{x}_q), \phi x_r^*)^2$$
$$+2\sec^2\psi \sum_{r=1}^{b}\sum_{p,q=1}^{c} g(\zeta(\hat{x}_p, h\hat{x}_q), \phi x_r^*)^2 + \sec^4\psi \sum_{r=1}^{b}\sum_{p,q=1}^{c} g(\zeta(h\hat{x}_p, h\hat{x}_q), \phi x_r^*)^2.$$

Using Corollary 4.1, the above relation reduces to

$$\sum_{r=1}^{b}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), \phi x_r^*)^2 = 2c\sum_{r=1}^{b} [\eta(x_r^*)]^2 + 2c\cos^2\psi \sum_{r=1}^{b} [(x_r^* \ln i)]^2. \tag{4.17}$$

Now, since $\eta(x_r^*) = 0$, for every $r = 1, 2, \cdots b$, (4.17) turns into

$$\sum_{r=1}^{b}\sum_{p,q=1}^{2c} g(\zeta(\hat{x}_p, \hat{x}_q), \phi x_r^*)^2 = 2c\cos^2\psi[||\nabla^\perp \ln i||^2]. \tag{4.18}$$

On the other hand

$$
\begin{aligned}
\sum_{r,q=1}^{2c}\sum_{p=1}^{2a+1} g(\zeta(x_p,\hat{x}_q),k\hat{x}_r)^2 &= \csc^2\psi\sum_{r,q=1}^{c}\sum_{p=1}^{a} g(\zeta(x_p,\hat{x}_q),k\hat{x}_r)^2\\
&+\csc^2\psi\sum_{r,q=1}^{c}\sum_{p=1}^{2a} g(\zeta(\phi x_p,\hat{x}_q),k\hat{x}_r)^2+\csc^2\psi\sec^2\psi\sum_{r,q=1}^{c}\sum_{p=1}^{a} g(\zeta(x_p,h\hat{x}_q),k\hat{x}_r)^2\\
&+\csc^2\psi\sum_{r,q=1}^{c} g(\zeta(\xi,\hat{x}_q),k\hat{x}_r)^2+\csc^2\psi\sec^2\psi\sum_{r,q=1}^{c}\sum_{p=1}^{a} g(\zeta(\phi x_p,h\hat{x}_q),k\hat{x}_r)^2\\
&+\csc^2\psi\sec^2\psi\sum_{r,q=1}^{c} g(\zeta(\xi,h\hat{x}_q),k\hat{x}_r)^2+\csc^2\psi\sec^2\psi\sum_{r,q=1}^{c}\sum_{p=1}^{a} g(\zeta(x_p,h\hat{x}_q),kh\hat{x}_r)^2\\
&+\csc^2\psi\sec^2\psi\sum_{r,q=1}^{c}\sum_{p=1}^{2a} g(\zeta(\phi x_p,h\hat{x}_q),kh\hat{x}_r)^2+\csc^2\psi\sec^2\psi\sum_{r,q=1}^{c} g(\zeta(\xi,h\hat{x}_q),k\hat{x}_r)^2\\
&+\csc^2\psi\sec^4\psi\sum_{r,q=1}^{c}\sum_{p=1}^{a} g(\zeta(x_p,h\hat{x}_q),kh\hat{x}_r)^2+\csc^2\psi\sec^4\psi\sum_{r,q=1}^{c} g(\zeta(\xi,h\hat{x}_q),kh\hat{x}_r)^2\\
&+\csc^2\psi\sec^4\psi\sum_{r,q=1}^{c}\sum_{p=1}^{a} g(\zeta(x_p,h\hat{x}_q),kh\hat{x}_r)^2.
\end{aligned}
$$

Using Lemma 4.4 in the above relation, we obtain

$$
\begin{aligned}
\sum_{r,q=1}^{2c}\sum_{p=1}^{2a+1} g(\zeta(x_p,\hat{x}_q),k\hat{x}_r)^2 &= c\csc^2\psi\sum_{p=1}^{a}\left[(\phi x_p\ln i)-\eta(x_p)\right]^2\\
+c\csc^2\psi\sum_{p=1}^{a}\left[(x_p\ln i)+\alpha\eta(x_p)\right]^2 &+2c\alpha^2\csc^2\psi+c\cot^2\psi\sum_{p=1}^{a}(x_p\ln i)^2\\
&+c\cot^2\psi\sum_{p=1}^{a}(\phi x_p\ln i)^2+c\cot^2\psi\sum_{p=1}^{a}(x_p\ln i)^2\\
&+c\cot^2\psi\sum_{p=1}^{a}(\phi x_p\ln i)^2+c\csc^2\psi\sum_{p=1}^{a}\left[(\phi x_p\ln i)-\eta(x_p)\right]^2\\
&+c\csc^2\psi\sum_{p=1}^{a}\left[(x_p\ln i)+\alpha\eta(x_p)\right]^2+2c\alpha^2\cot^2\psi.
\end{aligned}
$$

Since $\eta(x_p)=0$ for every $p=1,2,\cdots,a$, the above equation reduces to

$$
\sum_{r,q=1}^{2c}\sum_{p=1}^{2a+1} g(\zeta(x_p,\hat{x}_q),k\hat{x}_r)^2 = 2c(\cos^2\psi+\cot^2\psi)\|\nabla^I\ln i\|^2. \tag{4.19}
$$

Using (4.18) and (4.19) in (4.17), we get the inequality (4.14).

Proof of the equalty case is similar to the proof of the equality case of Theorem 3.1. □

Theorem 4. *Let $\Sigma = \Sigma_2 \times_i \Sigma_\psi$ be a $\mathcal{D}^\perp - \mathcal{D}^\psi$ mixed geodesic warped product submanifold of $\bar{\Sigma}$ such that ξ is tangent to $\Sigma_\perp$, where $\Sigma_2 = \Sigma_I \times \Sigma_\perp$. Then the squared norm of the second fundamental form satisfies*

$$||\zeta||^2 \geq 2c[(\csc^2\psi + \cot^2\psi) \parallel \nabla^I \ln i \parallel^2 + \cos^2\psi\{\parallel \nabla^\perp \ln i \parallel^2 - \alpha^2\}], \tag{4.20}$$

where $\nabla^I \ln i$ and $\nabla^\perp \ln i$ are the gradient of $\ln i$ along Σ_I and $\Sigma_\perp$, respectively, and for the case of equality, Σ_2 becomes totally geodesic and Σ_ψ becomes totally umbilical in $\bar{\Sigma}$.

5. Conclusions

This paper investigated different types of submanifolds in the context of a Lorentzian concircular structure manifold. We examined invariant, anti-invariant, proper slant, and pointwise slant submanifolds, and further explored two distinct categories of warped product submanifolds.

In the first category, we considered the fiber submanifold as an anti-invariant submanifold, while in the second category, the fiber submanifold was treated as a pointwise slant submanifold. Throughout our analysis, we established several fundamental results and derived important inequalities for the squared norm of the second fundamental form.

Our research not only provided a theoretical framework for understanding the properties and characteristics of these submanifold classes but also demonstrated the existence of such submanifold classes through specific examples. By examining these examples, we gained valuable insights into the behavior and geometric structures of the submanifolds within the Lorentzian concircular structure manifold.

Overall, this study contributes to the field of differential geometry by expanding our understanding of submanifolds and their relationships within a Lorentzian concircular structure manifold. The results and inequalities derived in this paper can serve as valuable tools for future research in this area, and we hope that they will inspire further investigations into the geometric properties of submanifolds in related contexts.

Author contributions

Tanumoy Pal: Conceptualization, Methodology, Investigation, Writing-original draft preparation, Writing-review and editing; Ibrahim Al-Dayel: Investigation, Writing-original draft preparation; Meraj Ali Khan: Conceptualization, Writing-review and editing; Biswabismita Bag: Methodology, Investigation, Writing-original draft preparation, Writing-review and editing; Shyamal Kumar Hui: Conceptualization, Methodology, Writing-review and editing, Foued Aloui: Investigation, Writing-original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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