AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 17984–17996.
DOI: 10.3934/math.2024876
Received: 13 February 2024
Revised: 03 May 2024
Accepted: 08 May 2024
Published: 27 May 2024

*Research article*

# Study on the controllability of delayed evolution inclusions involving fractional derivatives

**Yue Liang***

College of Science, Gansu Agricultural University, Lanzhou Gansu 730070, China

* **Correspondence:** Email: liangyuegsau@163.com.

**Abstract:** This paper dealt with the infinite controllability of delayed evolution inclusions with $\alpha$-order fractional derivatives in Fréchet spaces, where $\alpha \in (1,2)$. The controllability conclusion was acquired without any compactness for the nonlinear term, the cosine family, and the sine family. The investigation was based on a nonlinear alternative and the cosine family theory. An application of our findings was provided.

**Keywords:** fractional evolution inclusions; infinite controllability; the cosine family; fréchet spaces; mild solutions
**Mathematics Subject Classification:** 34K35, 93C25

## 1. Introduction

Fractional Cauchy problems are useful in physics to model anomalous diffusion [1]. Nigmatullin [2] introduced a generalization diffusion equation of the non-Markovian type

$$\partial_t^\alpha \omega(t,\varrho) = \frac{\partial^2 \omega(t,\varrho)}{\partial \varrho^2}, \quad 0 \le \alpha \le 2,$$

where $\partial_t^\alpha$ is the $\alpha$-order fractional derivative operator on $t$. The case $\alpha = 0$ corresponds to an elliptic equation. The case $\alpha = 1$ corresponds to a parabolic equation. The case $\alpha = 2$ corresponds to a hyperbolic equation.

Let $\Upsilon \subset \mathbb{R}^N$ be a bounded domain and $\partial\Upsilon$, the boundary of $\Upsilon$, be sufficiently smooth. We first demonstrate the linear IBVP

$$\begin{cases} {}^C\partial_t^\alpha \omega(t,\varrho) - \Delta\omega(t,\varrho) = g(t,\varrho), \quad (t,\varrho)\in(0,\infty)\times\Upsilon, \quad \alpha\in(1,2), \\ \omega|_{\partial\Upsilon} = 0, \\ \omega(0,\varrho) = \omega_0(\varrho), \quad \forall\varrho\in\Upsilon, \\ \omega'(0,\varrho) = \omega_1(\varrho), \quad \forall\varrho\in\Upsilon, \end{cases} \tag{1.1}$$

where ${}^C\partial_t^\alpha$ is the $\alpha$-order fractional partial derivative operator on $t$ in the Caputo sense, $\Delta$ denotes the Laplace operator, $g:(0,\infty)\times\Upsilon\to\mathbb{R}$ is a given linear function, $\omega_0(\varrho),\omega_1(\varrho)\in\mathbb{R}$ for all $\varrho\in\Upsilon$. Equation (1.1) has wide applications in viscoelastic models. When $\alpha\to1$, it is the heat equation. When $\alpha\to2$, it becomes the standard wave equation. For $\alpha\in(1,2)$, Eq (1.1) is called the super-diffusive equation. Hence, Eq (1.1) has significant backgrounds in physical applications.

Let $A=\Delta$ and

$$\mathcal{D}(A) = H^2(\Upsilon)\cap H_0^1(\Upsilon).$$

Then the IBVP(1.1) can be rewritten as an abstract IVP

$$\begin{cases} {}^C D_{0^+}^\alpha \omega(t) - A\omega(t) = g(t), \quad t>0, \quad \alpha\in(1,2), \\ \omega(0) = \omega_0, \\ \omega'(0) = \omega_1, \end{cases} \tag{1.2}$$

where ${}^C D_{0^+}^\alpha$ represents the Caputo fractional derivative operator. Let $\mathbb{E}=L^2(\Upsilon)$. It follows from Arendt et al. [3] that $A:\mathcal{D}(A)\subseteq\mathbb{E}\to\mathbb{E}$ is densely defined and closed in $\mathbb{E}$ generating a strongly continuous cosine family $\{G(t):t\geq0\}$ which is uniformly bounded. When $g:[0,\infty)\to\mathbb{E}$ is continuous and $\omega_0,\omega_1\in\mathbb{E}$, Zhou et al. [4] presented a new concept of mild solutions of the IVP(1.2) by employing the Laplace transform. Exact controllability results were proved on the interval $[0,c]$(where $c>0$ is a constant) for the IVP of the semilinear fractional evolution equation (without delays) in [4] when the nonlinear term $f$ is global Lipschitz continuous or satisfies certain compact conditions.

It is well known that the delay appears in wide fields, such as biology, physics, economics, etc. It is meaningful to study the fractional evolution equation with time delay. Let $\mathbb{E}$ be a general Banach space. In 2023, Yang in [5] investigated the fractional delayed control system in $\mathbb{E}$

$$\begin{cases} {}^L D_t^\alpha \omega(t) = A\omega(t) + f(t,\bar{\omega}_t) + Pv(t), \quad t\in(0,c], \ \alpha\in(1,2), \\ \bar{\omega}(0) = \varphi(t), \quad t\in[-r,0], \\ (\mathcal{H}_{2-\alpha}*\omega)'(0) = \omega_1, \end{cases}$$

where ${}^L D_t^\alpha$ is the fractional derivative operator on $t$ of order $\alpha$ in the Riemann-Liouville sense, $\bar{\omega}(t) = t^{2-\alpha}\omega(t)$ for $t\in(0,c]$ with $\bar{\omega}(0)=\lim\limits_{t\to0^+}\bar{\omega}(t)$ and $\bar{\omega}_t(\theta)=\bar{\omega}(t+\theta)$ for $t\in[0,c]$ and $\theta\in[-r,0], r>0$, $v\in L^2([0,c],\mathbb{V})$, $\mathbb{V}$ is another Banach space, $P\in\mathcal{L}(\mathbb{V},\mathbb{E})$ (here $\mathcal{L}(\mathbb{V},\mathbb{E}) := \{P:\mathbb{V}\to\mathbb{E}$ is a bounded operator$\}$, $\mathcal{L}(\mathbb{E}) := \mathcal{L}(\mathbb{E},\mathbb{E})$), $\varphi\in C([-r,0],\mathbb{E})$ with $\varphi(0)\in\mathcal{D}(A)$ and $\omega_1\in\mathbb{E}$. $\mathcal{H}_\varrho(t)=\frac{t^{\varrho-1}}{\Gamma(\varrho)}$ for $t,\varrho>0$. The symbol $*$ represents the convolution. When the sine family $\{W(t):t\geq0\}$, corresponding to the cosine family $\{G(t):t\geq0\}$ generated by $A$, is compact for $t>0$, the approximate controllability results were obtained.

In [6], Gou et al. studied the existence and approximate controllability of the Hilfer fractional evolution equations when the cosine family $\{G(t) : t \geq 0\}$, generated by the linear operator $A$, is exponentially bounded and continuous in the uniform operator topology for every $t > 0$. In [7], He et al. investigated the approximate controllability for a class of fractional stochastic wave equations when the sine family $\{W(t) : t \geq 0\}$, corresponding to the cosine family $\{G(t) : t \geq 0\}$ generated by $A$, is compact for every $t > 0$. However, if we remove the global Lipschitz condition on $f$ or the compactness assumption on $\{W(t) : t \geq 0\}$, how is the controllability of the fractional delayed evolution inclusions?

In the present work, we research the infinite controllability of fractional delayed evolution inclusions (FDEIs)

$$\begin{cases} {}^C D_{0^+}^{\alpha} \omega(t) - A\omega(t) \in F(t, \omega_t) + Pv(t), \quad a.e.\ t \geq 0, \\ \omega(t) = \phi(t), \quad t \in [-r, 0], \\ \omega'(0) = \omega_1, \end{cases} \tag{1.3}$$

where $F$ is a multi-valued mapping with compact values in $\mathbb{E}$, $v \in L^2_{loc}([0, \infty), \mathbb{V})$, $\mathbb{V}$ is another Banach space, $P \in \mathcal{L}(\mathbb{V}, \mathbb{E})$, $\phi \in C([-r, 0], \mathbb{E})$ and $\omega_1 \in \mathbb{E}$. For any $t \geq 0$, $\omega_t(\xi) = \omega(t + \xi)$ for $\xi \in [-r, 0]$.

The infinite controllability of evolution systems is demonstrated extensively. In [8], Benchohra et al. proved the controllability of first-order evolution equations in a semi-infinite time horizon. The infinite controllability of differential and integrodifferential inclusions had been discussed in [9, 10]. However, there are research articles on the infinite controllability of the fractional delayed evolution equations. Particularly, for $\alpha \in (1, 2)$, the infinite controllability of the $\alpha$-order Caputo fractional delayed evolution inclusions has not been studied.

In this paper, using a nonlinear alternative established by Frigon in [11], the infinite controllability of the FDEIs (1.3) is investigated in Fréchet spaces. The major results and features of our work are summarized below:

(1) The infinite controllability of the FDEIs (1.3) of order $\alpha \in (1, 2)$ in Fréchet spaces is demonstrated. It is new and novel even if $F$ is a single-valued mapping.

(2) Our result is obtained without the global Lipschitz continuity of the nonlinear term $F$. Hence, it greatly generalizes Theorem 4.1 of [4].

(3) We remove the assumption of compactness on the cosine family and the sine family, which are essential assumptions in [5–7]. Hence, Our work is an improvement of [5–7].

## 2. Preliminaries

Let $(\mathbb{E}, \|\cdot\|)$ and $(\mathbb{V}, \|\cdot\|)$ be Banach spaces. $C([-r, 0], \mathbb{E})$ is a Banach space of $\mathbb{E}$-valued continuous functions with $\|\phi\|_C = \sup\limits_{t \in [-r, 0]} \|\phi(t)\|$.

We first introduce symbols:

$\mathcal{K}(\mathbb{E})$ is the set of nonempty subsets of $\mathbb{E}$.

$\mathcal{K}_{cl}(\mathbb{E}) = \{Y \in \mathcal{K}(\mathbb{E}) : Y \text{ is closed }\}$,

$\mathcal{K}_{cp}(\mathbb{E}) = \{Y \in \mathcal{K}(\mathbb{E}) : Y \text{ is compact }\}$,

$\mathcal{K}_{cl,b}(\mathbb{E}) = \{Y \in \mathcal{K}(\mathbb{E}) : Y \text{ is closed and bounded }\}$.

For $\varrho > 0$, denote by

$$g_{\varrho}(t) = \frac{t^{\varrho-1}}{\Gamma(\varrho)},$$

where $\Gamma(\varrho) = \int_0^{\infty} e^{-\theta}\theta^{\varrho-1}d\theta$. For $n \geq 1$, it follows from [12, 13] that the Caputo fractional derivative is given by

$${}^{C}D_{0^+}^{\alpha}\eta(t) = {}^{L}D_{0^+}^{\alpha}\Big(\eta(t) - \sum_{k=0}^{n-1}\frac{\eta^{(k)}(0)}{k!}t^k\Big), \quad t > 0, \ \ n-1 < \alpha < n,$$

where ${}^{L}D_{0^+}^{\alpha}x(t) = \frac{d^n}{dt^n}(g_{n-\alpha} * x)(t)$ and $*$ means the convolution.

**Definition 2.1.** *[14] $\{G(t) : t \in \mathbb{R}\} \subset \mathcal{L}(E)$ is called a strongly continuous cosine family, if*
($i$) *$G(0) = I$;*
($ii$) *for any $\sigma_1, \sigma_2 \in \mathbb{R}$, $G(\sigma_1 + \sigma_2) + G(\sigma_1 - \sigma_2) = 2G(\sigma_1)G(\sigma_2)$;*
($iii$) *for each $\omega \in \mathbb{E}$, $G(\cdot)\omega$ is continuous.*

Define

$$W(t)\omega := \int_0^t G(\theta)\omega d\theta, \quad \omega \in \mathbb{E}, \ \ t \in \mathbb{R}.$$

Then, $\{W(t) : t \in \mathbb{R}\}$ is called the sine family corresponding to $\{G(t) : t \in \mathbb{R}\}$. Let

$$\mathcal{D}(A) = \{\omega \in \mathbb{E} : G(t)\omega \in C^2(\mathbb{R}, \mathbb{E})\}$$

and, for any $\omega \in \mathcal{D}(A)$,

$$A\omega = \frac{d^2}{dt^2}G(t)\omega|_{t=0}.$$

Then, $A$ is called the infinitesimal generator of the cosine family $\{G(t) : t \in \mathbb{R}\}$. Obviously, $A$ is a linear operator and it is closed and densely defined in $\mathbb{E}$.

($H1$) $A$ generates a strongly continuous cosine family $\{G(t) : t \geq 0\}$ in $\mathbb{E}$, and $\exists M \geq 1$ such that, for any $t \geq 0$,

$$\|G(t)\| \leq M.$$

Let

$$\mathcal{M}_{\tau}(\theta) = \sum_{m=0}^{\infty}\frac{(-\theta)^m}{m!\Gamma(1 - \tau(m+1))}, \quad \theta \in \mathbb{C}, \ \tau \in (0, 1),$$

where $\mathbb{C}$ is the imaginary line. Then $\mathcal{M}_{\tau}(\theta)$ is called Mainardi's Wright-type function.

**Lemma 2.1.** *[4, 15, 16] The function $\mathcal{M}_{\tau}(\theta)$ has the properties:*
($i$) $\mathcal{M}_{\tau}(\theta) \geq 0, \quad \forall \theta > 0$;
($ii$) $\int_0^{\infty}\theta^{\delta}\mathcal{M}_{\tau}(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\tau\delta)}, \quad \forall \delta \in (-1, \infty)$.

Let $\beta = \frac{\alpha}{2}$. Then $\beta \in (\frac{1}{2}, 1)$. By [4] we can get the following concept.

**Definition 2.2.** *[4] $\omega \in C([-r, \infty], \mathbb{E})$ is called the mild solution of the FDEIs* (1.3) *if*
($i$) *$\omega'(0) = \omega_1 \in \mathbb{E}$ and, for all $t \in [-r, 0]$, $\omega(t) = \phi(t)$;*
($ii$) *There exists $g \in L^1_{loc}([0, \infty), \mathbb{E})$ with $g(t) \in F(t, \omega_t)$, a.e. $t \geq 0$ satisfying*

$$\omega(t) = \Psi_{\beta}(t) + \int_0^t \mathcal{T}_{\beta}(t-\theta)g(\theta)d\theta + \int_0^t \mathcal{T}_{\beta}(t-\theta)Pv(\theta)d\theta, \quad t \geq 0, \tag{2.1}$$

*where*

$$\begin{aligned}\Psi_\beta(t) =& G_\beta(t)\phi(0) + \mathcal{S}_\beta(t)\omega_1,\\ G_\beta(t) =& \int_0^\infty \mathcal{M}_\beta(\theta)G(t^\beta\theta)d\theta,\\ \mathcal{S}_\beta(t) =& \int_0^t G_\beta(\theta)d\theta,\\ \mathcal{T}_\beta(t) =& \int_0^\infty \beta\theta t^{\beta-1}\mathcal{M}_\beta(\theta)W(t^\beta\theta)d\theta.\end{aligned}$$

**Lemma 2.2.** *[4] Let* $(H1)$ *hold. For* $\forall\omega \in \mathbb{E}$, *the families* $\{G_\beta(t) : t \geq 0\}, \{\mathcal{S}_\beta(t) : t \geq 0\}$ *and* $\{\mathcal{T}_\beta(t) : t \geq 0\}$ *satisfy*

$$\|G_\beta(t)\omega\| \leq M\|\omega\|, \quad \|\mathcal{S}_\beta(t)\omega\| \leq M\|\omega\|t$$

*and*

$$\|\mathcal{T}_\beta(t)\omega\| \leq \frac{M\|\omega\|}{\Gamma(2\beta)}t^{2\beta-1}.$$

For any $B_1, B_2 \in \mathcal{K}(\mathbb{E})$, the Hausdorff pseudometric $D_\nu, \nu \in \Lambda$ induced by $d_\nu$ is defined by

$$D_\nu(B_1, B_2) = \inf\{\vartheta > 0 : \forall x \in B_1, z \in B_2, \exists\bar{x} \in B_1, \bar{z} \in B_2 \text{ such that } d_\nu(x, \bar{z}) < \vartheta, d_\nu(z, \bar{x}) < \vartheta\}$$

with $\inf\emptyset = \infty$. Particularly, when $\mathbb{E}$ is locally-convex and complete, the set $B_1 \subset \mathbb{E}$ is bounded if $D_\nu(\{0\}, B_1) < \infty$ for every $\nu \in \Lambda$.

**Definition 2.3.** *[11] Let* $F : \mathbb{E} \to \mathcal{K}(\mathbb{E})$ *be a multi-valued map. F is named as an admissible contraction with constants* $\{\kappa_\nu\}_{\nu\in\Lambda}$ *if for* $\forall\nu \in \Lambda$, $\exists\kappa_\nu \in (0, 1)$ *satisfying*

$(i)$ *for* $\forall x, z \in \mathbb{E}$, $D_\nu(F(x), F(z)) \leq \kappa_\nu d_\nu(x, z)$;

$(ii)$ *for* $\forall\epsilon > 0, x \in \mathbb{E}$, $\exists z \in F(x)$ *satisfying*

$$d_\nu(x, z) \leq d_\nu(x, F(x)) + \epsilon.$$

**Lemma 2.3.** *[11] Let* $D \subset \mathbb{E}$ *be an open neighborhood of the origin in the Fréchet space* $\mathbb{E}$. *If* $Q : \overline{D} \to \mathcal{K}(\mathbb{E})$ *is an admissible contraction and Q is bounded, then either*

$(i)$ *Q admits fixed points,*

*or*

$(ii)$ *there is* $\mu \in [0, 1)$ *satisfying* $\omega \in \mu Q\omega$ *for* $\forall\omega \in \partial D$.

## 3. Controllability of the FDEIs (1.3)

**Definition 3.1.** *[10] For each* $b > 0$ *and any* $\bar{x} \in \mathbb{E}$, *if there is* $v \in L^2([0, b], \mathbb{V})$ *such that the mild solution* $\omega$ *of the FDEIs* (1.3) *corresponding to this v satisfies* $\omega(b) = \bar{x}$, *then we say that the FDEIs* (1.3) *is infinitely controllable on* $(0, \infty)$ .

We first define $H_d : \mathcal{K}(\mathbb{E}) \times \mathcal{K}(\mathbb{E}) \to \mathbb{R}^+ \cup \{\infty\}$ by

$$H_d(\mathbb{A}, \mathbb{B}) = max\{\sup_{\varsigma\in\mathbb{B}} d(\mathbb{A}, \varsigma), \sup_{\varpi\in\mathbb{A}} d(\varpi, \mathbb{B})\},$$

where $d(\mathbb{A},\varsigma) = \inf\limits_{\varpi\in\mathbb{A}} d(\varpi,\varsigma)$ and $d(\varpi,\mathbb{B}) = \inf\limits_{\varsigma\in\mathbb{B}} d(\varpi,\varsigma)$. Then $(\mathcal{K}_{cl,b}(\mathbb{E}), H_d)$ is a metric space. To prove our controllability result, we further make the assumptions:

$(H2)$ $F : [0,\infty)\times C([-r,0],\mathbb{E}) \to \mathcal{K}_{cp}(\mathbb{E})$ satisfies

(i) $F(\cdot,\omega)$ is strongly measurable for each $\omega \in C([-r,0],\mathbb{E})$;

(ii) $F(t,\cdot)$ is continuous for $a.e.\ t \in [0,\infty)$;

(iii) for every $\omega \in C([-r,0],\mathbb{E})$, there is $p \in L^2_{loc}([0,\infty),\mathbb{R}^+)$ satisfying, for $a.e.\ t \geq 0$,

$$\|F(t,\omega)\| \leq p(t).$$

$(H3)$ For any $R > 0$, there is $\rho \in L^1_{loc}([-r,\infty),\mathbb{R}^+)$ satisfying

$$H_d(F(t,\omega_1),F(t,\omega_2)) \leq \rho(t)\|\omega_1-\omega_2\|_C, \quad a.e.\ t\in[0,\infty)$$

for any $\omega_1,\omega_2 \in C([-r,0],\mathbb{E})$ with $\|\omega_1\|_C, \|\omega_2\|_C \leq R$.

For each $b > 0$, define $\Phi : L^2([0,b],\mathbb{V}) \to \mathbb{E}$ by

$$\Phi v = \int_0^b \mathcal{T}_\beta(b-\theta)Pv(\theta)d\theta.$$

$(H4)$ $\Phi$ is a linear operator and satisfies

(i) $\Phi^{-1}$ exists and takes values in $L^2([0,b],\mathbb{V}) \setminus Ker(\Phi)$;

(ii) $\exists M_1, M_2 > 0$ s.t.

$$\|P\| \leq M_1, \qquad \|\Phi^{-1}\| \leq M_2.$$

For each $b > 0$, we introduce a semi-norm in $C([-r,\infty),\mathbb{E})$ as

$$\|\omega\|_b = \sup_{t\in[-r,b]} e^{-\sigma L_b(t)}\|\omega(t)\|, \tag{3.1}$$

where $L_b(t) = \int_0^t l_b(s)ds$ and

$$l_b(t) = \max\{\frac{M}{\Gamma(2\beta)}(b-t)^{2\beta-1}\rho(t), \frac{M^2M_1M_2}{[\Gamma(2\beta)]^2}\int_0^b (b-\theta)^{2\beta-1}\rho(\theta)d\theta(b-t)^{2\beta-1}\}.$$

In the following we always choose $\sigma > 2$ large enough. Then $C([-r,\infty),\mathbb{E})$ is a Fréchet space with $\|\cdot\|_b$.

**Theorem 3.1.** *Let* $(H1)-(H4)$ *be satisfied. Then the FDEIs* (1.3) *is infinitely controllable on* $(0,\infty)$.

*Proof.* For each $b > 0$ and any $\omega \in C([-r,\infty),\mathbb{E})$, let $S^b_{F,\omega} := \{g \in L^1([0,b],\mathbb{E}) : g(t) \in F(t,\omega_t)\ a.e.\ t \in [0,b]\}$. By $(H4)$, we choose

$$v^b_\omega(t) = \Phi^{-1}[\bar{x} - \Psi_\beta(b) - \int_0^b \mathcal{T}_\beta(b-\theta)g(\theta)d\theta](t), \quad t\in[0,b],$$

where $g \in S^b_{F,\omega}$.

If $\omega \in C([-r,\infty),\mathbb{E})$ is a mild solution of the FDEIs (1.3) corresponding to the control $v^b_\omega$, by Definition 2.2, we easily check that $\omega(b) = \bar{x}$. Hence the FDEIs (1.3) is infinitely controllable on $(0,\infty)$ owing to Definition 3.1.

In the following we prove that the FDEIs (1.3) admits a mild solution in $C([-r,\infty),\mathbb{E})$. First, we define $h$ in $C([-r,\infty),\mathbb{E})$ by

$$h(t)=\begin{cases}\Psi_\beta(t)+\int_0^t \mathcal{T}_\beta(t-\theta)g(\theta)d\theta+\int_0^t \mathcal{T}_\beta(t-\theta)Pv_\omega^b(\theta)d\theta, & \forall t\geq 0,\\ \omega(t)=\phi(t), \quad t\in[-r,0], & \\ \omega'(0)=\omega_1, & \end{cases} \tag{3.2}$$

where $g\in S_{F,\omega}:=\{g\in L^1_{loc}([0,\infty),\mathbb{E}): g(t)\in F(t,\omega_t)\ a.e.\ t\geq 0\}$. Define $Q$ by

$$Q\omega=\{h: h \text{ is defined by } (3.2)\}. \tag{3.3}$$

Then $Q: C([-r,\infty),\mathbb{E})\to\mathcal{K}(C([-r,\infty),\mathbb{E}))$. Next, we are going to prove that $Q$ has fixed points.

For each $b>0$, denote by $C:=C([-r,b],\mathbb{E})$. If $\omega\in C$ is a possible mild solution of (1.3) on $[-r,b]$, that is, $\omega\in Q\omega$. Then there is $g\in S^b_{F,\omega}$ satisfying, for any $t\in[0,b]$,

$$\begin{aligned}\|\omega(t)\|\leq& M\|\phi\|_C+Mb\|\omega_1\|+\frac{M}{\Gamma(2\beta)}\int_0^t(t-\theta)^{2\beta-1}p(\theta)d\theta\\&+\frac{M}{\Gamma(2\beta)}\int_0^t(t-\theta)^{2\beta-1}\|Pv_\omega^b(\theta)\|d\theta\\ \leq& M\|\phi\|_C+Mb\|\omega_1\|+\frac{M}{\Gamma(2\beta)}\int_0^t(t-\theta)^{2\beta-1}p(\theta)d\theta\\&+\frac{MM_1M_2}{\Gamma(2\beta)}\int_0^t(t-\theta)^{2\beta-1}[\|\bar{x}\|+M\|\phi\|_C+Mb\|\omega_1\|+\frac{M}{\Gamma(2\beta)}\int_0^b(\theta-s)^{2\beta-1}p(s)ds]d\theta\\ \leq& M\|\phi\|_C+Mb\|\omega_1\|+\frac{MM_1M_2b^{2\beta}}{\Gamma(2\beta+1)}(\|\bar{x}\|+M\|\phi\|_C+Mb\|\omega_1\|)\\&+\max\{\frac{Mb^{2\beta-\frac{1}{2}}}{\Gamma(2\beta)\sqrt{4\beta-1}}\|p\|_{L^2([0,b],\mathbb{E})},\frac{M^2M_1M_2b^{4\beta-\frac{1}{2}}}{[\Gamma(2\beta)]^2 2\beta\sqrt{4\beta-1}\|p\|_{L^2([0,b],\mathbb{E})}}\}\\ :=& M^*.\end{aligned}$$

By (3.1), we can infer that

$$e^{-\sigma L_b(t)}\|\omega(t)\|\leq M^*, \quad \forall t\in[0,b].$$

By choosing $K=\max\{\|\phi\|_C, M^*\}$, we obtain that

$$\|\omega\|_b\leq K.$$

Then $Q$ is bounded.

Let $R=K+1$ be the constant in $(H3)$. Denote by

$$D=\{\omega\in C:\|\omega(t)\|<R, \quad \forall t\in[-r,b], \quad b>0\}.$$

Clearly, $D\subset C$ is an open neighborhood of zero. Next, we will show that $Q:\overline{D}\to\mathcal{K}(C)$ is an admissible contraction.

Let $\omega, z \in \overline{D}$ and $h \in Q\omega$. There is $g \in S^b_{F,\omega}$ satisfying

$$h(t) = \Psi_\beta(t) + \int_0^t \mathcal{T}_\beta(t-\theta)g(\theta)d\theta + \int_0^t \mathcal{T}_\beta(t-\theta)Pv^b_\omega(\theta)d\theta, \quad t \in [0,b].$$

It follows from $(H3)$ that

$$H_d(F(t,\omega_t), F(t,z_t)) \le \rho(t)\|\omega_t - z_t\|, \quad t \in [0,b].$$

Thus, there exists $\gamma \in F(t,z_t)$ satisfying

$$\|g(t) - \gamma\| \le \rho(t)\|\omega_t - z_t\|, \quad t \in [0,b].$$

Let

$$D_*(t) = \{\gamma \in \mathbb{E} : \|g(t) - \gamma\| \le \rho(t)\|\omega_t - z_t\|\}.$$

Since the Proposition III.4 of [8] yields that the set $\Omega(t) = D_*(t) \cap F(t,z_t)$ is measurable, it follows that there is $\bar{g} \in \Omega(t), t \in [0,b]$. Thus, $\bar{g} \in F(t,z_t)$ for all $t \in [0,b]$ and

$$\|g(t) - \bar{g}(t)\| \le \rho(t)\|\omega_t - z_t\|, \quad t \in [0,b].$$

Define

$$\bar{h}(t) = \Psi_\beta(t) + \int_0^t \mathcal{T}_\beta(t-\theta)\bar{g}(\theta)d\theta + \int_0^t \mathcal{T}_\beta(t-\theta)Pv^b_z(\theta)d\theta, \quad \forall t \in [0,b].$$

Then $\bar{h} \in Qz$ and

$$\begin{aligned}
\|h(t) - \bar{h}(t)\| \le& \|\int_0^t \mathcal{T}_\beta(t-\theta)[g(\theta) - \bar{g}(\theta)]d\theta\| \\
&+ \|\int_0^t \mathcal{T}_\beta(t-\theta)[Pv^b_\omega(\theta) - Pv^b_z(\theta)]d\theta\| \\
\le& \frac{M}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|g(\theta) - \bar{g}(\theta)\|d\theta \\
&+ \frac{MM_1}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|v^b_\omega(\theta) - v^b_z(\theta))\|d\theta \\
\le& \frac{M}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\rho(\theta)\|\omega_\theta - z_\theta\|d\theta \\
&+ \frac{MM_1M_2}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|\int_0^b \mathcal{T}_\beta(b-s)[g(s) - \bar{g}(s)]ds\|d\theta \\
\le& \frac{M}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\rho(\theta)\|\omega_\theta - z_\theta\|d\theta \\
&+ \frac{M^2M_1M_2}{[\Gamma(2\beta)]^2}\int_0^t (t-\theta)^{2\beta-1}\int_0^b (b-s)^{2\beta-1}\rho(s)\|\omega_s - z_s\|ds\|d\theta \\
\le& \frac{2}{\sigma}e^{\sigma L_b(t)}\|\omega - z\|_b.
\end{aligned}$$

Therefore,

$$\|h-\bar{h}\|_b \le \frac{2}{\sigma}\|\omega - z\|_b.$$

Analogously, by interchanging $\omega$ and $z$, we have

$$H_d(Q\omega, Qz) \le \frac{2}{\sigma}\|\omega - z\|_b.$$

On the other hand, for any $\omega \in C$, since $F(t,\omega_t)$ has compact values, by the definition of $Q$, we get $Q\omega \subset \mathcal{K}_{cp}(C)$. Hence there exists $\omega^* \in C$ satisfying $\omega^* \in Q\omega^*$.

Let $\hat{\omega} \in \overline{D}$ and $h \in Q\hat{\omega}$. For every $\epsilon > 0$, if $\omega^* \in Q\hat{\omega}$, we have

$$\begin{aligned}\|\hat{\omega}(t) - \omega^*(t)\| =& \|\hat{\omega}(t) - h(t) + h(t) - \omega^*(t)\| \\ \le& \|\hat{\omega}(t) - h(t)\| + \|h(t) - \omega^*(t)\| \\ \le& e^{\sigma L_b(t)}\|\hat{\omega} - Q\hat{\omega}\|_b + \|h(t) - \omega^*(t)\|.\end{aligned}$$

Since $h \in Q\hat{\omega}$ is arbitrary, let $h \in \{h \in C : \|h - \omega^*\|_b \le \epsilon\}$. We can achieve that

$$\|\hat{\omega} - \omega^*\|_b \le \|\hat{\omega} - Q\hat{\omega}\|_b + \epsilon.$$

If $\omega^* \notin Q\hat{\omega}$, then $\|\omega^* - Q\hat{\omega}\|_b \ne 0$. By means of the compactness of $Q\hat{\omega}$, there is $\omega^{**} \in Q\hat{\omega}$ satisfying

$$\|\omega^* - Q\hat{\omega}\|_b = \|\omega^* - \omega^{**}\|_b > 0.$$

Then,

$$\|\hat{\omega}(t) - \omega^{**}(t)\| \le e^{\sigma L_b(t)}\|\hat{\omega} - Q\hat{\omega}\|_b + \|h(t) - \omega^{**}(t)\|.$$

Consequently, we have

$$\|\hat{\omega} - \omega^{**}\|_b \le \|\hat{\omega} - Q\hat{\omega}\|_b + \epsilon.$$

Thus, we deduce that $Q : \overline{D} \to \mathcal{K}(C)$ is an admissible contraction. In view of the definition of $D$, there is no $\omega \in \partial D$ satisfying $\omega \in \mu Q\omega$ for all $\mu \in (0,1)$. Then according to Lemma 2.3, $Q$ has at least one fixed point $\omega \in C$ such that for each $b > 0$, $\omega(b) = \bar{x}$. Consequently, the FDEIs (1.3) is infinitely controllable on $(0,\infty)$. □

## 4. Applications

**Example 4.1.** Let $\Upsilon \subset \mathbb{R}^N$ be bounded and $\partial\Upsilon$ sufficiently smooth. We study the fractional partial differential inclusions

$$\begin{cases} {}^C\partial_t^{\frac{3}{2}}\omega(t,\varrho) - \Delta\omega(t,\varrho) \in \mathcal{G}(t,\varrho,\omega(t-r,\varrho)) + Pv(t), & (t,\varrho) \in (0,\infty)\times\Upsilon, \\ \omega|_{\partial\Upsilon} = 0, \\ \omega(t,\varrho) = \phi(t,\varrho), \quad t \in [-r,0],\ \varrho \in \Upsilon, \\ \omega(0,\varrho) = \omega_1(\varrho), \quad \varrho \in \Upsilon, \end{cases} \tag{4.1}$$

where ${}^C\partial_t^{\frac{3}{2}}$ represents the $\frac{3}{2}$-order Caputo fractional partial derivative operator, $P : L^2(\Upsilon) \to L^2(\Upsilon)$ is linear and bounded, $v \in L^2_{loc}([0,\infty), L^2(\Upsilon))$.

Let $\mathbb{E} = \mathbb{V} = L^2(\Upsilon)$ and let

$$A = \Delta,$$
$$\mathcal{D}(A) = H^2(\Upsilon) \cap H_0^1(\Upsilon).$$

It follows from Section 7.2 of [3] that $A$ generates a strongly continuous cosine family $\{G(t) : t \geq 0\}$ of uniformly bounded linear operators in $\mathbb{E}$ and, for all $t \geq 0$, $\|G(t)\| \leq 1$. Then $(H1)$ is achieved.

$A$ has eigenvalues $\lambda_m = m^2\pi^2, m \in \mathbb{N}$. $\vartheta_m(t) = \sqrt{\frac{2}{\pi}}\sin(m\pi t), m \in \mathbb{N}$ are corresponding eigenvectors. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots,$$

and $\lambda_m \to \infty$ as $m \to \infty$. Hence

$$A\omega = -\sum_{m=1}^{\infty} \lambda_m \langle \omega, \vartheta_m \rangle \vartheta_m, \quad \omega \in \mathcal{D}(A),$$

$$G(t)\omega = \sum_{m=1}^{\infty} \cos(\sqrt{\lambda_m} t) \langle \omega, \vartheta_m \rangle \vartheta_m, \quad t \geq 0,\ \omega \in \mathbb{E}$$

and

$$W(t)\omega = \sum_{m=1}^{\infty} \frac{1}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m} t) \langle \omega, \vartheta_m \rangle \vartheta_m, \quad t \geq 0,\ \omega \in \mathbb{E},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{E}$.

For any $v \in L^2_{loc}([0, \infty), \mathbb{V})$, we define $P$ on $\mathbb{V}$ by

$$Pv(t) = \sum_{m=1}^{\infty} a\lambda_m \langle \widehat{v}(t), \vartheta_m \rangle \vartheta_m, \quad t \geq 0,$$

where $a > 0$ and

$$\widehat{v}(t) = \begin{cases} \langle v(t), \vartheta_m \rangle, & m = 1, 2, \cdots, N, \\ 0, & m = N + 1, N + 2, \cdots. \end{cases}$$

Then $P : \mathbb{V} \to \mathbb{E}$. Since for any $v \in L^2_{loc}([0, \infty), \mathbb{V})$, we have

$$\|v\|_{L^2} = \Big( \sum_{m=1}^{\infty} \langle v(t), \vartheta_m \rangle^2 \Big)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned} \|Pv(t)\| =& \Big( \sum_{m=1}^{\infty} a^2 \lambda_m^2 \langle \widehat{v}(t), \vartheta_m \rangle^2 \Big)^{\frac{1}{2}} \\ \leq& aN\lambda_N \|v\|_{L^2}. \end{aligned}$$

Then there exists $M_1 > 0$ satisfying $\|P\| \leq M_1$.

For every $b > 0$, we define $\Phi : L^2([0, b], \mathbb{V}) \to \mathbb{E}$ by

$$\Phi v = \frac{3}{4} \int_0^b \int_0^{\infty} (b - s)^{\frac{1}{4}} \theta M_{\frac{3}{4}}(\theta) W((b - s)^{\frac{3}{4}} \theta) Pv(s) d\theta ds.$$

Let $\delta = E_{\frac{3}{2},1}(-\frac{1}{10})$. According to [4] and [17], for any $m \in \mathbb{N}$, we have

$$0 < 1 - \delta \leq 1 - E_{\frac{3}{2},1}(-\lambda_m) < 2.$$

Then the operator $\Phi$ is surjective. Thus, for any $\omega \in \mathbb{E}$, $\Phi^{-1} : \mathbb{E} \to L^2([0,b], \mathbb{V}) \setminus Ker(\Phi)$ exists and expresses by

$$(\Phi^{-1}\omega)(t,\tau) = \frac{1}{\tau}\sum_{m=1}^{\infty} \frac{\langle \omega, \vartheta_m \rangle \vartheta_m}{1 - E_{\frac{3}{2},1}(-\lambda_m)},$$

which implies

$$\|(\Phi^{-1}\omega)(t,\cdot)\| \leq \frac{1}{\tau(1-\delta)}\|\omega\|.$$

Consequently, we have

$$\|\Phi^{-1}\| \leq \frac{1}{\tau(1-\delta)} := M_2.$$

Hence, the condition $(H4)$ holds.

**Theorem 4.1.** *Let $\mathcal{G} : [0,\infty) \times \Upsilon \times C([-r,\infty) \times \Upsilon, \mathbb{R}) \to \mathcal{K}_{cp}(L^2(\Upsilon))$ satisfies the following condition*

$(F1)$ $(i)$ *For each $\omega \in C([-r,\infty) \times \Upsilon, \mathbb{R})$ and for any $\varrho \in \Upsilon$, $\mathcal{G}(\cdot, \varrho, \omega)$ is strongly measurable;*

$(ii)$ *For a.e. $t \in [0,\infty)$ and for any $\varrho \in \Upsilon$, $\mathcal{G}(t, \varrho, \cdot)$ is continuous;*

$(iii)$ *There exists $p_1 \in L^2_{loc}([0,\infty), \mathbb{R}^+)$ satisfying*

$$|\mathcal{G}(t,\varrho,\omega)| \leq p_1(t), \quad \forall t \geq 0, \ \ \varrho \in \Upsilon, \ \ \omega \in \mathbb{R};$$

$(iv)$ *For any $R > 0$, there is $p_2 \in L^1_{loc}([-r,\infty), \mathbb{R}^+)$ satisfying, for a.e. $t \in [0,\infty)$,*

$$H_d(\mathcal{G}(t,\varrho,\omega_1), \mathcal{G}(t,\varrho,\omega_2)) \leq p_2(t)|\omega_1 - \omega_2|$$

*for any $\omega_1, \omega_2 \in \mathbb{R}$ with $|\omega_1|, |\omega_2| \leq R$.*

*Then the inclusion* (4.1) *is infinitely controllable on* $(0,\infty)$.

*Proof.* Let

$$\omega(t)(\varrho) = \omega(t,\varrho),$$
$$\phi(t)(\varrho) = \phi(t,\varrho)$$

and

$$F(t,\omega_t)(\tau) = \mathcal{G}(t,\varrho,\omega(t-r,\varrho)).$$

Then, (4.1) can be rewritten as the abstract FDEIs (1.3). From the above arguments, we know that $(H1) - (H4)$ are fulfilled. Hence, by Theorem 3.1, the differential inclusion (4.1) is infinitely controllable on $(0,\infty)$. □

## 5. Conclusions

In the present work, by applying a nonlinear alternative of multivalued mapping which is established by Frigon [11], the infinite controllability theorem of the FDEIs (1.3) is established in Fr*é*chet spaces. All compactness conditions are removed in our theorem. The obtained result is new even if the nonlinear function $F$ is a single-valued mapping. The scheme established in this article is also useful for investigating the infinite controllability of the Riemann-Liouville (or the Hilfer) fractional evolution equations. In the future, we will further study the optimal control problems of the FDEIs (1.3).

## Use of AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

The research is supported by the NSF of Gansu (No. 22JR5RA875).

## Conflict of interest

The author declares that they have no conflicts of interest.

## References

1. K. X. Li, J. G. Peng, J. X. Jia, Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives, *J. Funct. Anal.*, **263** (2012), 476–510. https://doi.org/10.1016/j.jfa.2012.04.011
2. R. R. Nigmatullin, To the theoretical explanation of the "universal response", *Phys. Stat. Sol. B*, **123** (1984), 739–745. https://doi.org/10.1002/pssb.2221230241
3. W. Arendt, Vector-valued Laplace transforms and Cauchy problems, *Israel J. Math.*, **59** (1987), 327–352. https://doi.org/10.1007/BF02774144
4. Y. Zhou, J. W. He, New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$, *Evol. Equ. Control Theory*, **10** (2021), 491–509. https://doi.org/10.3934/eect.2020077
5. H. Yang, Existence and approximate controllability of Riemann-Liouville fractional evolution equations of order $1 < \mu < 2$ with weighted time delay, *Bull. Sci. Math.*, **187** (2023), 103303. https://doi.org/10.1016/j.bulsci.2023.103303
6. H. D. Gou, Y. X. Li, Existence and approximate controllability of Hilfer fractional evolution equations in Banach spaces, *J. Appl. Anal. Comput.*, **11** (2021), 2895–2920. https://doi.org/10.11948/20210053
7. J. W. He, L. Peng, Approximate controllability for a class of fractional stochastic wave equations, *Comput. Math. Appl.*, **78** (2019), 1463–1476. https://doi.org/10.1016/j.camwa.2019.01.012
8. M. Benchohra, S. Ntouyas, Controllability on infinite time horizon of nonlinear differential equations in Banach spaces with nonlocal conditions, *An. Stiint. Univ. Al. I. Cuza. Iasi. Mat.*, **47** (2001), 277–286.
9. M. Benchohra, S. Ntouyas, Controllability of neutral functional differential and integrodifferential inclusions in Banach spaces, *Italian J. Pure Appl. Math.*, **14** (2003), 95–112.
10. M. Benchohra, A. Ouahab, Controllability results for functional semilinear differential inclusions in Fréchet spaces, *Nonlinear Anal. Theor.*, **61** (2005), 405–423. https://doi.org/10.1016/j.na.2004.12.002

11. M. Frigon, Fixed point results for multivalued contractions on Gauge spces, In: *Set valued mappings with applications in nonlinear analysis*, 4 Eds., London: Taylor and Francis, 2002.
12. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam, Boston: Elsevier, 2006.
13. Y. Zhou, *Fractional evolution equations and inclusions: Analysis and control*, Academic Press, 2016. https://doi.org/10.1016/C2015-0-00813-9
14. C. C. Travis, G. F. Webb, Consine families and abstract nonlinear second order differential equations, *Acta Math. Hung.*, **32** (1978), 75–96. https://doi.org/10.1007/BF01902205
15. F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models*, Imperial College Press, 2010.
16. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
17. J. W. Hanneken, D. M. Vaught, B. N. Narahari Achar, Enumeration of the real zeros of the Mittag-Leffler function $E_\alpha(z)$, $1 < \alpha < 2$, In: *Advances in fractional calculus*, Dordrecht: Springer, 2007. https://doi.org/10.1007/978-1-4020-6042-7_2


AIMS Press