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Research article

On the controllability results of semilinear delayed evolution systems involving fractional derivatives in Banach spaces

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Abstract: The paper investigated the exact controllability of delayed fractional evolution systems of order $\alpha \in (1, 2)$ in abstract spaces. At first, the exact controllability result is obtained when the nonlinear term f is locally Lipschitz continuous. Then, the certain compactness conditions and the measure of noncompactness conditions were applied to demonstrate the exact controllability of the concerned problem. The discussion was based on the fixed point theorems and the cosine family theory.

Keywords: the cosine family theory; exact controllability; fractional evolution equations; fixed point theorem
Mathematics Subject Classification: 34K30, 34K35, 93C25

1. Introduction

It is well known that the semilinear evolution equations have a considerable practical background in physics, biology, engineering, and other fields (see [1, 2] and the reference therein). For fractional evolution equations, the existence as well as the controllability have become a hot topic in recent years (see, for example, [3–6]). In [3], Li et al. considered the existence and uniqueness of weak solutions and strong solutions of an inhomogeneous Cauchy problem of Riemann-Liouville fractional evolution equations of order $\alpha \in (1, 2)$ using an α-order fractional resolvent. In [4], by introducing an operator S_α in term of the generalized Mittag-Leffler function and the curve integral, Li et al. investigated the existence and uniqueness of (mild) solutions for a class of Caputo fractional Cauchy problem by means of the Banach fixed point theorem and the Schauder fixed point theorem. In [5], by using the Krasnoselskii fixed point theorem and the solution operator method, the existence of mild solutions to a class of fractional semilinear integro-differential equations of order $1 < \alpha < 2$ was demonstrated. In [6], Yang proved the existence and approximate controllability of the Sobolev-type fractional evolution equations of order $\alpha \in (1, 2)$ using the Sadovskii fixed point theorem and the resolvent operator theory.

In this article, we demonstrate the exact controllability of the delayed fractional control system in the Banach space E

$$\begin{cases} {}^C\partial_t^\alpha y(t) = Ay(t) + f(t, y_t) + Bv(t), \quad t \in U := [0, c],\ \alpha \in (1, 2), \\ y(t) = \phi(t), \quad \forall t \in [-r, 0], \\ y'(0) = y_1, \end{cases} \tag{1.1}$$

where ${}^C\partial_t^\alpha$ represents the fractional partial derivative operator on t of order α in the Caputo sense, $A : \mathcal{D}(A) \subseteq E \to E$ is a densely-defined and closed linear operator in E, $\mathcal{D}(A)$ is the domain of A equipped with the norm $\|x\|_{\mathcal{D}(A)} = \|x\| + \|Ax\|$ for $x \in \mathcal{D}(A)$, f is a given function which represents the nonlinear term, $v \in L^2(U, H)$, H is another Banach space, $y_1 \in E$ and $\phi : [-r, 0] \to E$ is continuous, $B : H \to E$ is a linear operator. For any $t \in U$, $y_t(\theta) = y(t + \theta)$ for any $\theta \in [-r, 0]$. r and c are given positive constants.

The observation of the electromagnetic, acoustic, and mechanical influence shows that there exist some transfer processes in a medium, which are not described by a usual diffusion equation. Fractional Cauchy problems are useful in physics to model such anomalous diffusion, see [3] and the references therein. Hence, it is significant to study the control system (1.1).

In 2021, the Laplace transform method and the cosine family theory were applied by Zhou et al. in [7] to present the suitable definition of mild solutions of the linear inhomogeneous Cauchy problem

$$\begin{cases} {}^C\partial_t^\alpha y(t) = Ay(t) + h(t), \quad t \in [0, c],\ \alpha \in (1, 2), \\ y(0) = y_0,\ \ y'(0) = y_1, \end{cases}$$

where $y_0 \in E$ and h is a given linear function. Then the controllability was considered for the corresponding semilinear fractional evolution system

$$\begin{cases} {}^C\partial_t^\alpha y(t) = Ay(t) + f(t, y(t)) + Bv(t), \quad t \in (0, c], \alpha \in (1, 2), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases}$$

where $f(t, y)$ is global Lipschitz continuous or satisfies a certain compactness condition. Obviously, the global Lipschitz continuity contains local Lipschitz continuity and first-order linear growth condition. The reverse is not true.

In 2023, Yang in [8] demonstrated the approximate controllability of the fractional control system

$$\begin{cases} {}^L\partial_t^\alpha y(t) = Ay(t) + F(t, \tilde{y}_t) + Bv(t), \quad t \in (0, c],\ \alpha \in (1, 2), \\ \tilde{y}(0) = \phi(t), \quad t \in [-r, 0], \\ (\mathcal{H}_{2-\alpha} * y)'(0) = y_1, \end{cases}$$

where ${}^L\partial_t^\alpha$ is the fractional derivative operator on t of order α in the Riemann-Liouville sense, $\tilde{y}(t) = t^{2-\alpha}y(t)$ for $t \in U$ with $\tilde{y}(0) = \lim_{t\to 0^+} \tilde{y}(t)$ and $\tilde{y}_t(\theta) = \tilde{y}(t + \theta)$ for $t \in U$ and $\theta \in [-r, 0]$, $\mathcal{H}_\varsigma(t) = \frac{t^{\varsigma-1}}{\Gamma(\varsigma)}$ for $t, \varsigma > 0$, $\Gamma(\cdot)$ is the standard Gamma function. The symbol $*$ represents the convolution. When the sine family $T(t)$, corresponding to the cosine family $\{G(t) : t \geq 0\}$ generated by A, is a compact operator for $t > 0$, the approximate controllability results were obtained.

The main features of this paper are summarized below:

(1) The exact controllability of the control system (1.1) is demonstrated in Theorem 3.1 when f is local Lipschitz continuous. Compactness conditions of the nonlinear term f or the cosine family $\{G(t) : t \geq 0\}$ or the sine family $\{T(t) : t \geq 0\}$ are essential assumptions in existing articles, but we removed it in Theorem 3.1.

(2) When f satisfies the certain compactness conditions or the measure of noncompactness conditions, the exact controllability of (1.1) is also obtained (see Theorems 3.2–3.4). These results are nature extensions of [7, 8].

2. Preliminaries

Let $C([-r, c], E)$ be the Banach space of E-valued continuous functions on $[-r, c]$ with the norm $\|y\|_{C([-r,c],E)} = \sup_{t\in[-r,c]} \|y(t)\|$. $\mathscr{B}(H, E)$ represents the Banach space of linear bounded operators from H to E with the norm $\|\cdot\|_{\mathscr{B}}$. $\mathscr{B}(E) := \mathscr{B}(E, E)$. We always suppose that $B \in \mathscr{B}(H, E)$, that is, $\exists \mathcal{M}_1 > 0$ such that $\|B\|_{\mathscr{B}} \leq \mathcal{M}_1$.

Definition 2.1. *[9] A family $\{G(t) : t \in \mathbb{R}\} \subset \mathscr{B}(E)$ is called the strongly continuous cosine family if it satisfies*

(i) $G(0) = I$;

(ii) for any $t, s \in \mathbb{R}$, $G(t + s) + G(t - s) = 2G(t)G(s)$;

(iii) for each $x \in E$, $t \mapsto G(t)x$ is continuous on $\mathbb{R}$.

Set

$$T(t)y := \int_0^t G(\theta)y d\theta, \quad \forall t \in \mathbb{R}, y \in E.$$

Then $\{T(t) : t \in \mathbb{R}\}$ is called the sine family corresponding to $\{G(t) : t \in \mathbb{R}\}$. Let

$$\mathcal{D}(A) = \{y \in E : G(t)y \in C^2(\mathbb{R}, E)\},$$

$$Ay = \frac{d^2}{dt^2}G(t)y|_{t=0}, \quad \forall y \in \mathcal{D}(A).$$

Then A generates a cosine family $\{G(t) : t \in \mathbb{R}\}$. Therefore, we make the following assumption on A.

$(P1)$ $A : \mathcal{D}(A) \subseteq E \to E$ generates a strongly continuous cosine family $\{G(t) : t \geq 0\}$ and

$$\|G(t)\|_{\mathscr{B}} \leq \mathcal{M}, \quad \forall t \geq 0,$$

where $\mathcal{M} \geq 1$ is a constant.

Let $\beta = \frac{\alpha}{2}$. Then $\beta \in (\frac{1}{2}, 1)$. We recall the function

$$M_\varrho(\theta) = \sum_{m=0}^{\infty} \frac{(-\theta)^m}{m!\Gamma(1 - \varrho(m + 1))}, \quad \theta \in \mathbb{C},\ \varrho \in (0, 1),$$

where $\mathbb{C}$ is the imaginary line.

By [7], the following definition and lemmas are achieved.

Definition 2.2. *[7] For each $v \in L^2(U, H)$, $y \in C([-r, c], E)$ is called the mild solution of (1.1) if $y(t) = \phi(t)$ for $t \in [-r, 0]$, $y'(0) = y_1$ and*

$$y(t) = \Psi_\beta(t) + \int_0^t \mathcal{P}_\beta(t-s)[f(s, y_s) + Bv(s)]ds, \quad t \in U,$$

where

$$\begin{aligned}\Psi_\beta(t) =& C_\beta(t)\phi(0) + \mathcal{K}_\beta(t)y_1,\\ C_\beta(t) =& \int_0^\infty M_\beta(\theta)G(t^\beta\theta)d\theta,\\ \mathcal{K}_\beta(t) =& \int_0^t C_\beta(\theta)d\theta,\\ \mathcal{P}_\beta(t) =& \int_0^\infty \beta\theta t^{\beta-1}M_\beta(\theta)T(t^\beta\theta)d\theta.\end{aligned}$$

Lemma 2.1. *[7] Let $(P1)$ hold. Then for any $t \geq 0$ and $y \in E$, we have*

$$\|C_\beta(t)y\| \leq \mathcal{M}\|y\|,$$

$$\|\mathcal{K}_\beta(t)y\| \leq \mathcal{M}\|y\|t,$$

$$\|\mathcal{P}_\beta(t)y\| \leq \frac{\mathcal{M}\|y\|}{\Gamma(2\beta)}t^{2\beta-1}.$$

Lemma 2.2. *[7] If $(P1)$ is fulfilled, then*
(i) $\{C_\beta(t) : t \geq 0\}$ is strongly continuous;
(ii) $\{\mathcal{K}_\beta(t) : t \geq 0\}$ and $\{\mathcal{P}_\beta(t) : t \geq 0\}$ are uniformly continuous;
(iii) $\mathcal{P}_\beta(t)$ is a compact operator for $t > 0$ if $T(t), t > 0$, is compact.

By Lemma 2.9 of [8], we can achieve the following lemma.

Lemma 2.3. *[8] If $(P1)$ holds and $T(t)$ is compact for $t > 0$, the operator $\Phi : L^2(U, E) \to C(U, E)$, defined by*

$$(\Phi\varpi)(t) = \int_0^t \mathcal{P}_\beta(t-s)\varpi(s)ds, \quad \forall\varpi \in L^2(U, E),$$

is compact.

For the nonlinear term f, we make the assumptions below.
$(P2)$ $f : U \times C([-r, 0], E) \to E$ is continuous and for each $R > 0$, there is $K(R) > 0$ satisfying

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq K(R)\|\psi_1 - \psi_2\|_{C([-r,0],E)}$$

for any $t \in U$ and $\psi_j \in C([-r, 0], E)$ with $\|\psi_j\|_{C([-r,0],E)} \leq R$, $j = 1, 2$.
$(P3)$ There is $\varphi \in L^2(U, \mathbb{R}^+)$ satisfying

$$\|f(t, \psi)\| \leq \varphi(t), \quad \forall t \in U,\ \psi \in C([-r, 0], E).$$

Lemma 2.4. *Let assumptions* $(P1)$ *and* $(P3)$ *hold. For each* $v \in L^2(U, H)$*, if* $y \in C([-r, c], E)$ *is the mild solution of (1.1) associated with* v*, there is* $R > 0$ *satisfying*

$$\|y\|_{C([-r,c],E)} \leq R.$$

Proof. By Definition 2.2, we have $y(t) = \phi(t)$ for $t \in [-r, 0]$. Then $\|y\|_{C([-r,0],E)} \leq \|\phi\|_{C([-r,0],E)}$.

If $t \in [0, c]$, by $(P1)$ and $(P3)$, we have

$$\begin{aligned}\|y(t)\| \leq& \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|f(\theta, y_\theta)\|d\theta + \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|Bv(\theta)\|d\theta \\ \leq& \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\varphi(\theta)d\theta + \frac{\mathcal{M}\mathcal{M}_1}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|v(\theta)\|d\theta \\ \leq& \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}\sigma}{\Gamma(2\beta)}\|\varphi\|_{L^2} + \frac{\mathcal{M}\mathcal{M}_1\sigma}{\Gamma(2\beta)}\|v\|_{L^2} \\ :=& R^*,\end{aligned}$$

where $\sigma = \frac{c^{2\beta-\frac{1}{2}}}{\sqrt{4\beta-1}}$. Choosing $R := R^* + \|\phi\|_{C([-r,0],E)}$, we get

$$\|y\|_{C([-r,c],E)} \leq R.$$

This completes the proof. □

Definition 2.3. *For each* $\bar{y} \in E$*, if there is* $v \in L^2(U, H)$ *such that the control system (1.1) has a mild solution* $y \in C([-r, c], E)$ *corresponding to* v *satisfying* $y(c) = \bar{y}$*, then the control system (1.1) is called exactly controllable on* $[-r, c]$.

Define an operator $\mathcal{W}$ by

$$\mathcal{W}v = \int_0^b \mathcal{P}_\beta(b-s)Bv(s)ds.$$

Obviously, $\mathcal{W} : L^2(U, H) \to E$ is a linear operator.

$(P4)$ $(i)\mathcal{W}^{-1}$ exists and takes values in $L^2(U, H) \setminus Ker\mathcal{W}$.

(ii) There is $\mathcal{M}_2 > 0$ such that

$$\|\mathcal{W}^{-1}\| \leq \mathcal{M}_2.$$

According to $(P4)$ and the definition of $\mathcal{W}$, we choose a control $v_y \in L^2(U, H)$ by

$$v_y(t) = \mathcal{W}^{-1}[\bar{y} - \Phi_\beta(c) - \int_0^c \mathcal{P}_\beta(c-\theta)f(\theta, y_\theta)d\theta](t), \quad t \in U. \tag{2.1}$$

If $y \in C([-r, c], E)$ is a mild solution of (1.1) associated with v_y, by (2.1) and Definition 2.2, we have

$$\begin{aligned}y(c) =& \Psi_\beta(c) + \int_0^c \mathcal{P}_\beta(c-\theta)[f(\theta, y_\theta) + Bv(\theta)]d\theta \\ =& \Psi_\beta(c) + \int_0^c \mathcal{P}_\beta(c-\theta)f(\theta, y_\theta)d\theta \\ &+ \int_0^c \mathcal{P}_\beta(c-\theta)B\mathcal{W}^{-1}[\bar{y} - \Phi_\beta(c) - \int_0^c \mathcal{P}_\beta(c-s)f(s, y_s)ds](\theta)d\theta \\ =& \bar{y}.\end{aligned}$$

This fact implies that (1.1) is exactly controllable according to Definition 2.3.

Lemma 2.5. *Let* $(P1), (P3)$ *and* $(P4)$ *hold. For any* $y \in C([-r, c], E)$*, there is* $\mathcal{K}_1 > 0$ *satisfying*

$$\|v_y\|_{L^2} \leq \mathcal{K}_1.$$

Proof. For any $y \in C([-r, c], E)$, by $(P1), (P3)$ and $(P4)$, we have

$$\begin{aligned}
\|v_y\|_{L^2} =& \left(\int_0^b \|v_y(t)\|^2 dt\right)^{\frac{1}{2}} \\
\leq& \mathcal{M}_2 c^{\frac{1}{2}} [\|\bar{y}\| + \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}}{\Gamma(2\beta)} \int_0^c (c-\theta)^{2\beta-1} \|f(\theta, y_\theta)\| d\theta] \\
\leq& \mathcal{M}_2 c^{\frac{1}{2}} [\|\bar{y}\| + \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}}{\Gamma(2\beta)} \int_0^c (c-\theta)^{2\beta-1} \varphi(\theta) d\theta] \\
\leq& \mathcal{M}_2 c^{\frac{1}{2}} [\|\bar{y}\| + \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}\sigma}{\Gamma(2\beta)} \|\varphi\|_{L^2}] \\
:=& \mathcal{K}_1.
\end{aligned}$$

This completes the proof. □

For any $R > 0$, put $B_C(R) := \{y \in C([-r, c], E) : \|y\|_{C([-r,c],E)} \leq R\}$.

Lemma 2.6. *Let* $(P1), (P2)$ *and* $(P4)$ *hold. For any* $y_1, y_2 \in B_C(R)$*, there is* $\mathcal{K}_2 > 0$ *satisfying*

$$\|v_{y_1}(t) - v_{y_2}(t)\| \leq \mathcal{K}_2 \|y_1 - y_2\|_{C([-r,c],E)}, \quad \forall t \in U.$$

Proof. For any $y_1, y_2 \in B_C(R)$, by $(P1), (P2)$ and $(P4)$, we have

$$\begin{aligned}
&\|v_{y_1}(t) - v_{y_2}(t)\| \\
\leq& \frac{\mathcal{M}\mathcal{M}_2}{\Gamma(2\beta)} \int_0^c (c-\theta)^{2\beta-1} \|f(\theta, y_{1\theta}) - f(\theta, y_{2\theta})\| d\theta \\
\leq& \frac{\mathcal{M}\mathcal{M}_2 K(R)}{\Gamma(2\beta)} \int_0^c (c-\theta)^{2\beta-1} \|y_{1\theta} - y_{2\theta}\| d\theta \\
\leq& \frac{\mathcal{M}\mathcal{M}_2 K(R) c^{2\beta}}{\Gamma(2\beta+1)} \|y_1 - y_2\|_{C([-r,c],E)}.
\end{aligned}$$

By choosing $\mathcal{K}_2 = \frac{\mathcal{M}\mathcal{M}_2 K(R) c^{2\beta}}{\Gamma(2\beta+1)}$, we obtain the desired conclusion. □

3. Controllability results

Theorem 3.1. *Let* $(P1) - (P4)$ *be fulfilled. Then the control system* (1.1) *is exactly controllable on* $[-r, c]$ *provided that*

$$\frac{\mathcal{M}K(R)c^{2\beta}}{\Gamma(2\beta+1)} \left(1 + \frac{\mathcal{M}\mathcal{M}_1\mathcal{M}_2 c^{2\beta}}{\Gamma(2\beta+1)}\right) < 1. \tag{3.1}$$

Proof. Defined $Q : C([-r,c],E) \to C([-r,c],E)$ by

$$(Qy)(t) = \begin{cases} \Psi_\beta(t) + \int_0^t \mathcal{P}_\beta(t-s)[f(s,y_s) + Bv_y(s)]ds, & t \in U, \\ \phi(t), & t \in [-r,0]. \end{cases} \tag{3.2}$$

By Definition 2.2 we know that the mild solution of (1.1) is equivalent to the fixed point of Q.

Let $R \geq \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}\sigma}{\Gamma(2\beta)}\|\varphi\|_{L^2} + \frac{\mathcal{M}\mathcal{M}_1\sigma}{\Gamma(2\beta)}\mathcal{K}_1 + \|\phi\|_{C([-r,0],E)}$. We first prove that $Q(B_C(R)) \subseteq B_C(R)$. For any $y \in B_C(R)$, it is easy to see

$$\|Qy\|_{C([-r,0],E)} \leq \|\phi\|_{C([-r,0],E)}. \tag{3.3}$$

For $t \in [0,c]$, by $(P1),(P3)$ and Lemma 2.5, we have

$$\begin{aligned} \|(Qy)(t)\| \leq & \|\Psi_\beta(t)\| + \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|f(\theta,y_\theta)\|d\theta + \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|Bv_y(\theta)\|d\theta \\ \leq & \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\varphi(\theta)d\theta + \frac{\mathcal{M}\mathcal{M}_1}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|v_y(\theta)\|d\theta \\ \leq & \mathcal{M}\|\phi(0)\| + \mathcal{M}c\|y_1\| + \frac{\mathcal{M}\sigma}{\Gamma(2\beta)}\|\varphi\|_{L^2} + \frac{\mathcal{M}\mathcal{M}_1\sigma}{\Gamma(2\beta)}\mathcal{K}_1. \end{aligned}$$

This fact, together with (3.3), yields that

$$\|Qy\|_{C([-r,c],E)} \leq R.$$

Thus, Q maps $B_C(R))$ into itself.

Then, we claim that $Q : B_C(R)) \to B_C(R))$ is a contraction mapping. For any $y_1, y_2 \in B_C(R))$, if $t \in [-r,0]$, $(Qy_1)(t) = (Qy_2)(t) = \phi(t)$. If $t \in [0,c]$, by $(P1),(P2)$ and Lemma 2.6, we achieve that

$$\begin{aligned} & \|(Qy_1)(t) - (Qy_2)(t)\| \\ \leq & \frac{\mathcal{M}}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|f(\theta,y_{1\theta}) - f(\theta,y_{2\theta})\|d\theta + \frac{\mathcal{M}\mathcal{M}_1}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|v_{y_1}(\theta) - v_{y_2}(\theta)\|d\theta \\ \leq & \frac{\mathcal{M}K(R)}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|y_{1\theta} - y_{2\theta}\|_{C([-r,0],E)}d\theta + \frac{\mathcal{M}\mathcal{M}_1}{\Gamma(2\beta)}\int_0^t (t-\theta)^{2\beta-1}\|v_{y_1}(\theta) - v_{y_2}(\theta)\|d\theta \\ \leq & \frac{\mathcal{M}c^{2\beta}}{\Gamma(2\beta+1)}(K(R) + \mathcal{M}_1\mathcal{K}_2)\|y_1 - y_2\|_{C([-r,c],E)}. \end{aligned}$$

According to (3.1) we deduce that $Q : B_C(R)) \to B_C(R))$ is a contraction mapping.

As such, Q has a unique fixed point $y \in B_C(R)) \subset C([-r,c],E)$, which is the mild solution of (1.1). □

Remark 3.1. *Since the global Lipschitz continuity contains local Lipschitz continuity (but the reverse is not true), Theorem 3.1 is an improvement of Theorem 4.1 of [7].*

Theorem 3.2. *Let* $(P1), (P3)$ *and* $(P4)$ *hold. If* $f : U \times C([-r, 0], E) \to E$ *is continuous, and the condition* $(P5)$ *is satisfied, where*

$(P5)$ *For any* $t \in U$ *and* $R > 0$*, the set*

$$\{\mathcal{P}_\beta(t-s)f(s,x) : s \in [0,t], \|x\|_{C([-r,0],E)} \leq R\}$$

is relatively compact in E, then the system (1.1) *is exactly controllable on* $[-r, c]$.

Proof. Put the operator $Q : C([-r, c], E) \to C([-r, c], E)$ as in (3.2). It follows from the proof of Theorem 3.1 that $Q(B_C(R)) \subseteq B_C(R)$ for some $R > 0$ and $Q : B_C(R) \to B_C(R)$ is continuous.

Next, we will show that Q is a compact operator, that is, $Q(B_C(R))$ is relatively compact. By the Ascoli-Arzela theorem, it suffices to prove that $Q(B_C(R))$ is equicontinuous and $\{(Qy)(t) : y \in B_C(R)\}$ is relatively compact for $t \in [-r, c]$. By employing Lemma 2.2$(i)(ii)$, the equicontinuity of $Q(B_C(R))$ can be verified using the proof of Theorem 4.2 of [7]. Since $(Qy)(t) \equiv \phi(t)$ for $t \in [-r, 0]$, it remains to prove that $\{(Qy)(t) : y \in B_C(R)\}$ is relatively compact for $t \in U$. By Lemma 2.5 and $(P5)$, we obtain that

$$\Xi(t) := \{\mathcal{P}_\beta(t-s)[f(s,y_s) + Bv_y(s)] : s \in [0,t], y \in B_C(R)\} \tag{3.4}$$

is relatively compact in E for $t \in U$. Hence, we can infer from (3.4) that

$$\int_0^t \mathcal{P}_\beta(t-s)[f(s,y_s) + Bv_y(s)ds \in t\,\overline{conv}\Xi(t), \quad \forall t \in U,$$

where $\overline{conv}\Xi(t)$ represents the convex closure of $\Xi(t)$. Therefore, $Q(B_C(R))$ is relatively compact and $Q : B_C(R) \to B_C(R)$ is completely continuous. According to Schauder's fixed point theorem, there is a function $y \in Q(B_C(R))$ satisfying $y = Qy$ and $y(c) = \bar{y}$. Thus the system (1.1) is exactly controllable on $[-r, c]$. □

Obviously, if f is uniformly bounded, the assumption $(P3)$ holds automatically. In this case, if we suppose that $T(t)$ is compact for $t > 0$, then the assumption $(P5)$ is fulfilled. Hence, according to Theorem 3.2, the corollary is acquired below.

Corollary 3.1. *Let* $(P1)$ *and* $(P4)$ *hold. In addition,* $f : U \times C([-r, 0], E) \to E$ *is continuous and uniformly bounded,* $T(t)$ *is compact for* $t > 0$*. Then the system (1.1) is exactly controllable on* $[-r, c]$.

Remark 3.2. *Obviously, if* f *is completely continuous, the assumption* $(P5)$ *hold automatically.*

Furthermore, if we achieve the compactness of $T(t)$ for $t > 0$, by Lemma 2.2(iii) and Lemma 2.3, the assumption $(P3)$ yields that

$$\{\int_0^t \mathcal{P}_\beta(t-s)[f(s,y_s) + Bv_y(s)ds : s \in [0,t], y \in B_C(R)\}$$

is relatively compact for $t \in U$. By the proof of Theorem 3.2, we can acquire the relative compactness of $\{(Qy)(t) : y \in B_C(R)\}$ for $t \in [-r, c]$. Hence, the uniform boundedness of f in Corollary 3.1 can be removed.

Theorem 3.3. *Let* $(P1), (P3)$ *and* $(P4)$ *hold. In addition,* $f : U \times C([-r, 0], E) \to E$ *is continuous and* $T(t)$ *is compact for* $t > 0$*. Then the system (1.1) is exactly controllable on* $[-r, c]$.

From Corollary 3.1 and Theorem 3.3, we can see that the assumption $(P5)$ and the compactness of $T(t)$ for $t > 0$ are too strong. It is of interesting to weaken or remove such assumptions. Next, we will apply the measure of noncompactness method to discuss the exact controllability of (1.1).

Let E be a Banach space and $D \subset E$ a bounded subset of E. We define

$$\chi(D) := \inf\{\epsilon > 0 : D \text{ has finite } \epsilon\text{-net in } E\}$$

the Hausdorff measure of noncompactness in E. If $D \subset C(U, E)$ is bounded, $D(t) := \{y(t) : y \in D\} \subset E$ is bounded for each $t \in U$ and $\chi(D(t)) \leq \chi(D)$.

Lemma 3.1. *[10] If $D \subset E$ is bounded, then there exists a countable subset $D_0 \subset D$ such that*

$$\chi(D) \leq 2\chi(D_0).$$

Lemma 3.2. *[11] If $D \subset C(U, E)$ is bounded and equicontinuous, then $\chi(D(t))$ is continuous for $t \in U$ and*

$$\chi(D) = \sup_{t \in U} \|\chi(D(t))\|.$$

Lemma 3.3. *[12] Let $D := \{y_n\} \subset C(U, E)$ be countable. If there is $\omega \in L^1(U)$ such that, for each $n \geq 1$, $y_n(t) \leq \omega(t)$ a.e., then $\chi(D(t))$ is Lebesgue integrable on U and*

$$\chi(\{\int_U y_n(t)dt\}) \leq 2 \int_U \chi(D(t))dt.$$

Furthermore, we make the assumptions below.

$(P6)$ $f : U \times C([-r, 0], E) \to E$ is continuous and there is a function $\eta \in L^2(U, \mathbb{R}^+)$ such that for each bounded subset $D_1 \subset C([-r, 0], E)$, we have

$$\chi(f(t, D_1)) \leq \eta(t) \sup_{-r \leq s \leq 0} \chi(D_1(s)), \quad \forall t \in U.$$

$(P7)$ There is a constant $L > 0$ such that

$$\chi(W^{-1}(D_2)(t)) \leq L\chi(D_2), \quad \forall t \in U$$

for each bounded subset $D_2 \subset E$.

Theorem 3.4. *Let $(P1), (P3), (P4), (P6)$ and $(P7)$ be fulfilled and*

$$\frac{M\sigma}{\Gamma(2\beta)}\Big(1 + \frac{2MLM_1c^{2\beta}}{\Gamma(2\beta + 1)}\Big)\|\eta\|_{Ł^2} < \frac{1}{4}. \tag{3.5}$$

Then, the system (1.1) is exactly controllable on $[-r, c]$.

Proof. From the Theorem 3.2 and its proof, we obtain that $Q : B_C(R) \to B_C(R)$, defined as in (3.2), is continuous and equicontinuous. Next, we will verify that $Q : B_C(R) \to B_C(R)$ is a condensing mapping using $(P6)$ and $(P7)$.

Since for any $\theta \in U$, we have

$$\begin{aligned}
&\sup_{-r\le\tau\le 0} \chi(\{y_\theta(\tau) : y \in B_C(R)\}) \\
&\le \sup_{0\le\tau\le 0} \chi(\{y(\tau) : y \in B_C(R)\}) \\
&\le \chi(B_C(R)),
\end{aligned} \tag{3.6}$$

according to $(P6)$, $(P7)$, (3.6) and (2.1), we can deduce that

$$\chi(\{Bv_y(s) : y \in B_C(R)\}) \le \frac{2\mathcal{M}\mathcal{M}_1 L\sigma}{\Gamma(2\beta)} \|\eta\|_{L^2} \chi(\{B_C(R)\}). \tag{3.7}$$

Since $Q(B_C(R))$ is bounded and equicontinuous, according to Lemma 3.1 and Lemma 3.2, there is a countable subset $\{y_n : n \ge 1\} \subset B_C(R)$ such that

$$\begin{aligned}
\chi(Q(B_C(R))) &\le 2\chi(\{(Qy_n) : n \ge 1\}) \\
&= 2 \sup_{t\in[-r,c]} \chi(\{(Qy_n)(t) : n \ge 1\}).
\end{aligned} \tag{3.8}$$

Obviously, for $t \in [-r, 0]$, $(Qy_n)(t) \equiv \phi(t)$ for $n \ge 1$, so

$$\chi(\{(Qy_n)(t) : n \ge 1\}) = 0.$$

For $t \in U$, by (3.6), (3.7) and Lemma 3.3, we can infer that

$$\begin{aligned}
\chi(\{(Qy_n)(t) : n \ge 1\}) &\le \frac{2\mathcal{M}}{\Gamma(2\beta)} \int_0^t (t-s)^{2\beta-1} \chi(\{f(s, y_{ns}) + Bv_{y_n}(s) : n \ge 1\}) ds \\
&\le \frac{2\mathcal{M}}{\Gamma(2\beta)} \int_0^t (t-s)^{2\beta-1} \eta(s) \sup_{-r\le\theta\le 0} \chi(\{y_{ns}(\theta) : n \ge 1\}) ds \\
&\quad + \frac{2\mathcal{M}}{\Gamma(2\beta)} \int_0^t (t-s)^{2\beta-1} \chi(\{Bv_{y_n}(s) : n \ge 1\}) ds \\
&\le \frac{2\mathcal{M}\sigma}{\Gamma(2\beta)} \left(1 + \frac{2\mathcal{M}\mathcal{M}_1 L c^{2\beta}}{\Gamma(2\beta+1)}\right) \|\eta\|_{Ł^2} \chi(B_C(R)).
\end{aligned}$$

The above facts together with (3.8) yield that

$$\begin{aligned}
\chi(Q(B_C(R))) &\le 2 \sup_{t\in[-r,0]} \chi(\{(Qy_n)(t) : n \ge 1\}) + 2 \sup_{t\in U} \chi(\{(Qy_n)(t) : n \ge 1\}) \\
&\le \frac{4\mathcal{M}\sigma}{\Gamma(2\beta)} \left(1 + \frac{2\mathcal{M}\mathcal{M}_1 L c^{2\beta}}{\Gamma(2\beta+1)}\right) \|\eta\|_{Ł^2} \chi(B_C(R)).
\end{aligned}$$

According to (3.5), we acquire that $Q : B_C(R) \to B_C(R)$ is a condensing mapping. By Sadovskii's fixed point theorem, there exists $y \in B_C(R)$ satisfying $y = Qy$ and $y(c) = \bar{y}$. Therefore, the system (1.1) is exactly controllable on $[-r, c]$. □

Remark 3.3. *In Theorem 3.4, we remove the assumption* $(P5)$ *and the compactness of the sine family* $\{T(t) : t \ge 0\}$*, which are essential assumptions in [7, 8]. Hence, our results extend many existing research works.*

Remark 3.4. *In [7], under the global Lipschitz condition or certain compactness conditions on nonlinear term f, the exact controllability of the Caputo fractional evolution equations without delay was investigated. Compared with the main results of [7], we consider the delayed control system (1.1). Theorem* 3.1 *is obtained when the nonlinear term f is locally Lipschitz continuous. Theorem 3.2 and Theorem 3.3 are achieved under certain compactness conditions. The measure of noncompactness conditions are applied in Theorem 3.4. Therefore, our conclusions improve the major results of [7].*

4. Applications

Let $\Omega \subset \mathbb{R}^N$ be an open subset with Dirichlet boundary conditions. We focus on the fractional delayed evolution system

$$\begin{cases} {}^C\partial_t^{\frac{3}{2}} y(t,z) = \Delta y(t,z) + f(t,z,y_t) + Bv(t,z), \quad t \in (0,1], \ z \in \Omega, \quad \alpha \in (1,2), \\ y(t,z) = \phi(t,z), \quad \forall t \in [-r,0], \\ y'(0,z) = y_1(z), \end{cases} \tag{4.1}$$

where ${}^C\partial_t^{\frac{3}{2}}$ represents the fractional partial derivative operator on t of order $\frac{3}{2}$ in the Caputo sense, Δ represents the Laplace operator, $v \in L^2([0,1]\times\Omega, L^2(\Omega))$ stands for the control function. f is the nonlinear function satisfying the following conditions.

$(A1)$ $f : [0,1]\times\Omega\times C([-r,0],L^2(\Omega)) \to L^2(\Omega)$ is continuous and for each $R > 0$, there exists a constant $K(R) > 0$ such that

$$\|f(t,z,\phi_1) - f(t,z,\phi_2)\|_{L^2(\Omega)} \le K(R)\|\phi_1 - \phi_2\|_{C([-r,0],L^2(\Omega))}$$

for any $t \in [0,1]$ and $\phi_j \in C([-r,0],L^2(\Omega))$ with $\|\phi_j\|_{C([-r,0],L^2(\Omega))} \le R, \ \ j = 1,2$.

$(A2)$ There is $\varphi \in L^2([0,1],\mathbb{R}^+)$ such that

$$\|f(t,z,\phi)\|_{L^2(\Omega)} \le \varphi(t)$$

for any $t \in [0,1], z \in \Omega$ and $\phi \in C([-r,0],L^2(\Omega))$.

Let $E = H = L^2(\Omega)$. We define

$$\mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

$$Ay = \Delta y.$$

Then, by [7, 13], $A : \mathcal{D}(A) \subseteq E \to E$ generates a strongly continuous cosine family $\{G(t) : t \ge 0\}$ in E satisfying $\|G(t)\|_{\mathcal{B}} \le 1$ for every $t \ge 0$.

Define $B : H \to E$ by

$$Bv = \sum_{n=1}^{\infty} \mu_n \langle \hat{v}, \varpi_n \rangle \varpi_n,$$

where

$$\hat{v} = \begin{cases} \langle v, \varpi_n \rangle, \quad n = 1,2,\cdots,\ell, \\ 0, \quad n = \ell+1, \ell+2, \cdots \end{cases}$$

for $\ell \in \mathbb{N}$. $\mu_n = n^2\pi^2$ is eigenvalues of A with the corresponding eigenvectors $\varpi_n(z) = \sqrt{\frac{2}{\pi}}\sin(n\pi z)$ for $n \in \mathbb{N}$. From [7], there is a positive constant $\mathcal{M}_1$ satisfying $\|B\|_{\mathcal{B}} \le \mathcal{M}_1$.

Define the operator $W : L^2([0,1], L^2(\Omega)) \to L^2(\Omega)$ by

$$Wv = \int_0^1 \mathcal{P}_{\frac{3}{4}}(1-s)Bv(s)ds,$$

where

$$\mathcal{P}_{\frac{3}{4}}(t) = t^{-\frac{1}{4}}\int_0^\infty \frac{3}{4}\theta M_{\frac{3}{4}}(\theta)T(t^{\frac{3}{4}}\theta)d\theta, \quad t \in [0,1]$$

and $\{T(t) : t \ge 0\}$ is the sine family corresponding to $\{G(t) : t \ge 0\}$ generated by A.

$(A3)$ W^{-1} exists and takes values in $L^2([0,1], L^2(\Omega)) \setminus KerW$ and there is $\mathcal{M}_2 > 0$ such that

$$\|W^{-1}\| \le M_2.$$

Thus, we can rewrite the fractional delayed evolution system (4.1) into the abstract fractional control system (1.1). By Theorem 3.1, we can achieve the cxact controllability result.

Theorem 4.1. *Assume that conditions* $(A1)-(A3)$ *hold. Then the fractional delayed evolution system* (4.1) *is exactly controllable provided that*

$$\frac{K(R)}{\Gamma(\frac{5}{2})}\left(1 + \frac{\mathcal{M}_1\mathcal{M}_2}{\Gamma(\frac{5}{2})}\right) < 1.$$

5. Conclusions

This paper deal with the exact controllability of the control system govern by evolution equations involving Caputo fractional derivatives of order $\alpha \in (1,2)$ in abstract spaces. At first, the exact controllability of the delayed system (1.1) is studied when f is local Lipschitz continuous. Then, the certain compactness conditions and the measure of noncompactness conditions are employed to demonstrate the exact controllability of (1.1) in this article. The results improve and generalize the conclusions of many researchers. In the future, the method can be applied to study the controllability of the Sobolev-type fractional evolution equations. By utilizing the compactness of the Sobolev operator, we can delete compactness conditions on the nonlinear term f and the sine family $\{T(t) : t \ge 0\}$.

Use of AI tools declaration

The author declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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