Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 17955–17970.
DOI: 10.3934/math.2024874
Received: 25 March 2024
Revised: 03 May 2024
Accepted: 16 May 2024
Published: 27 May 2024

Research article

On Hermite-Hadamard-type inequalities for second order differential inequalities with inverse-square potential

Hassen Aydi[1,2,*], **Bessem Samet**[3] **and Manuel De la Sen**[4]

[1] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

[2] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

[3] Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

[4] Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country, 48940-Leioa (Bizkaia), Spain

* **Correspondence:** Email: hassen.aydi@isima.rnu.tn.

Abstract: We consider the class of functions $u \in C^2((0,\infty))$ satisfying second-order differential inequalities in the form $u''(x) + \frac{k}{x^2}u(x) \geq 0$ for all $x > 0$. For this class of functions, we establish Hermite-Hadamard-type inequalities in both cases ($k = \frac{1}{4}$ and $0 < k < \frac{1}{4}$). We next extend our obtained results to the two-dimensional case. In the limit case $k \to 0^+$ we deriver some existing results from the literature that are related to convex functions and convex functions on the coordinates. In our approach, we make use of some tools from ordinary differential equations.

Keywords: Hermite-Hadamard-type inequalities; second order differential inequalities; convex functions; convex functions on the coordinates
Mathematics Subject Classification: 26A51, 26B25, 26D15

1. Introduction

The double Hermite-Hadamard inequality [13, 14] can be stated as follows: If $u : I \to \mathbb{R}$ is convex on I then for all $a_1, a_2 \in I$ with $a_1 < a_2$ we have

$$u\left(\frac{1}{2}\sum_{i=1}^{2} a_i\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} u(x)\,dx \leq \frac{1}{2}\sum_{i=1}^{2} u(a_i). \tag{1.1}$$

Inequalities of type (1.1) have many applications in theoretical and applied mathematics, such as convex analysis, optimization, numerical integrations, etc. As it is mentioned above, (1.1) holds for convex functions $u : I \to \mathbb{R}$. A natural question is to ask whether it is possible to derive inequalities of type (1.1) for other classes of functions. Several results in this direction were obtained for different types of convex functions, such as m-convex functions [7,10,25], s-convex functions [11,12,17,22,27], h-convex functions [5, 23, 24], logarithmically convex functions [19, 20, 26], convex functions with respect to a pair of functions [3,4,21,28] and convex functions on the coordinates [6,15,18].

The study of Hermite-Hadamard-type inequalities for second-order differential inequalities was first considered in [1,2,8,9]. Namely, in [8], Dragomir established inequalities of type (1.1) for the class of functions $u : I \to \mathbb{R}$ satisfying: For all $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$

$$u(x) \leq \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]}u(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]}u(b), \quad a \leq x \leq b,$$

where $\rho > 0$ is a constant. A function u satisfying the above condition is called trigonometrically ρ-convex. Moreover, in [1]- it was shown that, if $u \in C^2(I)$, then u is trigonometrically ρ-convex if and only if u satisfies the second order differential inequality

$$u''(x) + \rho^2 u(x) \geq 0, \quad x \in I.$$

Similar results were obtained in [9] for the class of functions $u : I \to \mathbb{R}$ satisfying: For all $a, b \in I$ with $a < b$

$$u(x) \leq \frac{\sinh[\rho(b-x)]}{\sinh[\rho(b-a)]}u(a) + \frac{\sinh[\rho(x-a)]}{\sinh[\rho(b-a)]}u(b), \quad a \leq t \leq b,$$

where $\rho > 0$ is a constant. A function u satisfying the above condition is called hyperbolic ρ-convex. On the other hand, it was proved in [2] that, if $u \in C^2(I)$, then u is hyperbolic ρ-convex if and only if u satisfies the second order differential inequality

$$u''(x) - \rho^2 u(x) \geq 0, \quad x \in I.$$

Motivated by the above cited works, we extend in this paper the (right) Hermite-Hadamard inequality (1.1) to the class of functions $u \in C^2((0,\infty))$ satisfying second-order differential inequalities of the form

$$u''(x) + \frac{k}{x^2}u(x) \geq 0, \quad x > 0, \tag{1.2}$$

where $0 < k \leq \frac{1}{4}$ is a constant. Remark that in the limit case, $k = 0$ (1.2) reduces to

$$u''(x) \geq 0, \quad x > 0,$$

which is equivalent to the convexity of u on $(0,\infty)$. Remark also that, if $u \leq 0$ on $(0,\infty)$ and u satisfies (1.2), then u is convex on $(0,\infty)$. However, if u satisfies (1.2) then u is not necessarily convex. For instance, the function $u(x) = \sqrt{x} + x$ is concave on $(0,\infty)$ and

$$u''(x) + \frac{1}{4x^2}u(x) = \frac{1}{4x} > 0, \quad x > 0.$$

The rest of the paper is arranged as follows: In Section 2 we fix some notations and establish some lemmas that will be used later. In Section 3, we establish Hermite-Hadamard-type inequalities for the class of functions $u \in C^2((0,\infty))$ satisfying (1.2). We discuss separately the cases of $k = \frac{1}{4}$ and $0 < k < \frac{1}{4}$. The obtained results in Section 3 are extended to the two-dimensional case in Section 4.

2. Auxiliary results

In this section, we establish some lemmas that will be useful in the proofs of our main results. We first fix some notations that will be used throughout this paper.

We denote by $M, N : (1,\infty) \to \mathbb{R}$ the functions defined by

$$M(s) = \frac{1}{\ln s}\left(s^{\frac{3}{2}} - \frac{3}{2}\ln s - 1\right),\ N(s) = \frac{1}{\ln s}\left(s^{-\frac{3}{2}} + \frac{3}{2}\ln s - 1\right).$$

For $0 < k \le \frac{1}{4}$ and $0 < a < b$, we introduce the terms

$$\lambda_k = \sqrt{1-4k},$$

$$A(a,b) = \frac{a^{\frac{3}{2}}\ln b - b^{\frac{3}{2}}\ln a}{\ln b - \ln a},\ B(a,b) = \frac{b^{\frac{3}{2}} - a^{\frac{3}{2}}}{\ln b - \ln a}$$

$$C_k(a,b) = \frac{1-\left(\frac{a}{b}\right)^{\frac{\lambda_k-3}{2}}}{1-\left(\frac{a}{b}\right)^{\lambda_k}},\ D_k(a,b) = \frac{1-\left(\frac{a}{b}\right)^{\frac{\lambda_k+3}{2}}}{1-\left(\frac{a}{b}\right)^{\lambda_k}}.$$

Lemma 2.1. *Let* $0 < a < b$ *and*

$$F(x) = A(a,b) + B(a,b)\ln x - x^{\frac{3}{2}}, \quad a \le x \le b. \tag{2.1}$$

Then the following holds:

(i) $F(a) = F(b) = 0$*;*
(ii) $F'(a) = \sqrt{a}M\left(\frac{b}{a}\right)$*,* $F'(b) = \sqrt{b}M\left(\frac{b}{a}\right)$*;*
(iii) $F(x) \ge 0$ *for all* $x \in [a,b]$.

Proof. (i) follows immediately from the definition of F. On the other hand, for all $x \in [a,b]$, we have

$$F'(x) = \frac{B(a,b) - \frac{3}{2}x^{\frac{3}{2}}}{x}. \tag{2.2}$$

Taking $x = a$ (resp. $x = b$) in (2.2), we obtain (ii). Furthermore, from (i) and Rolle's theorem, there exists at least $c \in (a,b)$ such that $F'(c) = 0$. However, from (2.2), we have $F'(x) = 0$ if and only if

$$x = \left(\frac{2}{3}B(a,b)\right)^{\frac{2}{3}}.$$

Consequently, $c = \left(\frac{2}{3}B(a,b)\right)^{\frac{2}{3}} \in (a,b)$ and from (2.2) F reaches its maximum value at $x = c$. Then due to (i) we deduce that $F(x) \ge 0$ for all $x \in [a,b]$, which proves (iii). □

Lemma 2.2. *Let* $0 < a < b$ *and*

$$f(x) = \frac{4}{9}\sqrt{x}F(x), \quad a \le x \le b, \tag{2.3}$$

where F *is the function defined by* (2.1). *Then the following holds:*

(i) $f(a) = f(b) = 0$;
(ii) $f'(a) = \frac{4a}{9} M\left(\frac{b}{a}\right)$, $f'(b) = \frac{-4b}{9} N\left(\frac{b}{a}\right)$;
(iii) $f(x) \geq 0$ *for all* $x \in [a,b]$;
(iv) $f''(x) + \dfrac{1}{4x^2} f(x) = -1$ *for all* $x \in (a,b)$.

Proof. (i)–(iii) are the immediate consequences of Lemma 2.1. (iv) follows from the definition of f and some elementary calculations. □

Lemma 2.3. *Let* $0 < k < \frac{1}{4}$, $0 < a < b$ *and*

$$G(x) = a^{\frac{\lambda_k+3}{2}} C_k(a,b) + b^{\frac{3-\lambda_k}{2}} D_k(a,b) x^{\lambda_k} - x^{\frac{\lambda_k+3}{2}}, \quad a \leq x \leq b. \tag{2.4}$$

Then the following holds:

(i) $G(a) = G(b) = 0$;
(ii) $G'(a) = \lambda_k a^{\lambda_k - 1} b^{\frac{3-\lambda_k}{2}} D_k(a,b) - \left(\frac{\lambda_k+3}{2}\right) a^{\frac{\lambda_k+1}{2}}$;
(iii) $G'(b) = b^{\frac{\lambda_k+1}{2}} \left(\lambda_k D_k(a,b) - \left(\frac{\lambda_k+3}{2}\right)\right)$;
(iv) $G(x) \geq 0$ *for all* $x \in [a,b]$.

Proof. (i) follows immediately from the definition of G. On the other hand, for all $x \in [a,b]$, we have

$$G'(x) = x^{\lambda_k - 1} \left(\lambda_k D_k(a,b) b^{\frac{3-\lambda_k}{2}} - \left(\frac{\lambda_k + 3}{2}\right) x^{\frac{3-\lambda_k}{2}}\right). \tag{2.5}$$

Taking $x = a$ (resp. $x = b$) in (2.5) we obtain (ii) and (iii). Furthermore, from (i) and Rolle's theorem, there exists at least $c \in (a,b)$ such that $G'(c) = 0$. However, from (2.5), $G'(x) = 0$ if and only if

$$x = \left(\frac{2\lambda_k D_k(a,b)}{\lambda_k + 3}\right)^{\frac{2}{3-\lambda_k}} b.$$

Consequently, $c = \left(\frac{2\lambda_k D_k(a,b)}{\lambda_k+3}\right)^{\frac{2}{3-\lambda_k}} b \in (a,b)$ and from (2.5), G reaches its maximum value at $x = c$. Then due to (i) we deduce that $G(x) \geq 0$ for all $x \in [a,b]$, which proves (iv). □

Lemma 2.4. *Let* $0 < k < \frac{1}{4}$, $0 < a < b$ *and*

$$g(x) = \frac{1}{k+2} x^{\frac{1-\lambda_k}{2}} G(x), \quad a \leq x \leq b, \tag{2.6}$$

where G is the function defined by (2.4). *Then the following holds:*

(i) $g(a) = g(b) = 0$;
(ii) $g'(a) = \frac{3+\lambda_k}{2(k+2)} a \left(\frac{2\lambda_k}{3+\lambda_k} \left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}} D_k(a,b) - 1\right)$;
(iii) $g'(b) = \frac{3+\lambda_k}{2(k+2)} b \left(\frac{2\lambda_k}{3+\lambda_k} D_k(a,b) - 1\right)$;
(iv) $g(x) \geq 0$ *for all* $x \in [a,b]$;
(v) $g''(x) + \dfrac{k}{x^2} g(x) = -1$ *for all* $x \in (a,b)$.

Proof. (i)–(iv) are the immediate consequences of Lemma 2.3. (v) follows from the definition of g and some elementary calculations. □

Lemma 2.5. *We have*

$$M(s) \geq 0, \;\; N(s) \geq 0$$

for all $s > 1$.

Proof. We have

$$M(s) = \frac{1}{\ln s}\varphi(s), \quad s > 1,$$

where

$$\varphi(s) = s^{\frac{3}{2}} - \frac{3}{2}\ln s - 1.$$

Remark that $\lim_{s\to 1^+} \varphi(s) = 0$. On the other hand, for all $s > 1$ we have

$$\varphi'(s) = \frac{3}{2}\sqrt{s} - \frac{3}{2}s^{-1} = \frac{3}{2}s^{-1}\left(s^{\frac{3}{2}} - 1\right) \geq 0,$$

which shows that φ is a nondecreasing function. Consequently, we have $\varphi(s) \geq 0$, which implies that $M(s) \geq 0$. Similarly, we have

$$N(s) = \frac{1}{\ln s}\psi(s), \quad s > 1,$$

where

$$\psi(s) = s^{-\frac{3}{2}} + \frac{3}{2}\ln s - 1.$$

Remark that $\lim_{s\to 1^+} \psi(s) = 0$. We also have

$$\psi'(s) = -\frac{3}{2}s^{-\frac{5}{2}} + \frac{3}{2}s^{-1} = \frac{3}{2}s^{-1}\left(1 - s^{-\frac{3}{2}}\right) \geq 0,$$

which shows that ψ is a nondecreasing function. Consequently, we have $\psi(s) \geq 0$, which implies that $N(s) \geq 0$. □

Lemma 2.6. *Let* $0 < k < \frac{1}{4}$ *and* $0 < a < b$. *Then the following holds:*

$$\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}} \frac{2\lambda_k}{\lambda_k + 3} D_k(a, b) - 1 \geq 0 \tag{2.7}$$

and

$$1 - \frac{2\lambda_k}{\lambda_k + 3} D_k(a, b) \geq 0. \tag{2.8}$$

Proof. We have

$$\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}} \frac{2\lambda_k}{\lambda_k + 3} D_k(a, b) - 1 = \left[1 - \left(\frac{a}{b}\right)^{\lambda_k}\right]^{-1} \gamma\left(\frac{b}{a}\right), \tag{2.9}$$

where

$$\gamma(s) = \frac{2\lambda_k}{\lambda_k + 3} s^{\frac{3-\lambda_k}{2}} + \frac{3 - \lambda_k}{3 + \lambda_k} s^{-\lambda_k} - 1, \quad s \geq 1.$$

Observe that $\gamma(1) = 0$. On the other hand, for all $s \geq 1$, we have

$$\gamma'(s) = \frac{\lambda_k(3-\lambda_k)}{\lambda_k+3} s^{-\lambda_k-1}\left(s^{\frac{\lambda_k+3}{2}} - 1\right) \geq 0,$$

which shows that γ is a nondecreasing function. Consequently, we have $\gamma(s) \geq 0$ for all $s \geq 1$, which implies that $\gamma\left(\frac{b}{a}\right) \geq 0$. Then by (2.9), we obtain (2.7). On the other hand, we have

$$1 - \frac{2\lambda_k}{\lambda_k+3} D_k(a,b) = \left[1 - \left(\frac{a}{b}\right)^{\lambda_k}\right]^{-1} \eta\left(\frac{a}{b}\right), \tag{2.10}$$

where

$$\eta(s) = \frac{3-\lambda_k}{\lambda_k+3} + \frac{2\lambda_k}{\lambda_k+3} s^{\frac{3+\lambda_k}{2}} - s^{\lambda_k}, \quad 0 < s \leq 1.$$

Remark that $\eta(1) = 0$, and for all $0 < s \leq 1$, we have

$$\eta'(s) = \lambda_k s^{\frac{1+\lambda_k}{2}}\left(1 - s^{\frac{\lambda_k-3}{2}}\right) \leq 0,$$

which shows that γ is a decreasing function. Consequently, we have $\eta\left(\frac{a}{b}\right) \geq \eta(1) = 0$. Then by (2.10) we obtain (2.8). □

3. Hermite-Hadamard-type inequalities

For $0 < k \leq \frac{1}{4}$, we consider the class of real-valued functions

$$C_k^{2,+}((0,\infty)) = \left\{u \in C^2((0,\infty)) : u''(x) + \frac{k}{x^2}u(x) \geq 0 \text{ for all } x > 0\right\}.$$

Remark that, in the limit case $k = 0$, we have

$$C_0^{2,+}((0,\infty)) = \left\{u \in C^2((0,\infty)) : u \text{ is convex on } (0,\infty)\right\}.$$

In this section we establish Hermite-Hadamard-type inequalities for functions $u \in C_k^{2,+}((0,\infty))$.

We first consider the case $k = \frac{1}{4}$.

Theorem 3.1. *If $u \in C_{1/4}^{2,+}((0,\infty))$, then for all $a, b \in \mathbb{R}$ with $0 < a < b$, we have*

$$\int_a^b u(x)\,dx \leq \frac{4}{9}\left(aM\left(\frac{b}{a}\right)u(a) + bN\left(\frac{b}{a}\right)u(b)\right). \tag{3.1}$$

Proof. Let $u \in C_{1/4}^{2,+}((0,\infty))$ and $0 < a < b$. By Lemma 2.2-(iv), we have

$$\int_a^b u(x)\,dx = -\int_a^b u(x)\left(f''(x) + \frac{1}{4x^2}f(x)\right)dx, \tag{3.2}$$

where f is the function defined by (2.3). Integrations by parts give us that

$$
\begin{aligned}
&-\int_a^b u(x)\left(f''(x)+\frac{1}{4x^2}f(x)\right)dx\\
&=-\left[f'(x)u(x)\right]_{x=a}^b+\int_a^b u'(x)f'(x)\,dx-\int_a^b f(x)\,\frac{1}{4x^2}u(x)\,dx\\
&=-f'(b)u(b)+f'(a)u(a)+\left[f(x)u'(x)\right]_{x=a}^b-\int_a^b f(x)u''(x)\,dx-\int_a^b f(x)\,\frac{1}{4x^2}u(x)\,dx\\
&=f'(a)u(a)-f'(b)u(b)+\big(f(b)u'(b)-f(a)u'(a)\big)-\int_a^b f(x)\left(u''(x)+\frac{1}{4x^2}u(x)\right)dx.
\end{aligned}
$$

On the other hand, from Lemma 2.2-(i) we know that $f(a)=f(b)=0$. Consequently, we have

$$
\begin{aligned}
-\int_a^b u(x)\left(f''(x)+\frac{1}{4x^2}f(x)\right)dx&=f'(a)u(a)-f'(b)u(b)\\
&\quad-\int_a^b f(x)\left(u''(x)+\frac{1}{4x^2}u(x)\right)dx.
\end{aligned}
\tag{3.3}
$$

From (3.2) and (3.3), we deduce that

$$
\int_a^b u(x)\,dx=f'(a)u(a)-f'(b)u(b)-\int_a^b f(x)\left(u''(x)+\frac{1}{4x^2}u(x)\right)dx.
$$

Since $f\geq 0$ by Lemma 2.2-(iii) and $u\in C^{2,+}_{1/4}((0,\infty))$, then

$$
\int_a^b f(x)\left(u''(x)+\frac{1}{4x^2}u(x)\right)dx\geq 0.
$$

Hence, it holds that

$$
\int_a^b u(x)\,dx\leq f'(a)u(a)-f'(b)u(b).
$$

Finally, using Lemma 2.2-(ii), we obtain (3.1). □

We now consider case $0<k<\frac{1}{4}$.

Theorem 3.2. *Let $0<k<\frac{1}{4}$. If $u\in C^{2,+}_k((0,\infty))$, then for all $a,b\in\mathbb{R}$ with $0<a<b$, we have*

$$
\begin{aligned}
&\int_a^b u(x)\,dx\\
&\leq\frac{\lambda_k+3}{2(k+2)}\left[a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)u(a)+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)u(b)\right].
\end{aligned}
\tag{3.4}
$$

Proof. Let $u\in C^{2,+}_k((0,\infty))$ and $0<a<b$. We shall follow the same approach used in the proof of Theorem 3.1. Namely, by Lemma 2.4-(v), we have

$$
\int_a^b u(x)\,dx=-\int_a^b u(x)\left(g''(x)+\frac{k}{x^2}g(x)\right)dx,
\tag{3.5}
$$

where g is the function defined by (2.6). Integrating by parts, we get

$$-\int_a^b u(x)\left(g''(x)+\frac{k}{x^2}g(x)\right)dx = g'(a)u(a)-g'(b)u(b)+g(b)u'(b)-g(a)u'(a) - \int_a^b g(x)\left(u''(x)+\frac{k}{x^2}u(x)\right)dx,$$

which implies by Lemma 2.4-(i) that

$$\begin{aligned}&-\int_a^b u(x)\left(g''(x)+\frac{k}{x^2}g(x)\right)dx\\ &= g'(a)u(a)-g'(b)u(b)-\int_a^b g(x)\left(u''(x)+\frac{k}{x^2}u(x)\right)dx.\end{aligned} \tag{3.6}$$

In view of (3.5) and (3.6), we have

$$\int_a^b u(x)\,dx = g'(a)u(a)-g'(b)u(b)-\int_a^b g(x)\left(u''(x)+\frac{k}{x^2}u(x)\right)dx.$$

Since $g \geq 0$ by Lemma 2.4-(iv) and $u \in C_k^{2,+}((0,\infty))$, then

$$\int_a^b g(x)\left(u''(x)+\frac{k}{x^2}u(x)\right)dx \geq 0.$$

Consequently, we get

$$\int_a^b u(x)\,dx \leq g'(a)u(a)-g'(b)u(b).$$

Finally, using Lemma 2.4-(ii)-(iii), we obtain (3.4). □

Remark 3.1. Remark that in the limit case $k = 0$, (3.4) reduces to the right Hermite-Hadamard inequality for convex functions

$$\frac{1}{b-a}\int_a^b u(x)\,dx \leq \frac{u(a)+u(b)}{2}.$$

4. The two-dimensional case

We now consider the class of real-valued functions

$$\begin{aligned}&C_k^{2,+}((0,\infty)^2)\\ &= \left\{u \in C^2((0,\infty)^2) : \frac{\partial^2 u}{\partial x^2}(x,y)+\frac{k}{x^2}u(x,y) \geq 0,\ \frac{\partial^2 u}{\partial y^2}(x,y)+\frac{k}{y^2}u(x,y) \geq 0,\ x,y>0\right\},\end{aligned}$$

where $0 < k \leq \frac{1}{4}$ and $(0,\infty)^2 = (0,\infty)\times(0,\infty)$. Notice that in the limit case $k = 0$, $C_0^{2,+}((0,\infty)^2)$ is the class of functions $u \in C^2((0,\infty)^2)$ such that u is convex on the coordinates in $(0,\infty)^2$ (see Dragomir [6]). In this section, we extend some of the obtained results in [6] to the class of functions $C_k^{2,+}((0,\infty)^2)$.

We first consider the case $k = \frac{1}{4}$.

Theorem 4.1. *If $u \in C^{2,+}_{1/4}((0,\infty)^2)$, then for all $a, b, c, d \in \mathbb{R}$ with $0 < a < b$ and $0 < c < d$ we have*

$$\begin{aligned}
&\int_a^b \int_c^d u(x,y)\,dx\,dy \\
&\leq \frac{2}{9} \int_a^b \left(cM\left(\frac{d}{c}\right) u(x,c) + dN\left(\frac{d}{c}\right) u(x,d) \right) dx \\
&\quad + \frac{2}{9} \int_c^d \left(aM\left(\frac{b}{a}\right) u(a,y) + bN\left(\frac{b}{a}\right) u(b,y) \right) dy \\
&\leq \frac{16ac}{81} M\left(\frac{d}{c}\right) M\left(\frac{b}{a}\right) u(a,c) + \frac{16bc}{81} M\left(\frac{d}{c}\right) N\left(\frac{b}{a}\right) u(b,c) \\
&\quad + \frac{16ad}{81} M\left(\frac{b}{a}\right) N\left(\frac{d}{c}\right) u(a,d) + \frac{16bd}{81} N\left(\frac{b}{a}\right) N\left(\frac{d}{c}\right) u(b,d).
\end{aligned} \tag{4.1}$$

Proof. Let $u \in C^{2,+}_{1/4}((0,\infty)^2)$ and $a, b, c, d \in \mathbb{R}$ with $0 < a < b$ and $0 < c < d$. By the definition of $C^{2,+}_{1/4}((0,\infty)^2)$, the function $u(x,\cdot) : y \mapsto u(x,\cdot)(y) = u(x,y)$ belongs to $C^{2,+}_{1/4}((0,\infty))$ for all $x > 0$. Then by Theorem 3.1 we have

$$\int_c^d u(x,y)\,dy \leq \frac{4}{9} \left(cM\left(\frac{d}{c}\right) u(x,c) + dN\left(\frac{d}{c}\right) u(x,d) \right).$$

for all $x > 0$. Integrating the above-mentioned inequality over (a,b) we get

$$\int_a^b \int_c^d u(x,y)\,dx\,dy \leq \frac{4}{9} \left(cM\left(\frac{d}{c}\right) \int_a^b u(x,c)\,dx + dN\left(\frac{d}{c}\right) \int_a^b u(x,d)\,dx \right). \tag{4.2}$$

Similarly, by the definition of $C^{2,+}_{1/4}((0,\infty)^2)$, the function $u(\cdot,y) : x \mapsto u(\cdot,y)(x) = u(x,y)$ belongs to $C^{2,+}_{1/4}((0,\infty))$ for all $y > 0$. Then by Theorem 3.1 we have

$$\int_a^b u(x,y)\,dx \leq \frac{4}{9} \left(aM\left(\frac{b}{a}\right) u(a,y) + bN\left(\frac{b}{a}\right) u(b,y) \right).$$

for all $y > 0$. Integrating the above inequality over (c,d), we obtain

$$\int_a^b \int_c^d u(x,y)\,dx\,dy \leq \frac{4}{9} \left(aM\left(\frac{b}{a}\right) \int_c^d u(a,y)\,dy + bN\left(\frac{b}{a}\right) \int_c^d u(b,y)\,dy \right). \tag{4.3}$$

Summing (4.2) and (4.3), we obtain

$$\begin{aligned}
\int_a^b \int_c^d u(x,y)\,dx\,dy \leq{}& \frac{2}{9} \int_a^b \left(cM\left(\frac{d}{c}\right) u(x,c) + dN\left(\frac{d}{c}\right) u(x,d) \right) dx \\
&+ \frac{2}{9} \int_c^d \left(aM\left(\frac{b}{a}\right) u(a,y) + bN\left(\frac{b}{a}\right) u(b,y) \right) dy,
\end{aligned}$$

which proves the first inequality in (4.1). On the other hand, since the function $u(\cdot,c) : x \mapsto u(\cdot,c)(x) = u(x,c)$ belongs to $C^{2,+}_{1/4}((0,\infty))$, then by Theorem 3.1 we have

$$\int_a^b u(x,c)\,dx \leq \frac{4}{9} \left(aM\left(\frac{b}{a}\right) u(a,c) + bN\left(\frac{b}{a}\right) u(b,c) \right).$$

Multiplying the above inequality by $cM\left(\frac{d}{c}\right)$ and using Lemma 2.5, we obtain

$$cM\left(\frac{d}{c}\right)\int_a^b u(x,c)\,dx \leq \frac{4}{9}cM\left(\frac{d}{c}\right)\left(aM\left(\frac{b}{a}\right)u(a,c) + bN\left(\frac{b}{a}\right)u(b,c)\right). \tag{4.4}$$

Similarly, we have

$$\int_a^b u(x,d)\,dx \leq \frac{4}{9}\left(aM\left(\frac{b}{a}\right)u(a,d) + bN\left(\frac{b}{a}\right)u(b,d)\right),$$

which implies after multiplication by $dN\left(\frac{d}{c}\right)$ (notice that by Lemma 2.5 we have $N(s) \geq 0$ for all $s > 1$) that

$$dN\left(\frac{d}{c}\right)\int_a^b u(x,d)\,dx \leq \frac{4}{9}dN\left(\frac{d}{c}\right)\left(aM\left(\frac{b}{a}\right)u(a,d) + bN\left(\frac{b}{a}\right)u(b,d)\right). \tag{4.5}$$

In the same manner, we obtain

$$aM\left(\frac{b}{a}\right)\int_c^d u(a,y)\,dy \leq \frac{4}{9}aM\left(\frac{b}{a}\right)\left(cM\left(\frac{d}{c}\right)u(a,c) + dN\left(\frac{d}{c}\right)u(a,d)\right) \tag{4.6}$$

and

$$bN\left(\frac{b}{a}\right)\int_c^d u(b,y)\,dy \leq \frac{4}{9}bN\left(\frac{b}{a}\right)\left(cM\left(\frac{d}{c}\right)u(b,c) + dN\left(\frac{d}{c}\right)u(b,d)\right). \tag{4.7}$$

Finally, the second inequality in (4.1) follows from (4.4)–(4.7). □

We next consider case $0 < k < \frac{1}{4}$.

Theorem 4.2. *Let $0 < k < \frac{1}{4}$. If $u \in C_k^{2,+}((0,\infty)^2)$, then for all $a,b,c,d \in \mathbb{R}$ with $0 < a < b$ and $0 < c < d$ we have*

$$\begin{aligned}
&\frac{4(k+2)}{\lambda_k+3}\int_a^b\int_c^d u(x,y)\,dx\,dy\\
&\leq c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)\int_a^b u(x,c)\,dx + d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)\int_a^b u(x,d)\,dx\\
&\quad +a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)\int_c^d u(a,y)\,dy + b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)\int_c^d u(b,y)\,dy\\
&\leq \left[\frac{(\lambda_k+3)ac}{k+2}\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)\right]u(a,c)\\
&\quad +\left[\frac{(\lambda_k+3)bc}{k+2}\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)\right]u(b,c)\\
&\quad +\left[\frac{(\lambda_k+3)ad}{k+2}\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)\right]u(a,d)\\
&\quad +\left[\frac{(\lambda_k+3)bd}{k+2}\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)\right]u(b,d).
\end{aligned} \tag{4.8}$$

Proof. Let $u \in C_k^{2,+}((0,\infty)^2)$ and $a,b,c,d \in \mathbb{R}$ with $0 < a < b$ and $0 < c < d$. By Theorem 3.2, we have

$$\begin{aligned}&\int_c^d u(x,y)\,dy\\ &\leq \frac{\lambda_k+3}{2(k+2)}\left[c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)u(x,c)+d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)u(x,d)\right]\end{aligned}$$

for all $x > 0$. Integrating the above inequality over (a,b), we obtain

$$\begin{aligned}\frac{2(k+2)}{\lambda_k+3}\int_a^b\int_c^d u(x,y)\,dx\,dy \quad &\leq c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)\int_a^b u(x,c)\,dx\\ &+d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)\int_a^b u(x,d)\,dx. \end{aligned} \tag{4.9}$$

Similarly, by Theorem 3.1, we have

$$\begin{aligned}&\int_a^b u(x,y)\,dx\\ &\leq \frac{\lambda_k+3}{2(k+2)}\left[a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)u(a,y)+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)u(b,y)\right]\end{aligned}$$

for all $y > 0$. Integrating the above inequality over (c,d), we obtain

$$\begin{aligned}\frac{2(k+2)}{\lambda_k+3}\int_a^b\int_c^d u(x,y)\,dx\,dy \quad &\leq a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)\int_c^d u(a,y)\,dy\\ &+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)\int_c^d u(b,y)\,dy. \end{aligned} \tag{4.10}$$

Summing (4.9) and (4.10), we obtain

$$\begin{aligned}&\frac{4(k+2)}{\lambda_k+3}\int_a^b\int_c^d u(x,y)\,dx\,dy\\ &\leq c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)\int_a^b u(x,c)\,dx+d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)\int_a^b u(x,d)\,dx\\ &\quad+a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)\int_c^d u(a,y)\,dy+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)\int_c^d u(b,y)\,dy,\end{aligned}$$

which proves the first inequality in (4.8). We now use Theorem 3.2 to get

$$\begin{aligned}&\int_a^b u(x,c)\,dx\\ &\leq \frac{\lambda_k+3}{2(k+2)}\left[a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)u(a,c)+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)u(b,c)\right].\end{aligned}$$

Multiplying the above inequality by $c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)$, it follows from Lemma 2.6 (see (2.7)) that

$$\begin{aligned}
&c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)\int_a^b u(x,c)\,dx \\
&\le \frac{\lambda_k+3}{2(k+2)}c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right) \\
&\quad\cdot\left[a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)u(a,c)+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)u(b,c)\right].
\end{aligned} \tag{4.11}$$

Using Theorem 3.2, we obtain

$$\begin{aligned}
&\int_a^b u(x,d)\,dx \\
&\le \frac{\lambda_k+3}{2(k+2)}\left[a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)u(a,d)+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)u(b,d)\right].
\end{aligned}$$

Multiplying the above inequality by $d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)$, it follows from Lemma 2.6 (see (2.8)) that

$$\begin{aligned}
&d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)\int_a^b u(x,d)\,dx \\
&\le \frac{\lambda_k+3}{2(k+2)}d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right) \\
&\quad\cdot\left[a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)u(a,d)+b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)u(b,d)\right].
\end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned}
&a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right)\int_c^d u(a,y)\,dy \\
&\le \frac{\lambda_k+3}{2(k+2)}a\left(\left(\frac{b}{a}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)-1\right) \\
&\quad\cdot\left[c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}}\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)-1\right)u(a,c)+d\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(c,d)\right)u(a,d)\right]
\end{aligned} \tag{4.13}$$

and

$$b\left(1-\frac{2\lambda_k}{\lambda_k+3}D_k(a,b)\right)\int_c^d u(b,y)\,dy$$

$$\leq \frac{\lambda_k+3}{2(k+2)} b\left(1-\frac{2\lambda_k}{\lambda_k+3} D_k(a,b)\right) \tag{4.14}$$
$$\cdot\left[c\left(\left(\frac{d}{c}\right)^{\frac{3-\lambda_k}{2}} \frac{2\lambda_k}{\lambda_k+3} D_k(c,d)-1\right) u(b,c)+d\left(1-\frac{2\lambda_k}{\lambda_k+3} D_k(c,d)\right) u(b,d)\right].$$

Finally the second inequality in (4.8) follows from (4.11)–(4.14). □

Remark 4.1. In the limit case $k=0$, we have

$$\lambda_0=1,\ D_0(a,b)=1+\frac{a}{b},\ D_0(c,d)=1+\frac{c}{d}.$$

In this case, (4.8) reduces to

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)} \int_a^b \int_c^d u(x,y)\,dx\,dy \\
&\leq \frac{1}{4(b-a)}\left(\int_a^b u(x,c)\,dx+\int_a^b u(x,d)\,dx\right)+\frac{1}{4(d-c)}\left(\int_c^d u(a,y)\,dy+\int_c^d u(b,y)\,dy\right) \\
&\leq \frac{u(a,c)+u(b,c)+u(a,d)+u(b,d)}{4}.
\end{aligned}$$

The above inequalities were derived previously by Dragomir [6] when u is convex on the coordinates.

5. Conclusions

Hermite-Hadamard-type inequalities are very useful in many branches of pure and applied mathematics. Such inequalities were established by many authors for different kinds of convex functions. The class of functions $u \in C^2((0,\infty))$ satisfying second-order differential inequalities of the form

$$u''(x)+\frac{k}{x^2}u(x) \geq 0, \quad x>0$$

is investigated in this paper. Namely, Hermite-Hadamard-type inequalities are established for this class of functions in both cases $k=\frac{1}{4}$ (see Theorem 3.1) and $0<k<\frac{1}{4}$ (see Theorem 3.2). Next, the obtained results are extended to the two-dimensional case by considering the class of functions $u=u(x,y) \in C^2((0,\infty)^2)$ satisfying the system of second order differential inequalities

$$\begin{cases}
\dfrac{\partial^2 u}{\partial x^2}(x,y)+\dfrac{k}{x^2}u(x,y) \geq 0, & x,y>0, \\
\dfrac{\partial^2 u}{\partial y^2}(x,y)+\dfrac{k}{y^2}u(x,y) \geq 0, & x,y>0.
\end{cases}$$

Hermite-Hadamard-type inequalities are derived for this class of functions in the case $k=\frac{1}{4}$ (see Theorem 4.1) and the case $0<k<\frac{1}{4}$ (see Theorem 4.2).

An interesting question consists of studying the class of functions $u \in C^2(\Omega)$ satisfying second-order differential inequalities of the form

$$\Delta u+\frac{k}{|x|^2}u(x) \geq 0, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n \backslash \{0\}$ and Δ is the Laplacian operator ($\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$). Notice that in the limit case $k = 0$, the above class of functions reduces to the class of subharmonic functions (see, e.g., [16]).

Author contributions

Hassen Aydi: Conceptualization, methodology, investigation, formal analysis, writing review and editing; Bessem Samet: Conceptualization, methodology, validation, investigation, writing original draft preparation; Manuel De La Sen: methodology, validation, formal analysis, investigation, writing review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

B. Samet is supported by Researchers Supporting Project number (RSP2024R4), King Saud University, Riyadh, Saudi Arabia. M. De La Sen is supported by the project: Basque Government IT1555-22.

Conflict of interest

The authors declare no conflict of interest.

References

1. M. S. S. Ali, On certain properties of trigonometrically ρ-convex functions, *Adv. Pure Math.*, **2** (2012), 337–340.
2. M. S. S. Ali, On certain properties for two classes of generalized convex functions, *Abstr. Appl. Anal.*, **2016** (2016), 4652038. https://doi.org/10.1155/2016/4652038
3. M. A. Ali, J. Soontharanon, H. Budak, T. Sitthiwirattham, M. Fečkon, Fractional Hermite-Hadamard inequality and error estimates for Simpson's formula through convexity with respect to pair of functions, *Miskolc Math. Notes*, **24** (2023), 553–568. https://doi.org/10.18514/MMN.2023.4214
4. M. A. Ali, F. Wannalookkhee, H. Budak, S. Etemad, S. Rezapour, New Hermite-Hadamard and Ostrowski-type inequalities for newly introduced co-ordinated convexity with respect to a pair of functions, *Mathematics*, **10** (2022), 3469. https://doi.org/10.3390/math10193469
5. S. S. Dragomir, Hermite-Hadamard type inequalities for generalized Riemann-Liouville fractional integrals of h-convex functions, *Math. Meth. Appl. Sci.*, **44** (2021), 2364–2380. https://doi.org/10.1002/mma.5893

6. S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwan. J. Math.*, **4** (2001), 775–788.

7. S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, *Tamkang J. Math.*, **3** (2002), 45–56. https://doi.org/10.5556/j.tkjm.33.2002.304

8. S. S. Dragomir, Some inequalities of Hermite-Hadamard type for trigonometrically ρ-convex functions, *Preprints*, **21** (2018), 2018020059. https://doi.org/10.20944/preprints201802.0059.v1

9. S. S. Dragomir, Some inequalities of Hermite-Hadamard type for hyperbolic p-convex functions, *Preprints*, **21** (2018), 2018020136. https://doi.org/10.20944/preprints201802.0136.v1

10. S. S. Dragomir, G. Toader, Some inequalities for m-convex functions, *Stud. U. Babeş-Bol. Mat.*, **38** (1993), 21–28.

11. T. S. Du, Y. Peng, Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals, *J. Comput. Appl. Math.*, **440** (2024), 115582. https://doi.org/10.1016/j.cam.2023.115582

12. G. Gulshan, H. Budak, R. Hussain, K. Nonlaopon, Some new quantum Hermite-Hadamard type inequalities for s-convex functions, *Symmetry*, **14** (2022), 870. https://doi.org/10.3390/sym14050870

13. J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.*, **58** (1893), 171–215.

14. C. Hermite, Sur deux limites d'une intégrale défine, *Mathesis*, **3** (1983), 1–82.

15. R. Hussain, A. Ali, A. Latif, G. Gulshan, Co-ordinated convex function of three variables and some analogues inequalities with applications, *J. Comput. Anal. Appl.*, **29** (2021), 505–517.

16. M. Jleli, B. Samet, On Hermite-Hadamard-type inequalities for subharmonic functions over circular ring domains, *Numer. Funct. Anal. Optim.*, **44** (2023), 1395–1408. https://doi.org/10.1080/01630563.2023.2259198

17. M. Khan, Y. M. Chu, T. Khan, J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, *Open Math.*, **15** (2017), 1414–1430. https://doi.org/10.1515/math-2017-0121

18. M. A. Latif, Hermite-Hadamard-type inequalities for coordinated convex functions using fuzzy integrals, *Mathematics,* **11** (2023), 2432. https://doi.org/10.3390/math11112432

19. M. A. Latif, S. S. Dragomir, E. Momoniat, On Hermite-Hadamard type integral inequalities for n-times differentiable m-and-logarithmically convex functions, *Filomat*, **30** (2016), 3101–3114. https://doi.org/10.2298/FIL1611101L

20. M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, *J. Math. Anal. Approx. Theory.*, **2** (2007), 126–131.

21. B. Samet, A convexity concept with respect to a pair of functions, *Numer. Funct. Anal. Optim.*, **43** (2022), 522–540. https://doi.org/10.1080/01630563.2022.2050753

22. M. Z. Sarikaya, M. E. Kiris, Some new inequalities of Hermite-Hadamard type for s-convex functions, *Miskolc Math. Notes*, **16** (2015), 491–501. https://doi.org/10.18514/MMN.2015.1099

23. M. Z. Sarikaya, A. Saglam, H. Yildrim, On some Hadamard-Type inequalities for h-convex functions, *J. Math. Inequal.*, **2** (2008), 335–341. https://doi.org/10.7153/jmi-02-30
24. S. Varoanec, On h-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086
25. B. Y. Xi, R. F. Bai, F. Qi, Hermite-Hadamard type inequalities for the m-and (α, m)-geometrically convex functions, *Aequationes Math.*, **84** (2012), 261–269.
26. B. Y. Xi, F. Qi, Some integral inequalities of Hermite-Hadamard type for s-log convex functions, *Acta Math. Sci. Ser. A (Chin. Ed.)*, **35** (2015), 515–524.
27. S. H. Wang, F. Qi, Hermite-Hadamard type inequalities for s-convex functions via Riemann-Liouville fractional integrals, *J. Comput. Anal. Appl.*, **22** (2017), 1124–1134.
28. L. L. Zhang, Y. Peng, T. S. Du, On multiplicative Hermite-Hadamard and Newton-type inequalities for multiplicatively (P, m)-convex functions, *J. Math. Anal. Appl.*, **534** (2024), 128117.