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Research article

The Helmholtz decomposition of vector fields for two-dimensional exterior Lipschitz domains

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Abstract: Let Ω be an exterior Lipschitz domain in $\mathbb{R}^2$. It is proved that the Helmholtz decomposition of the vector fields in $L_p(\Omega;\mathbb{R}^2)$ exists if p satisfies $|1/p-1/2| < 1/4+\varepsilon$ with some constant $\varepsilon = \varepsilon(\Omega) \in (0,1/4]$, where it is allowed to take $\varepsilon = 1/4$ if $\partial\Omega \in C^1$.

Keywords: exterior Lipschitz domains; Helmholtz decomposition
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1. Introduction

Let Ω be an *exterior* Lipschitz domain in $\mathbb{R}^2$, i.e., the complement of a bounded planar Lipschitz domain. The aim of this paper is to show the existence of the Helmholtz decomposition of the vector fields in $L_p(\Omega;\mathbb{R}^2)$ provided that p satisfies

$$\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{4}+\varepsilon \tag{1.1}$$

with some constant $\varepsilon = \varepsilon(\Omega) \in (0,1/4]$, where it is allowed to take $\varepsilon = 1/4$ if $\partial\Omega \in C^1$. Let $L_{p,\sigma}(\Omega)$ be the subspace of functions f in $L_p(\Omega;\mathbb{R}^2)$ such that $\int_\Omega f\cdot\nabla\varphi\,\mathrm{d}x = 0$ for every $\varphi \in \dot{H}^1_{p'}(\Omega)$. Notice that $L_{p,\sigma}(\Omega)$ is the closure of $C^\infty_{c,\sigma}(\Omega) := \{\phi \in C^\infty_c(\Omega;\mathbb{R}^2) : \operatorname{div}\phi = 0\}$ in $L_p(\Omega;\mathbb{R}^2)$, see [4, Proposition 2.5]. In addition, let $G_p(\Omega) := \operatorname{Im}\nabla_p = \nabla_p\dot{H}^1_p(\Omega)$ denote the space of gradients in $L_p(\Omega;\mathbb{R}^2)$, where $\nabla_p\colon \dot{H}^1_p(\Omega) \to L_p(\Omega;\mathbb{R}^2)$ is the linear map. We aim to prove the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an exterior Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1/4]$ such that for every p subject to* (1.1) *the Helmholtz decomposition*

$$L_p(\Omega;\mathbb{R}^2) = L_{p,\sigma}(\Omega) \oplus G_p(\Omega) \tag{1.2}$$

holds, where the direct sum is topological. In particular, if $\partial\Omega \in C^1$*, it is allowed to take* $\varepsilon = 1/4$*, i.e., the Helmholtz decomposition* (1.2) *is valid for every* $1 < p < \infty$.

The Helmholtz decomposition (1.2), a useful tool in the study of the incompressible Navier-Stokes equations, is well-known in the case that the boundary $\partial\Omega$ is smooth (see e.g., [9, Theorem 1.6] and [11, Theorem 1.4]). In the case of exterior domains with lower regularity (locally Lipschitz), it was proved by Lang and Méndez [7, Theorem 6.1] as well as Tolksdorf and the author [12, Proposition 2.3] that the Helmholtz decomposition (1.2) holds for all p satisfying $|1/p - 1/2| < 1/6 + \varepsilon$ with some constant $\varepsilon(\Omega) > 0$ provided that $\Omega \subset \mathbb{R}^d$ with $d \geq 3$. However, to the best of the author's knowledge, there seems to be no result on the Helmholtz decomposition (1.2) in the case of exterior planar Lipschitz domains.

It is well-known (cf. [2, Lemma III.1.2]) that the existence of the Helmholtz decomposition (1.2) is equivalent to the unique solvability of the following weak Neumann problem: Given $f \in L_p(\Omega; \mathbb{R}^2)$, consider the weak Neumann problem

$$\langle \nabla u, \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega \quad \text{for every } \varphi \in \dot{H}^1_{p'}(\Omega). \tag{1.3}$$

The present paper aims to prove the following theorem.

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^2$ *be an exterior Lipschitz domain. Then there exists* $\varepsilon = \varepsilon(\Omega) \in (0, 1/4]$ *having the following property: If p satisfies* (1.1)*, for every* $f \in L_p(\Omega)$ *Problem* (1.3) *admits a solution* $u \in \dot{H}^1_p(\Omega)$ *subject to the estimate*

$$\|\nabla u\|_{L_p(\Omega;\mathbb{R}^2)} \leq C\|f\|_{L_p(\Omega;\mathbb{R}^2)}$$

with some positive constant $C > 0$ *which depends only on* Ω *and p. In particular, the solution is unique in* $\dot{H}^1_p(\Omega; \mathbb{R}^2)$ *up to an additive constant. Furthermore, if* $\partial\Omega \in C^1$*, then it is allowed to take* $\varepsilon = 1/4$.

Remark 1.3. Let us make some comments on Theorem 1.2.

(1) The constant ε appearing in Theorem 1.2 arises from analyses of elliptic systems on *bounded* Lipschitz domains (i.e., analyses near the boundary $\partial\Omega$). More precisely, ε is given by $\varepsilon = \varepsilon_{\mathrm{Neu}} \in (0, 1/4]$, where $\varepsilon_{\mathrm{Neu}}$ is a constant arising in the result of the Neumann-Laplacian, see Lemma 3.3 below. This is due to the fact that we will construct a solution to (1.3) via a cut-off procedure so that, roughly speaking, a solution to (1.3) may be given as a sum of the solution to the weak Neumann problem in a bounded Lipschitz domain and the fundamental solution of the Laplace equation. Namely, the restriction (1.1) on p stems essentially from the roughness of the boundary $\partial\Omega$ but the unboundedness of the domain does not restrict the range of p, which is entirely different from the strategy in [7]. A similar observation for the large-time behavior of the three-dimensional Navier-Stokes flow in an exterior Lipschitz domain was made by the author [13].

(2) Concerning the higher-dimensional case, say, $\Omega \subset \mathbb{R}^d$ ($d \geq 3$), that was discovered by Lang and Méndez [7, Theorem 6.1] as well as Tolksdorf and the author [12, Proposition 2.3], in their papers, it was proved that the Helmholtz decomposition (1.2) exists provided that p satisfies $|1/p - 1/2| < 1/6 + \varepsilon$ with some constant $\varepsilon(\Omega) > 0$. Although the present paper is restricted to dealing with the two-dimensional case, it is easy to extend to the higher-dimensional case and obtain the same result as in [7, 12]. Nevertheless, the present paper provides a simpler proof than that of [7]

(notice that the proof in [12] relied on the result obtained in [7]). It should be emphasized that it seems to be difficult to extend the approach of Lang and Méndez [7] to the two-dimensional case since their approach heavily relied on potential theory, which causes several difficulties in the two-dimensional case due to a logarithm singularity of the fundamental solution to the Laplace equation.

This paper is organized as follows: In the next section, we introduce some notation and the function spaces that will be used throughout this paper. Section 3 is concerned with the solvability result of some elliptic systems in bounded domains. Then, in the last section, we give the proof of Theorem 1.2 via a cut-off technique introduced by Shibata [10]. It should be noted that some modifications are required in contrast to Shibata's argument [10] since we may not expect the $H^2_p(\Omega)$-regularity for the solution to elliptic systems due to the lack of the smoothness of the boundary. Note that, for the same reason, we may not use the approaches of Miyakawa [9] as well as Simader and Sohr [11] to prove Theorem 1.2.

2. Notation

As usual, let $\mathbb{N}$ and $\mathbb{R}$ be the set of all natural and real numbers, respectively. For scalar-valued functions u and v defined on $G \subset \mathbb{R}^2$, let $\langle u, v\rangle_G = \int_G u(x)v(x)\,\mathrm{d}x$. For vector-valued functions $U = (U_1, U_2)$ and $V = (V_1, V_2)$ defined on $G \subset \mathbb{R}^2$, let $\langle U, V\rangle_G = \sum_{j=1,2}\int_G U_j(x)V_j(x)\,\mathrm{d}x$. For $R > 0$, let $B_R(0) = \{x \in \mathbb{R}^2 \mid |x| < R\}$. For Banach spaces X and Y, let $\mathcal{L}(X, Y)$ be all bounded linear operators from X into Y, where we will write $\mathcal{L}(X) = \mathcal{L}(X, X)$ to simplify the notation. For a Banach space X, denote by X' the dual space of X. Throughout this paper, the letter C stands for a generic constant that does not depend on the quantities whenever there is no confusion.

Let X be a complex Banach space and let $\mathbb{R}^2$ be endowed with the Lebesgue measure. Let $C^\infty_c(G; X)$ be the set of all C^∞-functions on $\mathbb{R}^2$ whose supports are compact and contained in $D \subset \mathbb{R}^2$. For $1 \le p \le \infty$ and $G \subset \mathbb{R}^2$, let $L_p(G; X)$ be the Lebesgue space equipped with the norm $\|\cdot\|_{L_p(G;X)}$. For $1 \le p < \infty$ and $s \in \mathbb{R}$, let $H^s_p(\mathbb{R}^2; X)$ be the *inhomogeneous* Sobolev space endowed with the norm $\|\cdot\|_{H^s_p(\mathbb{R}^2;X)}$. The inhomogeneous Sobolev space on G is defined by the collection of all $u \in \mathcal{D}'(G; X) = (C^\infty_c(G; X))'$ such that there exists $v \in H^s_p(G; X)$ with $v|_G = u$. Furthermore, the norm $\|\cdot\|_{H^s_p(G;X)}$ is defined by the usual quotient norm:

$$\|u\|_{H^s_p(G;X)} = \inf\|v\|_{H^s_p(\mathbb{R}^2;X)}$$

where the infimum is taken over all $v \in H^s_p(\mathbb{R}^2; X)$ such that its restriction $v|_G$ to G coincides in $\mathcal{D}'(G; X)$ with u. In particular, if $u \in H^s_p(G; X)$ vanishes on the boundary ∂G then the space will be attached with the subscript 0, i.e., $H^s_{p,0}(G; X)$. For $1 \le p < \infty$ and $s \in \mathbb{R}$, let $\dot{H}^s_p(\mathbb{R}^2; X)$ be the *homogeneous* Sobolev space equipped with the norm $\|\cdot\|_{H^s_p(\mathbb{R}^2;X)}$. For $1 \le p < \infty$ and $G \subset \mathbb{R}^2$, we also define

$$\dot{H}^1_p(G) := \{[u] = u + \mathbb{R} : \; u \in L_{p,\mathrm{loc}}(G) \;\text{ and }\; \nabla u \in L_p(G; \mathbb{R}^2)\}$$

with the norm $\|\cdot\|_{\dot{H}^1_p(G)} = \|\nabla\cdot\|_{L_p(G)}$, where $u \in L_{p,\mathrm{loc}}(G)$ means $u \in L_p(G')$ for any bounded domain G' with $G' \subset G$. If $X = \mathbb{R}$, we often write $L_p(G) = L_p(G; \mathbb{R})$, $H^s_p(\mathbb{R}^2) = H^s_p(\mathbb{R}^2; \mathbb{R})$, and $\dot{H}^s_p(\mathbb{R}^2) = \dot{H}^{-1}_p(\mathbb{R}^2; \mathbb{R})$ for short. The Hölder conjugate exponent of p is denoted by p'.

3. Solvability result in bounded domains

In this section, we give the solvability result for elliptic systems in the case of bounded domains. Consider the following elliptic system:

$$\begin{cases} -\Delta u = f & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Here, $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain with $\partial\Omega = \Sigma \cup \Gamma$ with $\Sigma = \partial D \setminus \Gamma$ and $\Sigma \neq \emptyset$. We suppose that there exist some constants $0 < R_1 < R_2$ such that $\Sigma \subsetneq B_{R_1}(0)$ and $\Gamma \subset \mathbb{R}^2 \setminus \overline{B_{R_2}(0)}$. Let

$$L_{p,0}(D) := \{f \in L_p(D) : \langle f, 1\rangle_D = 0\}.$$

Then the result for (3.1) reads as follows.

Theorem 3.1. *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Suppose that there exist some constants $0 < R_1 < R_2$ such that $\Sigma \subsetneq B_{R_1}(0)$ and $\Gamma \subset \mathbb{R}^2 \setminus \overline{B_{R_2}(0)}$. Then there exists $\varepsilon = \varepsilon(D) \in (0, 1/4]$ having the following property: If p satisfies* (1.1)*, for every $s \in [0, 1/p)$ and for every $f \in L_{p,0}(D)$ Problem* (3.1) *admits a solution $u \in H^{1+s}_p(D)$ subject to the estimate*

$$\|u\|_{H^{1+s}_p(D)} \leq C\|f\|_{L_p(D)}$$

with some positive constant $C > 0$ which depends only on D, p, and s. Furthermore, if $\partial D \in C^1$, then it is allowed to take $\varepsilon = 1/4$.

To prove this theorem, we define the weak Dirichlet-Laplacian $\Delta^D_{p,s,w}$ on $L_p(D)$ as

$$\begin{aligned} \mathsf{D}(\Delta^D_{p,s,w}) &= \{u \in H^{1+s}_{p,0}(D) : \Delta u \in L_p(D)\}, \\ \Delta^D_{p,s,w} u &= \Delta u \end{aligned} \tag{3.2}$$

with $0 \leq s < 1/p$. Here, $\Delta u \in L_p(D)$ is understood in the sense of distributions. We also define the weak Neumann-Laplacian $\Delta^N_{p,s,w}$ on $L_p(D)$ as

$$\begin{aligned} \mathsf{D}(\Delta^N_{p,s,w}) &= \left\{u \in H^{1+s}_p(D) : \exists v \in L_p(D) \text{ s.t. } \forall \varphi \in H^{1+s}_{p',0}(D) : \langle \nabla u, \nabla \varphi\rangle_D = \langle v, \varphi\rangle_D\right\}, \\ \Delta^N_{p,s,w} u &= v \end{aligned} \tag{3.3}$$

with $0 \leq s < 1/p$. Notice that the Neumann boundary condition $\partial u/\partial \nu = 0$ on ∂D is interpreted in the sense that

$$\langle \Delta u, \varphi\rangle_D = -\langle \nabla u, \nabla \varphi\rangle_D \quad \text{for any } \varphi \in H^{1+s}_{p'}(D).$$

Mimicking the argument as in Sections 3–5 in [15], we may prove the following results.

Lemma 3.2. *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon_{\mathrm{Dir}} = \varepsilon_{\mathrm{Dir}}(D) \in (0, 1/4]$ having the following property: If p satisfies* (1.1)*, for every $s \in [0, 1/p)$ the operator $\Delta^D_{p,s,w}$ defined by* (3.2) *generates a C_0-semigroup of contractions on $L_p(D)$. Furthermore, if $\partial D \in C^1$, then it is allowed to take $\varepsilon_{\mathrm{Dir}} = 1/4$.*

Lemma 3.3. *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon_{\mathrm{Neu}} = \varepsilon_{\mathrm{Neu}}(D) \in (0, 1/4]$ having the following property: If p satisfies* (1.1)*, for every $s \in [0, 1/p)$ the operator $\Delta^N_{p,s,w}$ defined by* (3.3) *generates a C_0-semigroup of contractions on $L_p(D)$. Furthermore, if $\partial D \in C^1$, then it is allowed to take $\varepsilon_{\mathrm{Neu}} = 1/4$.*

Remark 3.4. Let us make a few comments on Lemmas 3.2 and 3.3.

(1) If $s = 0$, Lemma 3.2 was obtained by Wood [15, Proposition 4.1]. However, it is easy to extend its result to the case of $0 < s < 1/p$ due to [6, Theorem 1.3].

(2) In the case of $D \subset \mathbb{R}^d$ with $d \geq 3$, Lemma 3.3 with $s = 0$ was proved by Wood [15, Theorem 5.6]. It is not difficult to extend the result of [15, Theorem 5.6] to the case of $D \subset \mathbb{R}^2$ if one replaces [15, Theorem 2.6] by [8, Corollary 4.2].

For every $\lambda > 0$, consider the resolvent problem for the Laplacian with Dirichlet boundary condition

$$\begin{cases} \lambda u - \Delta u = f & \text{in } D, \\ \qquad\quad u = 0 & \text{on } \partial D \end{cases} \tag{3.4}$$

as well as the resolvent problem for the Laplacian with Neumann boundary condition

$$\begin{cases} \lambda u - \Delta u = f & \text{in } D, \\ \qquad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \tag{3.5}$$

By the Hille-Yosida theorem, we infer from Lemma 3.2 that for every $\lambda > 0$, there exists a unique solution u satisfying

$$\lambda \|u\|_{L_p(D)} \leq \|f\|_{L_p(D)}.$$

Let p and s be the same numbers as in Lemma 3.2. From the invertibility results in [8, Corollary 4.2], we have

$$\begin{aligned} \|u\|_{H^{1+s}_p(D)} &\leq C\|-\lambda u + f\|_{H^{-1+s}_p(D)} \\ &\leq C\|-\lambda u + f\|_{L_p(D)} \\ &\leq C\Big(\lambda\|u\|_{L_p(D)} + \|f\|_{L_p(D)}\Big) \\ &\leq C\|f\|_{L_p(D)} \end{aligned}$$

since $H^{-1+s}_p(D) \hookleftarrow L_p(D)$ due to $-1 + s < 0$. Hence, the solution $u \in H^{1+s}_{p,0}(D)$ to (3.4) verifies

$$\lambda\|u\|_{L_p(D)} + \|u\|_{H^{1+s}_p(D)} \leq C\|f\|_{L_p(D)} \quad \text{for all } \lambda > 0$$

provided that $f \in L_p(D)$. Likewise, we infer from Lemma 3.3 and the invertibility results in [8, Corollary 4.2] that there exists a unique solution $u \in H^{1+s}_p(D)$ satisfying

$$\lambda\|u\|_{L_p(D)} + \|u\|_{H^{1+s}_p(D)} \leq C\|f\|_{L_p(D)} \quad \text{for all } \lambda > 0$$

provided that $f \in L_{p,0}(D)$, where p and s are the same numbers as in Lemma 3.3. With the aforementioned preliminaries, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\varepsilon \in (0, 1/4]$ be $\varepsilon = \min\{\varepsilon_{\mathrm{Dir}}, \varepsilon_{\mathrm{Neu}}\}$, where $\varepsilon_{\mathrm{Dir}}$ and $\varepsilon_{\mathrm{Neu}}$ are the same numbers as in Lemmas 3.2 and 3.3, respectively. In addition, let p and s satisfy (1.1) and $0 \leq s < 1/p$, respectively. Assume that $f \in L_{p,0}(D)$. Then, for every $\lambda > 0$, there exists a unique solution $u_{\mathrm{Dir}} \in H^{1+s}_{p,0}(D)$ to

$$\begin{cases} \lambda u_{\mathrm{Dir}} - \Delta u_{\mathrm{Dir}} = f & \text{in } D, \\ u_{\mathrm{Dir}} = 0 & \text{on } \Sigma \cup \Gamma. \end{cases}$$

In addition, u satisfies the estimate

$$\lambda \|u_{\mathrm{Dir}}\|_{L_p(D)} + \|u_{\mathrm{Dir}}\|_{H^{1+s}_p(D)} \leq C\|f\|_{L_p(D)} \quad \text{for all } \lambda > 0. \tag{3.6}$$

By complex interpolation, we observe that for every $t \in (0, s)$, there holds

$$\|u_{\mathrm{Dir}}\|_{H^{1+t}_p(D)} \leq C\lambda^{\frac{s-t}{1+s}}\|f\|_{L_p(D)} \quad \text{for all } \lambda > 0.$$

Similarly, for every $\lambda > 0$, there exists a unique solution $u_{\mathrm{Neu}} \in H^{1+s}_p(D)$ to

$$\begin{cases} \lambda u_{\mathrm{Neu}} - \Delta u_{\mathrm{Neu}} = f & \text{in } D, \\ \dfrac{\partial u_{\mathrm{Neu}}}{\partial \nu} = 0 & \text{on } \Sigma \cup \Gamma. \end{cases}$$

Moreover, u satisfies the estimate

$$\lambda \|u_{\mathrm{Neu}}\|_{L_p(D)} + \|u_{\mathrm{Neu}}\|_{H^{1+s}_p(D)} \leq C\|f\|_{L_p(D)} \quad \text{for all } \lambda > 0.$$

By complex interpolation, we observe that for every $t \in (0, s)$, there holds

$$\|u_{\mathrm{Neu}}\|_{H^{1+t}_p(D)} \leq C\lambda^{\frac{s-t}{1+s}}\|f\|_{L_p(D)} \quad \text{for all } \lambda > 0. \tag{3.7}$$

Let $\zeta \in C^\infty_c(D; [0, 1])$ be a cut-off function such that $\zeta = 1$ if $|x| \leq R_1 + \epsilon$ and $\zeta = 0$ if $|x| \geq R_2 - \epsilon$, where $\epsilon := (R_1 + R_2)/3$. Set $v = \zeta u_{\mathrm{Dir}} + (1 - \zeta)u_{\mathrm{Neu}}$. We see that v solves

$$\begin{cases} \lambda v - \Delta v = f + \mathcal{R}_0 f & \text{in } D, \\ \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \Sigma, \\ v = 0 & \text{on } \Gamma, \end{cases}$$

where we have set

$$\mathcal{R}_0 f := \Delta\zeta(u_{\mathrm{Neu}} - u_{\mathrm{Dir}}) + 2\nabla\zeta \cdot \nabla(u_{\mathrm{Neu}} - u_{\mathrm{Dir}}).$$

From (3.6) and (3.7), there exists $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0$, we have

$$\|\mathcal{R}_0 f\|_{L_p(D)} \leq \frac{1}{2}\|f\|_{L_p(D)},$$

which together with a Neumann series argument implies that the operator $I + \mathcal{R}_0 \colon L_{p,0}(D) \to L_p(D)$ is invertible. Thus, we see that

$$\begin{cases} \lambda u - \Delta u = f & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{3.8}$$

is solvable for all $\lambda \geq \lambda_0$. Notice that the uniqueness of the solution follows from the duality argument. Indeed, suppose that $u \in H^{1+s}_p(D)$ solves (3.8) with f vanishing in D. For any $\lambda \geq \lambda_0$ and $\phi \in C^\infty_c(D)$, consider

$$\begin{cases} \lambda u_0 - \Delta u_0 = \phi & \text{in } D, \\ \dfrac{\partial u_0}{\partial \nu} = 0 & \text{on } \Sigma, \\ u_0 = 0 & \text{on } \Gamma. \end{cases}$$

Then we infer from the divergence theorem that

$$\langle u, \phi \rangle_D = \langle u, \lambda u_0 - \Delta u_0 \rangle_D = \langle \lambda u - \Delta u, u_0 \rangle_D = 0.$$

Since $\phi \in C^\infty_c(D)$ is arbitrary, we deduce that $u = 0$ in D, which gives the uniqueness assertion. In the following, let $\mathcal{A}_{p,s}$ be the operator defined by

$$\mathsf{D}(\mathcal{A}_{p,s}) = \left\{ u \in H^{1+s}_p(D) : \ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma \quad \text{and} \quad u = 0 \quad \text{on } \Gamma \right\}$$
$$\mathcal{A}_{p,s} u = \Delta u$$

with $0 \leq s < 1/p$. From the aforementioned argument, we see that the resolvent set $\rho(\mathcal{A}_{p,s})$ of $\mathcal{A}_{p,s}$ contains $[\lambda_0, \infty)$.

We next deal with the case $0 \leq \lambda < \lambda_0$. Since (3.8) may be written as $(\lambda I - \mathcal{A}_{p,s})u = f$, we find that (3.8) and

$$\Big(I + (\lambda - 2\lambda_0) R_{p,s}(\lambda_0) \Big)(2\lambda_0 I - \mathcal{A}_{p,s})u = f, \qquad R_{p,s}(\lambda_0) := (2\lambda_0 I - \mathcal{A}_{p,s})^{-1}$$

are equivalent. Hence, to prove that $[0, \lambda_0)$ is contained in $\rho(\mathcal{A}_{p,s})$, it suffices to show the invertibility of the operator $I + (\lambda - 2\lambda_0) R_{p,s}(\lambda_0)$. Since it follows from the Rellich-Kondrachov theorem (cf. [1, Theorem 6.3]) that $R_{p,s}(\lambda_0)$ is a compact operator from $H^{1+s}_p(D)$ into $L_p(D)$, we observe that $[0, \lambda_0) \subset \rho(\mathcal{A}_{p,s})$ follows from the Fredholm alternative theorem and the injection of $I + (\lambda - 2\lambda_0) R_{p,s}(\lambda_0)$. To see this, for any $\lambda \in [0, \lambda_0)$, take $u_1 \in \mathrm{Ker}\,(I + (\lambda - 2\lambda_0) R_{p,s}(\lambda_0))$, i.e.,

$$\Big(I + (\lambda - 2\lambda_0) R_{p,s}(\lambda_0) \Big) u_1 = 0 \quad \text{for any } u_1 \in L_{p,0}(D).$$

From the definition of $R_{p,s}(\lambda_0)$, we see that $u_1 \in \mathsf{D}(\mathcal{A}_{p,s})$ and w solves

$$\begin{cases} \lambda u_1 - \Delta u_1 = 0 & \text{in } D, \\ \dfrac{\partial u_1}{\partial \nu} = 0 & \text{on } \Sigma, \\ u_1 = 0 & \text{on } \Gamma. \end{cases} \tag{3.9}$$

We first deal with the case $2 \leq p < \infty$. In this case, we may assume $u_1 \in \mathsf{D}(\mathcal{A}_{2,s})$ due to the boundedness of D. Using the divergence theorem, we deduce from (3.9) that

$$0 = \lambda \|u_1\|^2_{L_2(D)} + \frac{1}{2} \|D(u_1)\|^2_{L_2(D)},$$

where we have set $D(u_1) := \nabla u_1 + [\nabla u_1]^\top$. By the Korn inequality (cf. [14, Theorem A.4]), we obtain

$$\lambda \|u_1\|_{L_2(D)}^2 + \frac{1}{C}\|\nabla u_1\|_{L_2(D)}^2 \leq 0$$

with some constant $C > 0$ depending only on D. Since $\lambda \in [0, \lambda_0)$, we see that $u_1 = 0$ in D due to the Dirichlet boundary condition $u_1 = 0$ on Γ. This completes the proof of $[0, \lambda_0) \subset \rho(\mathcal{A}_{p,s})$ in the case that p satisfies $2 \leq p < \infty$. The remaining case $1 < p < 2$ follows from the duality [5, Corollary A.4.3]. □

4. Proof of the main results

Without loss of generality, we may assume that the origin of the coordinates is located interior to the complement of the domain Ω. Then, we may take $R_0 > 0$ so large that there holds $\mathbb{R}^2 \setminus \Omega \subset B_R(0)$ for any $R > R_0$. Set $D_R := \Omega \cap B_{8R}(0)$ and $A_R := \{x \in \mathbb{R}^2 \mid 2R \leq |x| \leq 7R\}$.

Step 1: Construction of $\mathcal{U}_1$

Following the argument of Shibata [10, Section 3], we introduce cut-off functions as follows. Let $\psi_0, \chi_0 \in C_c^\infty(\mathbb{R}^2; [0,1])$ be the cut-off functions satisfying

$$\psi_0(x) = \begin{cases} 1 & \text{for } |x| \leq 4R, \\ 0 & \text{for } |x| \geq 5R, \end{cases} \qquad \chi_0(x) = \begin{cases} 1 & \text{for } |x| \leq 6R, \\ 0 & \text{for } |x| \geq 7R, \end{cases}$$

respectively. In addition, let $\psi_\infty, \chi_\infty \in C^\infty(\mathbb{R}^2; [0,1])$ be smooth functions defined by

$$\psi_\infty = 1 - \psi_0(x), \qquad \chi_\infty(x) = \begin{cases} \chi_0(9R - |x|) & \text{for } |x| \leq 9R, \\ 1 & \text{for } |x| > 9R, \end{cases}$$

respectively. Clearly, there holds $\psi_\infty(x) = 0$ for $|x| \leq 4R$ and $\psi_\infty(x) = 1$ for $|x| \geq 5R$, while $\chi_\infty(x) = 0$ for $|x| \leq 2R$ and $\chi_\infty(x) = 1$ for $|x| \geq 3R$. Let $\mathcal{U}_0$ be the inverse of $-\mathcal{A}_{p,s}$ and $\mathcal{U}_\infty$ be the solution operator to

$$\langle \nabla u, \nabla \varphi \rangle_{\mathbb{R}^2} = \langle F, \nabla \varphi \rangle_{\mathbb{R}^2} \qquad \text{for every } \varphi \in \dot{H}^1_{p'}(\mathbb{R}^2), \tag{4.1}$$

i.e., the solution u to this equation may be written as $u = \mathcal{U}_\infty(F)$, where F has been assumed to be $F \in L_p(\mathbb{R}^2; \mathbb{R}^2)$. Since the solution u to (4.1) is unique up to an additive constant, there is no loss of generality in assuming $\langle \mathcal{U}_\infty(F), 1 \rangle_D = 0$. Notice that the Fourier multiplier theorem and the Poincaré-Wirtinger inequality yield

$$\|\nabla \mathcal{U}_\infty(F)\|_{L_p(\mathbb{R}^2;\mathbb{R}^2)} + \|\mathcal{U}_\infty(F)\|_{L_p(G)} \leq C\|F\|_{L_p(\mathbb{R}^2;\mathbb{R}^2)}$$

for every bounded Lipschitz domain $G \subset \mathbb{R}^2$. Given $f \in L_p(\Omega; \mathbb{R}^2)$, define the linear operator $\mathcal{U}_1$ by

$$\mathcal{U}_1(f) = \chi_0 \mathcal{U}_0(\psi_0 f) + (1 - \chi_0)\mathcal{U}_\infty(\psi_\infty f).$$

Here, $\psi_\infty f$ may be regarded as a function defined on $\mathbb{R}^2$ since $\psi_\infty f$ vanishes in $B_R(0)$. In the following, for all smooth functions h defined on $\mathbb{R}^2$, the term $h\mathcal{U}_0(\psi_0 f)$ is regarded as a function that is extended by zero to all of $\mathbb{R}^2$. Then we see that $\mathcal{U}_1(f)$ satisfies

$$\langle \nabla \mathcal{U}_1(f), \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega + \langle \mathcal{R}_1(f), \varphi \rangle_\Omega \quad \text{for every } \varphi \in \dot{H}^1_{p'}(\Omega)$$

with

$$\mathscr{R}_1(f) := -2(\nabla\chi_0)\cdot\nabla(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)) - (\Delta\chi_0)(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)).$$

Here, we have used the identity

$$\langle \psi_0 f, \nabla(\chi_0\varphi)\rangle_\Omega + \langle \psi_\infty f, \nabla(\chi_\infty\varphi)\rangle_\Omega = \langle(\psi_0+\psi_\infty)f, \nabla\varphi\rangle_\Omega = \langle f, \nabla\varphi\rangle_\Omega,$$

which follows from $\psi_0 + \psi_\infty = 1$ and the fact that $\chi_0 = 1$ on $\mathrm{supp}(\psi_0)$ and $\chi_\infty = 1$ on $\mathrm{supp}(\psi_\infty)$. Since we may write

$$\mathscr{R}_1(f) = -(\nabla\chi_0)\cdot\nabla(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)) - \mathrm{div}\Big((\nabla\chi_0)(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f))\Big),$$

we have

$$\langle \mathscr{R}_1(f), \varphi\rangle_{\mathbb{R}^2} = \langle \mathcal{R}_1(f), \nabla\varphi\rangle_{\mathbb{R}^2} \qquad \text{for every } \varphi \in \dot{H}^1_{p'}(\mathbb{R}^2) \tag{4.2}$$

with

$$\mathcal{R}_1(f) := -\psi_0 f + (\nabla\chi_0)\mathcal{U}_0(\psi_0 f) + \chi_0\nabla\mathcal{U}_0(\psi_0 f) - (\nabla\chi_0)\mathcal{U}_\infty(\psi_\infty f) - \chi_0\nabla\mathcal{U}_\infty(\psi_\infty f). \tag{4.3}$$

In fact, it follows that

$$\begin{aligned}\langle \mathscr{R}_1(f), \varphi\rangle_{\mathbb{R}^2} &= -\langle \nabla(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)), (\nabla\chi_0)\varphi\rangle_{\mathbb{R}^2} + \langle (\nabla\chi_0)(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)), \nabla\varphi\rangle_{D_R} \\ &= -\langle \nabla\mathcal{U}_0(\psi_0 f), \nabla(\chi_0\varphi)\rangle_{D_R} + \langle \nabla\mathcal{U}_\infty(\psi_\infty f), \nabla(\chi_0\varphi)\rangle_{\mathbb{R}^2} \\ &\quad + \langle \chi_0\nabla(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_0 f)), \nabla\varphi\rangle_{D_R} + \langle (\nabla\chi_0)(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)), \nabla\varphi\rangle_{D_R}\end{aligned}$$

for any $\varphi \in \dot{H}^1_{p'}(\mathbb{R}^2)$. Here, we have used the identity

$$-\Big\langle \mathrm{div}\Big((\nabla\chi_0)(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f))\Big), \varphi\Big\rangle_{\mathbb{R}^2} = \langle (\nabla\chi_0)(\mathcal{U}_0(\psi_0 f) - \mathcal{U}_\infty(\psi_\infty f)), \nabla\varphi\rangle_{D_R},$$

which may be justified since $\mathrm{supp}(\nabla\chi_0)$ is contained in an annulus (i.e., a bounded domain). Since $\chi_0 = 1$ on $\mathrm{supp}(\psi_0)$ and $\chi_0 = 0$ on $\mathrm{supp}(\psi_\infty)$, we deduce that

$$\langle \nabla\mathcal{U}_0(\psi_0 f), \nabla(\chi_0\varphi)\rangle_{D_R} = \langle \psi_0 f, \nabla\varphi\rangle_\Omega, \qquad \langle \nabla\mathcal{U}_\infty(\psi_\infty f), \nabla(\chi_0\varphi)\rangle_{\mathbb{R}^2} = 0,$$

which yields the representation (4.3). Clearly, for any $f \in L_p(\Omega;\mathbb{R}^2)$, we have

$$\|\mathscr{R}_1(f)\|_{L_p(\mathbb{R}^2)} \le C\|f\|_{L_p(\Omega;\mathbb{R}^2)}, \qquad \|\mathcal{R}_1(f)\|_{L_p(\mathbb{R}^2;\mathbb{R}^2)} \le C\|f\|_{L_p(\Omega;\mathbb{R}^2)}.$$

We also infer from (4.2) that

$$\|\mathscr{R}_1(f)\|_{\dot{H}^{-1}_p(\mathbb{R}^2)} \le C\|f\|_{L_p(\Omega;\mathbb{R}^2)}.$$

In addition, we infer from $1 \in \dot{H}^1_p(\mathbb{R}^2)$ and (4.2) that $\langle \mathscr{R}_1(f), 1\rangle_{\mathbb{R}^2} = \langle \mathcal{R}_1(f), 0\rangle_{\mathbb{R}^2} = 0$.

For later, we introduce the function space

$$\mathcal{H}_p(\mathbb{R}^2) = \Big\{g \in L_p(\mathbb{R}^2) \cap \dot{H}^{-1}_p(\mathbb{R}^2) : \ \mathrm{supp}(g) \subset A_R, \ \langle g, 1\rangle_{\mathbb{R}^2} = 0\Big\}.$$

Clearly, from the aforementioned argument, we see that $\mathscr{R}_1(f) \in \mathcal{H}_p(\mathbb{R}^2)$ for any $f \in L_p(\Omega;\mathbb{R}^2)$.

Step 2: Construction of $\mathcal{U}_2$

Given $g \in \mathcal{H}_p(\mathbb{R}^2)$, we intend to construct the solution operator to

$$\langle \nabla u_2, \nabla \varphi \rangle_\Omega = \langle g, \varphi \rangle_\Omega \quad \text{for every } \varphi \in \dot{H}^1_{p'}(\Omega) \tag{4.4}$$

possessing the estimate

$$\|\nabla u_2\|_{L_p(\Omega;\mathbb{R}^2)} \le C\|g\|_{L_p(\mathbb{R}^2)}. \tag{4.5}$$

Notice that, by using this operator $\mathcal{U}_2$, it follows that $\mathcal{U}_1(f) - \mathcal{U}_2(\mathscr{R}_1(f))$ solves (1.3).

In the following, let $g \in \mathcal{H}_p(\mathbb{R}^2)$. Let $\mathcal{V}_\infty$ be the operator defined by

$$(\mathcal{V}_\infty f)(x) = -\int_{\mathbb{R}^2} \mathcal{E}(x-y)g(y)\,\mathrm{d}y,$$

where $\mathcal{E}(x-y)$ stands for the fundamental solution of the Laplace equation:

$$\mathcal{E}(x-y) = (2\pi)^{-1}\log|x-y|.$$

By [10, (82)], the formula

$$\langle \nabla \mathcal{V}_\infty(g), \nabla \varphi \rangle_{\mathbb{R}^2} = \langle g, \varphi \rangle_{\mathbb{R}^2} \quad \text{for every } \varphi \in \dot{H}^1_{p'}(\mathbb{R}^2)$$

may be justified. Furthermore, for every $g \in \mathcal{H}_p(\mathbb{R}^2)$, we have

$$\begin{aligned} \|\mathcal{V}_\infty(g)\|_{L_p(B_{9R}(0))} + \sup_{|x|\ge 9R}|x||\mathcal{V}_\infty(g)| &\le C\|g\|_{L_p(\mathbb{R}^2)}, \\ \|\nabla\mathcal{V}_\infty(g)\|_{L_p(\mathbb{R}^2;\mathbb{R}^2)} + \sup_{|x|\ge 9R}|x|^2|\mathcal{V}_\infty(g)| &\le C\|g\|_{L_p(\mathbb{R}^2)}, \\ \|\nabla^2\mathcal{V}_\infty(g)\|_{L_p(\mathbb{R}^2;\mathbb{R}^4)} &\le C\|g\|_{L_p(\mathbb{R}^2)}, \end{aligned} \tag{4.6}$$

see [10, (80)].

We next consider the following elliptic problem:

$$\begin{cases} -\Delta u_3 = g|_{D_R} & \text{in } D_R, \\ \dfrac{\partial u_3}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u_3 = 0 & \text{on } \partial B_{8R}(0). \end{cases} \tag{4.7}$$

By Theorem 3.1, we know that (4.7) admits a unique solution $u_3 \in H^{1+s}_p(D_R)$ provided that p satisfies (1.1) with some $\varepsilon \in (0, 1/4]$ and s satisfies $0 \le s < 1/p$. Notice that we may take $\varepsilon = 1/4$ if $\partial\Omega \in C^1$. Denote by $\mathcal{V}_0$ the solution operator to (4.7), i.e., $u_3 = \mathcal{V}_0(g)$. From Theorem 3.1, we see that $\mathcal{V}_0(g)$ solves

$$\langle \nabla \mathcal{V}_0(g), \nabla\varphi \rangle_{D_R} = \langle g, \varphi \rangle_{D_R} \quad \text{for every } \varphi \in \dot{H}^1_{p'}(D_R)$$

possessing the estimate

$$\|\mathcal{V}_0(g)\|_{H^{1+s}_p(D_R)} \le C\|f\|_{L_p(D_R)}. \tag{4.8}$$

Let $\mathcal{V}_{\infty,0}$ be $\mathcal{V}_{\infty,0}(g) = \mathcal{V}_\infty(g) + c_g$ with a constant c_g such that

$$\int_{6R\le|x|\le 7R} (\mathcal{V}_\infty(g) + c_g)\,\mathrm{d}x = 0.$$

Furthermore, $\mathcal{V}_{\infty,0}(g)$ verifies

$$\|\mathcal{V}_{\infty,0}(g)\|_{L_p(B_{9R}(0))} \le C_R\|g\|_{L_p(\mathbb{R}^2;\mathbb{R}^2)}.$$

For every $g \in \mathcal{H}_p(\mathbb{R}^2)$, let $\mathcal{U}_2(f) = \psi_0\mathcal{V}_0(g) + \psi_\infty\mathcal{V}_{\infty,0}(g)$. As noted before, for all smooth functions h defined on $\mathbb{R}^2$, the term $h\mathcal{V}_0(g)$ is regarded as a function that is extended by zero to all of $\mathbb{R}^2$. Clearly, there holds

$$\int_{6R\le|x|\le 7R} \mathcal{U}_2(g)\,\mathrm{d}x = 0 \tag{4.9}$$

since there holds $\mathcal{U}_2(g) = \mathcal{V}_{\infty,0}(g)$ for $6R \le |x| \le 7R$. Note that it follows from (4.6) and (4.8) that

$$\|\mathcal{U}_2(g)\|_{L_p(D_R)} + \|\nabla\mathcal{U}_2(g)\|_{H^s_p(\Omega)} \le C\|g\|_{L_p(\Omega)}. \tag{4.10}$$

We also see that $\mathcal{U}_2(g)$ satisfies

$$\langle\nabla\mathcal{U}_2(g), \nabla\varphi\rangle_\Omega = \langle g + \mathscr{R}_2(g), \varphi\rangle_\Omega \quad \text{for every } \varphi \in \dot{H}^1_{p'}(\Omega) \tag{4.11}$$

with

$$\mathscr{R}_2(g) := -2(\nabla\psi_0)\cdot\nabla(\mathcal{V}_0(g) - \mathcal{V}_{\infty,0}(g)) - (\Delta\psi_0)(\mathcal{V}_0(g) - \mathcal{V}_{\infty,0}(g)). \tag{4.12}$$

Similarly to (4.2), we set

$$\mathcal{R}_2(g) := \nabla(\psi_\infty\mathcal{V}_0(g)) + \nabla(\psi_0\mathcal{V}_{\infty,0}(g)), \tag{4.13}$$

so that there holds

$$\langle\mathscr{R}_2(g), \varphi\rangle_{\mathbb{R}^2} = \langle g, \varphi\rangle_{\mathbb{R}^2} - \langle\mathcal{R}_2(g), \nabla\varphi\rangle_{\mathbb{R}^2} \qquad \text{for every } \varphi \in \dot{H}^1_{p'}(\mathbb{R}^2). \tag{4.14}$$

Indeed, by

$$\begin{aligned}\mathscr{R}_2(g) &= 2(\nabla\psi_\infty)\cdot\nabla\mathcal{V}_0(g) + (\Delta\psi_\infty)\mathcal{V}_0(g) + 2(\nabla\psi_0)\cdot\nabla\mathcal{V}_{\infty,0}(g) + (\Delta\psi_0)\mathcal{V}_{\infty,0}(g)\\ &= \operatorname{div}\Big((\nabla\psi_\infty)\mathcal{V}_0(g) + \psi_\infty\nabla\mathcal{V}_0(g) + (\nabla\psi_0)\mathcal{V}_{\infty,0}(g) + \psi_0\nabla\mathcal{V}_{\infty,0}(g)\Big)\\ &\quad - \psi_\infty\Delta\mathcal{V}_0(g) - \psi_0\Delta\mathcal{V}_{\infty,0}(g),\end{aligned}$$

we have

$$\begin{aligned}\langle\mathscr{R}_2(g), \varphi\rangle_{\mathbb{R}^2} = \langle g, \varphi\rangle_{\mathbb{R}^2} &- \Big\langle(\nabla\psi_\infty)\mathcal{V}_0(g) + \psi_\infty\nabla\mathcal{V}_0(g), \nabla\varphi\Big\rangle_{D_R}\\ &+ \Big\langle(\nabla\psi_0)\mathcal{V}_{\infty,0}(g) + \psi_0\nabla\mathcal{V}_{\infty,0}(g), \nabla\varphi\Big\rangle_{B_{8R}(0)},\end{aligned}$$

which yields the representation of $\mathcal{R}_2(g)$. Since $1 \in \dot{H}^1_{p'}(\mathbb{R}^2)$ and $g \in \mathcal{H}_p(\mathbb{R}^2)$, the relation (4.14) gives $\langle\mathscr{R}_2(g), 1\rangle_{\mathbb{R}^2} = 0$. In addition, we also deduce from (4.13) and (4.14) that there holds $\mathscr{R}_2(g) \in \mathcal{H}_p(\mathbb{R}^2)$. Thus, to construct the solution operator to (4.4), it remains to verify the invertibility of $I + \mathscr{R}_2$ on $\mathcal{H}_p(\mathbb{R}^2)$.

Step 3: Invertibility of $I + \mathscr{R}_2$

To show the invertibility of $I+\mathscr{R}_2$ on $\mathcal{H}_p(\mathbb{R}^2)$, we first note that $\mathscr{R}_2$ is a compact operator on $\mathcal{H}_p(\mathbb{R}^2)$. In fact, we see that $\operatorname{supp}(\mathscr{R}_2(g)) \subset A_R$ and

$$\|\mathscr{R}_2(g)\|_{H^s_p(\mathbb{R}^2)} \leq C\|g\|_{L_p(\mathbb{R}^2)},$$

where the Kato-Ponce type inequality has been applied (cf. [3, Theorem 1]). Let $(g_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}_p(\mathbb{R}^2)$. Then we infer from the Rellich-Kondrachov theorem (cf. [1, Theorem 6.3]) that $H^s_p(\mathbb{R}^2)$ is compactly embedded into $L_p(A_R)$, and thus the operator $\mathscr{R}_2$ may be regarded as a compact operator from $\mathcal{H}_p(\mathbb{R}^2)$ into $L_p(A_R)$. Namely, there exists a subsequence $(\mathscr{R}_2(g_{j(k)}))_{k\in\mathbb{N}} \subset (\mathscr{R}_2(g_j))_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\|\mathscr{R}_2(g_{j(k)}) - R_g\|_{L_p(A_R)} = 0 \tag{4.15}$$

with some $R_g \in L_p(A_R)$. Let $\widetilde{R}_g$ be the zero extension of R_g to $\mathbb{R}^2$. Since $\operatorname{supp}(\mathscr{R}_2(g_{j(k)})) \subset A_R$, it follows from the Poincaré-Wirtinger inequality that

$$\begin{aligned} |\langle\mathscr{R}_2(g_{j(k)}) - \widetilde{R}_g, \varphi\rangle_{\mathbb{R}^2}| &\leq \|\mathscr{R}_2(g_{j(k)}) - R_g\|_{L_p(A_R)} \left\|\varphi - |A_R|^{-1}\int_{A_R}\varphi\,\mathrm{d}x\right\|_{L_{p'}(A_R)} \\ &\leq \|\mathscr{R}_2(g_{j(k)}) - R_g\|_{L_p(A_R)}\|\nabla\varphi\|_{L_{p'}(\mathbb{R}^2;\mathbb{R}^2)} \end{aligned} \tag{4.16}$$

for every $\varphi \in \dot{H}^1_{p'}(\mathbb{R}^2)$. Together with (4.15), we deduce that

$$\lim_{k\to\infty}\|\mathscr{R}_2(g_{j(k)}) - \widetilde{R}_g\|_{L_p(\mathbb{R}^2)\cap\dot{H}^{-1}_p(\mathbb{R}^2)} = 0.$$

Notice that, similarly to (4.16), we have $\widetilde{R}_g \in \dot{H}^{-1}_p(\mathbb{R}^2)$. In addition, there holds

$$\langle\widetilde{R}_g, 1\rangle_{\mathbb{R}^2} = \langle R_g, 1\rangle_{A_R} = \lim_{k\to\infty}\langle\mathscr{R}_2(g_{j(k)}), 1\rangle_{A_R} = 0$$

due to the construction of $\mathscr{R}_2$. Since $\operatorname{supp}(\widetilde{R}_g) \subset A_R$, we observe $\widetilde{R}_g \in \mathcal{H}_p(\mathbb{R}^2)$. Therefore, the operator $\mathscr{R}_2$ is a compact operator from $\mathcal{H}_p(\mathbb{R}^2)$ into itself.

We next verify the following lemma.

Lemma 4.1. *Assume that $\Omega \subset \mathbb{R}^2$ is an exterior Lipschitz domain and p satisfies* (1.1) *with some $0 < \varepsilon \leq 1/4$ as well as $p \geq 2$. Let $\mathcal{H}_p(\mathbb{R}^2)$, $\mathscr{R}_2$, and $\mathcal{U}_2$ be as above. If $g \in \mathcal{H}_p(\mathbb{R}^2)$ satisfies $(I + \mathscr{R}_2)g = 0$, then $\mathcal{U}_2(g) = 0$ in Ω.*

Proof. Let $\omega \in C^\infty_c(\mathbb{R}^2; [0, 1])$ be a function such that $\omega(x) = 1$ for $|x| \leq 1$ and $\omega(x) = 0$ for $|x| \geq 2$. Set $\omega_L(x) = \omega(x/L)$. By (4.10), we have $\mathcal{U}_2(g) \in H^{1+s}_{p,\mathrm{loc}}(\Omega)$, which implies $\omega_L\mathcal{U}_2(g) \in \dot{H}^1_p(\Omega)$. Here, $u \in H^{1+s}_{p,\mathrm{loc}}(\Omega)$ means $u \in H^{1+s}_p(\Omega')$ for any bounded domain Ω' with $\Omega' \subset \Omega$. Hence, the formula (4.11) together with $(I + \mathscr{R}_2)g = 0$ yields

$$\begin{aligned} 0 &= \langle(I + \mathscr{R}_2)g, \omega_L\mathcal{U}_2(g)\rangle_\Omega \\ &= \langle\nabla\mathcal{U}_2(g), \nabla(\omega_L\mathcal{U}_2(g))\rangle_\Omega \\ &= \langle\omega_L\nabla\mathcal{U}_2(g), \nabla\mathcal{U}_2(g)\rangle_\Omega + \langle(\nabla\omega_L)\cdot\nabla\mathcal{U}_2(g), \mathcal{U}_2(g)\rangle_\Omega. \end{aligned} \tag{4.17}$$

Let $L > 10R$. Then there holds $\mathrm{supp}(\nabla\psi_\infty) \cap \mathrm{supp}(\nabla\omega_L) = \emptyset$. Recalling the definition of $\mathcal{U}_2$, by (4.6), we obtain

$$\begin{aligned}|\langle(\nabla\omega_L)\cdot\nabla\mathcal{U}_2(g), \mathcal{U}_2(g)\rangle_\Omega| &\le |\langle(\nabla\omega_L)\cdot\nabla\mathcal{V}_\infty(g), c_g\rangle_{\mathbb{R}^2}| + |\langle(\nabla\omega_L)\cdot\nabla\mathcal{V}_\infty(g), \mathcal{V}_\infty(g)\rangle_{\mathbb{R}^2}| \\ &\le \frac{C_R}{L}\sup_{x\in\mathbb{R}^2}|\nabla\omega(x)|\int_{L\le|x|\le 2L}\|g\|_{L_p(\mathbb{R}^2)}\Big(|c_g| + \|g\|_{L_p(\mathbb{R}^2)}|x|^{-1}\Big)|x|^{-2}\,\mathrm{d}x,\end{aligned}$$

where a constant C_R may depend on R but is independent of L. We then observe

$$\lim_{L\to\infty}\langle(\nabla\omega_L)\cdot\nabla\mathcal{U}_2(g), \mathcal{U}_2(g)\rangle_\Omega = 0.$$

Thus, letting $L \to \infty$ in (4.17) implies that $\|\nabla\mathcal{U}_2(g)\|_{L_2(\Omega;\mathbb{R}^2)} = 0$. Hence, $\mathcal{U}_2(g)$ is a constant. In particular, we infer from (4.9) that $\mathcal{U}_2(g) = 0$. □

By Lemma 4.1 and the Fredholm alternative theorem, we may show the existence of the inverse of $I + \mathscr{R}_2$ on $\mathcal{L}(\mathcal{H}_p(\mathbb{R}^2))$ provided that p satisfies (1.1). Namely, we may show the following lemma.

Lemma 4.2. *Assume that $\Omega \subset \mathbb{R}^2$ is an exterior Lipschitz domain and p satisfies* (1.1) *with some $0 < \varepsilon \le 1/4$. Let $\mathscr{R}_2$ be the operator defined by* (4.12). *Then the inverse of $I + \mathscr{R}_2$ on $\mathcal{L}(\mathcal{H}_p(\mathbb{R}^2))$ exists.*

Proof. It suffices to show that the kernel of $I + \mathscr{R}_2$ is trivial. To this end, let g be an element of $\mathcal{H}_p(\mathbb{R}^2)$ such that $(I + \mathscr{R}_2)g = 0$.

We first deal with the case $p \ge 2$. By Lemma 4.1, we have $\mathcal{U}_2(g) = 0$. Recalling the definition of $\mathcal{U}_2$, there holds

$$\psi_0\mathcal{V}_0(g) + (1-\psi_0)\mathcal{V}_{\infty,0}(g) = 0 \quad \text{in } \Omega. \tag{4.18}$$

Notice that it follows from the definition of ψ_0 that $\mathcal{V}_{\infty,0}(g) = 0$ for $|x| \ge 5R$ and $\mathcal{V}_0(g) = 0$ for $x \in \Omega \cap B_{4R}(0)$. Let

$$V = \begin{cases}\mathcal{V}_0(g) & \text{for } 4R < |x| \le 8R,\\ 0 & \text{for } 0 \le |x| \le 4R.\end{cases}$$

Since $\mathcal{V}_0(g)$ is a solution to (4.7) and satisfies (4.8), it is clear that V solves

$$\begin{cases}-\Delta V = g|_{B_{8R}(0)} & \text{in } B_{8R}(0),\\ \quad V = 0 & \text{on } \partial B_{8R}(0).\end{cases} \tag{4.19}$$

Since $\mathcal{V}_{\infty,0}(g) = 0$ for $|x| \ge 5R$, we observe $\mathcal{V}_{\infty,0}(g) \in H^{1+s}_p(B_{8R}(0))$ as follows from (4.6). In addition, $\mathcal{V}_{\infty,0}(g)$ also solves (4.19). Thus, by the uniqueness of the solution to (4.19), we obtain $V = \mathcal{V}_{\infty,0}(g)$ in $B_{8R}(0)$, i.e., $\mathcal{V}_0(g) = \mathcal{V}_{\infty,0}(g)$ in $\Omega \cap B_{8R}$.

By virtue of the relation (4.18), we have $\mathcal{V}_{\infty,0}(g) = 0$, and thus there holds $g = \Delta\mathcal{V}_{\infty,0}(g) = 0$ in Ω. Recalling that $\mathrm{supp}(g) \subset A_R$, it is necessary to have $g = 0$ in $\mathbb{R}^2$.

Concerning the remaining case $p < 2$, we note that if $u_2 \in \dot{H}^1_p(\Omega)$ satisfies (4.4) with $g = 0$, then u_2 is a constant. In fact, for any $f \in L_{p'}(\Omega;\mathbb{R}^2)$, let $U_2 \in \dot{H}^1_{p'}(\Omega)$ be a solution to

$$\langle\nabla U_2, \nabla\varphi\rangle_\Omega = \langle f, \nabla\varphi\rangle_\Omega \qquad \text{for every } \varphi \in \dot{H}^1_p(\Omega).$$

Then, the aforementioned argument ensures the existence of U_2 since the inverse of $I + \mathscr{R}_2$ on $\mathcal{L}(\mathcal{H}_p(\mathbb{R}^2))$ exists. From the assumption, we know that $u_2 \in \dot{H}^1_p(\Omega)$ fulfills $\langle\nabla u_2, \nabla\varphi\rangle_\Omega = 0$ for every

$\varphi \in \dot{H}^1_{p'}(\Omega)$, and hence there holds $0 = \langle \nabla U_2, \nabla u_2\rangle_\Omega = \langle f, \nabla u_2\rangle_\Omega$. Then we deduce that u_2 is a constant since $f \in L_{p'}(\Omega; \mathbb{R}^2)$ is arbitrary.

Since $(I + \mathscr{R}_2)g = 0$, it follows from (4.11) that $\mathcal{U}_2(g) \in \dot{H}^1_p(\Omega)$ solves

$$\langle \nabla \mathcal{U}_2(g), \nabla\varphi\rangle_\Omega = 0 \qquad \text{for every } \varphi \in \dot{H}^1_{p'}(\Omega).$$

As seen before, we deduce that $\mathcal{U}_2(g)$ is a constant. In particular, $\mathcal{U}_2(g) = 0$ as follows from (4.9). Then, mimicking the argument as in the case $2 \le p$, we may show the invertibility of $I + \mathscr{R}_2$ on $\mathcal{L}(\mathcal{H}_p(\mathbb{R}^2))$. □

From Lemma 4.2, we may construct the solution operator to (4.4) with the desired estimate (4.5). Then, as noted before (cf. Step 2 in this section), we see that $\mathcal{U}_1(f) - \mathcal{U}_2(\mathscr{R}_1(f))$ satisfies (3.5) with the desired estimate. Hence, it remains to verify the uniqueness of the solution to (3.5), but this may be proved along the same line as in the latter part of the proof of Lemma 4.2. Thus, the proof of Theorem 1.2 is complete. Namely, Theorem 1.1 has been proved.

5. Conclusions

For an exterior Lipschitz domain $\Omega \subset \mathbb{R}^2$, we have proved the Helmholtz decomposition of the vector fields in $L_p(\Omega; \mathbb{R}^2)$ provided that p satisfies $|1/p - 1/2| < 1/4 + \varepsilon$ with some constant $\varepsilon = \varepsilon(\Omega) \in (0, 1/4]$. In particular, it is allowed to take $\varepsilon = 1/4$ if $\partial\Omega \in C^1$. We have presented a new proof of the Helmholtz decomposition of the vector fields for two-dimensional exterior domains, which is different from the previous approaches of Miyakawa [9] as well as Simader and Sohr [11].

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

References

1. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, **140** (2003).

2. G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, Springer Monographs in Mathematics, Springer, New York, **2** (2011).

3. L. Grafakos, S. Oh, The Kato-Ponce inequality, *Commun. Part. Diff. Equ.*, **39** (2014), 1128–1157. https://doi.org/10.1080/03605302.2013.822885

4. B. H. Haak, P. C. Kunstmann, On Kato's method for Navier-Stokes equations, *J. Math. Fluid Mech.*, **11** (2009), 492–535. https://doi.org/10.1007/s00021-008-0270-5

5. M. Haase, The functional calculus for sectorial operators, *Oper. Theory Adv. Appl.*, **169** (2006).

6. D. Jerison, C. E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, *J. Funct. Anal.*, **130** (1995), 161–219. https://doi.org/10.1006/jfan.1995.1067

7. J. Lang, O. Méndez, Potential techniques and regularity of boundary value problems in exterior non-smooth domains: Regularity in exterior domains, *Potential Anal.*, **24** (2006), 385–406. https://doi.org/10.1007/s11118-006-9008-2

8. D. Mitrea, Layer potentials and Hodge decompositions in two dimensional Lipschitz domains, *Math. Ann.*, **322** (2002), 75–101. https://doi.org/10.1007/s002080100266

9. T. Miyakawa, On nonstationary solutions of the Navier-Stokes equations in an exterior domain, *Hiroshima Math. J.*, **12** (1982), 115–140. https://doi.org/10.32917/hmj/1206133879

10. Y. Shibata, On the local wellposedness of free boundary problem for the Navier-Stokes equations in an exterior domain, *Commun. Pure Appl. Anal.*, **17** (2018), 1681–1721. https://doi.org/10.3934/cpaa.2018081

11. C. G. Simader, H. Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains*, Series on Advances in Mathematics for Applied Sciences, Mathematical Problems Relating to the Navier-Stokes Equations, River Edge, NJ, **11** (1992), 1–35. https://doi.org/10.1142/9789814503594_0001

12. P. Tolksdorf, K. Watanabe, The Navier-Stokes equations in exterior Lipschitz domains: L^p-theory, *J. Differ. Equations*, **269** (2020), 5765–5801. https://doi.org/10.1016/j.jde.2020.04.015

13. K. Watanabe, Decay estimates of gradient of the Stokes semigroup in exterior Lipschitz domains, *J. Differ. Equations*, **346** (2023), 277–312. https://doi.org/10.1016/j.jde.2022.11.045

14. M. Wilke, On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations in cylindrical domains, *Interface. Free Bound.*, **24** (2022), 487–531. https://doi.org/10.4171/ifb/480

15. I. Wood, Maximal L^p-regularity for the Laplacian on Lipschitz domains, *Math. Z.*, **255** (2007), 855–875. https://doi.org/10.1007/s00209-006-0055-6