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Research article

On some new frames along a space curve and integral curves with Darboux q-vector fields in $\mathbb{E}^3$

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Abstract: In this paper, to start we defined osculating q-frame, normal q-frame, and rectifying q-frame along a space curve in Euclidean 3-space $\mathbb{E}^3$ by using the Darboux vector field of the q-frame. We obtained the derivative equations of these new frames. Later, we defined some new integral curves of a space curve and called them $\overline{\mathsf{d}}_o$-direction curve, $\overline{\mathsf{d}}_n$-direction curve and $\overline{\mathsf{d}}_r$-direction curve. Finally, we gave some theorems and results related with these curves.

Keywords: q-frame; integral curve; osculating q-frame; normal q-frame; rectifying q-frame
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1. Introduction

The Frenet frame is the best known and the most used frame for characterizing curves in differential geometry. However, there exist other frames for characterizing curves. Some of these alternative frames are the Bishop frame (rotation minimizing frame), the Flc (Frenet-like curve) frame, the $\{\mathsf{N}, \mathsf{C}, \mathsf{W}\}$ frame, etc. The Bishop frame [2] is also very suitable for engineering applications [10] because of being defined at points where even curvatures vanish. That's why many studies have been done by using this frame. On the other hand, the Bishop frame cannot be calculated analytically [6]. Therefore, Dede [5] recently introduced a new moving frame called the Flc frame. This frame has much easier calculations and a more analytical form that the Bishop frame does not have. The other alternative frame, $\{\mathsf{N}, \mathsf{C}, \mathsf{W}\}$, has been defined by Scofield [14], providing a different approach. Uzunoğlu et al. [15] showed that the $\{\mathsf{N}, \mathsf{C}, \mathsf{W}\}$ frame has an important advantage according to the Frenet frame since the expression of slant helix characterization is more short with the new curvatures. As an alternative to the Frenet frame, Dede et al. [4] defined a new adapted frame along a space curve. They called this new frame as q-frame and obtained the relations between the Frenet frame and the q-frame.

By using the Darboux frame $\{\mathsf{T}, \mathsf{V}, \mathsf{U}\}$ along a regular curve α lying on an oriented surface M in $\mathbb{E}^3$, Hananoi et al. [9] defined the osculating Darboux vector field D_o, the normal Darboux vector field D_n, and the rectifying Darboux vector field D_r, where T is the unit tangent vector field of α, U is the unit normal vector field of M restricted to α, and $\mathsf{V} = \mathsf{U} \times \mathsf{T}$. Considering these vector fields, Önder [13] defined three special curves on a surface as D_i-Darboux slant helices, where $i \in \{o, n, r\}$. In recent days, Alkan et al. [1] defined osculator Darboux frame, normal Darboux frame, and rectifying Darboux frame.

Recently, many researchers [3, 7, 8, 11, 12, 16] have studied associated curves and have revealed the relationships between the main curve and the associated curves. The spherical indicators, the involute-evolute curve couple, the Bertrand curve couple, the Mannheim curve couple, and the integral curves are the most familiar ones among these associated curves. Integral curves are one of the interesting curves among these curves since they are tangent to the vector field at every point.

In this study, considering the Darboux vector field of the q-frame, we first define some new frames called the osculating q-frame, the normal q-frame, and the rectifying q-frame along a space curve in $\mathbb{E}^3$ and obtain their derivative equations. Second, by using some vector fields of these new q-frames, we define new integral curves called $\overline{\mathsf{d}}_o$-direction curve, $\overline{\mathsf{d}}_n$-direction curve, and $\overline{\mathsf{d}}_r$-direction curve of a space curve.

2. Preliminaries

Let $\alpha = \alpha(t)$ be a regular space curve with nondegenerate condition $\alpha' \times \alpha'' \neq 0$. Then, the Frenet frame vector fields are defined as

$$\mathsf{t} = \frac{\alpha'}{\|\alpha'\|}, \quad \mathsf{n} = \mathsf{b} \times \mathsf{t}, \quad \mathsf{b} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|},$$

where t, n, and b denote the tangent, the principal normal, and the binormal vector fields of the curve α and the prime denotes the derivative with respect to t. Then, we have the following Frenet equations:

$$\begin{cases} \mathsf{t}' = \nu\kappa\mathsf{n}, \\ \mathsf{n}' = \nu(-\kappa\mathsf{t} + \tau\mathsf{b}), \\ \mathsf{b}' = -\nu\tau\mathsf{n}, \end{cases}$$

where $\nu = \|\alpha'(t)\|$, and the curvature and the torsion functions are

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2},$$

respectively.

The q-frame $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$, which is an alternative to the Frenet frame along a space curve $\alpha = \alpha(t)$, is given by Dede et al. [4], and its vector fields are defined as

$$\mathsf{t} = \frac{\alpha'}{\|\alpha'\|}, \quad \mathsf{n}_q = \frac{\mathsf{t} \times \mathsf{k}}{\|\mathsf{t} \times \mathsf{k}\|}, \quad \mathsf{b}_q = \mathsf{t} \times \mathsf{n}_q,$$

where the vector $\mathsf{k} = (0, 0, 1)$ is the projection vector and the vectors n_q and b_q are the quasi-normal and the quasi-binormal vector, respectively. The relation matrix between the Frenet frame and the q-frame is given by

$$\begin{pmatrix} \mathsf{t} \\ \mathsf{n} \\ \mathsf{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \mathsf{t} \\ \mathsf{n}_q \\ \mathsf{b}_q \end{pmatrix},$$

where θ is the angle between the vectors n and n_q.

Let $\alpha = \alpha(s)$ be a space curve with arc length parameter s. Then, the q-frame equations are given by

$$\begin{pmatrix} \mathsf{t}' \\ \mathsf{n}_q' \\ \mathsf{b}_q' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{t} \\ \mathsf{n}_q \\ \mathsf{b}_q \end{pmatrix}, \tag{2.1}$$

where k_1, k_2, k_3 denote the q-curvatures of α [4]. The relations between the curvatures are

$$k_1 = \kappa cos\theta, \quad k_2 = -\kappa sin\theta, \quad k_3 = \theta' + \tau,$$

where κ, τ are the Frenet curvatures. The Darboux vector d_q of the q-frame $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ is given by

$$\mathsf{d}_q = k_3\mathsf{t} - k_2\mathsf{n}_q + k_1\mathsf{b}_q.$$

Thus, the instantaneous angular speed of the q-frame is calculated as

$$\|\mathsf{d}_q\| = \sqrt{k_1^2 + k_2^2 + k_3^2}.$$

Let us consider the vector fields d_o, d_n, and d_r which are called the osculating q-vector field, the normal q-vector field, and the rectifying q-vector field along α, respectively:

$$\mathsf{d}_o = k_3\mathsf{t} - k_2\mathsf{n}_q,$$

$$\mathsf{d}_n = -k_2\mathsf{n}_q + k_1\mathsf{b}_q,$$

$$\mathsf{d}_r = k_3\mathsf{t} + k_1\mathsf{b}_q.$$

Then, the unit osculating q-vector field, the unit normal q-vector field, and the unit rectifying q-vector field along α are given by, respectively:

$$\overline{\mathsf{d}}_o = \frac{k_3}{\sqrt{k_2^2 + k_3^2}}\mathsf{t} - \frac{k_2}{\sqrt{k_2^2 + k_3^2}}\mathsf{n}_q, \tag{2.2}$$

$$\overline{\mathsf{d}}_n = -\frac{k_2}{\sqrt{k_1^2 + k_2^2}}\mathsf{n}_q + \frac{k_1}{\sqrt{k_1^2 + k_2^2}}\mathsf{b}_q, \tag{2.3}$$

$$\overline{\mathsf{d}}_r = \frac{k_3}{\sqrt{k_1^2 + k_3^2}}\mathsf{t} + \frac{k_1}{\sqrt{k_1^2 + k_3^2}}\mathsf{b}_q. \tag{2.4}$$

3. Some new frames along a space curve

In this section, let us define some new frames along a space curve using the Darboux vector field of the q-frame and give some theorems related with these frames.

Definition 3.1. Let α be a space curve in $\mathbb{E}^3$, $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ be its q-frame, and $\overline{\mathsf{d}}_o$ be the unit osculating q-vector field along the curve α. Then, the orthonormal frame $\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$ is called the osculating q-frame along α, where $\mathsf{e}_o = \overline{\mathsf{d}}_o \times \mathsf{b}_q$.

Theorem 3.2. *Let $\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$ be the osculating q-frame along the curve α. Then, the derivative equations according to this frame can be found as*

$$\begin{cases} \overline{\mathsf{d}}_o' = -\rho_o \mathsf{e}_o, \\ \mathsf{b}_q' = \eta_o \mathsf{e}_o, \\ \mathsf{e}_o' = \rho_o \overline{\mathsf{d}}_o - \eta_o \mathsf{b}_q, \end{cases} \tag{3.1}$$

where $\rho_o = \frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2+k_3^2)}{k_2^2+k_3^2}$ and $\eta_o = \sqrt{k_2^2 + k_3^2}$ are the curvatures of α according to the osculating q-frame.

Proof. Since $\overline{\mathsf{d}}_o$ is the unit osculating q-vector field along the curve α, we get $\overline{\mathsf{d}}_o \in \mathrm{Sp}\{\mathsf{t}, \mathsf{n}_q\}$. So, $\overline{\mathsf{d}}_o \perp \mathsf{b}_q$, and, thus, $\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$ is an orthonormal frame along α, where $\mathsf{e}_o = \overline{\mathsf{d}}_o \times \mathsf{b}_q$. Since $\overline{\mathsf{d}}_o' \in \mathrm{Sp}\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$, we can write

$$\overline{\mathsf{d}}_o' = a_1 \overline{\mathsf{d}}_o + a_2 \mathsf{b}_q + a_3 \mathsf{e}_o.$$

Taking the inner product of both sides of this equation with $\overline{\mathsf{d}}_o$ yields $a_1 = \langle \overline{\mathsf{d}}_o', \overline{\mathsf{d}}_o \rangle = 0$ because of $\|\overline{\mathsf{d}}_o\| = 1$. So, we get

$$\overline{\mathsf{d}}_o' = a_2 \mathsf{b}_q + a_3 \mathsf{e}_o. \tag{3.2}$$

If we take the inner product of both sides of Eq (3.2) with b_q and e_o, respectively, we obtain $a_2 = \langle \overline{\mathsf{d}}_o', \mathsf{b}_q \rangle$ and $a_3 = \langle \overline{\mathsf{d}}_o', \mathsf{e}_o \rangle$. Substituting

$$\sin\phi = \frac{k_3}{\sqrt{k_2^2 + k_3^2}}, \quad \cos\phi = \frac{k_2}{\sqrt{k_2^2 + k_3^2}} \tag{3.3}$$

into Eq (2.2) gives

$$\overline{\mathsf{d}}_o = \sin\phi\, \mathsf{t} - \cos\phi\, \mathsf{n}_q. \tag{3.4}$$

If we differentiate this equation with respect to s and use Eq (2.1), we have

$$\overline{\mathsf{d}}_o' = (\phi' + k_1)\cos\phi\, \mathsf{t} + (\phi' + k_1)\sin\phi\, \mathsf{n}_q + (k_2 \sin\phi - k_3 \cos\phi)\mathsf{b}_q.$$

If we use Eq (3.3), we get $a_2 = 0$. So,

$$\overline{\mathsf{d}}_o' = (\phi' + k_1)[\cos\phi\, \mathsf{t} + \sin\phi\, \mathsf{n}_q]. \tag{3.5}$$

Moreover, using Eq (3.4), we obtain

$$\mathsf{e}_o = -\cos\phi\, \mathsf{t} - \sin\phi\, \mathsf{n}_q. \tag{3.6}$$

If we use Eqs (3.5) and (3.6), we obtain $a_3 = -(\phi' + k_1)$. Also, Eq.(3.3) gives $\tan\phi = \frac{k_3}{k_2}$, and by doing some calculations, we find

$$a_3 = -\left[\frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{k_2^2 + k_3^2}\right].$$

So, from Eq (3.2), we obtain

$$\overline{\mathsf{d}}_o' = -\left[\frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{k_2^2 + k_3^2}\right]\mathsf{e}_o. \tag{3.7}$$

In a similar way, we may write b_q' as a linear combination of the vectors $\overline{\mathsf{d}}_o$, b_q, and e_o, i.e.,

$$\mathsf{b}_q' = b_1\overline{\mathsf{d}}_o + b_2\mathsf{b}_q + b_3\mathsf{e}_o. \tag{3.8}$$

Taking the inner product of both sides of Eq (3.8) with $\overline{\mathsf{d}}_o$ yields $b_1 = \langle \mathsf{b}_q', \overline{\mathsf{d}}_o\rangle$. If we use Eqs (2.1), (3.3), and (3.4), we find $b_1 = 0$. Also, since $\|\mathsf{b}_q\| = 1$, we get $b_2 = 0$. Then, we have

$$\mathsf{b}_q' = b_3\mathsf{e}_o. \tag{3.9}$$

If we take the inner product of both sides of Eq (3.9) with e_o and use Eqs (2.1), (3.3), and (3.6), we obtain

$$\mathsf{b}_q' = \sqrt{k_2^2 + k_3^2}\,\mathsf{e}_o. \tag{3.10}$$

Moreover, since $\mathsf{e}_o' \in \mathrm{Sp}\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$, it can be written as

$$\mathsf{e}_o' = c_1\overline{\mathsf{d}}_o + c_2\mathsf{b}_q + c_3\mathsf{e}_o. \tag{3.11}$$

Taking the inner product of both sides of Eq (3.11) with $\overline{\mathsf{d}}_o$ gives $c_1 = \langle \mathsf{e}_o', \overline{\mathsf{d}}_o\rangle$. Using Eqs (2.1) and (3.6) yields

$$\mathsf{e}_o' = (\phi' + k_1)\sin\phi\,\mathsf{t} - (\phi' + k_1)\cos\phi\,\mathsf{n}_q - (k_2\cos\phi + k_3\sin\phi)\mathsf{b}_q. \tag{3.12}$$

If we use Eqs (3.3), (3.4), and (3.12), we obtain

$$c_1 = \frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{k_2^2 + k_3^2}.$$

From Eqs (3.3), (3.11), and (3.12), we get $c_2 = -\sqrt{k_2^2 + k_3^2}$. Also, since $\|\mathsf{e}_o\| = 1$, we have $c_3 = 0$. Then, from Eq (3.11), we find

$$\mathsf{e}_o' = \left[\frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{k_2^2 + k_3^2}\right]\overline{\mathsf{d}}_o - \sqrt{k_2^2 + k_3^2}\,\mathsf{b}_q. \tag{3.13}$$

If we denote $\rho_o = \frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{k_2^2 + k_3^2}$ and $\eta_o = \sqrt{k_2^2 + k_3^2}$, Eqs (3.7), (3.10), and (3.13) give the desired equations. Here, ρ_o and η_o are called the curvatures of α according to the osculating q-frame. □

Definition 3.3. Let $\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$ be the osculating q-frame along the space curve α. The curve α is called a b_q-slant helix relative to the osculating q-frame if the vector field b_q makes a constant angle with a fixed direction, i.e., $\langle \mathsf{b}_q, \mathsf{u}\rangle = \cos\psi$, where u is a constant unit vector and ψ is a constant angle.

Theorem 3.4. *Let $\{\overline{\mathsf{d}}_o, \mathsf{b}_q, \mathsf{e}_o\}$ be the osculating q-frame along the space curve α. The curve α is a b_q-slant helix relative to the osculating q-frame if, and only if, the expression $\frac{\eta_o}{\rho_o}$ is constant (for $\eta_o \neq 0$ and $\rho_o \neq 0$).*

Proof. Let α be a b_q-slant helix relative to the osculating q-frame. Then, $\langle \mathsf{b}_q, \mathsf{u} \rangle = \cos\psi = c \neq 0$, where u is a unit constant direction. So, it can be written as

$$\mathsf{u} = \lambda_1 \overline{\mathsf{d}}_o + c\mathsf{b}_q + \lambda_2 \mathsf{e}_o, \quad (\lambda_1, \lambda_2 \in \mathbb{R}).$$

If we differentiate this equation, we obtain

$$\mathsf{u}' = \lambda_1 \overline{\mathsf{d}}_o' + \lambda_1' \overline{\mathsf{d}}_o + c\mathsf{b}_q' + \lambda_2' \mathsf{e}_o + \lambda_2 \mathsf{e}_o'.$$

If we use Eq (3.1), we find

$$\begin{cases} \lambda_1' + \lambda_2 \rho_o = 0, \\ \lambda_2 \eta_o = 0, \\ \lambda_2' - \lambda_1 \rho_o + c\eta_o = 0. \end{cases}$$

Since $\eta_o \neq 0$ and $\rho_o \neq 0$, we have $\lambda_2 = 0$ and $\lambda_1 = constant$. Thus, we get $\frac{\eta_o}{\rho_o} = constant$.

Conversely, let $\frac{\eta_o}{\rho_o}$ be constant. Choosing $\frac{\eta_o}{\rho_o} = \frac{\cos\psi}{\sin\psi}$ and taking $\mathsf{u} = \cos\psi \overline{\mathsf{d}}_o + \sin\psi \mathsf{b}_q$ gives $\mathsf{u}' = 0$ by using Eq (3.1). So, the vector u is constant. Also, by taking the inner product of both sides of $\mathsf{u} = \cos\psi \overline{\mathsf{d}}_o + \sin\psi \mathsf{b}_q$ with b_q yields to $\langle \mathsf{u}, \mathsf{b}_q \rangle = \sin\psi$. Then, the constant vector u and the vector b_q make a constant angle, i.e., the curve α is a b_q-slant helix. □

Corollary 3.5. *A space curve α with $(k_2(s), k_3(s)) \neq (0, 0)$ is a b_q-slant helix if, and only if,*

$$\rho_1(s) = \frac{(k_3' k_2 - k_2' k_3) + k_1(k_2^2 + k_3^2)}{(k_2^2 + k_3^2)^{3/2}}$$

is a constant function.

Definition 3.6. Let α be a space curve in $\mathbb{E}^3$, $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ be its q-frame, and $\overline{\mathsf{d}}_n$ be the unit normal q-vector field along the curve α. Then, the orthonormal frame $\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$ is called the normal q-frame along α, where $\mathsf{e}_n = \overline{\mathsf{d}}_n \times \mathsf{t}$.

Theorem 3.7. *Let $\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$ be the normal q-frame along the curve α. Then, the derivative equations according to this frame can be obtained as*

$$\begin{cases} \overline{\mathsf{d}}_n' = -\rho_n \mathsf{e}_n, \\ \mathsf{t}' = \eta_n \mathsf{e}_n, \\ \mathsf{e}_n' = \rho_n \overline{\mathsf{d}}_n - \eta_n \mathsf{t}, \end{cases} \tag{3.14}$$

where $\rho_n = \frac{(k_2' k_1 - k_1' k_2) + k_3(k_1^2 + k_2^2)}{k_1^2 + k_2^2}$ and $\eta_n = \sqrt{k_1^2 + k_2^2}$ are the curvatures of α according to the normal q-frame.

Proof. Since $\overline{\mathsf{d}}_n$ is the unit normal q-vector field along α, from Eq (2.3), we have $\overline{\mathsf{d}}_n \in \mathrm{Sp}\{\mathsf{n}_q, \mathsf{b}_q\}$. Hence, $\overline{\mathsf{d}}_n \perp \mathsf{t}$. Let $\mathsf{e}_n = \overline{\mathsf{d}}_n \times \mathsf{t}$. Then, we obtain the orthonormal frame $\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$ along α. Since $\overline{\mathsf{d}}_n' \in \mathrm{Sp}\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$, it can be written as

$$\overline{\mathsf{d}}_n' = a_1\overline{\mathsf{d}}_n + a_2\mathsf{t} + a_3\mathsf{e}_n. \tag{3.15}$$

If we take the inner product of both sides of Eq (3.15) with $\overline{\mathsf{d}}_n$ and take into consideration $\|\overline{\mathsf{d}}_n\| = 1$, we get $a_1 = 0$. So, we have

$$\overline{\mathsf{d}}_n' = a_2\mathsf{t} + a_3\mathsf{e}_n. \tag{3.16}$$

Taking the inner product of both sides of Eq (3.16) with t and e_n, respectively, yields $a_2 = \langle\overline{\mathsf{d}}_n', \mathsf{t}\rangle$ and $a_3 = \langle\overline{\mathsf{d}}_n', \mathsf{e}_n\rangle$. In Eq (2.3), if we take

$$\sin\phi = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}, \quad \cos\phi = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}, \tag{3.17}$$

we obtain

$$\overline{\mathsf{d}}_n = -\sin\phi\,\mathsf{n}_q + \cos\phi\,\mathsf{b}_q. \tag{3.18}$$

Differentiating this equation with respect to s and using Eq (2.1) gives

$$\overline{\mathsf{d}}_n' = (k_1\sin\phi - k_2\cos\phi)\mathsf{t} - (\phi' + k_3)\cos\phi\,\mathsf{n}_q - (\phi' + k_3)\sin\phi\,\mathsf{b}_q.$$

If we use Eq (3.17), we get $a_2 = 0$. So, we have

$$\overline{\mathsf{d}}_n' = -(\phi' + k_3)[\cos\phi\,\mathsf{n}_q + \sin\phi\,\mathsf{b}_q]. \tag{3.19}$$

Also, using Eq (3.18) yields

$$\mathsf{e}_n = \cos\phi\,\mathsf{n}_q + \sin\phi\,\mathsf{b}_q. \tag{3.20}$$

Moreover, from Eqs (3.19) and (3.20), we have $a_3 = -(\phi' + k_3)$. By doing some calculations, we find

$$a_3 = -\left[\frac{(k_2'k_1 - k_1'k_2) + k_3(k_1^2 + k_2^2)}{k_1^2 + k_2^2}\right].$$

If we use Eq (3.16), we have

$$\overline{\mathsf{d}}_n' = -\left[\frac{(k_2'k_1 - k_1'k_2) + k_3(k_1^2 + k_2^2)}{k_1^2 + k_2^2}\right]\mathsf{e}_n. \tag{3.21}$$

Similarly, since $\mathsf{t}' \in \mathrm{Sp}\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$, we can write

$$\mathsf{t}' = b_1\overline{\mathsf{d}}_n + b_2\mathsf{t} + b_3\mathsf{e}_n. \tag{3.22}$$

If we take the inner product of both sides of Eq (3.22) with $\overline{\mathsf{d}}_n$, we obtain $b_1 = \langle\mathsf{t}', \overline{\mathsf{d}}_n\rangle$. Eqs (2.1), (3.17), and (3.18) give $b_1 = 0$. Also, since $\|\mathsf{t}\| = 1$, we have $b_2 = 0$. So, we get

$$\mathsf{t}' = b_3\mathsf{e}_n. \tag{3.23}$$

Taking the inner product of both sides of Eq (3.23) with e_n and using Eqs (2.1), (3.17), and (3.20) yields $b_3 = \sqrt{k_1^2 + k_2^2}$ and

$$\mathsf{t}' = \sqrt{k_1^2 + k_2^2}\,\mathsf{e}_n. \tag{3.24}$$

Additionally, since $\mathsf{e}_n' \in \mathrm{Sp}\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$, we can write

$$\mathsf{e}_n' = c_1\overline{\mathsf{d}}_n + c_2\mathsf{t} + c_3\mathsf{e}_n. \tag{3.25}$$

If we take the inner product of both sides of Eq (3.25) with $\overline{\mathsf{d}}_n$, we get $c_1 = \langle \mathsf{e}_n', \overline{\mathsf{d}}_n\rangle$. Eqs (2.1) and (3.20) give us

$$\mathsf{e}_n' = -(k_1\cos\phi + k_2\sin\phi)\mathsf{t} - (\phi' + k_3)\sin\phi\,\mathsf{n}_q + (\phi' + k_3)\cos\phi\,\mathsf{b}_q. \tag{3.26}$$

If we use Eqs (3.18) and (3.26), we find

$$c_1 = \frac{(k_2'k_1 - k_1'k_2) + k_3(k_1^2 + k_2^2)}{k_1^2 + k_2^2}.$$

Also, using Eqs (3.17), (3.25), and (3.26) yields $c_2 = -\sqrt{k_1^2 + k_2^2}$. In addition, since $\|\mathsf{e}_n\| = 1$, we have $c_3 = 0$. Thus, using Eq (3.25) gives

$$\mathsf{e}_n' = \left[\frac{(k_2'k_1 - k_1'k_2) + k_3(k_1^2 + k_2^2)}{k_1^2 + k_2^2}\right]\overline{\mathsf{d}}_n - \sqrt{k_1^2 + k_2^2}\,\mathsf{t}. \tag{3.27}$$

If we denote $\rho_n = \frac{(k_2'k_1 - k_1'k_2) + k_3(k_1^2 + k_2^2)}{k_1^2 + k_2^2}$ and $\eta_n = \sqrt{k_1^2 + k_2^2}$ and use Eqs (3.21), (3.24), and (3.27), we obtain the desired equations. □

Theorem 3.8. *Let $\{\overline{\mathsf{d}}_n, \mathsf{t}, \mathsf{e}_n\}$ be the normal q-frame along the space curve α. The curve α is a helix relative to the normal q-frame if, and only if, the expression $\frac{\eta_n}{\rho_n}$ is constant, for $\eta_n \neq 0$ and $\rho_n \neq 0$.*

Proof. Let α be a helix relative to the normal q-frame. Then, $\langle \mathsf{t}, \mathsf{u}\rangle = \cos\psi = c \neq 0$, where u is a unit constant direction. So, we can write

$$\mathsf{u} = \lambda_1\overline{\mathsf{d}}_n + c\mathsf{t} + \lambda_2\mathsf{e}_n, \quad (\lambda_1, \lambda_2 \in \mathbb{R}).$$

Differentiating this equation with respect to s and using Eq (3.14) gives

$$\begin{cases} \lambda_1' + \lambda_2\rho_n = 0, \\ \lambda_2\eta_n = 0, \\ \lambda_2' - \lambda_1\rho_n + c\eta_n = 0. \end{cases}$$

Since $\eta_n \neq 0$ and $\rho_n \neq 0$, we have $\lambda_2 = 0$ and $\lambda_1 = \mathit{constant}$. So, we obtain $\frac{\eta_n}{\rho_n} = \mathit{constant}$.

Conversely, let $\frac{\eta_n}{\rho_n}$ be constant. If we choose $\frac{\eta_n}{\rho_n} = \frac{\cos\psi}{\sin\psi}$ and take $\mathsf{u} = \cos\psi\,\overline{\mathsf{d}}_n + \sin\psi\,\mathsf{t}$, we find $\mathsf{u}' = 0$ with the help of Eq (3.14). Then, the vector u is constant. Additionally, if we take the inner product of both sides of $\mathsf{u} = \cos\psi\,\overline{\mathsf{d}}_n + \sin\psi\,\mathsf{t}$ with t, we get $\langle \mathsf{u}, \mathsf{t}\rangle = \sin\psi$. Consequently, the constant vector u and the tangent vector t make a constant angle, i.e., the curve α is a helix. □

Example 3.9. Let us consider the curve $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$ in $\mathbb{E}^3$. The q-frame apparatus of the curve α is

$$\mathsf{t} = \frac{1}{\sqrt{2}(1+t^2)}(1-t^2, 2t, 1+t^2), \quad \mathsf{n}_q = \frac{1}{1+t^2}(2t, t^2-1, 0), \quad \mathsf{b}_q = \frac{1}{\sqrt{2}(1+t^2)}(1-t^2, 2t, -1-t^2),$$

$$k_1 = -\frac{\sqrt{2}}{1+t^2}, \quad k_2 = 0, \quad k_3 = \frac{\sqrt{2}}{1+t^2},$$

with the projection vector $\mathsf{k} = (0, 0, 1)$. Also, the normal q-frame apparatus of the curve α is

$$\overline{\mathsf{d}}_n = \frac{1}{\sqrt{2}(1+t^2)}(t^2-1, -2t, 1+t^2), \quad \mathsf{t} = \frac{1}{\sqrt{2}(1+t^2)}(1-t^2, 2t, 1+t^2), \quad \mathsf{e}_n = \frac{1}{1+t^2}(-2t, 1-t^2, 0),$$

$$\rho_n = \eta_n = \frac{\sqrt{2}}{1+t^2}.$$

Since the expression $\frac{\eta_n}{\rho_n}$ is constant, the curve α is a helix relative to the normal q-frame.

Corollary 3.10. *A space curve α with $(k_1(s), k_2(s)) \neq (0, 0)$ is a helix if, and only if,*

$$\rho_2(s) = \frac{(k_2' k_1 - k_1' k_2) + k_3(k_1^2 + k_2^2)}{(k_1^2 + k_2^2)^{3/2}}$$

is a constant function.

Definition 3.11. Let α be a space curve in $\mathbb{E}^3$, $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ be its q-frame, and $\overline{\mathsf{d}}_r$ be the unit rectifying q-vector field along the curve α. Then, the orthonormal frame $\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$ is called the rectifying q-frame along α, where $\mathsf{e}_r = \overline{\mathsf{d}}_r \times \mathsf{n}_q$.

Theorem 3.12. *Let $\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$ be the rectifying q-frame along the curve α. Then, the derivative equations according to this frame can be calculated as*

$$\begin{cases} \overline{\mathsf{d}}_r' = -\rho_r \mathsf{e}_r, \\ \mathsf{n}_q' = \eta_r \mathsf{e}_r, \\ \mathsf{e}_r' = \rho_r \overline{\mathsf{d}}_r - \eta_r \mathsf{n}_q, \end{cases}$$

where $\rho_r = \frac{(k_3' k_1 - k_1' k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}$ and $\eta_r = \sqrt{k_1^2 + k_3^2}$ are the curvatures of α according to the rectifying q-frame.

Proof. Since $\overline{\mathsf{d}}_r$ is the unit rectifying q-vector field along α, we have $\overline{\mathsf{d}}_r \in \mathrm{Sp}\{\mathsf{t}, \mathsf{b}_q\}$. So, $\overline{\mathsf{d}}_r \perp \mathsf{n}_q$ and $\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$ is an orthonormal frame along α, where $\mathsf{e}_r = \overline{\mathsf{d}}_r \times \mathsf{n}_q$. Since $\overline{\mathsf{d}}_r' \in \mathrm{Sp}\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$, we can write

$$\overline{\mathsf{d}}_r' = a_1 \overline{\mathsf{d}}_r + a_2 \mathsf{n}_q + a_3 \mathsf{e}_r. \tag{3.28}$$

If we take the inner product of both sides of Eq (3.28) with $\overline{\mathsf{d}}_r$ and consider $\|\overline{\mathsf{d}}_r\| = 1$, we have $a_1 = 0$. So, it can be written as

$$\overline{\mathsf{d}}_r' = a_2 \mathsf{n}_q + a_3 \mathsf{e}_r. \tag{3.29}$$

Taking the inner product of both sides of Eq (3.29) with n_q and e_r, respectively, yields $a_2 = \langle \overline{\mathsf{d}}_r', \mathsf{n}_q\rangle$ and $a_3 = \langle \overline{\mathsf{d}}_r', \mathsf{e}_r\rangle$. In Eq (2.4), denoting

$$\sin\phi = \frac{k_3}{\sqrt{k_1^2 + k_3^2}}, \quad \cos\phi = \frac{k_1}{\sqrt{k_1^2 + k_3^2}}, \tag{3.30}$$

gives

$$\overline{\mathsf{d}}_r = \sin\phi\, \mathsf{t} + \cos\phi\, \mathsf{b}_q. \tag{3.31}$$

If we differentiate Eq (3.31) with respect to s and use Eq (2.1), we obtain

$$\overline{\mathsf{d}}_r' = (\phi' - k_2)\cos\phi\, \mathsf{t} + (k_1 \sin\phi - k_3\cos\phi)\mathsf{n}_q - (\phi' - k_2)\sin\phi\, \mathsf{b}_q.$$

So, from Eq (3.30), we get $a_2 = 0$ and

$$\overline{\mathsf{d}}_r' = (\phi' - k_2)[\cos\phi\, \mathsf{t} - \sin\phi\, \mathsf{b}_q]. \tag{3.32}$$

Moreover, from Eq (3.31), we have

$$\mathsf{e}_r = -\cos\phi\, \mathsf{t} + \sin\phi\, \mathsf{b}_q. \tag{3.33}$$

Also, if we use Eqs (3.32) and (3.33), we obtain $a_3 = -(\phi' - k_2)$. Doing some calculations yields

$$a_3 = -\left[\frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}\right]$$

and from Eq (3.29), it follows that

$$\overline{\mathsf{d}}_r' = -\left[\frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}\right]\mathsf{e}_r. \tag{3.34}$$

Similarly, since $\mathsf{n}_q' \in \mathrm{Sp}\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$, we can write

$$\mathsf{n}_q' = b_1\overline{\mathsf{d}}_r + b_2\mathsf{n}_q + b_3\mathsf{e}_r. \tag{3.35}$$

If we take the inner product of both sides of Eq (3.35) with $\overline{\mathsf{d}}_r$, we get $b_1 = \langle \mathsf{n}_q', \overline{\mathsf{d}}_r\rangle$. Using Eqs (2.1), (3.30), and (3.31) yields $b_1 = 0$. Also, since $\|\mathsf{n}_q\| = 1$, we have $b_2 = 0$. Then, we get

$$\mathsf{n}_q' = b_3\mathsf{e}_r. \tag{3.36}$$

Taking the inner product of both sides of Eq (3.36) with e_r gives $b_3 = \langle \mathsf{n}_q', \mathsf{e}_r\rangle$. If we use Eqs (2.1), (3.30), and (3.33), we obtain $b_3 = \sqrt{k_1^2 + k_3^2}$. So, we have

$$\mathsf{n}_q' = \sqrt{k_1^2 + k_3^2}\, \mathsf{e}_r. \tag{3.37}$$

Moreover, since $\mathsf{e}_r' \in \mathrm{Sp}\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$, it can be written as

$$\mathsf{e}_r' = c_1\overline{\mathsf{d}}_r + c_2\mathsf{n}_q + c_3\mathsf{e}_r. \tag{3.38}$$

If we take the inner product of both sides of Eq (3.38) with $\overline{\mathsf{d}}_r$, we have $c_1 = \langle \mathsf{e}_r', \overline{\mathsf{d}}_r \rangle$. From Eqs (2.1) and (3.33), we have

$$\mathsf{e}_r' = (\phi' - k_2)\sin\phi\, \mathsf{t} - (k_1\cos\phi + k_3\sin\phi)\mathsf{n}_q + (\phi' - k_2)\cos\phi\, \mathsf{b}_q. \tag{3.39}$$

If we use Eqs (3.31) and (3.39), we obtain

$$c_1 = \frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}.$$

From Eqs (3.30), (3.38), and (3.39), we find $c_2 = -\sqrt{k_1^2 + k_3^2}$. Also, we get $c_3 = 0$, since $\|\mathsf{e}_r\| = 1$. So, Eq (3.38) yields

$$\mathsf{e}_r' = \left[\frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}\right]\overline{\mathsf{d}}_r - \sqrt{k_1^2 + k_3^2}\,\mathsf{n}_q. \tag{3.40}$$

Thus, if we denote

$$\rho_r = \frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}$$

and

$$\eta_r = \sqrt{k_1^2 + k_3^2},$$

we obtain the desired equations from Eqs (3.34), (3.37), and (3.40). □

Definition 3.13. Let $\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$ be the rectifying q-frame along the space curve α. The curve α is called an n_q-slant helix relative to the rectifying q-frame if the vector field n_q makes a constant angle with a fixed direction, i.e., $\langle \mathsf{n}_q, \mathsf{u} \rangle = \cos\psi$, where u is a constant unit vector and ψ is a constant angle.

Theorem 3.14. *Let $\{\overline{\mathsf{d}}_r, \mathsf{n}_q, \mathsf{e}_r\}$ be the rectifying q-frame along the space curve α. The curve α is an n_q-slant helix relative to the rectifying q-frame if, and only if, the expression $\frac{\eta_r}{\rho_r}$ is constant (for $\eta_r \neq 0$ and $\rho_r \neq 0$).*

Proof. The proof of the theorem can be done in a similar way to the proof of Theorem 3.4. □

Corollary 3.15. *A space curve α with $(k_1(s), k_3(s)) \neq (0, 0)$ is an n_q-slant helix if, and only if,*

$$\rho_3(s) = \frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{(k_1^2 + k_3^2)^{3/2}}$$

is a constant function.

4. Some integral curves with Darboux q-vector fields

In this section, let us define some new integral curves associated with a space curve using the osculating, the normal, and the rectifying q-frame vector fields.

Definition 4.1. Let α be a space curve in $\mathbb{E}^3$, $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ be the q-frame along α, and $\overline{\mathsf{d}}_o$ be the unit osculating q-frame vector field of α. The integral curve of the vector field $\overline{\mathsf{d}}_o$ is called the $\overline{\mathsf{d}}_o$-direction curve of α. Namely, if $\gamma(s)$ is the $\overline{\mathsf{d}}_o$-direction curve of α, then $\gamma(s) = \int \overline{\mathsf{d}}_o(s)$ or $\overline{\mathsf{d}}_o(s) = \gamma'(s)$.

Now, let us find the Frenet apparatus $\{\mathsf{t}_\gamma, \mathsf{n}_\gamma, \mathsf{b}_\gamma, \kappa_\gamma, \tau_\gamma\}$ of γ. Since γ is the $\overline{\mathsf{d}}_o$-direction curve of α, it can be written $\gamma' = \overline{\mathsf{d}}_o$. So, the tangent vector t_γ of γ is

$$\mathsf{t}_\gamma = \overline{\mathsf{d}}_o = \frac{1}{\sqrt{k_2^2 + k_3^2}}(k_3\mathsf{t} - k_2\mathsf{n}_q).$$

If we differentiate this equation with respect to s and use Eq (2.1), we obtain

$$\overline{\mathsf{d}}_o' = \frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{(k_2^2 + k_3^2)^{3/2}}(k_2\mathsf{t} + k_3\mathsf{n}_q). \tag{4.1}$$

Since $\rho_o = \frac{(k_3'k_2 - k_2'k_3) + k_1(k_2^2 + k_3^2)}{k_2^2 + k_3^2}$, Eq (4.1) can be rewritten as

$$\overline{\mathsf{d}}_o' = -\rho_o \frac{\mathsf{b}_q'}{\|\mathsf{b}_q'\|}.$$

So, we have $\|\overline{\mathsf{d}}_o'\| = \varepsilon\rho_o$, where $\varepsilon = \pm 1$. Since $\mathsf{n}_\gamma = \frac{\overline{\mathsf{d}}_o'}{\|\overline{\mathsf{d}}_o'\|}$, we get

$$\mathsf{n}_\gamma = -\varepsilon \frac{\mathsf{b}_q'}{\|\mathsf{b}_q'\|}.$$

Additionally, since $\mathsf{b}_\gamma = \mathsf{t}_\gamma \times \mathsf{n}_\gamma$, the binormal vector is obtained as $\mathsf{b}_\gamma = \varepsilon\mathsf{b}_q$. Moreover, the curvature and the torsion of γ can be found as

$$\kappa_\gamma = \|\mathsf{t}_\gamma'\| = \varepsilon\rho_o, \qquad \tau_\gamma = -\langle \mathsf{b}_\gamma', \mathsf{n}_\gamma \rangle = \eta_o.$$

Then, the following corollary can be given.

Corollary 4.2. *Let γ be the $\overline{\mathsf{d}}_o$-direction curve of a space curve α. Then, the Frenet apparatus $\{\mathsf{t}_\gamma, \mathsf{n}_\gamma, \mathsf{b}_\gamma, \kappa_\gamma, \tau_\gamma\}$ of γ can be obtained as*

$$\mathsf{t}_\gamma = \overline{\mathsf{d}}_o, \qquad \mathsf{n}_\gamma = -\varepsilon \frac{\mathsf{b}_q'}{\|\mathsf{b}_q'\|}, \qquad \mathsf{b}_\gamma = \varepsilon\mathsf{b}_q, \qquad \kappa_\gamma = \varepsilon\rho_o, \qquad \tau_\gamma = \eta_o,$$

where $\varepsilon = \pm 1$.

Corollary 4.3. *γ is a general helix if, and only if, α is a b_q-slant helix.*

Definition 4.4. Let α be a space curve in $\mathbb{E}^3$, $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ be the q-frame along α, and $\overline{\mathsf{d}}_n$ be the unit normal q-frame vector field of α. The integral curve of the vector field $\overline{\mathsf{d}}_n$ is called the $\overline{\mathsf{d}}_n$-direction curve of α. Namely, if $\zeta(s)$ is the $\overline{\mathsf{d}}_n$-direction curve of α, then $\zeta(s) = \int \overline{\mathsf{d}}_n(s)$ or $\overline{\mathsf{d}}_n(s) = \zeta'(s)$.

Similarly, let us obtain the Frenet apparatus $\{\mathsf{t}_\zeta, \mathsf{n}_\zeta, \mathsf{b}_\zeta, \kappa_\zeta, \tau_\zeta\}$ of the curve ζ. Taking into account the definition of the $\overline{\mathsf{d}}_n$-direction curve gives

$$\mathsf{t}_\zeta = \overline{\mathsf{d}}_n = \frac{1}{\sqrt{k_1^2 + k_2^2}}(-k_2\mathsf{n}_q + k_1\mathsf{b}_q).$$

Differentiating this equation with respect to s and applying Eq (2.1) yields

$$\overline{\mathsf{d}}_n' = \frac{(k_1'k_2 - k_2'k_1) - k_3(k_1^2 + k_2^2)}{(k_1^2 + k_2^2)^{3/2}}(k_1\mathsf{n}_q + k_2\mathsf{b}_q). \tag{4.2}$$

Since $\rho_n = \frac{(k_2'k_1 - k_1'k_2) + k_3(k_1^2+k_2^2)}{k_1^2+k_2^2}$, Eq (4.2) can be rewritten as

$$\overline{\mathsf{d}}_n' = -\rho_n \frac{\mathsf{t}'}{\|\mathsf{t}'\|}.$$

Then, we get $\|\overline{\mathsf{d}}_n'\| = \varepsilon\rho_n$, where $\varepsilon = \pm 1$. Thus, since $\mathsf{n}_\zeta = \frac{\overline{\mathsf{d}}_n'}{\|\overline{\mathsf{d}}_n'\|}$, we obtain

$$\mathsf{n}_\zeta = -\varepsilon \frac{\mathsf{t}'}{\|\mathsf{t}'\|}.$$

Also, the binormal vector b_ζ is given by $\mathsf{b}_\zeta = \varepsilon\mathsf{t}$. Besides, the curvatures of the $\overline{\mathsf{d}}_n$-direction curve ζ can be obtained as

$$\kappa_\zeta = \|\mathsf{t}_\zeta'\| = \varepsilon\rho_n, \quad \tau_\zeta = -\langle \mathsf{b}_\zeta', \mathsf{n}_\zeta\rangle = \eta_n.$$

Then, we can give the following corollary.

Corollary 4.5. *Let ζ be the $\overline{\mathsf{d}}_n$-direction curve of a space curve α. Then, the Frenet apparatus $\{\mathsf{t}_\zeta, \mathsf{n}_\zeta, \mathsf{b}_\zeta, \kappa_\zeta, \tau_\zeta\}$ of ζ can be found as*

$$\mathsf{t}_\zeta = \overline{\mathsf{d}}_n, \quad \mathsf{n}_\zeta = -\varepsilon\frac{\mathsf{t}'}{\|\mathsf{t}'\|}, \quad \mathsf{b}_\zeta = \varepsilon\mathsf{t}, \quad \kappa_\zeta = \varepsilon\rho_n, \quad \tau_\zeta = \eta_n,$$

where $\varepsilon = \pm 1$.

Corollary 4.6. *ζ is a general helix if, and only if, α is a general helix.*

Definition 4.7. Let α be a space curve in $\mathbb{E}^3$, $\{\mathsf{t}, \mathsf{n}_q, \mathsf{b}_q, \mathsf{k}\}$ be the q-frame along α, and $\overline{\mathsf{d}}_r$ be the unit rectifying q-frame vector field of α. The integral curve of the vector field $\overline{\mathsf{d}}_r$ is called the $\overline{\mathsf{d}}_r$-direction curve of α. Namely, if $\varphi(s)$ is the $\overline{\mathsf{d}}_r$-direction curve of α, then $\varphi(s) = \int \overline{\mathsf{d}}_r(s)$ or $\overline{\mathsf{d}}_r(s) = \varphi'(s)$.

In a similar way, let us calculate the Frenet apparatus $\{\mathsf{t}_\varphi, \mathsf{n}_\varphi, \mathsf{b}_\varphi, \kappa_\varphi, \tau_\varphi\}$ of φ. From the definition of the $\overline{\mathsf{d}}_r$-direction curve, we have

$$\mathsf{t}_\varphi = \overline{\mathsf{d}}_r = \frac{1}{\sqrt{k_1^2 + k_3^2}}(k_3\mathsf{t} + k_1\mathsf{b}_q).$$

If we differentiate this equation with respect to s, we find

$$\overline{\mathsf{d}}_r' = \frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{(k_1^2 + k_3^2)^{3/2}}(k_1\mathsf{t} - k_3\mathsf{b}_q). \tag{4.3}$$

Since $\rho_r = \frac{(k_3'k_1 - k_1'k_3) - k_2(k_1^2 + k_3^2)}{k_1^2 + k_3^2}$, from Eq (4.3) we obtain

$$\overline{\mathsf{d}}_r' = -\rho_r \frac{\mathsf{n}_q'}{\|\mathsf{n}_q'\|}.$$

Then, we have $\|\overline{\mathsf{d}}_r'\| = \varepsilon\rho_r$, where $\varepsilon = \pm 1$. Since $\mathsf{n}_\varphi = \frac{\overline{\mathsf{d}}_r'}{\|\overline{\mathsf{d}}_r'\|}$, we get

$$\mathsf{n}_\varphi = -\varepsilon \frac{\mathsf{n}_q'}{\|\mathsf{n}_q'\|}.$$

Also, the binormal vector b_φ of the curve φ is found as $\mathsf{b}_\varphi = \varepsilon \mathsf{n}_q$. In addition, the curvature and the torsion of the $\overline{\mathsf{d}}_r$-direction curve φ can be obtained as

$$\kappa_\varphi = \|\mathsf{t}_\varphi'\| = \varepsilon\rho_r, \qquad \tau_\varphi = -\langle \mathsf{b}_\varphi', \mathsf{n}_\varphi \rangle = \eta_r.$$

Then, the following corollary can be given.

Corollary 4.8. *Let φ be the $\overline{\mathsf{d}}_r$-direction curve of a space curve α. Then, the Frenet apparatus $\{\mathsf{t}_\varphi, \mathsf{n}_\varphi, \mathsf{b}_\varphi, \kappa_\varphi, \tau_\varphi\}$ of φ can be obtained as*

$$\mathsf{t}_\varphi = \overline{\mathsf{d}}_r, \qquad \mathsf{n}_\varphi = -\varepsilon \frac{\mathsf{n}_q'}{\|\mathsf{n}_q'\|}, \qquad \mathsf{b}_\varphi = \varepsilon \mathsf{n}_q, \qquad \kappa_\varphi = \varepsilon\rho_r, \qquad \tau_\varphi = \eta_r,$$

where $\varepsilon = \pm 1$.

Corollary 4.9. *φ is a general helix if, and only if, α is an n_q-slant helix.*

5. Conclusions

In this study, we obtained the derivative equations of the osculating q-frame, the normal q-frame, and the rectifying q-frame which have been defined along a space curve by using the Darboux vector field of the q-frame in Euclidean 3-space. Then, we defined some new slant helices and new integral curves and gave their characterizations.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

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