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Research article

H-Toeplitz operators on the Dirichlet type space

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Abstract: In this paper, we conducted a study of H-Toeplitz operators on the Dirichlet type space $\mathfrak{D}_t$, which included several aspects. To begin, we established the matrix representation of the H-Toeplitz operator S_φ with respect to the orthonormal basis of $\mathfrak{D}_t$. Subsequently, we characterized the compactness of S_φ in terms of the symbol φ. Furthermore, we developed a new method to investigate the algebraic properties of H-Toeplitz operators, including self-adjointness, diagonality, co-isometry, partial isometry as well as commutativity.

Keywords: Dirichlet type spaces; H-Toeplitz operators; compactness; self-adjointness
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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $dA = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. Set

$$dA_t(z) = (t+1)(1-|z|^2)^t dA(z), \quad t > -1.$$

The Sobolev space $L_t^{2,1}$ is the completion of the space of smooth functions f on $\mathbb{D}$ such that

$$\|f\|_t = \left\{ \left| \int_{\mathbb{D}} f\, dA_t \right|^2 + \int_{\mathbb{D}} \left(\left| \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}} \right|^2 \right) dA_t \right\}^{1/2} < \infty.$$

Clearly, $L_t^{2,1}$ is a Hilbert space with the inner product

$$\langle f, g\rangle_t = \int_{\mathbb{D}} f\, dA_t \int_{\mathbb{D}} \bar{g}\, dA_t + \int_{\mathbb{D}} \left(\frac{\partial f}{\partial z}\overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \bar{z}}\overline{\frac{\partial g}{\partial \bar{z}}} \right) dA_t.$$

The Dirichlet type space $\mathfrak{D}_t$ consists of all analytic functions $f \in L_t^{2,1}$ with $f(0) = 0$. The space $\mathfrak{D}_t$ has been widely investigated; see [1–4], for example. Note that $\mathfrak{D}_t$ is a closed subspace of $L_t^{2,1}$ and hence, $\mathfrak{D}_t$ is a Hilbert space. It is well-known that the Dirichlet type space $\mathfrak{D}_t$ corresponds to several important spaces at specific values of t: the Hardy space ($t = 1$), the Dirichlet space ($t = 0$), and the weighted Bergman space ($t > 1$).

Let $\mathbb{N}$ be the set of all positive integers. For $z \in \mathbb{D}$ and $k \in \mathbb{N}$, let

$$e_k(z) = \frac{\sqrt{\Gamma(k+t+1)}}{\sqrt{k\,k!\,\Gamma(t+2)}} z^k,$$

then $\{e_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $\mathfrak{D}_t$. $\mathfrak{D}_t$ is a reproducing kernel space with reproducing kernel given by

$$K_z^t(w) = \sum_{k=1}^{\infty} e_k(w)\overline{e_k(z)} = \sum_{k=1}^{\infty} \frac{\Gamma(k+t+1)}{k\,k!\,\Gamma(t+2)} w^k \overline{z}^k, \tag{1.1}$$

where Γ denotes the gamma function.

Let P be the orthogonal projection from $L_t^{2,1}$ onto $\mathfrak{D}_t$, which by the property of reproducing kernel K_w^t can be expressed as

$$P(f)(w) = \langle f, K_w^t \rangle_t, \quad w \in \mathbb{D} \tag{1.2}$$

for all $f \in L_t^{2,1}$.

Denote

$$\mathcal{M} = \left\{ \varphi \,\middle|\, \varphi, \frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial \overline{z}} \in L^{\infty}(\mathbb{D}) \right\},$$

where the derivatives are taken in the sense of distribution.

For any $\varphi \in \mathcal{M}$, the multiplication operator $M_\varphi : L_t^{2,1} \to L_t^{2,1}$ is defined as $M_\varphi(f) = \varphi f$ for $f \in L_t^{2,1}$. Given $\varphi \in \mathcal{M}$, the Toeplitz operator $T_\varphi : \mathfrak{D}_t \to \mathfrak{D}_t$ and the Hankel operator $H_\varphi : \mathfrak{D}_t \to \mathfrak{D}_t$ with symbol φ are defined by

$$T_\varphi = PM_\varphi \quad \text{and} \quad H_\varphi = PM_\varphi J,$$

respectively. Here, $J : \mathfrak{D}_t \to \overline{\mathfrak{D}_t}$ denotes the flip operator given by $J(e_k) = \overline{e_k}$ for all $k \in \mathbb{N}$, where $\overline{\mathfrak{D}_t} := \left\{\overline{f} : f \in \mathfrak{D}_t\right\}$. It can be checked that T_φ and H_φ induced by $\varphi \in \mathcal{M}$ are bounded operators on $\mathfrak{D}_t$.

The harmonic Dirichlet space $\mathfrak{D}_h$ is the closed subspace of $L_t^{2,1}$ consisting of all harmonic functions f with $f(0) = 0$. It is well-known that $\mathfrak{D}_h = \mathfrak{D}_t \oplus \overline{\mathfrak{D}_t}$. Consider the dilation operator $K : \mathfrak{D}_t \to \mathfrak{D}_h$ defined by

$$K(e_{2k}) = e_k \quad \text{and} \quad K(e_{2k-1}) = \overline{e_k}$$

for all $k \in \mathbb{N}$. It can be observed that K is bounded on $\mathfrak{D}_t$ with $\|K\| = 1$. Moreover, the adjoint K^* of the operator K is given by

$$K^*(e_k) = e_{2k} \quad \text{and} \quad K^*(\overline{e_k}) = e_{2k-1}$$

for all $k \in \mathbb{N}$. Hence, $K^*K = I$ on $\mathfrak{D}_t$ and $KK^* = I$ on $\mathfrak{D}_h$.

With these notations, we introduce the H-Toeplitz operator on the Dirichlet type space $\mathfrak{D}_t$, which is defined as follows.

Definition 1.1. *For $\varphi \in \mathcal{M}$, the H-Toeplitz operator $S_\varphi : \mathfrak{D}_t \to \mathfrak{D}_t$ with symbol φ is defined by*

$$S_\varphi(f) = PM_\varphi K(f)$$

for all $f \in \mathfrak{D}_t$.

The H-Toeplitz operator is closely related to both the Toeplitz and Hankel operators. In fact, for any $\varphi \in \mathcal{M}$ and $k \in \mathbb{N}$, we have

$$S_\varphi(e_{2k}) = PM_\varphi K(e_{2k}) = PM_\varphi(e_k) = T_\varphi(e_k) \tag{1.3}$$

and

$$S_\varphi(e_{2k-1}) = PM_\varphi K(e_{2k-1}) = PM_\varphi(\overline{e_k}) = PM_\varphi J(e_k) = H_\varphi(e_k). \tag{1.4}$$

Motivated by the notions of Toeplitz, Hankel and Slant Toeplitz operators, Arora and Paliwal [5] introduced the H-Toeplitz operators on the Hardy space, where they established the necessary and sufficient conditions under which H-Toeplitz operators become partial isometric, co-isometric, Hilbert-Schmidt and hyponormal. Moreover, they demonstrated that any H-Toeplitz operator is unitarily equivalent to a direct sum of a Toeplitz operator and a Hankel operator. The concept of H-Toeplitz operators is significant because it connects closely with a class of Hankel operators and a class of Toeplitz operators where the original operators are neither Hankel nor Toeplitz.

In recent years, H-Toeplitz operators on the Bergman space have been investigated by some specialists. Gupta and Singh [6] initiated the study of H-Toeplitz operators on the Bergman space, where the fundamental properties of the H-Toeplitz operators have been systematically studied, such as compactness, Fredholmness, co-isometry, partial isometry and commutativity. Later, Kim and Lee [7] established the contractivity and expansivity criteria for H-Toeplitz operators. Moreover, Liang et al. [8] studied the commutativity of H-Toeplitz operators with quasi-homogeneous symbols. In the recent paper [9], Ding and Chen characterized when the product of two H-Toeplitz operators with a bounded and a quasi-homogeneous symbol, respectively, becomes an H-Toeplitz operator. They also characterized when the product of an H-Toeplitz operator and a Toeplitz operator equals to another H-Toeplitz operator with bounded harmonic symbols.

It is well-known that Hardy, Bergman and Dirichlet spaces are the three most important classical Hilbert spaces of analytic functions in the unit disk. Despite the fruitful results achieved in the realm of H-Toeplitz operators on Hardy spaces and Bergman spaces, the theory of H-Toeplitz operators on Dirichlet spaces is largely unexplored. On the other hand, there is no any result in the literatures about H-Toeplitz operators on the weighted versions of these classical spaces. For the full generality and potential applicability, the main purpose of this article is to fill in these blanks by studying several fundamental properties of H-Toeplitz operators on the Dirichlet type space.

Before we mention the novelties of our work, it is worthwhile to recall from [10, 11] that the study of Toeplitz operators and Hankel operators on the Dirichlet space are essentially different from that on the Hardy space and the Bergman space. Moreover, nontrivial self-adjoint Toeplitz operator with C^1-symbol and non-scalar Toeplitz operator satisfying $T_\varphi^* = T_{\overline{\varphi}}$ do not exist on the Dirichlet space [12–14]. Since H-Toeplitz operators connect closely with Hankel operators and Toeplitz operators, it is natural to predict from the results mentioned in the literatures above that many techniques in the study of H-Toeplitz operators on the Hardy space and the Bergman space are not available on the Dirichlet space. For instance, one of the important steps to establish many properties (e.g., co-isometry and partial isometry) of an H-Toeplitz operator on the Hardy space and the Bergman space is using the adjoint of the H-Toeplitz operator, where the adjoint can be expressed as a composition of several specific operators. However, this cannot be done on the Dirichlet type space. To overcome this difficulty, our strategy is to establish an equivalent form of the Dirichlet Toeplitz operators under unitary conditions.

This new form behaves much better than the original and avoids the need to compute the adjoint of the H-Toeplitz operator.

This paper is organized as follows. In Section 2, we obtain the matrix representation of the H-Toeplitz operator with the polynomial harmonic symbol under the orthonormal basis of the Dirichlet type space $\mathfrak{D}_t$. In Section 3, we mainly characterize the compactness of H-Toeplitz operators. In Section 4, several algebraic properties of H-Toeplitz operators are investigated, including self-adjointness, diagonality, co-isometry, partial isometry as well as commutativity.

2. The matrix representation of the H-Toeplitz operator

In this section, we will present the matrix representation of the H-Toeplitz operator induced by the polynomial harmonic symbol under the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of the Dirichlet type space $\mathfrak{D}_t$. Leveraging the established relationships between the H-Toeplitz operator and both the Toeplitz and Hankel operators as outlined in Eqs (1.3) and (1.4), we will initially provide the matrix representations for the Toeplitz and Hankel operators.

We begin with the following lemma which will be needed in subsequent results.

Lemma 2.1. *Suppose $t > -1$ and $z \in \mathbb{D}$. For any n, m and $k \in \mathbb{N}$, the following identities hold in the Dirichlet type space $\mathfrak{D}_t$:*

(a) $\langle z^n, z^m\rangle_t = \begin{cases} \dfrac{n\, n!\,\Gamma(t+2)}{\Gamma(n+t+1)}, & \text{if } n = m,\\ 0, & \text{otherwise}.\end{cases}$

(b) $\langle \overline{z}^n, z^m\rangle_t = 0$ *and* $\langle z^k\overline{z}^n, z^m\rangle_t = \begin{cases} \dfrac{m\,(n+m)!\,\Gamma(t+2)}{\Gamma(n+m+t+1)}, & \text{if } k = n+m,\\ 0, & \text{otherwise}.\end{cases}$

(c) $P(\overline{z}^n z^m) = \begin{cases} \dfrac{m!\,\Gamma(m-n+t+1)}{(m-n)!\,\Gamma(m+t+1)} z^{m-n}, & \text{if } m > n,\\ 0, & \text{if } m \le n.\end{cases}$

Proof. By integration in polar coordinates, we have

$$\begin{aligned}\langle z^n, z^m\rangle_t &= nm(1+t)\int_{\mathbb{D}} z^{n-1}\,\overline{z}^{m-1}(1-|z|^2)^t dA_t(z)\\ &= n^2(1+t)\int_0^1 r^{n-1}(1-r)^t dr\\ &= \begin{cases} \dfrac{n\, n!\,\Gamma(t+2)}{\Gamma(n+t+1)}, & \text{if } n = m,\\ 0, & \text{otherwise}.\end{cases}\end{aligned}$$

This proves (a). Now, we show (b) in a similar fashion. The first equality is obvious. For the second one, we deduce that

$$\langle z^k\overline{z}^n, z^m\rangle_t = \int_{\mathbb{D}} \frac{\partial(z^k\overline{z}^n)}{\partial z}\overline{\left(\frac{\partial z^m}{\partial z}\right)} dA_t(z)$$

$$
\begin{aligned}
&= km(1+t)\int_{\mathbb{D}} z^{k-1}\overline{z}^{n+m-1}(1-|z|^2)^t dA(z)\\
&= m(1+t)(n+m)\int_0^1 r^{n+m-1}(1-r)^t dr\\
&= \begin{cases} \dfrac{m\,(n+m)!\,\Gamma(t+2)}{\Gamma(n+m+t+1)}, & \text{if } k=n+m,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Next, we show equality (c). By (1.2) and integration in polar coordinates, we get

$$
\begin{aligned}
P(\overline{z}^n z^m) = \langle \overline{z}^n z^m, K_z^t\rangle_t &= \int_{\mathbb{D}} \frac{\partial(\overline{w}^n w^m)}{\partial w}\overline{\left(\frac{\partial K_z^t(w)}{\partial w}\right)} dA_t(w)\\
&= m(1+t)\int_{\mathbb{D}} \overline{w}^n w^{m-1}\left(\sum_{k=1}^{\infty}\frac{\Gamma(k+t+1)}{\Gamma(t+2)k!} z^k\overline{w}^{k-1}\right)(1-|w|^2)^t dA(w)\\
&= m(1+t)\int_{\mathbb{D}} w^{m-1}\left(\sum_{k=1}^{\infty}\frac{\Gamma(k+t+1)}{\Gamma(t+2)k!} z^k\overline{w}^{n+k-1}\right)(1-|w|^2)^t dA(w)\\
&= m(1+t)\frac{\Gamma(m-n+t+1)}{\Gamma(t+2)(m-n)!} z^{m-n}\int_{\mathbb{D}} |w|^{2(m-1)}\,(1-|w|^2)^t dA(w)\\
&= m(1+t)\frac{\Gamma(m-n+t+1)}{\Gamma(t+2)(m-n)!} z^{m-n}\int_0^1 r^{m-1}(1-r)^t dr\\
&= \begin{cases} \dfrac{m!\,\Gamma(m-n+t+1)}{(m-n)!\,\Gamma(m+t+1)} z^{m-n}, & \text{if } m>n,\\ 0, & \text{if } m\le n.\end{cases}
\end{aligned}
$$

This ends the proof of Lemma 2.1. □

According to Lemma 2.1, we can find the matrix representations of Toeplitz operator T_φ and of Hankel operator H_φ on $\mathfrak{D}_t$ with symbol

$$
\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j \in \mathcal{M}, \quad z\in\mathbb{D},\ a_i, b_j\in\mathbb{C}.
$$

For any $m,\ n\in\mathbb{N}$, the (m,n)-th entry of the matrix representation of T_φ with respect to the orthonormal basis $\{e_k\}_{k=1}^\infty$ of $\mathfrak{D}_t$ is given by

$$
\begin{aligned}
\left\langle T_\varphi(e_n), e_m\right\rangle_t &= \left\langle PM_\varphi(e_n), e_m\right\rangle_t = \langle \varphi e_n, e_m\rangle_t\\
&= \frac{\sqrt{\Gamma(n+t+1)\,\Gamma(m+t+1)}}{\sqrt{n\,m\,n!\,m!}\,\Gamma(t+2)}\langle \varphi z^n, z^m\rangle_t\\
&= \frac{\sqrt{\Gamma(n+t+1)\,\Gamma(m+t+1)}}{\sqrt{n\,m\,n!\,m!}\,\Gamma(t+2)}\left(\sum_{i=0}^{\infty} a_i\left\langle z^{i+n}, z^m\right\rangle_t + \sum_{j=1}^{\infty} b_j\left\langle \overline{z}^j z^n, z^m\right\rangle_t\right)\\
&= \begin{cases} \dfrac{\sqrt{m\,m!\,\Gamma(n+t+1)}}{\sqrt{n\,n!\,\Gamma(m+t+1)}} a_{m-n}, & \text{if } m\ge n,\\ \dfrac{\sqrt{m\,n!\,\Gamma(m+t+1)}}{\sqrt{n\,m!\,\Gamma(n+t+1)}} b_{n-m}, & \text{if } m<n.\end{cases}
\end{aligned} \tag{2.1}
$$

Therefore, the matrix representation of T_φ is explicitly given by

$$T_\varphi = \begin{bmatrix} a_0 & \frac{1}{\sqrt{2+t}}\, b_1 & \frac{\sqrt{2}}{\sqrt{(3+t)(2+t)}}\, b_2 & \frac{\sqrt{6\Gamma(2+t)}}{\sqrt{\Gamma(5+t)}}\, b_3 & \cdots \\ \frac{2}{\sqrt{2+t}}\, a_1 & a_0 & \frac{\sqrt{2}}{\sqrt{3+t}}\, b_1 & \frac{\sqrt{6}}{\sqrt{(4+t)(3+t)}}\, b_2 & \cdots \\ \frac{3\sqrt{2}}{\sqrt{(3+t)(2+t)}}\, a_2 & \frac{3}{\sqrt{2(3+t)}}\, a_1 & a_0 & \frac{\sqrt{3}}{\sqrt{4+t}}\, b_1 & \cdots \\ \frac{4\sqrt{6\Gamma(2+t)}}{\sqrt{\Gamma(5+t)}}\, a_3 & \frac{2\sqrt{6}}{\sqrt{(4+t)(3+t)}}\, a_2 & \frac{4}{\sqrt{3(4+t)}}\, a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{2.2}$$

and the matrix representation of its adjoint T_φ^* is given by

$$T_\varphi^* = \begin{bmatrix} \overline{a_0} & \frac{2}{\sqrt{2+t}}\, \overline{a_1} & \frac{3\sqrt{2}}{\sqrt{(3+t)(2+t)}}\, \overline{a_2} & \frac{4\sqrt{6\Gamma(2+t)}}{\sqrt{\Gamma(5+t)}}\overline{a_3} & \cdots \\ \frac{1}{\sqrt{2+t}}\overline{b_1} & \overline{a_0} & \frac{3}{\sqrt{2(3+t)}}\overline{a_1} & \frac{2\sqrt{6}}{\sqrt{(4+t)(3+t)}}\, \overline{a_2} & \cdots \\ \frac{\sqrt{2}}{\sqrt{(3+t)(2+t)}}\overline{b_2} & \frac{\sqrt{2}}{\sqrt{3+t}}\overline{b_1} & \overline{a_0} & \frac{4}{\sqrt{2(3+t)}}\overline{a_1} & \cdots \\ \frac{\sqrt{6\Gamma(2+t)}}{\sqrt{\Gamma(5+t)}}\overline{b_3} & \frac{\sqrt{6}}{\sqrt{(4+t)(3+t)}}\overline{b_2} & \frac{\sqrt{3}}{\sqrt{4+t}}\overline{b_1} & \overline{a_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Next, we find the matrix representation of the Hankel operator. The (m, n)-th entry of the matrix representation of H_φ with respect to the orthonormal basis $\{e_k\}_{k=1}^\infty$ of $\mathfrak{D}_t$ is given by

$$\begin{aligned} \left\langle H_\varphi(e_n), e_m \right\rangle_t &= \left\langle PM_\varphi J(e_n), e_m \right\rangle_t = \left\langle PM_\varphi(\overline{e_n}), e_m \right\rangle_t = \langle \varphi \overline{e_n}, e_m \rangle_t \\ &= \frac{\sqrt{\Gamma(n+t+1)\,\Gamma(m+t+1)}}{\sqrt{nm\, n!m!}\,\Gamma(t+2)} \left\langle \varphi \overline{z}^n, z^m \right\rangle_t \\ &= \frac{\sqrt{\Gamma(n+t+1)\,\Gamma(m+t+1)}}{\sqrt{nm\, n!m!}\,\Gamma(t+2)} \left(\sum_{i=0}^{\infty} a_i \left\langle z^i \overline{z}^n, z^m \right\rangle_t + \sum_{j=1}^{\infty} b_j \left\langle \overline{z}^{j+n}, z^m \right\rangle_t \right) \\ &= \frac{(n+m)!\, \sqrt{m\,\Gamma(n+t+1)\,\Gamma(m+t+1)}}{\sqrt{n\, n!\, m!}\,\Gamma(m+n+t+1)} a_{m+n} \end{aligned} \tag{2.3}$$

for $m, n \in \mathbb{N}$.

Thus, the matrix representation of H_φ in explicit form is given by

$$H_\varphi = \begin{bmatrix} \frac{2}{2+t}\, a_2 & \frac{3}{(3+t)\sqrt{2+t}}\, a_3 & \frac{4!\,\sqrt{\Gamma(4+t)\,\Gamma(2+t)}}{3\sqrt{2}\,\Gamma(5+t)}\, a_4 & \frac{5!\,\sqrt{\Gamma(5+t)\,\Gamma(2+t)}}{2\sqrt{4!}\,\Gamma(6+t)}\, a_5 & \cdots \\ \frac{6}{(3+t)\sqrt{2+t}}\, a_3 & \frac{12}{(4+t)(3+t)}\, a_4 & \frac{5!\,\sqrt{\Gamma(4+t)\,\Gamma(3+t)}}{3\sqrt{2}\,\Gamma(6+t)}\, a_5 & \frac{6!\,\sqrt{\Gamma(5+t)\,\Gamma(3+t)}}{2\sqrt{4!}\,\Gamma(7+t)}\, a_6 & \cdots \\ \frac{4!\,\sqrt{\Gamma(2+t)}}{(4+t)\sqrt{2\,\Gamma(4+t)}}\, a_4 & \frac{5!\,\sqrt{\Gamma(3+t)\,\Gamma(4+t)}}{2\sqrt{2}\,\Gamma(6+t)}\, a_5 & \frac{6!\,\Gamma(4+t)}{3!\,\Gamma(7+t)}\, a_6 & \frac{7!\,\sqrt{\Gamma(5+t)\,\Gamma(4+t)}}{8\sqrt{3}\,\Gamma(8+t)}\, a_7 & \cdots \\ \frac{5!\,\sqrt{\Gamma(2+t)}}{(5+t)\sqrt{6\,\Gamma(5+t)}}\, a_5 & \frac{6!\,\sqrt{\Gamma(3+t)\,\Gamma(5+t)}}{2\sqrt{6}\,\Gamma(7+t)}\, a_6 & \frac{7!\,\sqrt{\Gamma(4+t)\,\Gamma(5+t)}}{6\sqrt{3}\,\Gamma(8+t)}\, a_7 & \frac{8!\,\Gamma(5+t)}{4!\,\Gamma(9+t)}\, a_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.4}$$

Note that the matrix of Hankel operator H_φ is independent of co-analytic term $\sum_{j=1}^\infty b_j \overline{z}^j$ of the symbol function φ. By a direct calculation, the matrix representation of its adjoint H_φ^* is of the following form:

$$H_\varphi^* = \begin{bmatrix} \frac{2}{2+t}\,\overline{a_2} & \frac{6}{(3+t)\sqrt{2+t}}\,\overline{a_3} & \frac{4!\sqrt{\Gamma(2+t)}}{(4+t)\sqrt{2\Gamma(4+t)}}\,\overline{a_4} & \frac{5!\sqrt{\Gamma(2+t)}}{(5+t)\sqrt{6\Gamma(5+t)}}\,\overline{a_5} & \cdots \\ \frac{3}{(3+t)\sqrt{2+t}}\,\overline{a_3} & \frac{12}{(4+t)(3+t)}\,\overline{a_4} & \frac{5!\sqrt{\Gamma(3+t)\Gamma(4+t)}}{3\sqrt{2}\Gamma(6+t)}\,\overline{a_5} & \frac{6!\sqrt{\Gamma(3+t)\Gamma(5+t)}}{2\sqrt{6}\Gamma(7+t)}\,\overline{a_6} & \cdots \\ \frac{4!\sqrt{\Gamma(4+t)\Gamma(2+t)}}{3\sqrt{2}\Gamma(5+t)}\,\overline{a_4} & \frac{5!\sqrt{\Gamma(4+t)\Gamma(3+t)}}{3\Gamma(6+t)}\,\overline{a_5} & \frac{6!\Gamma(4+t)}{3!\Gamma(7+t)}\,\overline{a_6} & \frac{7!\sqrt{\Gamma(4+t)\Gamma(5+t)}}{6\sqrt{3}\Gamma(8+t)}\,\overline{a_7} & \cdots \\ \frac{5!\sqrt{\Gamma(5+t)\Gamma(2+t)}}{2\sqrt{4!}\Gamma(6+t)}\,\overline{a_5} & \frac{6!\sqrt{\Gamma(5+t)\Gamma(3+t)}}{2\sqrt{4!}\Gamma(7+t)}\,\overline{a_6} & \frac{7!\sqrt{\Gamma(5+t)\Gamma(4+t)}}{8\sqrt{3}\Gamma(8+t)}\,\overline{a_7} & \frac{8!\Gamma(5+t)}{4!\Gamma(9+t)}\,\overline{a_8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Observe that $H_\varphi^* = H_{\widehat{\varphi}}^\top$ for $\widehat{\varphi}(z) = \sum_{i=1}^\infty \overline{a_i} z^i + \sum_{j=1}^\infty \overline{b_j}\bar{z}^j$ (each b_j can be zero), where $H_{\widehat{\varphi}}^\top$ denotes the transpose of the matrix representation of $H_{\widehat{\varphi}}$.

Next we find the matrix representation of H-Toeplitz operator S_φ. Clearly, it follows from (1.3), (1.4), (2.1) and (2.3) that

$$\left\langle S_\varphi(e_{2n}), e_m \right\rangle_t = \left\langle T_\varphi(e_n), e_m \right\rangle_t = \begin{cases} \dfrac{\sqrt{m\, m!\,\Gamma(n+t+1)}}{\sqrt{n\, n!\,\Gamma(m+t+1)}} a_{m-n}, & \text{if } m \geq n, \\ \dfrac{\sqrt{m\, n!\,\Gamma(m+t+1)}}{\sqrt{n\, m!\,\Gamma(n+t+1)}} b_{n-m}, & \text{if } m < n, \end{cases} \tag{2.5}$$

and

$$\left\langle S_\varphi(e_{2n-1}), e_m \right\rangle_t = \left\langle H_\varphi(e_n), e_m \right\rangle_t = \frac{(n+m)!\sqrt{m\,\Gamma(n+t+1)\,\Gamma(m+t+1)}}{\sqrt{n\, n!\, m!}\,\Gamma(m+n+t+1)} a_{m+n}, \tag{2.6}$$

where $m, n \in \mathbb{N}$. Thus, the matrix representation of S_φ with respect to the orthonormal basis $\{e_k\}_{k=1}^\infty$ of $\mathfrak{D}_t$ is given by

$$S_\varphi = \begin{bmatrix} \frac{2}{2+t}\,a_2 & a_0 & \frac{3}{(3+t)\sqrt{2+t}}\,a_3 & \frac{1}{\sqrt{2+t}}\,b_1 & \cdots \\ \frac{6}{(3+t)\sqrt{2+t}}\,a_3 & \frac{2}{\sqrt{2+t}}\,a_1 & \frac{12}{(4+t)(3+t)}\,a_4 & a_0 & \cdots \\ \frac{4!\sqrt{\Gamma(2+t)}}{(4+t)\sqrt{2\Gamma(4+t)}}a_4 & \frac{3\sqrt{2}}{\sqrt{(3+t)(2+t)}}\,a_2 & \frac{5!\sqrt{\Gamma(3+t)\Gamma(4+t)}}{2\sqrt{2}\Gamma(6+t)}\,a_5 & \frac{3}{\sqrt{2(3+t)}}\,a_1 & \cdots \\ \frac{5!\sqrt{\Gamma(2+t)}}{(5+t)\sqrt{6\Gamma(5+t)}}\,a_5 & \frac{4\sqrt{6\Gamma(2+t)}}{\sqrt{\Gamma(5+t)}}\,a_3 & \frac{6!\sqrt{\Gamma(3+t)\Gamma(5+t)}}{2\sqrt{6}\Gamma(7+t)}\,a_6 & \frac{2\sqrt{6}}{\sqrt{(4+t)(3+t)}}\,a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{2.7}$$

and the matrix representation of its adjoint is given by

$$S_\varphi^* = \begin{bmatrix} \frac{2}{2+t}\,\overline{a_2} & \frac{6}{(3+t)\sqrt{2+t}}\,\overline{a_3} & \frac{4!\sqrt{\Gamma(2+t)}}{(4+t)\sqrt{2\Gamma(4+t)}}\,\overline{a_4} & \frac{5!\sqrt{\Gamma(2+t)}}{(5+t)\sqrt{6\Gamma(5+t)}}\,\overline{a_5} & \cdots \\ \overline{a_0} & \frac{2}{\sqrt{2+t}}\,\overline{a_1} & \frac{3\sqrt{2}}{\sqrt{(3+t)(2+t)}}\,\overline{a_2} & \frac{4\sqrt{6\Gamma(2+t)}}{\sqrt{\Gamma(5+t)}}\,\overline{a_3} & \cdots \\ \frac{3}{(3+t)\sqrt{2+t}}\,\overline{a_3} & \frac{12}{(4+t)(3+t)}\,\overline{a_4} & \frac{5!\sqrt{\Gamma(3+t)\Gamma(4+t)}}{2\sqrt{2}\Gamma(6+t)}\,\overline{a_5} & \frac{6!\sqrt{\Gamma(3+t)\Gamma(5+t)}}{2\sqrt{6}\Gamma(7+t)}\,\overline{a_6} & \cdots \\ \frac{1}{\sqrt{2+t}}\,\overline{b_1} & \overline{a_0} & \frac{3}{\sqrt{2(3+t)}}\,\overline{a_1} & \frac{2\sqrt{6}}{\sqrt{(4+t)(3+t)}}\,\overline{a_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Remark 2.1. *It can be seen from* (2.2), (2.4) *and* (2.7) *that the matrix representations of the Toeplitz operator T_φ and the Hankel operator H_φ can be obtained by deleting every odd and even column of* *the H-Toeplitz operator S_φ, respectively. The matrix of S_φ is an upper triangular matrix if the symbol $\varphi \in \mathcal{M}$ is co-analytic. However, it cannot be lower triangular. Additionally, it is worthwhile to mention that an $n \times n$ Dirichlet type H-Toeplitz matrix defined as follows has $2n$ degree of freedom rather than n^2. Consequently, for large n, it is relatively easy to solve the system of linear equations when the coefficient matrix is a Dirichlet type H-Toeplitz matrix.*

Definition 2.1. *Let $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j \in \mathcal{M}$ with $z \in \mathbb{D}$ and $a_i, b_j \in \mathbb{C}$. We define an infinite matrix $(c_{m,n})$ as a Dirichlet type H-Toeplitz matrix if its (m,n)-th entry satisfies the following relation:*

$$c_{m,n} = \begin{cases} \dfrac{\sqrt{m\, m!\, \Gamma(l+t+1)}}{\sqrt{l\, l!\, \Gamma(m+t+1)}} a_{m-l}, & \text{if } n = 2l \text{ and } m \geq l, \\ \dfrac{\sqrt{m\, l!\, \Gamma(m+t+1)}}{\sqrt{l\, m!\, \Gamma(l+t+1)}} b_{l-m}, & \text{if } n = 2l \text{ and } m < l, \\ \dfrac{(l+m)!\, \sqrt{m\, \Gamma(l+t+1)\, \Gamma(m+t+1)}}{\sqrt{l\, l!\, m!}\, \Gamma(m+l+t+1)} a_{m+l}, & \text{if } n = 2l-1, \end{cases}$$

where m, n and l are all in $\mathbb{N}$.

3. The compactness of H-Toeplitz operators

This section is mainly concerned with the compactness of H-Toeplitz operators. It is well-known that compact operators behave like operators on finite-dimensional vector spaces and play a fundamental role in operator theory.

The following proposition follows easily from the definition of the H-Toeplitz operator S_φ.

Proposition 3.1. *Suppose that $a, b \in \mathbb{C}$ and $\varphi, \psi \in \mathcal{M}$. Then*

(a) $S_{a\varphi + b\psi} = aS_\varphi + bS_\psi$;
(b) S_φ is a bounded linear operator on $\mathfrak{D}_t$ with $\|S_\varphi\|_t \leq \left\|\frac{\partial \varphi}{\partial z}\right\|_\infty + \|\varphi\|_\infty$.

Let $L_a^2(dA_t)$ be the weighted Bergman space on $\mathbb{D}$, which consists of all analytic functions in $L^2(\mathbb{D}, dA_t)$. We use the notations $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle_2$ to represent the norm and inner product in $L_a^2(dA_t)$, respectively.

Similar to the proof of [15, Lemma 12], we have the following result.

Lemma 3.1. *The identity operator $\mathfrak{I}$ from $\mathfrak{D}_t$ into $L_a^2(dA_t)$ defined by $\mathfrak{I}f = f$ for any $f \in \mathfrak{D}_t$ is compact.*

From Lemma 3.1, we conclude that for any sequence $\{f_k\}_k$ converging weakly to 0 in $\mathfrak{D}_t$ (write $f_k \xrightarrow{w} 0$ for short), the sequence $\{\|f_k\|_2\}_k$ converges to 0 as $k \to \infty$.

The next lemma will be utilized in the compactness of H-Toeplitz operators.

Lemma 3.2. *For any $\varphi \in \mathcal{M}$, $S_\varphi^* - K^* P_h M_{\overline{\varphi}}$ is compact on $\mathfrak{D}_t$, where P_h is the orthogonal projection from $L_t^{2,1}$ onto $\mathfrak{D}_h$.*

Proof. For any $f, g \in \mathfrak{D}_t$, we have

$$\begin{aligned} \left\langle (S_\varphi^* - K^* P_h M_{\overline{\varphi}})(f), g \right\rangle_t &= \left\langle f, S_\varphi(g) \right\rangle_t - \left\langle K^* P_h M_{\overline{\varphi}}(f), g \right\rangle_t \\ &= \left\langle f, PM_\varphi K(g) \right\rangle_t - \left\langle P_h M_{\overline{\varphi}}(f), K(g) \right\rangle_t \end{aligned}$$

$$
\begin{aligned}
&= \langle f, \varphi K(g)\rangle_t - \langle \overline{\varphi} f, K(g)\rangle_t \\
&= \left\langle \frac{\partial f}{\partial z}, K(g)\frac{\partial \varphi}{\partial z}\right\rangle_2 + \left\langle \frac{\partial f}{\partial z}, \varphi\frac{\partial (K(g))}{\partial z}\right\rangle_2 \\
&\quad - \left\langle f\frac{\partial \overline{\varphi}}{\partial z}, \frac{\partial (K(g))}{\partial z}\right\rangle_2 - \left\langle \overline{\varphi}\frac{\partial f}{\partial z}, \frac{\partial (K(g))}{\partial z}\right\rangle_2 \\
&= \left\langle \frac{\partial f}{\partial z}, \frac{\partial \varphi}{\partial z}K(g)\right\rangle_2 - \left\langle f, \frac{\partial \varphi}{\partial \overline{z}}\frac{\partial (K(g))}{\partial z}\right\rangle_2 .
\end{aligned}
$$

Since $\mathfrak{D}_t$ is contained in $L_a^2(dA_t)$, it follows that

$$
\begin{aligned}
\left|\left\langle (S_\varphi^* - K^* P_h M_{\overline{\varphi}})(f), g\right\rangle_t\right| &\leq \left|\left\langle \frac{\partial f}{\partial z}, \frac{\partial \varphi}{\partial z}K(g)\right\rangle_2\right| + \left|\left\langle f, \frac{\partial \varphi}{\partial \overline{z}}\frac{\partial (K(g))}{\partial z}\right\rangle_2\right| \\
&\leq \left\|\frac{\partial f}{\partial z}\right\|_2 \left\|\frac{\partial \varphi}{\partial z}K(g)\right\|_2 + \|f\|_2 \left\|\frac{\partial \varphi}{\partial \overline{z}}\frac{\partial (K(g))}{\partial z}\right\|_2 \\
&\leq \|\varphi\|_{\mathcal{M}} \left(\|f\|_t \|K(g)\|_2 + \|f\|_2 \|K(g)\|_t\right) \\
&\leq \|\varphi\|_{\mathcal{M}} \left(\|f\|_t \|g\|_2 + \|f\|_2 \|g\|_t\right),
\end{aligned}
$$

where

$$
\|\varphi\|_{\mathcal{M}} = \operatorname*{esssup}_{z\in\mathbb{D}} \max\left\{|\varphi|, \left|\frac{\partial \varphi}{\partial z}\right|, \left|\frac{\partial \varphi}{\partial \overline{z}}\right|\right\}.
$$

Let $\{f_k\}_k$ be any sequence converging weakly to 0 in $\mathfrak{D}_t$. Taking $f = f_k$ and $g = (S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k$ in the above, we obtain

$$
\begin{aligned}
&\left\|(S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k\right\|_t^2 \\
&\leq \|\varphi\|_{\mathcal{M}} \left[\|f_k\|_t \left\|(S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k\right\|_2 + \|f_k\|_2 \left\|(S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k\right\|_t\right].
\end{aligned}
$$

Note that $(S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k \xrightarrow{w} 0$ in $\mathfrak{D}_t$ as $k \to \infty$. It follows from Lemma 3.1 that $\|f_k\|_2$ and $\left\|(S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k\right\|_2 \to 0$ as $k \to \infty$. Therefore,

$$
\left\|(S_\varphi^* - K^* P_h M_{\overline{\varphi}}) f_k\right\|_t \to 0
$$

as $k \to \infty$, which implies that $S_\varphi^* - K^* P_h M_{\overline{\varphi}}$ is compact on $\mathfrak{D}_t$. This completes the proof of the lemma. □

Remark 3.1. *Let $\check{P}_h$ denote the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto the harmonic Bergman space L_h^2. We have known that the adjoint S_φ^* of an H-Toeplitz operator S_φ is equal to $K^* M_{\overline{\varphi}}$ (resp., $K^* \check{P}_h M_{\overline{\varphi}}$) on the Hardy space [5] (resp., Bergman space [6]). However, there is no analogues identity on the Dirichlet type space. The situation is different from that of Hardy space and Bergman space.*

Lemma 3.3. *[16, Proposition 7.2] If $\varphi \in L^1(\mathbb{D}, dA_t)$ is harmonic, then $\check{T}_\varphi$ is compact on $L_a^2(dA_t)$ if and only if $\varphi = 0$.*

We apply the above results to show the compactness of the H-Toeplitz operator on $\mathfrak{D}_t$.

Theorem 3.1. *Suppose $t > -1$ and $\varphi \in \mathcal{M}$ is co-analytic. Then, S_φ is a compact operator on $\mathfrak{D}_t$ if and only if $\varphi = 0$.*

Proof. If $\varphi = 0$, then S_φ is trivially compact on $\mathfrak{D}_t$.

Conversely, assume that S_φ is compact on $\mathfrak{D}_t$. We are going to show that $\varphi = 0$. Otherwise, if $\varphi \neq 0$, then $\check{T}_\varphi$ is not compact on $L_a^2(dA_t)$ by Lemma 3.3. Hence, there is a sequence $\{f_k\}_k \subseteq L_a^2(dA_t)$, $\|f_k\|_2 = 1$, $f_k \xrightarrow{w} 0$ such that $\left\|\check{T}_\varphi(f_k)\right\|_2 \not\to 0$ as $k \to \infty$. Thus, $\|\varphi f_k\|_2 \not\to 0$, that is,

$$\int_{\mathbb{D}} |\varphi|^2 |f_k|^2 \, dA_t \not\to 0$$

as $k \to \infty$.

Note that S_φ is compact on $\mathfrak{D}_t$, so is S_φ^* by [16, Theorem 1.16]. We deduce that $K^* P_h M_{\overline{\varphi}}$ is also compact on $\mathfrak{D}_t$ by Lemma 3.2. This implies that $S_\varphi K^* P_h M_{\overline{\varphi}}$ is compact on $\mathfrak{D}_t$. Let

$$F_k := \int_0^z f_k(w) \, dw,$$

then $F_k \xrightarrow{w} 0$ in $\mathfrak{D}_t$ and $\|F_k\|_t = 1$. So, $\left\|S_\varphi K^* P_h M_{\overline{\varphi}}(F_k)\right\|_t \to 0$, that is,

$$\left\| |\varphi|^2 F_k \right\|_t \to 0$$

as $k \to \infty$. Thus, we have

$$\left| \left\langle |\varphi|^2 F_k, F_k \right\rangle_t \right| \leq \left\| |\varphi|^2 F_k \right\|_t \|F_k\|_t = \left\| |\varphi|^2 F_k \right\|_t \to 0$$

as $k \to \infty$. However,

$$\begin{aligned} \left\langle |\varphi|^2 F_k, F_k \right\rangle_t &= \left\langle \frac{\partial \left(|\varphi|^2 F_k \right)}{\partial z}, \frac{\partial F_k}{\partial z} \right\rangle_2 \\ &= \left\langle F_k \frac{\partial |\varphi|^2}{\partial z}, \frac{\partial F_k}{\partial z} \right\rangle_2 + \left\langle |\varphi|^2 \frac{\partial F_k}{\partial z}, \frac{\partial F_k}{\partial z} \right\rangle_2 \\ &= \left\langle \varphi F_k \frac{\partial \overline{\varphi}}{\partial z}, f_k \right\rangle_2 + \left\langle |\varphi|^2 f_k, f_k \right\rangle_2 \not\to 0, \end{aligned}$$

since

$$\left| \left\langle \varphi F_k \frac{\partial \overline{\varphi}}{\partial z}, f_k \right\rangle_2 \right| \leq \|\varphi\|_{\mathcal{M}}^2 \|F_k\|_2 \|f_k\|_t \to 0$$

and

$$\left\langle |\varphi|^2 f_k, f_k \right\rangle_2 = \int_{\mathbb{D}} |\varphi|^2 |f_k|^2 \, dA_t \not\to 0$$

as $k \to \infty$. This contradiction shows that $\varphi = 0$. This ends the proof of Theorem 3.1. $\square$

Lemma 3.4. *Suppose $t > -1$ and $z, w \in \mathbb{D}$. The dilation operator $K : \mathfrak{D}_t \to \mathfrak{D}_h$ satisfies*

$$K(K_z^t)(w) = \sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{k\Gamma(t+2)\sqrt{2\,(2k)!\,k!}} \overline{z}^{2k} w^k + \sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{k\,k!\,(2k-1)\,(2k-1)!}} \overline{z}^{2k-1} \overline{w}^k.$$

Proof. For $z, w \in \mathbb{D}$, by (1.1) and the definition of K, we obtain

$$\begin{aligned} K(K_z^t)(w) &= \sum_{k=1}^{\infty} \overline{e_{2k}(z)} K(e_{2k})(w) + \sum_{k=1}^{\infty} \overline{e_{2k-1}(z)} K(e_{2k-1})(w) \\ &= \sum_{k=1}^{\infty} \overline{e_{2k}(z)} e_k(w) + \sum_{k=1}^{\infty} \overline{e_{2k-1}(z)}\, \overline{e_k(w)} \\ &= \sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t+1)\,\Gamma(k+t+1)}}{k\,\Gamma(t+2)\sqrt{2\,k!\,(2k)!}} \bar{z}^{2k} w^k \\ &\quad + \sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t)\,\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{k\,k!\,(2k-1)\,(2k-1)!}} \bar{z}^{2k-1} \bar{w}^k. \end{aligned}$$

This finishes the proof of the lemma. □

For $t > -1$ and $z, w \in \mathbb{D}$, denote

$$h_z^t(w) := K(k_z^t)(w),$$

where

$$k_z^t(w) = \frac{K_z^t(w)}{\left\|K_z^t\right\|_t}$$

is the normalized reproducing kernel of $\mathfrak{D}_t$. Let $\partial\mathbb{D}$ be the boundary of the unit disk $\mathbb{D}$. Next, we will discuss the boundary behavior of h_z^t.

Lemma 3.5. *For $t \geq 0$ and $z \in \mathbb{D}$, we have $h_z^t \to 0$ as $z \to \partial\mathbb{D}$.*

Proof. For $t > 0$, by the Stirling's formula, we have

$$\left\|K_z^t\right\|_t^2 = \left\langle K_z^t, K_z^t \right\rangle_t = K_z^t(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+t+1)}{k\,k!\,\Gamma(t+2)} |z|^{2k} \sim \sum_{k=1}^{\infty} \frac{\Gamma(k+t)}{k!\,\Gamma(t)} |z|^{2k} = \frac{1}{(1-|z|^2)^t}, \tag{3.1}$$

where the notation " $\sim$ " is used to denote that the ratio of the two sides tends to 1 as $k \to \infty$.

For $t = 0$, we have

$$\left\|K_z^t\right\|_t^2 = \left\langle K_z^t, K_z^t \right\rangle_t = K_z^t(z) = \sum_{k=1}^{\infty} \frac{|z|^{2k}}{k} = \log \frac{1}{1-|z|^2}.$$

We conclude that

$$h_z^t = K(k_z^t) = \frac{K(K_z^t)}{\left\|K_z^t\right\|_t} \to 0$$

as $z \to \partial\mathbb{D}$. □

Proposition 3.2. *Suppose $t \geq 0$ and $\varphi \in \mathcal{M}$. Then, S_φ is not bounded below on $\mathfrak{D}_t$.*

Proof. Given $z \in \mathbb{D}$, by the dominated convergence theorem, Lemma 3.5, and (3.1), we have

$$
\begin{aligned}
&\left\|S_{\varphi}(k_z^t)\right\|_t^2 \\
&= \left\|PM_{\varphi}(h_z^t)\right\|_t^2 \leq \left\|\varphi h_z^t\right\|_t^2 \\
&= \left|\int_{\mathbb{D}} \varphi h_z^t \, dA_t\right|^2 + \int_{\mathbb{D}} \left(\left|\frac{\partial(\varphi h_z^t)}{\partial w}\right|^2 + \left|\frac{\partial(\varphi h_z^t)}{\partial \overline{w}}\right|^2\right) dA_t \\
&= \left|\int_{\mathbb{D}} \varphi h_z^t \, dA_t\right|^2 + \int_{\mathbb{D}} \left| h_z^t(w)\frac{\partial \varphi(w)}{\partial w} + \frac{\varphi(w)}{\left\|K_z^t\right\|_t} \sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t+1)\,\Gamma(k+t+1)}}{\Gamma(t+2)\,\sqrt{2\,k!\,(2k)!}} \overline{z}^{2k} w^{k-1}\right|^2 dA_t(w) \\
&\quad + \int_{\mathbb{D}} \left| h_z^t(w)\frac{\partial \varphi(w)}{\partial \overline{w}} + \frac{\varphi(w)}{\left\|K_z^t\right\|_t} \sum_{k=1}^{\infty} \frac{\sqrt{k\,\Gamma(2k+t)\,\Gamma(k+t+1)}}{\Gamma(t+2)\,\sqrt{k!\,(2k-1)\,(2k-1)!}} \overline{z}^{2k-1} \overline{w}^{k-1}\right|^2 dA_t(w) \\
&\to 0
\end{aligned}
$$

as $z \to \partial\mathbb{D}$, from which we deduce that S_{φ} is not bounded below on $\mathfrak{D}_t$. □

Recall that for a bounded linear operator T defined on a Hilbert space, the approximated point spectrum of operator T is defined as the set

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

See [17]. Thus, for the H-Toeplitz operator S_{φ} defined on $\mathfrak{D}_t$, Proposition 3.2 implies that $0 \in \sigma_{ap}(S_{\varphi})$ for $t \geq 0$ and $\varphi \in \mathcal{M}$.

At the end of this section, we explore the question of when an H-Toeplitz operator is Fredholm. For more details concerning Fredholm operators; see [18, CHAPTER XI §2].

The subsequent proposition illustrates the property of the Fredholm operator on $\mathfrak{D}_t$ from the perspective of weakly convergent nets.

Proposition 3.3. *Suppose $t > -1$. If T is a Fredholm operator on $\mathfrak{D}_t$, then, there is no $\{h_z\}_{z\in\mathbb{D}}$ of unit vectors in $\mathfrak{D}_t$ such that $h_z \xrightarrow{w} 0$ as $z \to \partial\mathbb{D}$ and* $\lim \|Th_z\|_t = 0$.

Proof. Suppose there is $\{h_z\}_{z\in\mathbb{D}}$ of unit vectors in $\mathfrak{D}_t$ such that $h_z \xrightarrow{w} 0$ as $z \to \partial\mathbb{D}$ and $\lim \|Th_z\|_t = 0$. We shall provide a proof by contradiction. Since T is Fredholm, there exists a bounded operator B and a compact operator E on $\mathfrak{D}_t$ such that $BT = I + E$. Then,

$$\left|I - \|BTh_z\|_t\right| = \left|\|h_z\|_t - \|BTh_z\|_t\right| \leq \|Eh_z\|_t \to 0$$

as $z \to \partial\mathbb{D}$ by the compactness of E. This implies that $\|BTh_z\|_t \to 1$ as $z \to \partial\mathbb{D}$, which contradicts to the assumption $\lim \|Th_z\|_t = 0$. □

Theorem 3.2. *Suppose $t \geq 0$ and $\varphi \in \mathcal{M}$. Then, there is no nonzero H-Toeplitz operator S_{φ} on $\mathfrak{D}_t$ which is Fredholm.*

Proof. Assume that the H-Toeplitz operator S_{φ} is Fredholm on $\mathfrak{D}_t$ for some $\varphi \in \mathcal{M}$. Take the net $\left\{k_z^t\right\}_{z\in\mathbb{D}}$ of normalized kernels on $\mathfrak{D}_t$. Then, $k_z^t \to 0$ weakly and also $\left\|S_{\varphi}k_z^t\right\|_t \to 0$ as $z \to \partial\mathbb{D}$ by the proof of Proposition 3.2. This contradicts the fact that S_{φ} is a Fredholm operator by Proposition 3.3. It follows that S_{φ} is a Fredholm operator on $\mathfrak{D}_t$ if and only if $\varphi = 0$ in $\mathcal{M}$. □

Recall that the essential spectrum of a bounded linear operator T is given by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

For $t \geq 0$ and $\varphi \in \mathcal{M}$, we derive that the essential spectrum of H-Toeplitz operator S_φ on $\mathfrak{D}_t$ is nonempty by the above theorem, since $0 \in \sigma_e(S_\varphi)$ in this case.

4. Algebraic properties of H-Toeplitz operators on $\mathfrak{D}_t$

In this section, we investigate some algebraic properties of H-Toeplitz operators on $\mathfrak{D}_t$, which include self-adjointness, diagonality, co-isometry, partial isometry as well as commutativity.

Let

$$\mathfrak{H} = \left\{\varphi \in \mathcal{M} : \varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j,\ z \in \mathbb{D} \text{ and } a_i, b_j \in \mathbb{C}\right\}.$$

In the next theorem, we develop a new method to demonstrate that a nonzero H-Toeplitz operator S_φ induced by $\varphi \in \mathfrak{H}$ can never be a self-adjoint operator on $\mathfrak{D}_t$.

Theorem 4.1. *Let $t > -1$ and $\varphi \in \mathfrak{H}$. Then the H-Toeplitz operator S_φ is self-adjoint on $\mathfrak{D}_t$ if and only if $\varphi = 0$.*

Proof. Let $\varphi \in \mathfrak{H}$ defined by $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j$, where $z \in \mathbb{D}$ and $a_i, b_j \in \mathbb{C}$. The backward implication is trivial. Now, suppose that S_φ is self-adjoint. Then $(S_\varphi^* - S_\varphi)f = 0$ for any $f \in \mathfrak{D}_t$. Taking $f(z) = e_1(z) = z$, we apply the reproducing property of K_z^t, Lemma 3.4, and (1.4) to get

$$\begin{aligned}
S_\varphi^*(e_1)(z) &= \left\langle S_\varphi^*(e_1), K_z^t \right\rangle_t = \left\langle e_1, S_\varphi(K_z^t) \right\rangle_t \\
&= \left\langle e_1, PM_\varphi K(K_z^t) \right\rangle_t = \left\langle e_1, \varphi \cdot K(K_z^t) \right\rangle_t \\
&= \int_{\mathbb{D}} \overline{\frac{\partial\left(\varphi \cdot K(K_z^t)\right)}{\partial w}(w)}\, dA_t(w) \\
&= \int_{\mathbb{D}} \overline{K(K_z^t)(w)}\, \overline{\frac{\partial \varphi}{\partial w}(w)}\, dA_t(w) + \int_{\mathbb{D}} \overline{\varphi(w)}\, \overline{\frac{\partial (K(K_z^t))}{\partial w}(w)}\, dA_t(w) \\
&= \int_{\mathbb{D}} \left(\sum_{i=1}^{\infty} i\, \overline{a_i}\, \overline{w}^{i-1}\right)\left(\sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{k\Gamma(t+2)\sqrt{2\,(2k)!\,k!}} z^{2k}\overline{w}^k \right.\\
&\quad \left. + \sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{k\,k!\,(2k-1)\,(2k-1)!}} z^{2k-1} w^k\right) dA_t(w) \\
&\quad + \int_{\mathbb{D}} \left(\sum_{i=0}^{\infty} \overline{a_i}\, \overline{w}^i + \sum_{j=1}^{\infty} \overline{b_j} w^j\right)\left(\sum_{k=1}^{\infty} \frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{2\,(2k)!\,k!}} z^{2k}\overline{w}^{k-1}\right) dA_t(w) \\
&= \sum_{k=1}^{\infty} \frac{(k+1)\,\sqrt{(k-1)!\,\Gamma(2k+t)\,\Gamma(k+t+1)}\,\overline{a_{k+1}}}{\Gamma(k+t+2)\sqrt{(2k-1)\,(2k-1)!}} z^{2k-1} + \frac{\sqrt{t+2}\,\overline{a_0}}{2} z^2 \\
&\quad + \sum_{k=1}^{\infty} \frac{\sqrt{k!\,\Gamma(2k+t+3)}\,\overline{b_k}}{\sqrt{2}\,(k+1)\,(2k+2)!\,\Gamma(k+t+2)} z^{2k+2},
\end{aligned}$$

and

$$\begin{aligned}
S_\varphi(e_1)(z) &= H_\varphi(e_1)(z) = PM_\varphi J(e_1)(z) = P(\varphi\overline{e_1})(z) \\
&= \left\langle \varphi\overline{e_1}, K_z^t \right\rangle_t = \int_{\mathbb{D}} \frac{\partial(\varphi\overline{e_1})}{\partial w}(w)\, \overline{\frac{\partial K_z^t}{\partial w}(w)}\, dA_t(w) \\
&= \int_{\mathbb{D}} \left(\sum_{i=1}^{\infty} i a_i w^{i-1} \overline{w} \right) \left(\sum_{k=1}^{\infty} \frac{\Gamma(k+t+1)}{k!\,\Gamma(t+2)} z^k \overline{w}^{k-1} \right) dA_t(w) \\
&= \sum_{k=1}^{\infty} \frac{(k+1)\, a_{k+1}}{k+t+1} z^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= (S_\varphi^* - S_\varphi)(e_1)(z) \\
&= \left(\frac{\sqrt{t+2}\,\overline{a_0}}{2} - \frac{3\,a_3}{t+3} \right) z^2 \\
&\quad + \sum_{k=1}^{\infty} \left(\frac{(k+1)\,\sqrt{(k-1)!\,\Gamma(2k+t)\,\Gamma(k+t+1)}\,\overline{a_{k+1}}}{\Gamma(k+t+2)\,\sqrt{(2k-1)\,(2k-1)!}} - \frac{2k\,a_{2k}}{2k+t} \right) z^{2k-1} \\
&\quad + \sum_{k=1}^{\infty} \left(\frac{\sqrt{k!\,\Gamma(2k+t+3)}\,\overline{b_k}}{\sqrt{2}\,(k+1)\,(2k+2)!\,\Gamma(k+t+2)} - \frac{(2k+3)\,a_{2k+3}}{2k+t+3} \right) z^{2k+2}.
\end{aligned}$$

This implies that

$$\overline{a_0} = \frac{6a_3}{\sqrt{t+2}\,(t+3)}, \tag{4.1}$$

$$\overline{a_{k+1}} = \frac{2k\,\Gamma(k+t+2)\,\sqrt{(2k-1)\,(2k-1)!}}{(k+1)\,(2k+t)\,\sqrt{(k-1)!\,\Gamma(2k+t)\,\Gamma(k+t+1)}} a_{2k}, \quad k \in \mathbb{N}, \tag{4.2}$$

and

$$\overline{b_k} = \frac{\sqrt{2}\,(k+1)\,(2k+3)!\,\Gamma(k+t+2)}{(2k+t+3)\,\sqrt{k!\,\Gamma(2k+t+3)}} a_{2k+3}, \quad k \in \mathbb{N}. \tag{4.3}$$

Taking $f(z) = e_2(z) = \frac{\sqrt{t+2}}{2} z^2$ and by the reproducing property of K_z^t, Lemma 3.4, and (1.3), we deduce that

$$\begin{aligned}
S_\varphi^*(e_2)(z) &= \left\langle S_\varphi^*(e_2), K_z^t \right\rangle_t = \left\langle e_2, S_\varphi(K_z^t) \right\rangle_t \\
&= \left\langle e_2, PM_\varphi K(K_z^t) \right\rangle_t = \left\langle e_2, \varphi \cdot K(K_z^t) \right\rangle_t \\
&= \int_{\mathbb{D}} \frac{\partial e_2}{\partial w}(w)\, \overline{\frac{\partial(\varphi \cdot K(K_z^t))}{\partial w}(w)}\, dA_t(w) \\
&= \sqrt{t+2} \int_{\mathbb{D}} w \overline{\frac{\partial \varphi}{\partial w}(w)}\, \overline{K(K_z^t)(w)}\, dA_t(w) + \sqrt{t+2} \int_{\mathbb{D}} w \overline{\varphi(w)}\, \overline{\frac{\partial(K(K_z^t))}{\partial w}(w)}\, dA_t(w)
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{t+2}\int_{\mathbb{D}} w\left(\sum_{i=1}^{\infty} i\,\overline{a_i}\,\overline{w}^{i-1}\right)\Bigg(\sum_{k=1}^{\infty}\frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{k\Gamma(t+2)\sqrt{2\,(2k)!\,k!}}z^{2k}\overline{w}^{k} \\
&\quad + \sum_{k=1}^{\infty}\frac{\sqrt{\Gamma(2k+t)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{k\,k!\,(2k-1)\,(2k-1)!}}z^{2k-1}w^{k}\Bigg)\,dA_t(w) \\
&\quad + \sqrt{t+2}\int_{\mathbb{D}} w\left(\sum_{i=0}^{\infty}\overline{a_i}\,\overline{w}^{i} + \sum_{j=1}^{\infty}\overline{b_j}w^{j}\right) \\
&\quad \times\left(\sum_{k=1}^{\infty}\frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{2\,(2k)!\,k!}}z^{2k}\overline{w}^{k-1}\right)dA_t(w) \\
&= \overline{a_1}z^2 + \sum_{k=1}^{\infty}\frac{(k+1)\,(k+2)\,\sqrt{k!\,\Gamma(2k+t)\,\Gamma(k+t+1)}\,\overline{a_{k+2}}}{\Gamma(k+t+3)\,\sqrt{k\,(2k-1)\,(2k-1)!}}z^{2k-1} \\
&\quad + \frac{\sqrt{\Gamma(t+5)}\,\overline{a_0}}{4\sqrt{6\,\Gamma(t+2)}}z^4 + \sum_{k=1}^{\infty}\frac{\sqrt{(t+2)\,\Gamma(2k+t+5)}\,\overline{b_k}}{(k+2)\,\sqrt{2\,(2k+4)!}}z^{2(k+2)},
\end{aligned}
$$

and

$$
\begin{aligned}
S_{\varphi}(e_2)(z) = T_{\varphi}(e_1)(z) = PM_{\varphi}(e_1)(z) &= \left\langle \varphi e_1, K_z^t\right\rangle_t \\
&= \int_{\mathbb{D}}\frac{\partial(\varphi e_1)}{\partial w}(w)\,\overline{\frac{\partial K_z^t}{\partial w}(w)}\,dA_t(w) \\
&= \int_{\mathbb{D}}\left(\sum_{i=1}^{\infty} i a_{i-1}w^{i-1} + \sum_{j=1}^{\infty} b_j\overline{w}^{j}\right)\left(\sum_{k=1}^{\infty}\frac{\Gamma(k+t+1)}{k!\,\Gamma(t+2)}z^{k}\overline{w}^{k-1}\right)dA_t(w) \\
&= \sum_{k=1}^{\infty} a_{k-1}z^k.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
0 &= (S_{\varphi}^{*} - S_{\varphi})(e_2)(z) \\
&= (\overline{a_1} - a_1)\,z^2 + \left(\frac{\sqrt{\Gamma(t+5)}\,\overline{a_0}}{4\sqrt{6\,\Gamma(t+2)}} - a_3\right)z^4 \\
&\quad + \sum_{k=1}^{\infty}\left(\frac{(k+1)\,(k+2)\,\sqrt{k!\,\Gamma(2k+t)\,\Gamma(k+t+1)}\,\overline{a_{k+2}}}{\Gamma(k+t+3)\,\sqrt{k\,(2k-1)\,(2k-1)!}} - a_{2k-2}\right)z^{2k-1} \\
&\quad + \sum_{k=1}^{\infty}\left(\frac{\sqrt{(t+2)\,\Gamma(2k+t+5)}\,\overline{b_k}}{(k+2)\,\sqrt{2\,(2k+4)!}} - a_{2k+3}\right)z^{2(k+2)}.
\end{aligned}
$$

This implies that

$$
\overline{a_1} = a_1, \tag{4.4}
$$

$$
\overline{a_0} = \frac{4\sqrt{6\,\Gamma(t+2)}}{\sqrt{\Gamma(t+5)}}a_3, \tag{4.5}
$$

$$\overline{a_{k+2}} = \frac{\Gamma(k+t+3)\sqrt{k\,(2k-1)\,(2k-1)!}}{(k+1)\,(k+2)\sqrt{k!\,\Gamma(2k+t)\,\Gamma(k+t+1)}}a_{2k-2}, \quad k \in \mathbb{N}, \tag{4.6}$$

and

$$\overline{b_k} = \frac{(k+2)\sqrt{2\,(2k+4)!}}{\sqrt{(t+2)\,\Gamma(2k+t+5)}}a_{2k+3}, \quad k \in \mathbb{N}. \tag{4.7}$$

By (4.1) and (4.5), we get $\overline{a_0} = a_3 = 0$. This together with (4.2), (4.3), (4.6) and (4.7) further implies that

$$a_i = 0,\ b_j = 0, \quad \text{for any } i \in \{0\} \cup \mathbb{N} - \{1\} \text{ and } j \in \mathbb{N}. \tag{4.8}$$

It remains to show $a_1 = 0$. Taking

$$f(z) = e_3(z) = \frac{\sqrt{(t+2)(t+3)}}{3\sqrt{2}}z^3,$$

a similar argument shows that

$$\begin{aligned}
S_\varphi^*(e_3)(z) &= \left\langle S_\varphi^*(e_3), K_z^t \right\rangle_t = \left\langle e_3, S_\varphi(K_z) \right\rangle_t = \overline{\left\langle e_3, PM_\varphi K(K_z^t) \right\rangle_t} \\
&= \left\langle e_3, \varphi \cdot K(K_z^t) \right\rangle_t = \int_{\mathbb{D}} \frac{\partial e_3}{\partial w}(w)\, \overline{\frac{\partial\left(\varphi \cdot K(K_z^t)\right)}{\partial w}(w)}\, dA_t(w) \\
&= \frac{\sqrt{(t+2)(t+3)}}{\sqrt{2}} \int_{\mathbb{D}} w^2 \overline{\frac{\partial \varphi}{\partial w}(w)}\, \overline{K(K_z^t)(w)}\, dA_t(w) \\
&\quad + \frac{\sqrt{(t+2)(t+3)}}{\sqrt{2}} \int_{\mathbb{D}} w^2 \overline{\varphi(w)}\, \overline{\frac{\partial(K(K_z^t))}{\partial w}(w)}\, dA(w) \\
&= \frac{\sqrt{(t+2)(t+3)}}{\sqrt{2}} \int_{\mathbb{D}} w^2 \left(\sum_{i=1}^\infty i\, \overline{a_i}\, \overline{w}^{i-1} \right) \left(\sum_{k=1}^\infty \frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{k\Gamma(t+2)\sqrt{2\,(2k)!\,k!}} z^{2k}\overline{w}^k \right. \\
&\quad \left. + \sum_{k=1}^\infty \frac{\sqrt{\Gamma(2k+t)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{k\,k!\,(2k-1)\,(2k-1)!}} z^{2k-1} w^k \right) dA_t(w) \\
&\quad + \frac{\sqrt{(t+2)(t+3)}}{\sqrt{2}} \int_{\mathbb{D}} w^2 \left(\sum_{i=0}^\infty \overline{a_i}\, \overline{w}^i + \sum_{j=1}^\infty \overline{b_j} w^j \right) \\
&\quad \times \left(\sum_{k=1}^\infty \frac{\sqrt{\Gamma(2k+t+1)\Gamma(k+t+1)}}{\Gamma(t+2)\sqrt{2\,(2k)!\,k!}} z^{2k}\overline{w}^{k-1} \right) dA_t(w) \\
&= \frac{\sqrt{3\,(t+2)\,(t+4)}\,\overline{a_1}}{8} z^4 + \cdots,
\end{aligned}$$

and

$$\begin{aligned}
S_\varphi(e_3)(z) &= H_\varphi(e_2)(z) = PM_\varphi J(e_2)(z) \\
&= P(\varphi\overline{e_2})(z) = \left\langle \varphi\overline{e_2}, K_z^t \right\rangle_t
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} \frac{\partial(\varphi\overline{e_2})}{\partial w}(w)\,\overline{\frac{\partial K_z^t}{\partial w}(w)}\,dA_t(w) \\
&= \int_{\mathbb{D}} \left(\sum_{i=1}^{\infty} \frac{\sqrt{t+2}\,i\,a_i}{2} w^{i-1}\overline{w}^2\right)\left(\sum_{k=1}^{\infty} \frac{\Gamma(k+t+1)}{k!\,\Gamma(t+2)} z^k\overline{w}^{k-1}\right) dA_t(w) \\
&= \sum_{k=1}^{\infty} \frac{(k+1)(k+2)\,\sqrt{t+2}\,a_{k+2}}{2(k+t+2)(k+t+1)} z^k.
\end{aligned}
$$

Hence, we obtain

$$
0 = (S_\varphi^* - S_\varphi)(e_3)(z) = \left(\frac{\sqrt{3\,(t+2)\,(t+4)}\,\overline{a_1}}{8} - \frac{15\,\sqrt{t+2}\,a_6}{(t+6)\,(t+5)}\right) z^4 + \cdots,
$$

which implies

$$
a_1 = \frac{40\,\sqrt{3}}{(t+6)\,(t+5)\,\sqrt{t+4}}\overline{a_6} = 0.
$$

This together with (4.8) shows that $\varphi = 0$, completing the proof of the theorem. □

Recall that an operator T is diagonal on the Dirichlet type space $\mathfrak{D}_t$ if and only if $\left\langle Te_i, e_j\right\rangle_t = 0$ for all positive integers $i \neq j$.

Theorem 4.2. *Let $t > -1$ and $\varphi \in \mathfrak{H}$. Then, S_φ is a diagonal operator on $\mathfrak{D}_t$ if and only if $\varphi = 0$.*

Proof. Let $\varphi \in \mathfrak{H}$ defined by $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j$, where $z \in \mathbb{D}$ and $a_i, b_j \in \mathbb{C}$. The forward implication is trivial. Suppose conversely that S_φ is a diagonal operator on $\mathfrak{D}_t$. Then, for $m, n \in \mathbb{N}$ such that $m \neq n$, we have $\left\langle S_\varphi(e_n), e_m\right\rangle_t = 0$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{D}_t$. Then, the following two cases arise. If $n = 2k$ for some $k \in \mathbb{N}$, by (2.5), we get

$$
\left\langle S_\varphi(e_{2k}), e_m\right\rangle_t = \left\langle T_\varphi(e_k), e_m\right\rangle_t = \begin{cases} \dfrac{\sqrt{m\,m!\,\Gamma(k+t+1)}}{\sqrt{k\,k!\,\Gamma(m+t+1)}} a_{m-k}, & \text{if } m \geq k, \\ \dfrac{\sqrt{m\,k!\,\Gamma(m+t+1)}}{\sqrt{k\,m!\,\Gamma(k+t+1)}} b_{k-m}, & \text{if } k > m. \end{cases}
$$

If $n = 2k-1$ for some $k \in \mathbb{N}$, by (2.6), we obtain

$$
\left\langle S_\varphi(e_{2k-1}), e_m\right\rangle_t = \left\langle H_\varphi(e_k), e_m\right\rangle_t = \frac{(k+m)!\,\sqrt{m\,\Gamma(k+t+1)\,\Gamma(m+t+1)}}{\sqrt{k\,k!\,m!}\,\Gamma(m+k+t+1)} a_{m+k}.
$$

The above cases indicate that $a_i = 0$ and $b_j = 0$ for all $i \geq 0$, $j \geq 1$. Hence, $\varphi = 0$. □

Let $\check{P}$ be the Bergman projection from $L^2(\mathbb{D}, dA_t)$ onto the weighted Bergman space $L_a^2(dA_t)$. For any $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator $\check{T}_\varphi$ on $L_a^2(dA_t)$ is defined by

$$
\check{T}_\varphi = \check{P} M_\varphi.
$$

Note that the adjoint of $\check{T}_\varphi$ satisfies $\check{T}_\varphi^* = \check{T}_{\overline{\varphi}}$.

Let

$$\check{e}_k(z) = \frac{\sqrt{\Gamma(k+t+2)}}{\sqrt{k!\,\Gamma(t+2)}} z^k, \quad z \in \mathbb{D}.$$

Then, $\{\check{e}_k\}_{k=0}^{\infty}$ forms an orthonormal basis of $L_a^2(dA_t)$. Define an operator $U : \mathfrak{D}_t \to L_a^2(dA_t)$ by

$$U(e_k) = \check{e}_{k-1}$$

and linearly extending it to $\mathfrak{D}_t$. Then, U is a unitary operator such that

$$Uf = f^{'}$$

for each $f \in \mathfrak{D}_t$.

In the next result, we see that a Toeplitz operator induced by a co-analytic symbol in $\mathcal{M}$ on the Dirichlet type space $\mathfrak{D}_t$ is unitarily equivalent to that on the weighted Bergman space $L_a^2(dA_t)$.

Lemma 4.1. *Let $\varphi \in \mathcal{M}$ be a co-analytic function. Then, $T_\varphi = U^*\check{T}_\varphi U$.*

Proof. Recall that $\langle\cdot,\cdot\rangle_2$ denotes the inner product in $L_a^2(dA_t)$. Let $\varphi \in \mathcal{M}$ be a co-analytic function. For any $f, g \in \mathfrak{D}_t$, a direct calculation gives

$$\begin{aligned}
\left\langle T_\varphi f, g\right\rangle_t &= \left\langle PM_\varphi f, g\right\rangle_t = \langle \varphi f, g\rangle_t \\
&= \left\langle \frac{\partial(\varphi f)}{\partial z}, \frac{\partial g}{\partial z}\right\rangle_2 \\
&= \left\langle f\frac{\partial \varphi}{\partial z}, \frac{\partial g}{\partial z}\right\rangle_2 + \left\langle \varphi\frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right\rangle_2 \\
&= \left\langle \check{T}_\varphi Uf, Ug\right\rangle_2 \\
&= \left\langle U^*\check{T}_\varphi Uf, g\right\rangle_t.
\end{aligned}$$

This gives the desired result. □

In the next theorem, we apply Lemma 4.1 to establish a criterion of co-isometry for the H-Toeplitz operator on $\mathfrak{D}_t$.

Theorem 4.3. *Suppose $t > -1$ and $\varphi \in \mathcal{M}$ is a nonzero, co-analytic function on $\mathbb{D}$. Then, S_φ is a co-isometry on $\mathfrak{D}_t$ if and only if $\varphi = 1$ on $\mathbb{D}$.*

Proof. Let $\varphi \in \mathcal{M}$ be a nonzero, co-analytic function on $\mathbb{D}$. Then, by Lemma 4.1,

$$\begin{aligned}
S_\varphi S_\varphi^*\left(z^k\right) &= \left(PM_\varphi K\right)\left(K^*M_\varphi^*P\right)\left(z^k\right) \\
&= PM_\varphi T_\varphi^*\left(z^k\right) \\
&= PM_\varphi\left(U^*\check{T}_\varphi U\right)^*\left(z^k\right) \\
&= PM_\varphi U^*\check{T}_{\overline{\varphi}}U\left(z^k\right) \\
&= PM_\varphi U^*\left(k\overline{\varphi}z^{k-1}\right) \\
&= PM_\varphi\left(\overline{\varphi}z^k\right) \\
&= T_{|\varphi|^2}\left(z^k\right)
\end{aligned}$$

for arbitrary $k \in \mathbb{N}$. Since the polynomials are dense in $\mathfrak{D}_t$, it follows that

$$S_\varphi S_\varphi^* = T_{|\varphi|^2}. \tag{4.9}$$

Assume that S_φ is a co-isometry on $\mathfrak{D}_t$, that is, $S_\varphi S_\varphi^* = I$. Thus, by (4.9), we have $T_{1-|\varphi|^2} = 0$. Since $1 - \overline{\varphi}$ is analytic, it follows that

$$T_{1-\varphi} T_{1-\overline{\varphi}} = 0.$$

Similar to [15, Corollary 10], we conclude that either $1 - \varphi = 0$ or $1 - \overline{\varphi} = 0$, which gives that $\varphi = 1$ on $\mathbb{D}$.

Conversely, if $\varphi = 1$ on $\mathbb{D}$, then $S_\varphi S_\varphi^* = T_1 = I$ by (4.9), which means that S_φ is a co-isometry on $\mathfrak{D}_t$. This completes the proof of the theorem. □

Let $\mathcal{B}(\mathfrak{D}_t)$ denote the algebra consisting of all bounded linear operators on the Dirichlet type space $\mathfrak{D}_t$. We are going to show that the map $\varphi \mapsto S_\varphi$ is one-to-one if the domain is $\mathfrak{H}$, which is given in the following.

Lemma 4.2. *The map $\gamma : \mathfrak{H} \to \mathcal{B}(\mathfrak{D}_t)$ defined by $\gamma(\varphi) = S_\varphi$ is one-to-one.*

Proof. Let $\varphi, \psi \in \mathfrak{H}$, which are defined by

$$\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j, \quad z \in \mathbb{D},\ a_i, b_j \in \mathbb{C},$$

and

$$\psi(z) = \sum_{i=0}^{\infty} a_i' z^i + \sum_{j=1}^{\infty} b_j' \overline{z}^j, \quad z \in \mathbb{D},\ a_i', b_j' \in \mathbb{C},$$

respectively. If $S_\varphi = S_\psi$, then $S_{\varphi-\psi}(e_k) = 0$ for all $k \in \mathbb{N}$. In particular, $S_{\varphi-\psi}(e_2) = 0$, that is, $PM_{\varphi-\psi}K(e_2) = 0$. More precisely,

$$P\left(\sum_{i=0}^{\infty}(a_i - a_i')z^{i+1} + \sum_{j=1}^{\infty}(b_j - b_j')\overline{z}^j z\right) = 0.$$

Applying Lemma 2.1, we derive that

$$\sum_{i=0}^{\infty}(a_i - a_i')z^{i+1} = 0.$$

Therefore, $a_i = a_i'$ for all $i \geq 0$. Moreover, $S_{\varphi-\psi}(e_4) = 0$, thus we obtain

$$P\left(\sum_{i=0}^{\infty}(a_i - a_i')z^{i+2} + \sum_{j=1}^{\infty}(b_j - b_j')\overline{z}^j z^2\right) = 0.$$

Using Lemma 2.1 again, we get

$$\frac{2}{t+2}(b_1 - b_1')z = 0,$$

hence $b_1 = b_1'$. Continuing the above process for e_6, e_8, e_{10} and so on, we obtain $b_j = b_j'$ for all $j \geq 2$, and then $\varphi = \psi$. This proves the desired result. □

In the next result, we give a necessary and sufficient condition for an H-Toeplitz operator to be a partial isometry on the Dirichlet type space $\mathfrak{D}_t$.

Theorem 4.4. *Suppose $t > -1$ and $\varphi \in \mathcal{M}$ is a nonzero, co-analytic function on $\mathbb{D}$. Then S_φ is a partial isometry on $\mathfrak{D}_t$ if and only if $\varphi = 1$ on $\mathbb{D}$.*

Proof. If $\varphi = 1$ on $\mathbb{D}$, then S_φ is a co-isometry by Theorem 4.3. Thus, S_φ is a partial isometry.

Conversely, suppose S_φ is a partial isometry on $\mathfrak{D}_t$. Then, by [17, Theorem 2.3.3], we have $S_\varphi S_\varphi^* S_\varphi = S_\varphi$. In view of (4.9), we get

$$T_{|\varphi|^2} S_\varphi = S_\varphi,$$

or equivalently,

$$T_{1-|\varphi|^2} S_\varphi = 0.$$

Since $\varphi \neq 0$, we have $S_\varphi \neq 0$ by Lemma 4.2. Thus, $T_{1-|\varphi|^2} = 0$. The desired result is then obtained by proceeding as in the proof of Lemma 4.3. □

As an operator on the Hilbert space, S_φ is a partial isometry if and only if S_φ^* is a partial isometry for $\varphi \in \mathcal{M}$; see [19, Proposition 4.38]. Thus, combining Theorem 4.3 with Theorem 4.4, we get the following corollary.

Corollary 4.1. *Suppose $t > -1$ and $\varphi \in \mathcal{M}$ is a nonzero, co-analytic function on $\mathbb{D}$. Then, the following statements are equivalent:*

(a) S_φ^* is a isometry on $\mathfrak{D}_t$.
(b) S_φ^* is a partial isometry on $\mathfrak{D}_t$.
(c) $\varphi = 1$ on $\mathbb{D}$.

For any fixed positive integer M, define

$$H_M = \operatorname{span}\left\{z^l, 1 \leq l \leq 2M\right\}.$$

Then H_M is a closed subspace of the Dirichlet type space $\mathfrak{D}_t$. In fact, the following theorem reveals that it is the kernel of H-Toeplitz operator with some co-analytic symbol.

Theorem 4.5. *Suppose $t > -1$ and M is a fixed positive integer. Let $\varphi(z) = \sum_{l=M}^{\infty} a_l \bar{z}^l \in \mathcal{M}$. Then, the subspace H_M of $\mathfrak{D}_t$ is the kernel of the H-Toeplitz operator S_φ.*

Proof. Consider positive integers i, j satisfying $M \leq i < \infty$ and $1 \leq j \leq 2M$. If $j = 2k$ for some $k \in \mathbb{N}$, then by Lemma 2.1

$$S_{\bar{z}^i}(z^j) = PM_{\bar{z}^i}K\left(z^{2k}\right) = \begin{cases} \frac{\sqrt{2\,k!\,(2k)!}\,\Gamma(k-i+t+1)}{\sqrt{\Gamma(2k+t+1)\,\Gamma(k+t+1)}\,(k-i)!} z^{k-i}, & \text{if } k > i, \\ 0, & \text{if } k \leq i. \end{cases}$$

Note that $M \leq i < \infty$ and $1 \leq k \leq M$, then $S_{\bar{z}^i}(z^j)$ is equal to 0 in the case of $j = 2k$. If $j = 2k - 1$ for some $k \in \mathbb{N}$, similarly, we get

$$S_{\bar{z}^i}(z^j) = PM_{\bar{z}^i}K\left(z^{2k-1}\right) = \frac{\sqrt{(2k-1)\,(2k-1)!\,\Gamma(k+t+1)}}{\sqrt{k\,k!\,\Gamma(2k+t)}} P(\bar{z}^{i+k}) = 0.$$

Hence, $S_{\overline{z}^i}(z^j) = 0$ for the positive integers i, j satisfying $M \le i < \infty$ and $1 \le j \le 2M$. Now, for $\varphi(z) = \sum_{l=M}^{\infty} a_l \overline{z}^l \in \mathcal{M}$, by Proposition 3.1(a) and a limiting argument, we see that

$$S_{\varphi}(z^j) = \sum_{l=M}^{\infty} a_l S_{\overline{z}^l}(z^j).$$

Hence, we have $S_{\varphi}(z^j) = 0$ for all $1 \le j \le 2M$. Therefore, we conclude that H_M is the kernel of S_{φ}. □

Taking the symbol as a polynomial harmonic function for the H-Toeplitz operator, we can prove its kernel is infinite-dimensional.

Theorem 4.6. *If $\psi \in \mathcal{M}$ is a polynomial harmonic function, then* $\dim \ker S_{\psi} = \infty$.

Proof. Observe that if ψ is a co-analytic function in $\mathcal{M}$, then $S_{\psi}\left(zf(z^2)\right) = 0$ for suitable choice of function $f \in \mathfrak{D}_t$. This implies that $\ker S_{\psi} \neq \{0\}$. Now, suppose that M, $N \in \mathbb{N}$ are arbitrary given integers, set $\psi(z) = \sum_{s=0}^{N} a_s z^s + \sum_{m=1}^{M} b_m \overline{z}^m$. Let $\alpha = \max\{M, N\}$ and choose $f(z) = \sum_{i=\alpha}^{\infty} c_i z^{2i+1} \in \mathfrak{D}_t$. We can obtain

$$\begin{aligned}
S_{\psi} f(z) &= PM_{\psi}K\left(\sum_{i=\alpha}^{\infty} c_i z^{2i+1}\right) \\
&= PM_{\psi}\left(\sum_{i=\alpha}^{\infty} \frac{\sqrt{(2i+1)(2i+1)!\,\Gamma(i+t+2)}}{\sqrt{(i+1)(i+1)!\,\Gamma(2i+t+2)}} c_i \overline{z}^{i+1}\right) \\
&= P\left(\left(\sum_{s=0}^{N} a_s z^s + \sum_{m=1}^{M} b_m \overline{z}^m\right)\left(\sum_{i=\alpha}^{\infty} \frac{\sqrt{(2i+1)(2i+1)!\,\Gamma(i+t+2)}}{\sqrt{(i+1)(i+1)!\,\Gamma(2i+t+2)}} c_i \overline{z}^{i+1}\right)\right) \\
&= P\left(\left(\sum_{s=0}^{N} a_s z^s\right)\left(\sum_{i=\alpha}^{\infty} \frac{\sqrt{(2i+1)(2i+1)!\,\Gamma(i+t+2)}}{\sqrt{(i+1)(i+1)!\,\Gamma(2i+t+2)}} c_i \overline{z}^{i+1}\right)\right) = 0,
\end{aligned}$$

where the last equality follows from Lemma 2.1. Similarly, for all $n \in \mathbb{N}$, $S_{\psi}\left(z^{2n} f(z)\right) = 0$. Hence, if a nonzero function $g \in \ker S_{\psi}$, then $\sum_{k=1}^{n} \lambda_k z^{2k} g \in \ker S_{\psi}$ for $n \in \mathbb{N}$ and $\lambda_k \in \mathbb{C}$. In particular, the set $\left\{z^{2n} g : n \in \mathbb{N}\right\}$ is a linear independent set. In fact, suppose $\sum_{k=1}^{n} \lambda_k z^{2k} g(z) = 0$ but $g \neq 0$, then $\sum_{k=1}^{n} \lambda_k z^{2k}$ vanishes on a positive measure set so that $\lambda_k = 0$ for $k = 1, 2, \cdots, n$. This shows that $\left\{z^2 g, z^4 g, \cdots, z^{2n} g\right\}$ is linear independent. This is true for all $n \in \mathbb{N}$ and all such functions in $\ker S_{\psi}$, so $\ker S_{\psi}$ is infinite dimensional. □

It is well-known that the C^*-algebra generated by self-adjoint operators is abelian and hence its algebraic structure is primitive. As examples of non-self-adjoint operators, the C^*-algebra generated by H-Toeplitz operators is complicated. Therefore, it is of great importance to study the condition for commutativity of H-Toeplitz operators.

The subsequent theorem characterizes when two H-Toeplitz operators with analytic symbols commute on $\mathfrak{D}_t$ under certain conditions.

Theorem 4.7. *Suppose $t > -1$. Let $\varphi = \sum_{i=1}^{\infty} a_i z^i$ and $\psi = \sum_{j=1}^{\infty} b_j z^j$ in $\mathcal{M}$, where $z \in \mathbb{D}$, a_i, $b_j \neq 0$ for all i, $j \in \mathbb{N}$ and $\frac{b_1}{a_1} = \frac{b_{2i+1}}{a_{2i+1}}$ for all $i \in \mathbb{N}$. If $\frac{b_{i+k}}{a_{i+k}} \ge \frac{b_{2i}}{a_{2i}}$ for all i, $k \in \mathbb{N}$, then S_{φ} and S_{ψ} commute on $\mathfrak{D}_t$ if and only if φ and ψ are linearly dependent.*

Proof. We show the forward implication only because the reverse implication is trivial. Suppose $S_\varphi S_\psi = S_\psi S_\varphi$. In particular, $S_\varphi S_\psi(z) = S_\psi S_\varphi(z)$, that is,

$$PM_\varphi KP\left(\sum_{j=1}^{\infty} b_j z^j \overline{z}\right) = PM_\psi KP\left(\sum_{i=1}^{\infty} a_i z^i \overline{z}\right).$$

Hence, by Lemma 2.1,

$$PM_\varphi K\left(\sum_{j=1}^{\infty} \frac{(j+1)b_{j+1}}{j+1+t} z^j\right) = PM_\psi K\left(\sum_{i=1}^{\infty} \frac{(i+1)a_{i+1}}{i+1+t} z^i\right).$$

Using Lemma 2.1 again,

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \frac{(2k+1)\sqrt{2(2k)!\,\Gamma(k+t+1)}}{(2k+1+t)\sqrt{k!\,\Gamma(2k+t+1)}} b_{2k+1} a_i z^{i+k} \\
&\quad+ \sum_{k=1}^{\infty}\sum_{i>k}^{\infty} \frac{2k\sqrt{(2k-1)(2k-1)!\,\Gamma(k+t+1)}}{(2k+t)\sqrt{k\,k!\,\Gamma(2k+t)}} b_{2k} a_i z^{i-k} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{(2k+1)\sqrt{2(2k)!\,\Gamma(k+t+1)}}{(2k+1+t)\sqrt{k!\,\Gamma(2k+t+1)}} a_{2k+1} b_j z^{j+k} \\
&\quad+ \sum_{k=1}^{\infty}\sum_{j>k}^{\infty} \frac{2k\sqrt{(2k-1)(2k-1)!\,\Gamma(k+t+1)}}{(2k+t)\sqrt{k\,k!\,\Gamma(2k+t)}} a_{2k} b_j z^{j-k}.
\end{aligned}$$

Then, comparing the coefficients of z in the above equation, we get

$$\sum_{k=1}^{\infty} \frac{2k\sqrt{(2k-1)(2k-1)!\,\Gamma(k+t+1)}}{(2k+t)\sqrt{k\,k!\,\Gamma(2k+t)}} (b_{2k}a_{k+1} - a_{2k}b_{k+1})\, z = 0,$$

which implies that $\frac{b_{i+1}}{a_{i+1}} = \frac{b_{2i}}{a_{2i}}$ for each $i \in \mathbb{N}$ by the hypothesis $\frac{b_{i+k}}{a_{i+k}} \geq \frac{b_{2i}}{a_{2i}}$. Similarly, comparing the coefficients of z^2, we get

$$\frac{3\sqrt{4\,\Gamma(t+2)}}{(3+t)\sqrt{\Gamma(t+3)}} (b_3a_1 - a_3b_1)\, z^2 + \sum_{k=1}^{\infty} \frac{2k\sqrt{(2k-1)(2k-1)!\,\Gamma(k+t+1)}}{(2k+t)\sqrt{k\,k!\,\Gamma(2k+t)}} (b_{2k}a_{k+2} - a_{2k}b_{k+2}) z^2 = 0,$$

which means that $\frac{b_{i+2}}{a_{i+2}} = \frac{b_{2i}}{a_{2i}}$ for each $i \in \mathbb{N}$ by the hypothesis $\frac{b_{i+k}}{a_{i+k}} \geq \frac{b_{2i}}{a_{2i}}$ again. Continuing in this fashion, we obtain that $\frac{b_{i+k}}{a_{i+k}} = \frac{b_2}{a_2}$ for each $i, k \in \mathbb{N}$. Therefore, $b_i = \lambda a_i$ for each integer $i \geq 1$, where $\lambda = \frac{b_2}{a_2}$ is a constant. It follows that $\psi = \lambda\varphi$. □

More generally, we use the same trick in Theorem 4.7 to obtain an equivalent condition for the commutativity of H-Toeplitz operators with polynomial harmonic symbols.

Theorem 4.8. *Suppose $t > -1$. Let $\varphi = \sum_{i=1}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j$ and $\psi = \sum_{m=1}^{\infty} c_m z^m + \sum_{n=1}^{\infty} d_n \overline{z}^n$ in $\mathcal{M}$, where a_i, b_j, c_m, $d_n \neq 0$ for i, j, m, $d \in \mathbb{N}$ and $\frac{c_1}{a_1} = \frac{c_{2i+1}}{a_{2i+1}}$ for all $i \in \mathbb{N}$. If $\frac{a_{i+k}}{c_{i+k}} \geq \frac{a_{2i}}{c_{2i}}$ and $\frac{b_j}{d_j} \geq \frac{a_{2(j+k)+1}}{c_{2(j+k)+1}}$ for all i, j, $k \in \mathbb{N}$, then S_φ and S_ψ commute on $\mathfrak{D}_t$ if and only if φ and ψ are linearly dependent.*

5. Conclusions

In this research, we conduct a study of H-Toeplitz operators on the Dirichlet type space $\mathfrak{D}_t$. Specifically, the compactness, self-adjointness, diagonality, co-isometry, partial isometry and commutativity of H-Toeplitz operators on $\mathfrak{D}_t$ are characterized.

Author contributions

Peiying Huang and Yiyuan Zhang: Conceptualization, Formal analysis, Methodology, Writing-original draft, Validation, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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