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Research article

Stability for Cauchy problem of first order linear PDEs on $\mathbb{T}^m$ with forced frequency possessing finite uniform Diophantine exponent

Xinyu Guan[1,*] **and Nan Kang**[2]

[1] School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan, Shandong 250014, China
[2] Huaihai Industry Group Co., Ltd., Changzhi, Shanxi 046000, China

* **Correspondence:** Email: guanxinyumath@163.com.

Abstract: In this paper, we studied the stability of the Cauchy problem for a class of first-order linear quasi-periodically forced PDEs on the m-dimensional torus:

$$\begin{cases} \partial_t u + (\xi + f(x, \omega t, \xi)) \cdot \partial_x u = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

where $\xi \in \mathbb{R}^m, x \in \mathbb{T}^m, \omega \in \mathbb{R}^d$, in the case of multidimensional Liouvillean forced frequency. We proved that for each compact set $O \in \mathbb{R}^m$ there exists a Cantor subset O_γ of O with positive Lebesgue measure such that if $\xi \in O_\gamma$, then for a perturbation f being small in some analytic Sobolev norm, there exists a bounded and invertible quasi-periodic family of linear operator $\Psi(\omega t)$, such that the above PDEs are reduced by the transformation $v := \Psi(\omega t)^{-1}[u]$ into the following PDE:

$$\partial_t v + (\xi + m_\infty(\omega t)) \cdot \partial_x v = 0,$$

provided that the forced frequency $\omega \in \mathbb{R}^d$ possesses finite *uniform Diophantine exponent*, which allows Liouvillean frequency. The reducibility can immediately cause the stability of the above Cauchy problem, that is, the analytic Sobolev norms of the Cauchy problem are controlled uniformly in time. The proof is based on a finite dimensional Kolmogorov-Arnold-Moser (KAM) theory for quasi-periodically forced linear vector fields with multidimensional Liouvillean forced frequency. As we know, the results on Liouvillean frequency existing in the literature deal with two-dimensional frequency and exploit the theory of continued fractions to control the small divisor problem. The results in this paper partially extend the analysis to higher-dimensional frequency and impose a weak nonresonance condition, i.e., the forced frequency ω possesses finite *uniform Diophantine exponent*. Our result can be regarded as a generalization of analytic cases in the work [R. Feola, F. Giuliani, R. Montalto and M, Procesi, Reducibility of first order linear operators on tori via Moser's theorem, *J. Funct. Anal.*, 2019] from Diophantine frequency to Liouvillean frequency.

Keywords: rotations reducibility; stability of Cauchy-problem; hyperbolic PDEs; Liouvillean frequency; uniform Diophantine exponent
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1. Introduction

It is well known that the problem of reducibility and stability of Sobolev norms for quasi-periodically forced linear operators on the torus is a classical one in the theory of infinite dimensional dynamical systems, which has received new attention in the past few years. Roughly speaking, given a linear operator with coefficients that depend on time in a quasi-periodic way, we say that it is reducible if there exists a bounded change of variables depending quasi-periodically on time (say mapping $H^s \to H^s$ for all times), which makes its coefficients constant. Actually, the notion of reducibility has been first introduced for ordinary differential equations (ODEs), going back to Bogolyubov [11] and Moser [32]. Also, there is a large literature around the reducibility of ODEs by means of the KAM tools. Regarding recent developments in this direction, we invite the reader to consult [4,16,22,25,26] and the references therein. Such kind of reducibility results for PDEs using KAM machinery have been well developed, see [7–10,13,18,21,30,31]. Due to the large amount of work in reducible KAM theory for non linear partial differential equations (PDEs), we do not list them, and refer to the introduction in [20] for more details.

Investigating the reducibility of linear operators is important when reducibility problems of nonlinear PDEs are considered. Very recently, Feola et al. [20] considered the reducibility of linear equations of the form

$$\partial_t u + (\zeta + a_{(\omega,\zeta)}(\omega t, x)) \cdot \partial_x u = 0, \tag{1.1}$$

on the torus $x \in \mathbb{T}^d$ where $\omega \in \mathbb{R}^\nu$ and $\zeta \in \mathbb{R}^d$ are parameters (and the dependence on these parameters is denoted by the corresponding subscripts) while $a_{(\omega,\zeta)} \in C^\infty(\mathbb{T}^{\nu+d}, \mathbb{R}^d)$. They proved that if for any $s_1 \in \mathbb{Z}_+$ large enough and $\|a\|_{s_1} = \|a_{(\omega,\zeta)}\|_{H^{s_1}(\mathbb{T}^{\nu+d},\mathbb{R}^d)}$ small enough, then there exists a Borl set $\mathcal{O}_\infty \subset \mathcal{O}_0$ (a bounded domain in $\mathbb{R}^{\nu+d}$) with positive Lebesgue measure such that for any $(\omega, \zeta) \in \mathcal{O}_\infty$, there is a quasi-periodic family of bounded and invertible linear operators $\Psi(\omega t) = \Psi_{\omega,\zeta}(\omega t)$ and the equation (1.1) is reduced by $v = \Psi(\omega t)^{-1}[u]$ into constant coefficients

$$\partial_t v + m_{\omega,\zeta} \cdot \partial_x v = 0,$$

where $|m_{\omega,\zeta} - \zeta| = O(\|a\|_{s_1})$ uniformly in $(\omega, \zeta) \in \mathcal{O}_0$, see Theorem 1 in [20]. As a direct consequence, the Sobolev norms of the solutions of the Cauchy problem associated to (1.1) are controlled uniformly in time. The main strategies in [20] are divided into two steps: Firstly, the problem of reducing equations (1.1) to constant coefficients, by the identification between first order operators and vector fields, can be formulated as the problem of finding a family of diffeomorphism $\Psi_\xi : \theta \mapsto \theta + h_\xi(\theta)$ that conjugates a weakly perturbed constant vector field

$$(\xi + f_\xi(\theta)) \cdot \partial_\theta$$

on $\mathbb{T}^{\nu+d}$ to constant Diophantine vector fields $\alpha_\xi \cdot \partial_\theta$. Here, ξ is a parameter ranging in a bounded domain $\mathcal{O} \subset \mathbb{R}^{\nu+d}$ and $f_\xi,\ h_\xi \in C^\infty(\mathbb{T}^{\nu+d}, \mathbb{R}^{\nu+d})$. In other words, the authors obtained a new version of J. Moser's theorem on straightening vector fields on tori (called the tame Moser theorem), see Theorem 2 in [20]. Second, the tame Moser theorem is proven via an iterative KAM-type scheme in the Sobolev spaces H^s. Approximations by analytic functions are not used. Instead, the authors approached the problem in the spirit of the Nash-Moser theory, where one employs interpolation and smoothing estimates in order to control the loss of regularity due to the presence of small divisors.

Consider two classes of first order, linear quasi-periodically forced PDEs as the following:

$$\begin{cases} \partial_t u + (\xi + f(x, \omega t, \xi)) \cdot \partial_x u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.2}$$

where $x \in \mathbb{T}^m,\ \omega \in \mathbb{R}^d,\ \xi \in \mathcal{O} \subset \mathbb{R}^m$, $f\ :\ \mathbb{T}^{m+d} \to \mathbb{R}^m$ and $u_0\ :\ \mathbb{T}^m \to \mathbb{R}$ are both C^ω functions, and

$$\begin{cases} \partial_t u + (\rho + f(x, \omega t)) \cdot \partial_x u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.3}$$

where $x \in \mathbb{T},\ \omega \in \mathbb{R}^d,\ \rho \in \mathbb{R}$, $f\ :\ \mathbb{T}^{1+d} \to \mathbb{R}$ and $u_0\ :\ \mathbb{T} \to \mathbb{R}$ are both C^ω functions. We say (1.2) and (1.3) satisfy the Cauchy problem with initial condition $u(x, 0) = u_0(x)$ if $u(x, t)$ is the solution of (1.2) and (1.3) and satisfies $u(x, 0) = u_0(x)$. Furthermore, we say this Cauchy problem is stable if the norm of $u(x, \cdot)$ can be controlled by the norm of $u_0(x)$ for each $t \in \mathbb{R}$. Inspired by the works in [20], we study the stability of the Cauchy problem of (1.2) and (1.3) in analytic Sobolev norm under multi-Liouvillean forced frequency. To this end, we have to discuss the rotations reducibility* of PDEs

$$\partial_t u + (\xi + f(x, \omega t, \xi)) \cdot \partial_x u = 0, \tag{1.4}$$

and

$$\partial_t u + (\rho + f(x, \omega t)) \cdot \partial_x u = 0 \tag{1.5}$$

since it is quite difficult to study the reducibility of PEDs in the case of Liouvillean forced frequency, and such rotations reducibility is able to ensure the stability of solutions of Cauchy problems (1.2) and (1.3). See Theorems 1 and 2 in Section 3 for more detail. As done in [20], we use the identification between derivation operators and vector fields in order to change rotations reducible for PDEs (1.4) and (1.5) into a corresponding rotations reducible for analytic vector field on $\mathbb{T}^{m+d}$

$$\omega \cdot \frac{\partial}{\partial \varphi} + (\xi + f(\theta, \varphi, \xi)) \cdot \frac{\partial}{\partial \theta} \tag{1.6}$$

and for analytic vector field on $\mathbb{T}^{1+d}$

$$\omega \cdot \frac{\partial}{\partial \varphi} + (\rho + f(\theta, \varphi)) \cdot \frac{\partial}{\partial \theta}. \tag{1.7}$$

Under appropriate hypotheses on the size of f, rotations reducibility of vector field (1.6) or (1.7) follows a KAM-type iterative scheme with multidimensional Liouvillean forced frequency in the analytic

*Equation (1.4) is said to be rotations reducible, if there exists a quasi-periodic transformation $u = \Phi(\omega t)[v]$ such that system (1.4) is transformed into $\partial_t v + m(\omega t, \xi)) \cdot \partial_x v = 0$ and m is close to constant.

Sobolev spaces H^s. See Propositions 3.1 and 3.2 in Section 3 for more detail; these results can be regarded as the generalization of V.I. Arnold's [1] C^ω reducible result on analytic vector fields in the case of Diophantine frequency to C^∞ rotations reducible in the case of multidimensional Liouvillean forced frequency.

Let us briefly review recent developments in KAM theory with Liouvillean forced frequency. In 2011, Avila et al. [4] considered the following linear skew-products

$$\begin{aligned}(\alpha, A): \quad & \mathbb{T}\times\mathbb{R}^2\times\mathbb{T}\times\mathbb{R}^2\\ & (x,\omega)\to(x+\alpha, A(x)\cdot\omega)\end{aligned} \tag{1.8}$$

where $\alpha \in \mathbb{R}$ and $A : \mathbb{T} \to SL(2;\mathbb{R})$ is analytic, and gave a KAM scheme for the rotations reducibility of $SL(2;\mathbb{R})$ cocycles with Liouvillean frequency by using the technique of continued fractions. Subsequently, X. Hou and J. You [24] further proved that a quasi-periodic linear differential equation (1.8) in $sl(2,\mathbb{R})$ with two frequencies $(1,\alpha)$ is almost reducible provided that the coefficients are analytic and close to a constant. In the case that α is Diophantine, they got the non-perturbative reducibility. Reducibility and the rotations reducibility for an arbitrary irrational α under some assumption on the rotation number were also obtained. For more references, see [38–44]. Starting from the works [4] and [24], the question of reducibility on quasi-periodically forced systems beyond Diophantine or Brjuno condition † has been one of the central themes of the subject—when can the dynamics of a given system be related to those of a linear model, as for example periodic or quasi-periodic motion on a torus? However, very little is known about reducibility (or linearization) of quasi-periodically forced nonlinear flows on a m-dimensional torus under a non-resonance condition on forced frequency weaker than the Diophantine or Brjuno condition.

An analytic quasi-periodically forced (qpf) m-torus flow is the flow

$$\dot{\theta} = f(\theta,\varphi), \qquad \dot{\varphi} = \omega$$

defined by an analytic vector field of the form

$$\omega\cdot\frac{\partial}{\partial\varphi} + f(\theta,\varphi)\cdot\frac{\partial}{\partial\theta}, \tag{1.9}$$

where $f : \mathbb{T}^m\times\mathbb{T}^d \to \mathbb{R}^m$ is analytic, and $\omega \in \mathbb{R}^d$ is rationally independent. We denote by (ω, f) the flow of the vector field (1.9) for simplicity. Recently, in the case of $m = 1$, Krikorian et al. [27] proved that the flow $(\omega, \rho + f)$ with $\omega = (1,\alpha)$, where $f : \mathbb{T}\times\mathbb{T}^2 \to \mathbb{R}$ is analytic, α satisfies not super-Liouvillean condition

$$\beta(\alpha) := \limsup_{n>0}\frac{\ln\ln q_{n+1}}{\ln q_n} < \infty \tag{1.10}$$

†If there exist $\gamma > 0$ and $\tau > d$ such that

$$|\langle k,\omega\rangle| \geq \frac{\gamma}{|k|^\tau}, \quad k\in\mathbb{Z}^d\setminus\{0\},$$

then we say the frequency ω satisfies Diophantine conditions. Slightly weaker than Diophantine conditions can be often required as Brjuno conditions, which are defined by

$$\sum_{n\geq 0} 2^{-n}\max_{0<|k|\leq 2^n, k\in\mathbb{Z}^d}\ln\frac{1}{|\langle k,\omega\rangle|} < \infty.$$

If the frequency ω does not satisfy Brjuno condition, we call it Liouvillean.

with $\{\frac{p_n}{q_n}\}$ being the continued fraction approximate to α and its rotation number ρ_f is Diophantine with respect to the basic frequency ω [‡], is C^∞ rotations reducible provided that f is sufficiently small. Their inspiration comes from reducibility theory of quasi-periodic $SL(2,\mathbb{R})$ cocycle [2–6,14,15,19,24] and their breakthrough is solving a homological equations of variable coefficient taking advantage of diagonally dominant operators. This work generalizes the linear result [4, 24] to general nonlinear quasi-periodically forced circle flows. Subsequently, using the same Liouvillean non-resonant condition, i.e., $\omega = (1,\alpha)$ satisfies not super-Liouvillean condition (1.10), J. Wang and J. You [35] proved the boundedness of solutions for non-linear quasi-periodic differential equations. In addition, there are many interesting works on the Stoker's problem (existence of response solutions) for nonlinear mechanical models with Liouvillean forced frequency. Since the Stoker's problem is different from that of the reducible problem, its Liouvillean forced frequency can be much weaker. Readers can consult [12, 29, 36, 43] for related results.

All the above reducible results about quasi-periodically forced (qpf) flows with Liouvillean forced frequency are mainly concerned with two-dimensional frequency. However, for multidimensional Liouvillean forced frequency, the theory of high-dimensional continued fractions is not completely satisfactory. Thus, we must look for more appropriate methods to deal with the case of multidimensional Liouvillean forced frequency. In this direction, there are some works on ODEs or PDEs, see [12,34,37,38]. In particular, Xu et al. [38] provided quasi-periodic solutions for Hamiltonian PDE with frequency vector $\omega = (\overline{\omega}_1, \overline{\omega}_2) \in \mathbb{R}^2 \times \mathbb{R}^{d-2}$, $\overline{\omega}_1 = (1,\alpha)$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, which is called weak Liouvillean frequency, i.e., for $\gamma > 0, \tau > d+1$, there is

$$\begin{cases} \beta(\alpha) := \limsup\limits_{n>0} \dfrac{\ln\ln q_{n+1}}{\ln q_n} < \infty, \\ |\langle k_1, \overline{\omega}_1\rangle + \langle k_2, \overline{\omega}_2\rangle| \geq \frac{\gamma}{(|k_1|+|k_2|)^\tau},\ \ for\ k_1 \in \mathbb{Z}^2,\ k_2 \in \mathbb{Z}^{d-2}\backslash\{0\}, \end{cases} \tag{1.11}$$

where $\{\frac{p_n}{q_n}\}$ is the continued fraction approximate to α. We note that in (1.11) only $\overline{\omega}_1$ is allowed to be Liouvillean, not the entire frequency ω.

In the case of multidimensional Liouvillean forced frequency, a key step the stability of the Cauchy problem (1.2) or (1.3) is to study rotations reducibility of the qpf m-torus flow $(\omega, \rho + f)$. Even under Brjuno conditions, this problem becomes more complex in higher-dimensional tori ($m \geq 2$) because there is no conception of rotation number in higher-dimensional tori. The corresponding conception are rotation vectors, which are highly relying on the orbit of flow. In most cases, the rotation vector is not unique but a set. Thus, it is difficult to classify the quasi-periodically forced tori using the characters of rotation vector. In the case $m \geq 1$, $d \geq 2$, W. Si and J. Si [33] proved that the qpf flow $(\omega, \rho + f)$, where $\rho \in \mathbb{R}^m$, $f : \mathbb{T}^m \times \mathbb{T}^d \to \mathbb{T}^m$ is analytic and (ω, ρ) satisfies the Brjuno-Rüssmann's non-resonant condition

$$|\langle k, \omega\rangle + \langle l, \rho\rangle| \geq \frac{\gamma}{\Delta(|k| + |l|)}, \quad \text{for all } 0 \neq (k,l) \in \mathbb{Z}^d \times \mathbb{Z}^m,$$

is C^ω reducible provided that f is sufficiently small and satisfies some non-degeneracy condition (see [33] for the definition of Δ).

‡that is

$$|\langle k, \omega\rangle + l\rho_f| \geq \frac{\gamma}{|k|^\tau}, \text{ for all } 0 \neq (k,l) \in \mathbb{Z}^d \times \mathbb{Z},$$

with $\gamma > 0$ and $\tau > 2$.

In this paper, we will prove that, in the case $m \geq 1$ and $d \geq 2$, the analytic qpf m-torus flow $(\omega, \xi + f)$ can be C^∞ rotations reducible when ω possesses finite uniform Diophantine exponent (see (2.3) below), f is sufficiently small and $\xi \in O_\gamma$, where $O_\gamma \subset O$ is Cantor subset of compact set O in $\mathbb{R}^m$ with positive Lebesgue measure. In the case $m = 1$ and $d \geq 2$, the analytic qpf circle flow $(\omega, \rho + f)$ can be C^∞ rotations reducible when ω possesses finite uniform Diophantine exponent (see (2.3) below), its rotation number ρ_f is Diophantine with respect to the forced frequency ω, and f is sufficiently small.

Essentially, the Liouvillean property in multidimensional Liouvillean frequency (1.11) comes from 2-dimensional frequency $\bar{\omega}_1$. In this paper, we introduce multidimensional Liouvillean frequency by using finite uniform Diophantine exponent, which can holistically describe the Liouvillean property of a frequency. We can also prove that our Liouvillean frequency set contains all the Liouvillean frequencies in (1.11). This is the main novelty of our paper.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, definitions, and relevant concepts, which will be used subsequently. In Section 3, we first introduce the results of rotations reducibility for qpf m-torus flow $(\omega, \xi + f)$ in the case $m \geq 1$ and $d \geq 2$ and qpf circle flow $(\omega, \rho + f)$ in the case $m = 1$ and $d \geq 2$, i.e., Proposition 3.1 and Proposition 3.2. Then we give the proof of the main result, Theorem 1. In Section 4, we give the proof of Propositions 3.1 and 3.2. In Section 5, we give an Appendix that introduces the definition of CD-bridge.

2. Preliminaries

In this section, we first give some notations, definitions, and relevant concepts, which will be used subsequently..

2.1. Analytic function space

Denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of integers and positive integers, respectively. Let $\mathbb{T}^l = \mathbb{R}^l/2\pi\mathbb{Z}^l$ be the standard l-dimensional real torus. Given $m \in \mathbb{Z}_+$, we consider a real valued function $f \in L^2(\mathbb{T}^d, \mathbb{R}^m)$

$$f(\varphi) = \sum_{k \in \mathbb{Z}^d} f_k e^{\mathrm{i}\langle k, \varphi \rangle}.$$

We define the sets

$$W_r(\mathbb{T}^d) := \{\varphi \in (\mathbb{T}^d \oplus \mathrm{i}\mathbb{R}^d) : |\mathrm{Im}\varphi| < r\},$$
$$W_{s,r}(\mathbb{T}^m \times \mathbb{T}^d) := \{(\theta, \varphi) \in (\mathbb{T}^m \oplus \mathrm{i}\mathbb{R}^m) \times (\mathbb{T}^d \oplus \mathrm{i}\mathbb{R}^d) : |\mathrm{Im}\theta| < s,\ |\mathrm{Im}\varphi| < r\}.$$

For an analytic function $f(\varphi) = \sum_{k \in \mathbb{Z}^d} f_k e^{\mathrm{i}\langle k, \varphi \rangle}$ defined in $W_r(\mathbb{T}^d)$, we define the spaces of analytic functions

$$H^r := H^r(W_r(\mathbb{T}^d), \mathbb{R}^m) = \left\{ f \ : \ \|f\|_r^2 = \sum_{k \in \mathbb{Z}^d} |f_k|^2 e^{2|k|r} < +\infty \right\},$$

where $r > 0$. For any $N > 0$, we define the truncation and projection operators $\mathcal{T}_N$, $\mathcal{R}_N$ on $H^{r,a}$

$$\mathcal{T}_N f(\varphi) = \sum_{|k| < N} f_k e^{\mathrm{i}\langle k, \varphi \rangle}, \ \ \mathcal{R}_N f(\varphi) = \sum_{|k| \geq N} f_k e^{\mathrm{i}\langle k, \varphi \rangle}.$$

If we consider two variables $(\theta, \varphi) \in \mathbb{T}^{m+d}$, we may consider a real valued function $f(\theta, \varphi) \in L^2(\mathbb{T}^{m+d}, \mathbb{R}^m)$ as a φ-dependent family of functions $f(\theta, \cdot) \in L^2(\mathbb{T}^m, \mathbb{R}^m)$ with the Fourier series expansion

$$f(\theta, \varphi) = \sum_{l \in \mathbb{Z}^m} f_l(\varphi) e^{\mathrm{i}\langle l, \theta \rangle} = \sum_{(l,k) \in \mathbb{Z}^{m+d}} f_l^k e^{\mathrm{i}(\langle l, \theta \rangle + \langle k, \varphi \rangle)}$$

In this case, we describe the spaces of analytic functions defined in $W_{s,r}(\mathbb{T}^m \times \mathbb{T}^d)$

$$\begin{aligned} H^{s,r} := H^{s,r}(W_{s,r}(\mathbb{T}^m \times \mathbb{T}^d), \mathbb{R}^m) &= \Big\{ f \,:\, \|f\|_{s,r}^2 = \sum_{l \in \mathbb{Z}^m} \|f_l(\varphi)\|_r^2 e^{2|l|s} \\ &= \sum_{l \in \mathbb{Z}^m, k \in \mathbb{Z}^d} |f_l^k|^2 e^{2|l|s + 2|k|r} < +\infty, \Big\}, \end{aligned}$$

where $s > 0$, $r > 0$. It is clear that the spaces H^r and $H^{s,r}$ are Banach algebra under their norms, see [28]. Fix $m \in \mathbb{Z}_+$ and let O be a compact subset of $\mathbb{R}^m$. For a function $f : O \to H^r$, we define the norm of f as

$$\|f\|_{r,O}^2 = \sum_{k \in \mathbb{Z}^d} |f_k|_O^2 e^{2|k|r},$$

where

$$|f_k|_O = \sup_{\xi \in O} |f_k(\xi)| + \sup_{\xi \in O} |\frac{d f_k(\xi)}{d\xi}|.$$

For a function $f : O \to H^{s,r}$, we define the norm of f as

$$\|f\|_{s,r,O}^2 = \sum_{l \in \mathbb{Z}^m, k \in \mathbb{Z}^d} |f_l^k|_O^2 e^{2|l|s + 2|k|r},$$

where

$$|f_l^k|_O = \sup_{\xi \in O} |f_l^k(\xi)| + \sup_{\xi \in O} |\frac{d f_l^k(\xi)}{d\xi}|.$$

Here, the derivative with respect to ξ is in the sense of Whitney.

For an analytic function $f(\theta, \phi)$ defined in $W_{s,r}(\mathbb{T}^m \times \mathbb{T}^d)$, its supremum norm is defined by

$$|f(\theta, \phi)|_{s,r} = \sup_{\mathrm{Im}\theta < s, \mathrm{Im}\varphi < r} |f(\theta, \phi)|.$$

By the attenuation of Fourier coefficients f_l^k, i.e., $|f_l^k| \le |f|_{s,r} e^{|l|s + |k|r}$, one can check that

$$\|f\|_{s,r}^2 = \sum_{l \in \mathbb{Z}^m, k \in \mathbb{Z}^d} |f_l^k|^2 e^{2|l|s + 2|k|r} \le |f|_{s,r}^2.$$

In addition, one can easily check that $|f|_{s,r}^2 \le \|f\|_{s,r}^2$, which implies $\|f\|_{s,r} = |f|_{s,r}$.

2.2. Diffeomorphisms of the torus

Consider a diffeomorphisms of the m-dimensional torus

$$\Phi : \mathbb{T}^m \to \mathbb{T}^m,\ \theta \mapsto \theta + h(\theta) = \widehat{\theta}, \tag{2.1}$$

where $h : \mathbb{T}^m \to \mathbb{R}^m$ is an analytic function with $\|h\|_s \leq 1/2$. We denote the inverse of Φ by

$$\Phi^{-1} : \mathbb{T}^m \to \mathbb{T}^m,\ \ \Phi^{-1} : \theta \mapsto \theta + \tilde{h}(\theta),$$

with $\tilde{h}$ an analytic function. Using the same notation, we denote transformations like (2.1) with the corresponding linear operators acting on H^r as

$$\Phi : H^r \to H^r,\ \ f(\theta) \mapsto \Phi f(\theta) := f(\theta + h(\theta)).$$

Similarly, we consider the action of Φ on the vector fields on $\mathbb{T}^m$ by the pushforward. Explicitly, we denote by $T(\mathbb{T}^m)$ the tangent space of $\mathbb{T}^m$.

Now given a vector field $X : \mathbb{T}^m \to T(\mathbb{T}^m)$

$$X(\theta) = \sum_{j=1}^{m} X_j(\theta)\frac{\partial}{\partial\theta_j},\ X_1, \cdots, X_m \in C^{\omega}(\mathbb{T}^m, \mathbb{R}),$$

its pushforward is

$$(\Phi_* X)(\theta) = d\Phi(\Phi^{-1}(\theta))[X(\Phi^{-1}(\theta))] = \sum_{i=1}^{m} \Phi^{-1}(X_i + \sum_{j=1}^{m} \frac{\partial X_i}{\partial\theta_j} X_j)\frac{\partial}{\partial\widehat{\theta_i}}.$$

2.3. Linear operators

A C^{ω} vector field $X(\theta) = \sum_{j=1}^{m} X_j(\theta)\frac{\partial}{\partial\theta_j}$ induces a linear operator acting on the space of functions $f : \mathbb{T}^m \to \mathbb{R}$, that we denote by $X(\theta) \cdot \partial_\theta = \sum_{j=1}^{m} X_j(\theta)\frac{\partial}{\partial\theta_j}$. More precisely, the action of such a linear operator is given by

$$H^r \to H^r, \quad u(\theta) \mapsto X(\theta) \cdot \partial_\theta u(\theta).$$

2.4. The fibred rotation number

We consider the case of $m = 1$. Suppose (ω, f) is a qpf circle flow defined by

$$\omega \cdot \partial_\varphi + f(\theta, \varphi) \cdot \partial_\theta.$$

Let

$$\tilde{\rho}(\omega, f) = \lim_{t\to\infty} \frac{\hat{\Phi}^t_\varphi(\hat{\theta})}{t}$$

be the fibred rotation number associated with (ω, f), where $\hat{\Phi}^t_\varphi(\hat{\theta}) : \mathbb{R}^1_+ \times \mathbb{R}^1 \times \mathbb{T}^d \to \mathbb{R}^1$, via $(t, \hat{\theta}, \varphi) \mapsto \hat{\Phi}^t_\varphi(\hat{\theta})$ denotes the lift of the flow (ω, f) of the first variable θ. The limit exists and is independent of $(\hat{\theta}, \varphi)$ [23]. As a direct consequence of the definition, we have the following well-known results:

Lemma 2.1. *([23]) Let $\omega \in \mathbb{R}^d,\ \rho \in \mathbb{R}$ and $\|f(\theta, \varphi)\|_{C^0} \leq \epsilon$, then*

$$|\tilde{\rho}(\omega, \rho + f) - \tilde{\rho}(\omega, \rho)| \leq \epsilon.$$

Lemma 2.2. *([23]) Suppose (ω, f) is a qpf circle flow and $H \in C_0(\mathbb{T}\times\mathbb{T}^d, \mathbb{T}\times\mathbb{T}^d)$ is a homeomorphism homotopic to the identity that projects to the identity on the second factor. Then the fibred rotation numbers of (ω, f) and $H \circ (\omega, f) \circ H^{-1}$ are the same.*

2.5. Uniform Diophantine exponent

In this subsection, we recall the definition of uniform Diophantine exponent of forced frequency $\omega_0 \in \mathbb{R}^d$ introduced in [17].

• Recall the *Diophantine exponent* denoted by $\omega(\omega_0)$ that the supremum of all positive real numbers κ such that there exists $k \in \mathbb{Z}^d \setminus \{0\}$ and $|\langle k, \omega_0\rangle| \leq |k|^{-\kappa}$.

• Recall the *uniform Diophantine exponent* denoted by $\hat{\omega}(\omega_0)$ that the supremum of all positive real numbers κ such that for any sufficiently large N, there exists $k \in \mathbb{Z}^d \setminus \{0\}$ such that $|k| \leq N$ and $|\langle k, \omega_0\rangle| \leq N^{-\kappa}$. In other words, if we define the set

$$\begin{aligned} S \ : \ = \ & \{\kappa | for\ all\ N\ large\ enough,\ there\ exists\ k \in \mathbb{R}^d \backslash \{0\}, \\ & 0 < |k| \leq N\ such\ that\ |\langle k, \omega_0\rangle| \leq N^{-\kappa}\}, \end{aligned} \tag{2.2}$$

then $\hat{\omega}(\omega_0) = \sup S$.

By the definitions of $\hat{\omega}(\omega_0)$ and $\omega(\omega_0)$, one obtains that $\hat{\omega}(\omega_0) \leq \omega(\omega_0)$, and while $\omega(\omega_0)$ measures how small linear forms with integer coefficients of a given size can become when evaluated at ω_0. It is clear that if $\omega(\omega_0) < \infty$ then ω_0 is Diophantine vector and if $\omega(\omega_0) = \infty$ then ω_0 is not Diophantine vector. For *uniform Diophantine exponent* $\hat{\omega}(\omega_0)$, if $\hat{\omega}(\omega_0) = \infty$, then ω_0 is not Diophantine vector. But if $\hat{\omega}(\omega_0) < \infty$, then ω_0 also contains Liouvillean frequency. For example, $\hat{\omega}(\omega_0)$ is always finite if $\omega_0 = (1, \alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ because $|\langle k, \omega_0\rangle| > 1/2q_n$, for $0 < |k| \leq q_n$, where $\frac{p_n}{q_n}$ is the continued fraction approximated to α.

In this paper, we strengthen the condition $\hat{\omega}(\omega_0) < \infty$ to exist a τ_1 and an increasing sequence K_n such that

$$|\langle k, \omega_0\rangle| \geq \frac{1}{K_n^{\tau_1}},\ \forall\ 0 < |k| \leq K_n, \tag{2.3}$$

where the sequence $\{K_n\}_{n\geq 0}$ satisfies

(1) For the given constant $\tau_1 > 0$, there exists a positive constant $M > d/2$ such that

$$K_{n+1} > K_n^M; \tag{2.4}$$

(2) There exists a positive constant N such that

$$K_n^N \geq \ln K_{n+1}. \tag{2.5}$$

Remark 2.1. *Assumptions (2.4) and (2.5) are really technical conditions summarized from not super-Liouvillean condition in [27]. This property can also be reflected in high dimensional Liouvillean frequencies. One of the purposes of this paper is an observation of this property in higher dimensions.*

2.6. Liouvillean frequency vectors

Definition 2.1 (Diophantine condition). *Given a frequency $\omega \in \mathbb{R}^d$, we call ω Diophantine if there exists a sequence $\{K_n\}_{n\geq 0}$ with $K_n = K_{n-1}^{\widehat{\beta}}$ and $1 < \widehat{\beta} < 2$ such that*

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{K_n^{\tau_1}},\ \ \forall 0 < |k| < K_n,$$

where $\gamma \in (0, 1)$, $\tau_1 > d - 1$.

Definition 2.2 ($MN\tau_1$-*admissibility*)**.** *For a given sequence* $\{K_n\}_{n\geq 0}$, $N, \tau_1 > 0$ *and* $M > d/2$, *we call* $\{K_n\}_{n\geq 0}$ *is* ($MN\tau_1$)*-admissible if it satisfies the inequalities (2.4) and (2.5). The set of all such sequences is denoted by*

$$\mathcal{K}(\tau_1, M, N) = \{K | K = \{K_n\}_{n\geq 0} \text{ and } K \text{ is } (MN\tau_1) - admissible\}.$$

Definition 2.3 (Liouvillean vectors set)**.** *For given* $K = \{K_n\}_{n\geq 0} \in \mathcal{K}(\tau_1, M, N)$, *we define the set*

$$O_K = \{\omega \in D || \langle k, \omega \rangle| \geq \frac{C}{K_n^{\tau_1}}, \ \forall\ 0 < |k| \leq K_n\}, \tag{2.6}$$

where $D \in \mathbb{R}^d$ *is compact and C is a positive number. Setting*

$$\mathcal{L} := \bigcup_{\tau_1, M, N > 0} \bigcup_{K \in \mathcal{K}(\tau_1, M, N)} O_K,$$

we call $\mathcal{L}$ *Liouvillean vectors set.*

We will show that the set $\mathcal{L}$ allows many Liouvillean vectors. In the following discussions, we always assume $C \equiv 1$ occurring in the frequency set (2.6) without loss of generality.

Remark 2.2. *If frequency* ω_0 *belongs to Liouvillean vectors set* $\mathcal{L}$, *then convergence speed of small divisor tending to* 0 *can be much faster than Diophantine case that is because if* $\omega_0 \in \mathcal{L}$, $K_{n+1} = O(e^{K_n})$ *increases much faster than the Diophantine case* $K_{n+1} = O(K_n^{\widehat{\beta}})$.

Remark 2.3. *We can prove that all the vectors in* $\mathcal{L}$ *have finite uniform Diophantine exponent. In fact, by definition of* ω_0, *we have that if* $\omega_0 \in \mathcal{L}$, *then there exists K, which is* ($MN\tau_1$)*-admissible, such that* $\omega_0 \in O_K$. *For the set* S *defined in (2.2), it is clear that* S *is no-empty because* $0 \in S$, *and it has the following two characters:*

1. If $\kappa_1 \in S$, *then* $\kappa \in S$ *for all* $\kappa \leq \kappa_1$. *That is because if* $\kappa_1 \in S$, *then for all N large enough, there exists* $k \in \mathbb{R}^d \backslash \{0\}$, $0 < |k| \leq N$, *such that* $|\langle k, \omega_0 \rangle| \leq N^{-\kappa_1}$, *which implies* $|\langle k, \omega_0 \rangle| \leq N^{-\kappa}$ *for all* $\kappa \leq \kappa_1$. *Thus,* $\kappa \in S$.

2. If $\kappa_1 \notin S$, *then* $\kappa \notin S$ *for all* $\kappa \geq \kappa_1$. *That is because if* $\kappa_1 \notin S$, *then there exists N large enough, for all* $k \in \mathbb{R}^d \backslash \{0\}$, $0 < |k| \leq N$, *such that* $|\langle k, \omega_0 \rangle| > N^{-\kappa_1}$, *which implies* $|\langle k, \omega_0 \rangle| > N^{-\kappa}$ *for all* $\kappa \geq \kappa_1$. *Thus,* $\kappa \notin S$.

Moreover, $\omega_0 \in O_K$, *then* $\tau_1 \notin S$. *Thus* $0 \leq \hat{\omega}(\omega_0) = \sup S \leq \tau_1$.

Remark 2.4. *Now, we illustrate that the Liouvillean vectors set* $\mathcal{L}$ *allows many Liouvillean vectors.*

In two-dimensional case, we consider $\omega = (1, \alpha)$ *in* $\mathbb{R}^2$. *Let* $\mathcal{A} = 8$ *and* $(Q_n)_{n\in\mathbb{N}}$ *be the selected subsequence in Lemma A.1 of Appendix. By Corollary 2.1 in [27], if* $\beta(\alpha) < \infty$ *we have the following claims:*

(1) $Q_n > Q_{n-1}^{\mathcal{A}}$. *That is because for* $n = 1$, *it is clear that* $Q_1 > Q_0^{\mathcal{A}}$ *since* $Q_0 = 1$. *If* $n \geq 2$ *and* $\overline{Q}_{n-1} \leq Q_{n-1}^{\mathcal{A}}$, *then one has* $Q_n \geq \overline{Q}_{n-1} \geq Q_{n-1}^{\mathcal{A}}$. *If* $n \geq 2$ *and* $\overline{Q}_{n-1} > Q_{n-1}^{\mathcal{A}}$, *then* $(\overline{Q}_{n-1}, Q_n)$ *and* (Q_n, Q_{n+1}) *are both* $CD(\mathcal{A}, \mathcal{A}, \mathcal{A}^3)$ *bridges. Thus, we get* $Q_n > Q_{n-1}^{\mathcal{A}}$.

(2) $\ln Q_{n+1} < Q_n^U$, *where* $U = \beta(\alpha) + \frac{4\ln\mathcal{A}}{\ln 2}$. *That is because*

$$\frac{\ln\ln Q_{n+1}}{\ln Q_n} \leq \frac{4\ln\mathcal{A} + \ln\ln\overline{Q}_n}{\ln Q_n} \leq \beta(\alpha) + \frac{4\ln\mathcal{A}}{\ln 2},$$

which implies $\ln Q_{n+1} < Q_n^U$.

We also have $\langle k,\omega\rangle \geq \frac{1}{2Q_n}$ *for all* $0<|k|<Q_n$, *which implies* $\omega \in \cup_{K\in\mathcal{K}(1,\mathcal{A},U)} O_K$. *Thus* $\mathcal{L}$ *includes some frequencies beyond Brjuno in two-dimensional case.*

In multidimensional case, $\mathcal{L}$ *includes all the frequency satisfying condition (1.11). That is because if* $\omega = (\overline{\omega}_1, \overline{\omega}_2) \in \mathbb{R}^2 \times \mathbb{R}^{d-2}$ *satisfies condition (1.11), then*

$$|\langle k_1, \overline{\omega}_1\rangle + \langle k_2, \overline{\omega}_2\rangle| \geq \frac{\gamma}{Q_n^\tau}, \ for\ all\ |k_1| + |k_2| \leq Q_n,\ |k_2| \neq 0$$

and

$$|\langle k_1, \overline{\omega}_1\rangle + \langle k_2, \overline{\omega}_2\rangle| \geq \frac{1}{2Q_n}, \ for\ all\ |k_1| \leq Q_n,\ |k_2| = 0,$$

where $(Q_n)_{n\in\mathbb{N}}$ *is the selected subsequence in Lemma A.1 of Appendix. If* $\tau_1 = \max\{1,\tau\}$, *then* $\omega \in \cup_{K\in\mathcal{K}(\tau_1,\mathcal{A},U)} O_K$, *and so* $\omega \in \mathcal{L}$.

3. Reducibility

For any integer $m \geq 1$ and $d \geq 2$, we have the following Proposition.

Proposition 3.1. *Let* O *be a bounded closed set in* $\mathbb{R}^m$ *not containing zero,* $r>0$, $s>0$, $\tau > m+d-1$, $\gamma > 0$, $\omega \in \mathcal{L}$ *and* $\xi \in O$. *Consider a* C^1*-smooth family of vector fields on* $\mathbb{T}^{m+d}$

$$X := \omega \cdot \frac{\partial}{\partial\varphi} + (\xi + f(\theta,\varphi,\xi)) \cdot \frac{\partial}{\partial\theta}$$

where $f(\cdot;\xi) \in H^{s,r}(W_{s,r}(\mathbb{T}^m \times \mathbb{T}^d), \mathbb{R}^m)$. *Then, there exists a sufficiently small constant* $\epsilon = \epsilon(\tau,\tau_1,\gamma,s,r,M,N,O)$ *such that if*

$$\|f(\theta,\varphi)\|_{s,r,O} \leq \epsilon, \tag{3.1}$$

then there exists a set

$$O_\infty = \{\xi \in O : |\langle k,\omega\rangle| + \langle l, \rho_\infty(\xi)\rangle \geq \frac{\gamma_\infty}{(|k|+|l|)^\tau},\ \forall (k,l) \in \mathbb{Z}^{d+m}, |l| \neq 0, \}, \tag{3.2}$$

where $\rho_\infty(\xi) : O_\infty \to \mathbb{R}^m$ *is a* C^1 *function and there exists a map*

$$h(\theta,\varphi,\xi) : \mathbb{T}^{m+d} \times O_\infty \to \mathbb{R}^m,\ \sup_{\varphi\in\mathbb{T}^d} \|h(\theta,\cdot)\|_{\frac{s}{2},O_\infty} \leq C\epsilon, \tag{3.3}$$

which is C^∞*-smooth in* φ, *analytic in* θ *and* C^1*-smooth in* ξ, *so that for all* $\xi \in O_\infty$ *the pushforward of vector fields* X *by diffeomorphism* $\Psi : (\varphi,\theta) \mapsto (\varphi, \theta + h(\theta,\varphi)) = (\varphi, \widehat{\theta})$ *is*

$$\begin{aligned}\Psi_* X &= \omega \cdot \frac{\partial}{\partial\varphi} + \Psi^{-1}(\omega \cdot \partial_\varphi h + (\xi + f) + (\xi + f)\cdot\partial_\varphi h) \cdot \frac{\partial}{\partial\widehat{\theta}} \\ &= \omega \cdot \frac{\partial}{\partial\varphi} + (\rho_\infty(\xi) + m_\infty(\varphi,\xi)) \cdot \frac{\partial}{\partial\widehat{\theta}}.\end{aligned}$$

where $m_\infty : \mathbb{T}^d \times O_\infty \to \mathbb{R}^m$, $(\varphi,\xi) \mapsto m_\infty(\varphi,\xi)$ *is* C^1*-smooth in* ξ, *analytic in* φ *and for each* $\xi \in O_\infty$ *satisfies*

$$\sup_{\varphi\in\mathbb{T}^d} |m_\infty(\varphi,\xi)| \leq C\epsilon.$$

For $m = 1$ and any integer $d \geq 2$, we have the following Proposition.

Proposition 3.2. *Let $\rho \in \mathbb{R}$, $\omega \in \mathcal{L}$, $r > 0$, $s > 0$, $\tau > d$ and $\gamma > 0$. We consider a vector field on $\mathbb{T}^{d+1}$*

$$X := \omega \cdot \frac{\partial}{\partial \varphi} + (\rho + f(\theta, \varphi)) \cdot \frac{\partial}{\partial \theta}$$

where $f(\theta, \varphi) \in H^{s,r}(W_{s,r}(\mathbb{T} \times \mathbb{T}^d), \mathbb{R}^m)$. Assume that $\tilde{\rho}(\omega, \rho + f(\theta, \varphi)) = \rho_f \in DC_\omega(\gamma, \tau)$ in the sense

$$|\langle k, \omega \rangle + l\rho_f| \geq \frac{\gamma}{(|k| + |l|)^\tau}, \forall (k, l) \in \mathbb{Z}^d \times \mathbb{Z}, \ l \neq 0.$$

Then there exists a sufficiently small constant $\epsilon = \epsilon(\tau, \tau_1, \gamma, s, r, M, N)$ such that if $\|f(\theta, \varphi)\|_{s,r} \leq \epsilon$, and there exists a map

$$h(\theta, \varphi) : \mathbb{T}^{1+d} \to \mathbb{R}, \ \sup_{\varphi \in \mathbb{T}^m} \|h(\theta, \cdot)\|_{\frac{s}{2}} \leq C\epsilon, \tag{3.4}$$

which is C^∞-smooth in φ, analytic in θ, so that the pushforward of vector fields X by diffeomorphism $\Psi : (\varphi, \theta) \mapsto (\varphi, \theta + h(\theta, \varphi)) = (\varphi, \widehat{\theta})$ is

$$\Psi_* X = \omega \cdot \frac{\partial}{\partial \varphi} + (\rho_f + m_\infty(\varphi)) \cdot \frac{\partial}{\partial \widehat{\theta}},$$

where $m_\infty(\varphi)$ is C^∞-smooth in φ and satisfies

$$\sup_{\varphi \in \mathbb{T}^d} |m_\infty(\varphi)| \leq C\epsilon.$$

Remark 3.1. The proof of Proposition 3.1 or Proposition 3.2 is based on a KAM iterative scheme under the assumption that forced frequency has finite uniform Diophantine exponent. As we know, the results existing in the literature deal with two-dimensional frequency, and exploit the theory of continued fractions to control the small divisor problem. The results in this paper extend the analysis to higher dimensional frequency with finite uniform Diophantine exponent, allowing a class of Liouvillean frequencies. More specifically, Proposition 3.1 can be regarded as a generalization of the work in [20] from Diophantine frequency to Liouvillean frequency and work in [33] from Brjuno frequency to Liouvillean frequency, while Theorem 1 can be regarded as a generalization of the work in [27] from two-dimensional not super-Liouvillean forced frequency to high-dimensional forced frequency with finite uniform Diophantine exponent. The overall strategy of this paper comes from the literature [27, 33], but the method is still in the spirit of [4, 17, 24], which, however, has to overcome essential obstructions in techniques for dealing with the questions considered here.

Remark 3.2. Compared with the results in [20], which reduce the variables (θ, φ), our results only reduce variable θ because of the difficulty from Liouvillean frequency. Even so, the analytic norms of the solutions of the Cauchy problem can also be controlled uniformly in time. This is the following Theorem 1.

Theorem 1. *Let $\omega \in \mathcal{L}$ and consider the transport equation*

$$\partial_t u + (\xi + f(x, \omega t, \xi)) \cdot \partial_x u = 0. \tag{3.5}$$

Then if $f(\cdot;\xi) \in H^{s,r}(W_{s,r}(\mathbb{T}^m \times \mathbb{T}^d), \mathbb{R}^m)$ *is* C^1*-smooth with respect to* ξ *and (3.1) is fulfilled, for* $\xi \in \mathcal{O}_\infty$ *(see (3.2)), under the change of variable* $u = \Psi(\omega t)[v] = v(x + h(x, \omega t))$ *defined in (3.3), the PDE (3.5) transforms into the equation with coefficients independent of spatial variable* x

$$\partial_t v + (\xi + m_\infty(\omega t, \xi)) \cdot \partial_x v = 0.$$

As a consequence, for $u_0 \in H^{\frac{s}{2}}$*, the only solution of the Cauchy problem*

$$\begin{cases} \partial_t u + (\xi + f(x, \omega t, \xi)) \cdot \partial_x u = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

satisfies $\|u(x,t)\|_{\frac{s}{4}} \leq \|u_0(x)\|_{\frac{s}{2}}$ *for any* $t \in \mathbb{R}$*, i.e., this Cauchy problem is stable.*

Proof. Let $\omega \in \mathcal{L}$ and $\xi \in \mathcal{O}_\infty$. Under the change of coordinates $u = \Psi(\omega t)[v] = v(x + h(x, \omega t))$, the equation transforms (3.5) into the PDE

$$\partial_t v + \Psi(\omega t)^{-1}(\omega \cdot \partial_\varphi h + \xi + f + (\xi + f) \cdot \partial_x h) \cdot \partial_x v = 0.$$

In view of Proposition 3.1, one obtains that

$$\partial_t v + (\xi + m_\infty(\omega t, \xi)) \cdot \partial_x v = 0,$$

where

$$m_\infty(\omega t, \xi) = \Psi(\omega t)^{-1}(\omega \cdot \partial_\varphi h + f + (\xi + f) \cdot \partial_x h).$$

Let $v = \sum_{l \in \mathbb{Z}^m} v_l(t) e^{\mathrm{i}\langle l, x\rangle}$. Such a PDE with coefficients independent of a spatial variable can be integrated explicitly, implying that for any $l \in \mathbb{Z}^m$, $v_l(t) = v_l(0) e^{\mathrm{i}\langle \xi + \int_0^t m_\infty(\omega s) ds, l\rangle}$ and

$$\begin{aligned} \|v(x,t)\|_s^2 &= \sum_{l \in \mathbb{Z}^m} |v_l(0) e^{\mathrm{i}\langle \xi t + \int_0^t m_\infty(\omega s) ds, l\rangle}|^2 e^{2s|l|} \\ &= \sum_{l \in \mathbb{Z}^m} |v_l(0)|^2 e^{2s|l|} \\ &= \|v(0)\|_s^2. \end{aligned}$$

By Proposition 3.1 and inverse mapping theorem, we get $\Psi(\omega t)^{-1}[u] = u(x + \tilde{h}(x, \omega t))$, where

$$\sup_{\varphi \in \mathbb{T}^d} \|\tilde{h}(x, \cdot)\|_{\frac{s}{2} - \sigma} \leq C\epsilon$$

with $\sigma = s/12$. Then, given $u_0(x) \in H^s(\mathbb{T}^m, \mathbb{R})$, one gets that for each $t \in \mathbb{R}$

$$\begin{aligned} \|u(x,t)\|_{\frac{s}{4}} &= \|\Psi(\omega t)[v]\|_{\frac{s}{4}} = \|v(x + h(x, \omega t), t)\|_{\frac{s}{4}} \\ &\leq \|v(x,t)\|_{\frac{s}{4}+\sigma} = \|v(0)\|^2_{\frac{s}{4}+\sigma} = \|\Psi^{-1}(\omega t)[u_0]\|_{\frac{s}{4}+\sigma} \\ &= \|u_0(x + \tilde{h}(x, \omega t))\|_{\frac{s}{4}+\sigma} \\ &\leq \|u_0(x)\|_{\frac{s}{4}+2\sigma} \\ &\leq \|u_0(x)\|_{\frac{s}{2}}. \end{aligned}$$

This completes the proof. □

Similarly, we can prove the following theorem by using Proposition 3.2.

Theorem 2. *Let $\rho \in \mathbb{R}$ and $\omega \in \mathcal{L}$ and consider the transport equation*

$$\partial_t u + (\rho + f(x, \omega t)) \cdot \partial_x u = 0. \tag{3.6}$$

Then, if $\tilde{\rho}(\omega, \rho + f(\theta, \varphi)) = \rho_f \in DC_\omega(\gamma, \tau)$, $f \in H^{s,r}(W_{s,r}(\mathbb{T} \times \mathbb{T}^d), \mathbb{R})$ and $\|f(\theta, \varphi)\|_{s,r} \leq \epsilon$, under the change of variable $u = \Psi(\omega t)[v] = v(x + h(x, \omega t))$ defined in (3.4), the PDE (3.6) transforms into the equation with coefficients independent of spatial variable x

$$\partial_t v + (\rho_f + m_\infty(\omega t)) \cdot \partial_x v = 0.$$

As a consequence, for $u_0 \in H^{\frac{s}{2}}$, the only solution of the Cauchy problem

$$\begin{cases} \partial_t u + (\rho + f(x, \omega t)) \cdot \partial_x u = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

satisfies $\|u(x, t)\|_{\frac{s}{4}} \leq \|u_0(x)\|_{\frac{s}{2}}$ for any $t \in \mathbb{R}$, i.e., this Cauchy problem is stable.

4. Proofs of Propositions 3.1 and 3.2

In this section, we give the proofs of Propositions 3.1 and 3.2 by a modified KAM iterative scheme. To better understand this iterative scheme, we draw a block diagram as follows:

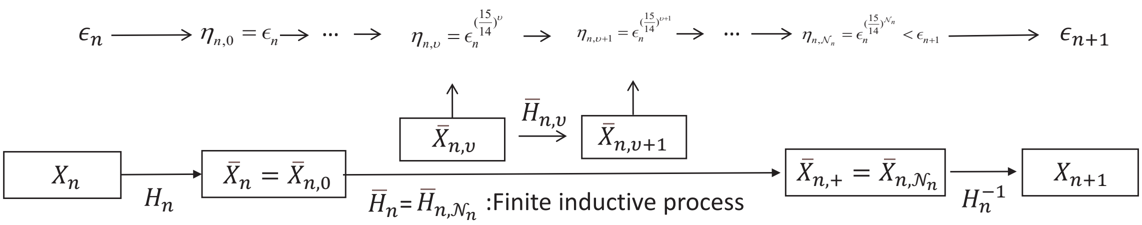

Figure 1. Iterative scheme.

4.1. Proof of Proposition 3.1

Since $\omega \in \mathcal{L}$, we assume that M, N, τ_1 are defined as in (2.3). And we take c and α satisfying $c > \frac{16\alpha(3\tau+2+\frac{d}{2})}{\tau N}$ and $\max\{\frac{M(2\tau_1-d-2)}{2M-d}, 1\} < \alpha < M$. For r_0, s_0, $\gamma > 0$, $\tau > m + d - 1$, we suppose $K_0 > 0$ large enough such that

$$\frac{K_0^{\frac{M-\alpha}{16M}}}{\ln K_0} > c\tau N,$$

and ϵ_0 is sufficiently small. We further define the following sequences for $n \geq 1$:

$$\begin{aligned} \Delta_n &= \tfrac{s_0}{10 \cdot 2^{n-1}}, \quad & \gamma_n &= \gamma_0(1 - \textstyle\sum_{j=2}^{n+1} 2^{-j}), \\ r_n &= \tfrac{r_0}{4K_n^\alpha}, \quad & s_n &= s_{n-1} - \Delta_n, \\ \epsilon_n &= \tfrac{\epsilon_{n-1}}{K_{n+1}^{2^{n+1}c\tau N}}, \quad & \widetilde{\epsilon}_n &= \textstyle\sum_{i=0}^{n-1} \epsilon_i. \end{aligned} \tag{4.1}$$

Let

$$O_{n-1}=\{\xi\in O_{n-2}:|\langle k,\omega\rangle|+\langle l,\rho_{n-1}(\xi)\rangle\geq\frac{\gamma_{n-1}}{(|k|+|l|)^{\tau}},\ \forall|l|\neq0,\ |k|+|l|\leq T^{(n-1)}\},\tag{4.2}$$

where $T^{(n-1)}=(\frac{\gamma^3}{32\epsilon_{n-1}^{\frac{1}{2}}})^{\frac{1}{3\tau+2+\frac{d}{2}}}$ and $O_{-1}=O$.

4.1.1. One KAM step

In $(n-1)$-th step, the vector fields X_{n-1} can be written as

$$X_{n-1}=\omega\cdot\partial_{\varphi}+(\rho_{n-1}(\xi)+g_{n-1}(\varphi,\xi)+f_{n-1}(\theta,\varphi,\xi))\cdot\partial_{\theta},\tag{4.3}$$

where $\xi\in O_{n-1}$, $[g_{n-1}]_{\varphi}=0$ and

$$\|g_{n-1}(\varphi,\xi)\|_{r_{n-1},O_{n-1}}\leq4\tilde{\epsilon}_{n-1},$$
$$\|f_{n-1}(\theta,\varphi,\xi)\|_{s_{n-1},r_{n-1},O_{n-1}}\leq\epsilon_{n-1}.$$

In what follows, we divide KAM step into three parts. In the proof, we do not write ξ explicit for simplicity if there is no confusion. First, we give a Lemma that is used to eliminate the lower-frequency terms.

Lemma 4.1. *If we denote $\bar{s}_n=s_{n-1}-\frac{\Delta_n}{3}$, $\bar{r}_n=\frac{r_0}{K_n^{\alpha}}$, then there exists $h_{n-1}(\varphi)$ with $\|h_{n-1}(\varphi)\|_{\bar{r}_n,O_{n-1}}\leq K_n^{\frac{M\tau_1+(d+1)\alpha}{M}}$, such that the transformation $H_{n-1}:\theta=\overline{\theta}+h_{n-1}(\varphi)(\mod 2\pi)$ conjugates the vector fields (4.3) into*

$$\overline{X}_{n-1}=(H_{n-1})_*X_{n-1}=\omega\cdot\partial_{\varphi}+(\rho_{n-1}+\bar{g}_{n-1}(\varphi)+\bar{f}_{n-1}(\bar{\theta},\varphi))\cdot\partial_{\bar{\theta}},\tag{4.4}$$

where

$$\|\bar{g}_{n-1}(\varphi)\|_{\bar{r}_n,O_{n-1}}\leq\epsilon_{n-1}^{\frac{1}{2}},$$
$$\|\bar{f}_{n-1}(\bar{\theta},\varphi)\|_{\bar{s}_n,\bar{r}_n,O_{n-1}}\leq\epsilon_{n-1}.$$

Proof. Using transformation $\theta=\bar{\theta}+h_{n-1}(\varphi)(\mod 2\pi)$, the vector field (4.3) becomes

$$\overline{X}_{n-1}=\omega\cdot\partial_{\varphi}+(\rho_{n-1}-\partial_{\omega}h_{n-1}(\varphi)+g_{n-1}(\varphi)+f_{n-1}(\bar{\theta}+h_{n-1}(\varphi),\varphi))\cdot\partial_{\bar{\theta}}.\tag{4.5}$$

The vector field (4.5) becomes

$$\overline{X}_{n-1}=\omega\cdot\partial_{\varphi}+\tilde{f}_{n-1}(\bar{\theta},\varphi)\cdot\partial_{\bar{\theta}},$$

where $\tilde{f}_{n-1}(\bar{\theta},\varphi)=\rho_{n-1}+\mathcal{R}_{K_n}g_{n-1}(\varphi)+f_{n-1}(\bar{\theta}+h_{n-1}(\varphi),\varphi)$, if homological equation $\partial_{\omega}h_{n-1}(\varphi)=\mathcal{T}_{K_n}g_{n-1}(\varphi)$ is solvable. In view of $|\langle k,\omega\rangle|>\frac{1}{K_n^{\tau_1}}$ for all $|k|\leq K_n$, one can check that

$$\|h_{n-1}(\varphi)\|^2_{\frac{r_{n-1}}{2},O_{n-1}}\leq\sum_{0<|k|\leq K_n}|h_{n-1,k}|^2_{O_{n-1}}e^{2|k|\frac{r_{n-1}}{2}}$$

$$
\begin{aligned}
&\le \sum_{0<|k|\le K_n} \left|\frac{g_{n-1,k}}{\langle k,\omega\rangle}\right|^2_{O_{n-1}} e^{2|k|\frac{r_{n-1}}{2}} \\
&\le K_n^{2\tau_1}\|g_{n-1}(\varphi)\|^2_{r_{n-1},O_{n-1}} \sum_{0<|k|\le K_n} e^{-|k|\frac{r_{n-1}}{2}} \\
&\le K_n^{2\tau_1}\|g_{n-1}(\varphi)\|^2_{r_{n-1},O_{n-1}} \left(\frac{2}{r_{n-1}}\right)^d \\
&< K_n^{\frac{2M\tau_1+d\alpha}{M}},
\end{aligned}
$$

since $K_{n-1} < K_n^{\frac{1}{M}}$ and ϵ_0 sufficiently small. In order to control $f_{n-1}(\bar{\theta} + h_{n-1}(\varphi), \varphi)$, we should illustrate that $\mathrm{Im}h_{n-1}(\varphi)$ can be well controlled. Let $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ for $\varphi_1 \in \mathbb{T}^d$ and $\varphi_2 \in \mathbb{R}^d$ and define

$$
h^1_{n-1}(\varphi_1) = \sum_{0<|k|\le K_n} \frac{g_{n-1,k}}{\mathrm{i}\langle k,\omega\rangle} e^{\mathrm{i}\langle k,\varphi_1\rangle}.
$$

Notice that $g_{n-1}(\varphi)$ is real analytic, which implies $\mathrm{Im}h^1_{n-1}(\varphi_1) = 0$. Since $\frac{1}{K_j^{\frac{\alpha(2m-d)-M(2\tau_1-d-2)}{2M}}}$ goes to zero much faster than Δ_j, we can assume $\frac{1}{K_j^{\frac{\alpha(2m-d)-M(2\tau_1-d-2)}{2M}}} < \frac{\Delta_j}{3}$ for all $j \ge 0$ without loss of generality. Thus, applying Cauchy-Schwarz inequality, we can get

$$
\begin{aligned}
\sup_{\varphi\in\mathbb{T}^d_{\bar{r}_n}} |\mathrm{Im}h_{n-1}(\varphi)| &= \sup_{\varphi\in\mathbb{T}^d_{\bar{r}_n}} \left|\sum_{0<|k|\le K_n} \frac{g_{n-1,k}}{\langle k,\omega\rangle}\left(e^{\mathrm{i}\langle k,\varphi_1+\mathrm{i}\varphi_2\rangle} - e^{\mathrm{i}\langle k,\varphi_1\rangle}\right)\right| \\
&\le \sum_{0<|k|\le K_n} \left|\frac{g_{n-1,k}}{\langle k,\omega\rangle}\right|_{O_{n-1}} \sup_{\varphi\in\mathbb{T}^d_{\bar{r}_n}} |e^{-\langle k,\varphi_2\rangle} - 1| \\
&\le \sum_{0<|k|\le K_n} \left|\frac{g_{n-1,k}}{\langle k,\omega\rangle}\right|_{O_{n-1}} |k|\bar{r}_n \\
&\le K_n \frac{r_0}{K_n^\alpha} \sqrt{\sum_{|k|>0} |g_{n-1,k}|^2} \sqrt{\sum_{0<|k|\le K_n} \langle k,\omega\rangle^{-2}} \\
&\le CK_n^{\tau_1+\frac{d}{2}+1-\alpha} K_n^{\frac{\alpha d}{2}} \|g_{n-1}(\varphi)\|_{r_{n-1},O_{n-1}} \le K_n^{\tau_1+\frac{d}{2}+1-\alpha} K_n^{\frac{\alpha d}{2M}} \\
&\le C\frac{1}{K_n^{\frac{\alpha(2m-d)-M(2\tau_1-d-2)}{2M}}} \le \frac{\Delta_n}{3}.
\end{aligned}
$$

Letting $\bar{g}_{n-1}(\varphi) = \mathcal{R}_{k_n} g_{n-1}(\varphi)$ and $\bar{f}_{n-1}(\bar{\theta},\varphi) = f_{n-1}(\bar{\theta} + h_{n-1}(\varphi), \varphi)$, we have

$$
\begin{aligned}
\|\bar{f}_{n-1}(\bar{\theta},\varphi)\|_{\bar{s}_n,\bar{r}_n,O_{n-1}} &= \|f_{n-1}(\bar{\theta} + h_{n-1}(\varphi),\varphi)\|_{\bar{s}_n,\bar{r}_n,O_{n-1}} \le \|f_{n-1}(\theta,\varphi)\|_{s_{n-1},r_{n-1},O_{n-1}} \le \epsilon_{n-1}, \\
\|\bar{g}_{n-1}(\varphi)\|^2_{\bar{s}_n,\bar{r}_n,O_{n-1}} &= \|\mathcal{R}_{k_n} g_{n-1}(\varphi)\|^2_{\bar{r}_n,O_{n-1}} \le \epsilon_0 e^{-K_n(r_{n-1}-\bar{r}_n)} \le \epsilon_0 e^{-K_n^{\frac{M-\alpha}{4M}}} \le \epsilon^{\frac{1}{2}}_{n-1},
\end{aligned}
$$

if $K_0 > (2c\tau N \ln K_0)^{\frac{2M}{M-\alpha}}$. The proof is completed.

□

In what follows, we should make the perturbation smaller taking advantage of diagonally dominant operators such that the order of perturbation reach that of the next KAM step. But we can not achieve this using one transformation. So we have to make several transformations. We give the following Lemma.

Lemma 4.2. *Denote* $r_{n+}=\frac{r_0}{2K_n^\alpha}$, $s_{n+}=\bar{s}_n-\frac{\Delta}{3}$. *Under the assumptions of Lemma 4.1 and* $\xi\in O_{n-1}$, *there exists a transformation* $\overline{H}_{n-1}$ *with estimates*

$$\begin{aligned}&\|\overline{H}_{n-1}-id\|_{s_{n+},r_{n+},O_{n-1}}\le 4\epsilon_{n-1}^{\frac{3}{4}},\\ &\|D(\overline{H}_{n-1}-id)\|_{s_{n+},r_{n+},O_{n-1}}\le 4\epsilon_{n-1}^{\frac{3}{4}},\end{aligned}$$

such that $\overline{H}_{n-1}$ *conjugates the vector field (4.4) to*

$$\overline{X}_{n-1,+}=\omega\cdot\partial_\varphi+(\rho_{n-1}+\bar{g}_{n-1,+}(\varphi)+\bar{f}_{n-1,+}(\bar{\theta}_+,\varphi))\cdot\partial_{\bar{\theta}_+} \tag{4.6}$$

with

$$\begin{aligned}&\|\bar{g}_{n-1,+}(\varphi)\|_{r_{n+},O_{n-1}}\le 2\epsilon_{n-1}^{\frac{1}{2}},\\ &\|\bar{f}_{n-1,+}(\bar{\theta}_+,\varphi)\|_{s_{n+},r_{n+},O_{n-1}}\le\epsilon_n,\\ &\|\bar{g}_{n-1,+}(\varphi)-\bar{g}_{n-1}(\varphi)\|_{r_{n+},O_{n-1}}\le 4\epsilon_{n-1}.\end{aligned}$$

Proof. In order to find $\overline{H}_{n-1}$, we divide $\overline{H}_{n-1}$ into $\mathcal{N}$ transformations where $\mathcal{N}$ is determined later. Letting $\tilde{r}=\bar{r}_n=\frac{r_0}{K_n^\alpha}$, $\tilde{s}=\bar{s}_n=s_{n-1}-\frac{\Delta}{3}$, $\tilde{\eta}=2\epsilon_{n-1}$, we define the following sequences

$$\begin{aligned}&\tilde{r}_0=\tilde{r},\quad \tilde{s}_0=\tilde{s},\quad \tilde{\eta}_\nu=\tilde{\eta}^{(\frac{3}{2})^\nu},\\ &\sigma_1=\frac{\tilde{r}}{4},\quad \sigma_{\nu+1}=\frac{1}{2^\nu}\sigma_1,\\ &\delta_1=\frac{\Delta_n}{6},\quad \delta_{\nu+1}=\frac{1}{2^\nu}\delta_1,\\ &\tilde{r}_\nu=\tilde{r}_{\nu-1}-\sigma_\nu,\quad \tilde{s}_\nu=\tilde{s}_{\nu-1}-\delta_\nu,\\ &T_\nu=\frac{1}{\sigma_\nu}\ln\frac{1}{\tilde{\eta}_{\nu-1}}.\end{aligned}$$

In the proof, we do not write n explicitly for simplicity. At $(\nu-1)$-th step, we write vector field as

$$\overline{X}_{\nu-1}=\omega\cdot\partial_\varphi+(\rho+\bar{g}_{\nu-1}(\varphi)+\bar{f}_{\nu-1}(\bar{\theta}_{\nu-1},\varphi))\cdot\partial_{\bar{\theta}_{\nu-1}}, \tag{4.7}$$

where

$$\begin{aligned}&\|\bar{g}_{\nu-1}(\varphi)\|_{\tilde{r}_{\nu-1}}\le 2\epsilon_{n-1}^{1/2},\\ &\|\bar{f}_{\nu-1}(\bar{\theta}_{\nu-1},\varphi)\|_{\tilde{s}_{\nu-1},\tilde{r}_{\nu-1}}\le\tilde{\eta}_{\nu-1}.\end{aligned}$$

Applying transformation $\phi_{\nu-1}:(\bar{\theta}_{\nu-1},\varphi)=(\bar{\theta}_\nu+h_\nu(\bar{\theta}_\nu,\varphi),\varphi)$, the vector field (4.7) becomes

$$\overline{X}_\nu=\omega\cdot\partial_\varphi+(\rho+\bar{g}_\nu(\varphi)+(1+\frac{\partial h_\nu}{\partial\bar{\theta}_\nu})^{-1}(\bar{f}_{\nu-1}(\bar{\theta}_\nu+h_\nu(\bar{\theta}_\nu,\varphi),\varphi)-\bar{f}_{\nu-1}(\bar{\theta}_\nu,\varphi))\cdot\partial_{\bar{\theta}_\nu}, \tag{4.8}$$

where $\bar{g}_\nu(\varphi)=\bar{g}_{\nu-1}(\varphi)+[\bar{f}_{\nu-1}(\bar{\theta}_\nu,\varphi)]_{\bar{\theta}_\nu}$, if the homological equation

$$\partial_\omega h_\nu^i+\langle\frac{\partial h_\nu^i}{\partial\bar{\theta}_\nu},\rho+\bar{g}_\nu(\varphi)\rangle=\bar{f}_{\nu-1}^i(\bar{\theta}_\nu,\varphi)-[\bar{f}_{\nu-1}^i(\bar{\theta}_\nu,\varphi)]_{\bar{\theta}_\nu},\ \forall\ 1\le i\le m, \tag{4.9}$$

is solved. Since homological equation (4.9) may have no analytic solution, we solve its approximate equation for each $1\le i\le m$.

$$\partial_\omega h_\nu^i+\langle\frac{\partial h_\nu^i}{\partial\bar{\theta}_\nu},\rho\rangle+\mathcal{T}_{T_\nu}\langle\frac{\partial h_\nu^i}{\partial\bar{\theta}_\nu},\bar{g}_\nu(\varphi)\rangle=\mathcal{T}_{T_\nu}(\bar{f}_{\nu-1}(\bar{\theta}_\nu,\varphi)-\bar{[}f_{\nu-1}(\bar{\theta}_\nu,\varphi)]_{\theta_\nu}). \tag{4.10}$$

Let

$$\bar{f}^i_{\nu-1}(\bar{\theta}_\nu,\varphi)=\sum_{l\in\mathbb{Z}^m}\bar{f}^i_{\nu-1,l}(\varphi)e^{\mathrm{i}\langle l,\bar{\theta}_\nu\rangle},\quad \bar{f}^i_{\nu-1,l}(\varphi)=\sum_{k\in\mathbb{Z}^d}\bar{f}^{ik}_{\nu-1,l}e^{\mathrm{i}\langle k,\varphi\rangle},$$
$$h^i_\nu(\bar{\theta}_\nu,\varphi)=\sum_{0<|l|\le T_\nu}h^i_{\nu,l}(\varphi)e^{\mathrm{i}\langle l,\bar{\theta}_\nu\rangle},\quad h^i_{\nu,l}(\varphi)=\sum_{|k|<T_\nu-|l|}h^{ik}_{\nu,l}e^{\mathrm{i}\langle k,\varphi\rangle}.$$

In order to solve equation (4.10), it is equivalent to solve

$$\partial_\omega h^i_{\nu,l}(\varphi)+\mathrm{i}\langle l,\rho\rangle h^i_{\nu,l}(\varphi)+\mathcal{T}_{T_\nu-|l|}h^i_{\nu,l}(\varphi)\langle \mathrm{i}l,\bar{g}_\nu(\varphi)\rangle=\mathcal{T}_{T_\nu-|l|}\bar{f}^i_{\nu-1,l}(\varphi), \tag{4.11}$$

for $0<|l|<T_\nu$.

For any fixed l, (4.11) can be written as a matrix equation

$$(A_l+G_l)\bar{h}_l=\bar{f}_l,$$

where

$$\bar{h}_l=(h^{ik}_{\nu,l})^T_{|k|<T_\nu-|l|},\quad A_l=diag(\mathrm{i}\langle k,\omega\rangle+\mathrm{i}\langle l,\rho\rangle:|k|<T_\nu-|l|),$$
$$\bar{f}_l=(\bar{f}^{ik}_{\nu-1,l})^T_{|k|<T_\nu-|l|},\quad G_l=\mathrm{i}\langle l,\bar{g}_{\nu,p-q}\rangle_{|p|,|q|<T_\nu-|l|},$$

with $\bar{g}_{\nu,k}$ being the Fourier coefficients of Fourier expansion of $\bar{g}_\nu(\varphi)$. If we denote

$$\Omega_{l,r'}=\mathrm{diag}(\cdots,e^{|k|r'},\cdots)$$

for any $r'\le\bar{r}_n$, then we have

$$(\tilde{A}_{l,r'}+\tilde{G}_{l,r'})\tilde{h}_{l,r'}=\tilde{f}_{l,r'},$$

where

$$\tilde{A}_{l,r'}=\Omega_{l,r'}A_l\Omega^{-1}_{l,r'},\quad \tilde{G}_{l,r'}=\Omega_{l,r'}G_l\Omega^{-1}_{l,r'},$$
$$\tilde{h}_{l,r'}=\Omega_{l,r'}\bar{h}_l,\quad \tilde{f}_{l,r'}=\Omega_{l,r'}\bar{f}_l.$$

Letting $\mathcal{N}=[2^nc_1\tau N\ln K_n]+1$ with $c_1=\frac{c}{16(3\tau+2+\frac{d}{2})\ln 3}-\frac{\alpha}{\tau N\ln 3}$, we have

$$\begin{aligned} T_\nu &= \frac{1}{\sigma_\nu}\ln\frac{1}{\tilde{\eta}_{\nu-1}}\le\frac{4K_n^\alpha 3^{\nu-1}}{r_0}\ln\frac{1}{2\epsilon_{n-1}}\\ &\le \frac{4K_n^\alpha 3^{2^nc_1\tau N\ln K_n}}{r_0}\ln\frac{1}{2\epsilon_{n-1}}\\ &\le \frac{4K_n^{\alpha+2^nc_1\tau N\ln 3}}{r_0}\ln\frac{1}{2\epsilon_{n-1}}\\ &\le \epsilon_{n-1}^{-\frac{1}{16(3\tau+2+\frac{d}{2})}-\frac{1}{16(3\tau+2+\frac{d}{2})}}\\ &\le (\frac{\gamma^3}{32\epsilon_{n-1}^{\frac{1}{2}}})^{\frac{1}{3\tau+2+\frac{d}{2}}}=T^{(n-1)}, \end{aligned}\tag{4.12}$$

since ϵ_0 sufficiently small. In view of $\xi\in O_{n-1}$ and (4.12), we get

$$\|\tilde{A}^{-1}_{l,r'}\|\le\max_{k\le T_\nu}\sup_{\xi\in O_{n-1}}(\frac{1}{|\langle k,\omega\rangle+\langle l,\rho\rangle|}+\frac{|\langle l,\frac{d\rho}{d\xi}\rangle|}{|\langle k,\omega\rangle+\langle l,\rho\rangle|^2})$$

$$\leq \quad 8\frac{T_\nu^{3\tau+1}}{\gamma^3} \leq 8\frac{(T^{n-1})^{3\tau+1}}{\gamma^3}$$

for all $0 < |k| + |l| < T_\nu$, $|l| \neq 0$. Meanwhile, since the (k_1, k_2)-th variable of $\tilde{G}_{l,r'}$ is $\mathrm{i}e^{(|k_1|-|k_2|)r'}\langle l, \bar{g}_{\nu,k_1-k_2}\rangle$, we obtain that

$$\begin{aligned}
\|\tilde{G}_{l,r'}\| &\leq T_\nu \max_{|k_2|\leq T_\nu} \sum_{|k_1|\leq T_\nu} e^{(|k_1-k_2|)r'}|\bar{g}_{\nu,k_1-k_2}| \\
&\leq T_\nu \max_{|k_2|\leq T_\nu} \sqrt{\sum_{|k_1|>0} e^{(2|k_1-k_2|)r'}|\bar{g}_{\nu,k_1-k_2}|^2}\sqrt{\sum_{|k_1|\leq T_\nu} 1} \\
&\leq T_\nu^{1+\frac{d}{2}}\|\bar{g}_{\nu-1}\|_{r',O_{n-1}},
\end{aligned} \tag{4.13}$$

which implies the diagonally dominant operators $\tilde{A}_{l,r'} + \tilde{G}_{l,r'}$ have a bounded inverse and the following estimate

$$\|(\tilde{A}_{l,r'} + \tilde{G}_{l,r'})^{-1}\| \leq \|\tilde{A}_{l,r'}^{-1}\|\|(I + \tilde{A}_{l,r'}^{-1}\tilde{G}_{l,r'})\| < 2T_\nu^\tau$$

for $r' = \tilde{r}_{\nu-1} - \sigma_\nu$. One can check that

$$\begin{aligned}
\|h_\nu^i\|^2_{\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2},\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2},O_{n-1}} &\leq \sum_{|l|<T_\nu}\sum_{|k|\leq|l|-T\nu} |h_{\nu,l}^{ik}|^2_{O_{n-1}} e^{2|k|(\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2})}e^{2|l|(\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2})} \\
&\leq 4T_\nu^{4a}\sum_{|l|<T_\nu}\langle \tilde{h}_{l,\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}}, \tilde{h}_{l,\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}}\rangle e^{2|l|(\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2})} \\
&\leq 4T_\nu^{4a}\sum_{|l|<T_\nu}\langle (\tilde{A}_{l,r'} + \tilde{G}_{l,r'})^{-1}\tilde{f}_{l,\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}}, (\tilde{A}_{l,r'} + \tilde{G}_{l,r'})^{-1}\tilde{f}_{l,\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}}\rangle \times \\
&\quad \times e^{2|l|(\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2})} \\
&\leq 8T_\nu^{4a+2\tau}\sum_{|l|<T_\nu}\langle \tilde{f}_{l,\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}}, \tilde{f}_{l,\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}}\rangle e^{2|l|(\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2})} \\
&\leq 8T_\nu^{4a+2\tau}\sum_{|l|<T_\nu}\sum_{|k|<T_\nu-|l|} |f_{\nu-1,l}^k|^2_{O_{n-1}} e^{2|k|(\tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2})}e^{2|l|(\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2})} \\
&\leq \frac{T_\nu^{4a+2\tau}2^{d+4+\tau}\tilde{\eta}_{\nu-1}^2}{\gamma\sigma_\nu^{d+2+\tau}} \leq \frac{1}{m^2}\tilde{\eta}_{\nu-1}^{\frac{3}{2}}
\end{aligned}$$

since ϵ_0 sufficiently small. And the error

$$P_\nu = \mathcal{R}_{T_\nu}(\bar{g}_\nu(\varphi)\frac{\partial h_\nu}{\partial \bar{\theta}_\nu}) - \mathcal{R}_{T_\nu}(\bar{f}_{\nu-1}(\bar{\theta}_\nu, \varphi) - [f_{\nu-1}(\bar{\theta}_\nu, \varphi)]_{\theta_\nu})$$

satisfies

$$\|P_\nu\|_{\tilde{s}_{\nu-1}-\delta_\nu,\tilde{r}_{\nu-1}-\sigma_\nu,O_{n-1}} \leq \frac{2}{5}\eta_{\nu-1}^{\frac{3}{2}}.$$

Now, the system (4.8) becomes

$$\overline{X}_\nu = \omega\cdot\partial_\varphi + (\rho + \bar{g}_\nu(\varphi) + \bar{f}_\nu(\bar{\theta}_\nu, \varphi))\cdot\partial_{\bar{\theta}_\nu},$$

where $\bar{g}_\nu(\varphi) = \bar{g}_{\nu-1}(\varphi) + [\bar{f}_{\nu-1}(\bar{\theta}_\nu, \varphi)]_{\bar{\theta}_\nu}$ and

$$\bar{f}_\nu(\bar{\theta}_\nu, \varphi) = (1 + \frac{\partial h_\nu}{\partial \bar{\theta}_\nu})^{-1}(P_\nu + \bar{f}_{\nu-1}(\bar{\theta}_\nu + h_\nu(\bar{\theta}_\nu, \varphi), \varphi) - \bar{f}_{\nu-1}(\bar{\theta}_\nu, \varphi)).$$

By the mean value theorem and Cauchy estimate, we have for each $\xi \in O_{n-1}$ that

$$\begin{aligned} & \|\bar{f}_{\nu-1}(\bar{\theta}_\nu + h_\nu(\bar{\theta}_\nu, \varphi), \varphi) - \bar{f}_{\nu-1}(\bar{\theta}_\nu, \varphi)\|_{\tilde{s}_\nu, \tilde{r}_\nu, O_{n-1}} \\ = \ & \|\frac{\partial \bar{f}_{\nu-1}(\bar{\theta}_\nu + s h_\nu(\bar{\theta}_\nu, \varphi), \varphi)}{\partial \theta} h_\nu(\bar{\theta}_\nu, \varphi)\|_{\tilde{s}_\nu, \tilde{r}_\nu, O_{n-1}} \\ \le \ & \frac{2}{\delta_\nu}\|\bar{f}_{\nu-1}\|_{\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2}, \tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}, O_{n-1}} \|h_\nu\|_{\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2}, \tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}, O_{n-1}} \\ \le \ & \frac{2\tilde{\eta}_{\nu-1}}{\delta_\nu}\|h_\nu\|_{\tilde{s}_{\nu-1}-\frac{\delta_\nu}{2}, \tilde{r}_{\nu-1}-\frac{\sigma_\nu}{2}, O_{n-1}} \le \frac{\eta_{\nu-1}^{\frac{3}{2}}}{5} \end{aligned}$$

where $s \in (0, 1)$. This implies

$$\|\bar{f}_\nu(\bar{\theta}_\nu, \varphi)\|_{\tilde{s}_\nu, \tilde{r}_\nu, O_{n-1}} \le (1 + \eta_{\nu-1}^{\frac{3}{4}})(\frac{\eta_{\nu-1}^{\frac{3}{2}}}{5} + \frac{2\eta_{\nu-1}^{\frac{3}{2}}}{5} + \frac{\eta_{\nu-1}^{\frac{3}{2}}}{5}) \le \tilde{\eta}_\nu.$$

Since $1 \le \nu \le \mathcal{N}$, we should estimate $\bar{f}_{\mathcal{N}}(\bar{\theta}_{\mathcal{N}}, \varphi)$. First, we have the following estimate

$$(\frac{3}{2})^{\mathcal{N}} - 1 \ge (\frac{3}{2})^{2^n c_1 \tau N \ln K_n} - 1 \ge K_n^{2^{n-1} c_1 \tau N \ln(3/2)}.$$

By (2.5), we have

$$\begin{aligned} \|\bar{f}_{\mathcal{N}}(\bar{\theta}_{\mathcal{N}}, \varphi)\|_{\tilde{s}_{\mathcal{N}}, \tilde{r}_{\mathcal{N}}, O_{n-1}} &\le \tilde{\eta}^{\frac{3}{2}^{\mathcal{N}}} = \tilde{\eta} e^{-(\frac{3}{2}^{\mathcal{N}}-1)\ln\frac{1}{\tilde{\eta}}} \\ &\le \tilde{\eta} e^{-(K_n^{2^{n-1} c_1 \tau N \ln(3/2)}) 2^{n+1} c \tau N} \\ &\le \tilde{\eta}\frac{1}{K_{n+1}^{2^{n+1} c \tau N}} < \epsilon_n, \end{aligned}$$

in view of $\epsilon_0 < e^{-2c\tau N}$. Let $\bar{\theta}_{\mathcal{N}} = \bar{\theta}_+$ and

$$\overline{H}_\nu(\bar{\theta}_\nu, \varphi) = \phi_1 \circ \cdots \circ \phi_{\nu-1} \circ \phi_\nu(\bar{\theta}_\nu, \varphi)$$

for $1 \le \nu \le \mathcal{N}$. Then, $\overline{H}_\nu(\bar{\theta}_\nu, \varphi)$ is analytic in $W_{\tilde{s}_\nu, \tilde{r}_\nu}(\mathbb{T}^m \times \mathbb{T}^d)$ and

$$\|\partial_{\bar{\theta}_\nu}(\pi \circ \overline{H}_\nu(\bar{\theta}_\nu, \varphi))\|_{\tilde{s}_\nu, \tilde{r}_\nu, O_{n-1}} \le \prod_{j=1}^{\nu}(1 + \tilde{\eta}_j^{\frac{3}{4}}),$$

where $\pi : \mathbb{T}^m \times \mathbb{T}^d \to \mathbb{T}^m$ denotes the natural projection to the first variable. If we rewrite $\overline{H}_{\mathcal{N}}(\bar{\theta}_+, \varphi) = (\bar{\theta}_+ + \tilde{h}(\bar{\theta}_+, \varphi) \mod 2\pi, \varphi)$, then we have

$$\begin{aligned} &\|\tilde{h}(\bar{\theta}_+, \varphi)\|_{\tilde{s}_{\mathcal{N}-1}-\frac{\delta_{\mathcal{N}}}{2}, \tilde{r}_{\mathcal{N}-1}-\frac{\sigma_{\mathcal{N}}}{2}, O_{n-1}} \le \textstyle\sum_{\nu=1}^{\mathcal{N}} \prod_{j=1}^{\nu-1} \|h_\nu\|_{\tilde{s}_\nu, \tilde{r}_\nu, O_{n-1}} < 2\tilde{\eta}^{\frac{3}{4}} < 4\epsilon_{n-1}^{\frac{3}{4}}, \\ &\|\tfrac{\partial \tilde{h}(\bar{\theta}_+, \varphi)}{\partial \bar{\theta}_+}\|_{\tilde{s}_{\mathcal{N}}, \tilde{r}_{\mathcal{N}}, O_{n-1}} < 4\epsilon_{n-1}^{\frac{3}{4}}. \end{aligned} \tag{4.14}$$

In conclusion, let $\tilde{s}_{\mathcal{N}} = s_{n+}$, $\tilde{r}_{\mathcal{N}} = r_{n+}$, $\bar{g}_{\mathcal{N}}(\varphi) = \bar{g}_{n-1,+}(\varphi)$ and $\bar{f}_{\mathcal{N}}(\bar{\theta}_+,\varphi) = \bar{f}_{n-1,+}(\bar{\theta}_+,\varphi)$. Then $\overline{H}_{n-1} = \overline{H}_{\mathcal{N}}(\bar{\theta}_+,\varphi)$ conjugates (4.4) to

$$\overline{X}_{n-1,+} = \omega \cdot \partial_\varphi + (\rho_{n-1} + \bar{g}_{n-1,+}(\varphi) + \bar{f}_{n-1,+}(\bar{\theta}_+,\varphi)) \cdot \partial_{\bar{\theta}_+},$$

with the estimates

$$\begin{aligned}
&\|\bar{g}_{n-1,+}(\varphi)\|_{r_{n+},O_{n-1}} \leq 2\epsilon_{n-1}^{\frac{1}{2}},\\
&\|\bar{f}_{n-1,+}(\bar{\theta}_+,\varphi)\|_{s_{n+},r_{n+},O_{n-1}} \leq \epsilon_n,\\
&\|\bar{g}_{n-1,+}(\varphi) - \bar{g}_{n-1}(\varphi)\|_{r_{n+},O_{n-1}} \leq \sum_\nu \|\bar{f}_\nu\|_{s_n,r_n,O_{n-1}} \leq \sum_\nu \tilde{\eta}_\nu \leq 4\epsilon_{n-1}.
\end{aligned}$$

□

The following Lemma is the end of one KAM step.

Lemma 4.3. *Under the assumptions of Lemma 4.1 and Lemma 4.2, there exists $\widetilde{H}_{n-1}$ with estimates*

$$\begin{aligned}
&\|\widetilde{H}_{n-1} - id\|_{s_n,r_n,O_{n-1}} \leq 4\epsilon_{n-1}^{\frac{3}{4}},\\
&\|D(\widetilde{H}_{n-1} - id)\|_{s_n,r_n,O_{n-1}} \leq 4\epsilon_{n-1}^{\frac{3}{4}},
\end{aligned}$$

such that $\widetilde{H}_{n-1}$ conjugates vector field (4.3) to

$$X_n = \omega \cdot \partial_\varphi + (\rho_n + g_n(\varphi) + f_n(\theta_+,\varphi)) \cdot \partial_{\theta_+},$$

with the following estimates

$$\begin{aligned}
&\|g_n(\varphi)\|_{r_n,O_n} \leq 4\tilde{\epsilon}_n,\\
&\|f_n(\theta_+,\varphi)\|_{s_n,r_n,O_n} \leq \epsilon_n.
\end{aligned}$$

Proof. In the Lemma 4.1, we eliminate the non-resonant terms of $g_{n-1}(\varphi)$ and, as a result, the transformation we obtain is not close to the identity. In order to get rotations reducibility results, we need to inverse the first step, which means conjugating back by the transformation of the first step. Applying the inverse transformation H_{n-1}^{-1}:

$$\bar{\theta}_+ = \theta_+ - h_{n-1}(\varphi),$$

the vector field (4.6) can be conjugated to

$$X_n = \omega \cdot \partial_\varphi + (\rho_n + g_n(\varphi) + f_n(\theta_+,\varphi)) \cdot \partial_{\theta_+},$$

where $\rho_n = \rho_{n-1} + [\bar{g}_{n-1,+}(\varphi)]_\varphi$ and

$$g_n(\varphi) = \mathcal{T}_{K_n} g_{n-1}(\varphi) + \bar{g}_{n-1,+}(\varphi) - [\bar{g}_{n-1,+}(\varphi)]_\varphi,\ f_n(\theta_+,\varphi) = \bar{f}_{n-1,+}(\theta_+ - h_{n-1}(\varphi),\varphi).$$

We have

$$\|g_n(\varphi) - g_{n-1}(\varphi)\|_{r_n,O_{n-1}} \leq \|\bar{g}_{n-1,+}(\varphi) - \bar{g}_{n-1}(\varphi)\|_{r_{n+},O_{n-1}} \leq 4\epsilon_{n-1},$$

which implies $\|g_n(\varphi)\|_{r_n,O_{n-1}} \leq 4\tilde{\epsilon}_n$. And it is obvious that

$$\|f_n(\theta_+,\varphi)\|_{s_n,r_n,O_n} \leq \epsilon_n.$$

Let $\widetilde{H}_{n-1} = H_{n-1} \circ \overline{H}_{n-1} \circ H_{n-1}^{-1}$, then $\widetilde{H}_{n-1} = \theta_+ + \tilde{h}(\theta_+ - h_{n-1}(\varphi),\varphi)$. By (4.14), we have

$$\begin{aligned}
&\|\widetilde{H}_{n-1} - id\|_{s_n,r_n,O_{n-1}} \leq \|\tilde{h}\|_{s_{n+},r_{n+},O_{n-1}} \leq 4\epsilon_{n-1}^{\frac{3}{4}},\\
&\|D(\widetilde{H}_{n-1} - id)\|_{s_n,r_n,O_{n-1}} \leq \|D\tilde{h}\|_{s_{n+},r_{n+},O_{n-1}} \leq 4\epsilon_{n-1}^{\frac{3}{4}}.
\end{aligned}$$

4.1.2. Iteration lemma

Lemmas 4.1, 4.2, and 4.3 can be summarized as the following iterative lemma.

Lemma 4.4. *Suppose $\omega \in \mathcal{L}$ and $\gamma_n,\ r_n,\ s_n,\ \epsilon_n$ are defined in (4.1) for $n \geq 0$. Let ϵ_0 sufficiently small. Then the following holds for all $n \geq 1$: If the vector field*

$$X_n = \omega \cdot \partial_\varphi + (\rho_n(\xi) + g_n(\varphi, \xi) + f_n(\theta, \varphi, \xi)) \cdot \partial_\theta \tag{4.15}$$

satisfies

$$\begin{aligned} &\|g_n(\varphi, \xi)\|_{r_n, O_n} \leq 4\tilde{\epsilon}_n, \\ &\|f_n(\theta, \varphi, \xi)\|_{s_n, r_n, O_n} \leq \epsilon_n, \\ &|\rho_n - \rho_{n-1}|_{O_n} \leq 4\epsilon_{n-1}, \end{aligned}$$

where $[g_n(\varphi, \xi)]_\varphi = 0$ and

$$O_n = \{\xi \in O_{n-1} : |\langle k, \omega\rangle| + \langle l, \rho_n(\xi)\rangle \geq \frac{\gamma_n}{(|k| + |l|)^\tau},\ \forall |l| \neq 0,\ |k| + |l| \leq T^{(n)}\}.$$

Then there exists a subset $O_{n+1} \subset O_n$, where

$$O_{n+1} = O_n \setminus \bigcup_{\substack{T^{(n)} < |k|+|l| \leq T^{(n+1)} \\ |l| \neq 0}} \Gamma_{kl}^{n+1}(\gamma_{n+1}) \tag{4.16}$$

with

$$\Gamma_{kl}^{n+1}(\gamma_{n+1}) = \{\xi \in O_n : |\langle k, \omega\rangle| + \langle l, \rho_{n+1}(\xi)\rangle < \frac{\gamma_{n+1}}{(|k| + |l|)^\tau}\},$$

and a change of variables $\widetilde{H}_n : \mathbb{T}^m \times \mathbb{T}^d \to \mathbb{T}^m \times \mathbb{T}^d$ with estimates

$$\begin{aligned} &\|\widetilde{H}_n - id\|_{s_{n+1}, r_{n+1}, O_{n+1}} \leq 4\epsilon_n^{\frac{3}{4}}, \\ &\|D(\widetilde{H}_n - id)\|_{s_{n+1}, r_{n+1}, O_{n+1}} \leq 4\epsilon_n^{\frac{3}{4}}, \end{aligned}$$

such that it transforms the vector field (4.15) to

$$\omega \cdot \partial_\varphi + (\rho_{n+1}(\xi) + g_{n+1}(\varphi, \xi) + f_{n+1}(\theta, \varphi, \xi)) \cdot \partial_\theta,$$

with

$$\begin{aligned} &\|g_{n+1}(\varphi, \xi)\|_{r_{n+1}, O_{n+1}} \leq 4\tilde{\epsilon}_{n+1}, \\ &\|f_{n+1}(\theta, \varphi, \xi)\|_{s_{n+1}, r_{n+1}, O_{n+1}} \leq \epsilon_{n+1}, \\ &|\rho_{n+1} - \rho_n|_{O_{n+1}} \leq 4\epsilon_n, \end{aligned}$$

and $[g_{n+1}(\varphi, \xi)]_\varphi = 0$.

Proof. Actually, Lemma 4.4 is an immediate corollary of Lemmas 4.1-4.3. The only point we need to illustrate is the definition of O_{n+1}. If $\xi \in O_n$ and $0 < |k| + |l| \leq T^{(n)}$, then

$$\begin{aligned}
&|\langle k, \omega\rangle + \langle l, \rho_{n+1}(\xi)\rangle| \\
\geq\ &|\langle k, \omega\rangle + \langle l, \rho_n(\xi)\rangle| - |l||\rho_{n+1}(\xi) - \rho_n(\xi)| \\
\geq\ &\tfrac{\gamma_n}{(|k|+|l|)^\tau} - 4T^{(n)}\epsilon_n \\
\geq\ &\tfrac{\gamma_n - \epsilon_n^{\frac{1}{2}}}{(|k|+|l|)^\tau} > \tfrac{\gamma_{n+1}}{(|k|+|l|)^\tau}.
\end{aligned}$$

Thus, by the definition of (4.16), we have

$$O_{n+1} = \{\xi \in O_n : |\langle k, \omega\rangle| + \langle l, \rho_{n+1}(\xi)\rangle \geq \frac{\gamma_{n+1}}{(|k| + |l|)^\tau},\ \forall |l| \neq 0,\ 0 < |k| + |l| \leq T^{n+1}\},$$

which satisfies the definition (4.2). □

4.1.3. Measure estimation

By the definition of O_n, we get $O_\gamma = \bigcap_{n\geq 0} O_n$.

By (4.16), we have

$$O \setminus O_\gamma = \bigcup_{n=0}^{\infty} \bigcup_{\substack{T^{(n)}<|k|+|l|\leq T^{(n+1)} \\ |l|\neq 0}} \Gamma_{kl}^{n+1}(\gamma_{n+1}).$$

In what follows, we start to estimate the measure of set $\Gamma_{kl}^{n+1}(\gamma_{n+1})$. In view of

$$\begin{aligned}
\frac{d\rho_n(\xi)}{d\xi} &= \frac{d\rho_0(\xi)}{d\xi} + \sum_{i=1}^{n}(\frac{d\rho_i(\xi)}{d\xi} - \frac{d\rho_{i-1}(\xi)}{d\xi}) \\
&= \frac{d\rho_0(\xi)}{d\xi} + \sum_{i=1}^{n}(\frac{dg_i(\varphi, \xi)}{d\xi} - \frac{dg_{i-1}(\varphi, \xi)}{d\xi})
\end{aligned}$$

and $\frac{d\rho_0(\xi)}{d\xi} = I$, we have

$$\text{rank}\left\{\frac{d\rho_n(\xi)}{d\xi}\right\} = m.$$

It follows from Lemma 1.2 in [33] that

$$\Gamma_{kl}^{n+1}(\gamma_{n+1}) \leq C\frac{\gamma_n}{|l|(|k| + |l|)^\tau}.$$

Then, we have the following estimation

$$\begin{aligned}
\text{meas}(O \setminus O_\gamma) &\leq C\sum_{n\geq 0}\ \sum_{T^{(n)}<|k|+|l|\leq T^{(n+1)}} C\frac{\gamma_n}{|l|(|k| + |l|)^\tau} \\
&\leq C\gamma\sum_{i=1}^{\infty}\frac{\ln i}{i^{\tau-(m+d-2)}} \\
&\leq O(\gamma)
\end{aligned}$$

provided $\tau > m + d - 1$.

4.1.4. Convergence

Select ϵ_0 sufficiently small and $r_0 = r$, $s_0 = s$. If $\rho \in \mathcal{O} = \bigcap_{n\geq 0} \mathcal{O}_n$ and $\|f\|_{s,r,\mathcal{O}_\gamma} < \epsilon_0$, then the vector field

$$X = \omega \cdot \partial_\varphi + (\xi + f(\theta, \varphi)) \cdot \partial_\theta, \tag{4.17}$$

is C^∞ rotations linearizable. We can use Lemma 4.2 to system (4.17). Thus we can get $H_0 \in \mathcal{H}^{s_1,r_1,a}$ which conjugates (4.17) to

$$X_1 = \omega \cdot \partial\varphi + (\rho_1(\xi) + g_1(\varphi, \xi) + f_1(\theta, \varphi, \xi)) \cdot \partial\theta,$$

with

$$\|g_1(\varphi, \xi)\|_{r_1,\mathcal{O}_1} \leq 2\tilde{\epsilon}_1,$$
$$\|f_1(\theta, \varphi, \xi)\|_{s_1,r_1,\mathcal{O}_1} \leq \epsilon_1,$$

Then we apply Lemma 4.4 inductively, we can get $\widetilde{H}_i \in \mathcal{H}^{s_{i+1},r_{i+1},a}$, $i = 1, \cdots, n$, such that $H^{(n)} = \widetilde{H}_0 \circ \widetilde{H}_1 \circ \cdots \circ \widetilde{H}_n$ conjugates (4.17) to

$$X_{n+1} = \omega \cdot \partial\varphi + (\rho_{n+1}(\xi) + g_{n+1}(\varphi, \xi) + f_{n+1}(\theta, \varphi, \xi)) \cdot \partial\theta,$$

with

$$\|g_{n+1}(\varphi, \xi)\|_{r_{n+1},\mathcal{O}_{n+1}} \leq 2\tilde{\epsilon}_{n+1},$$
$$\|f_{n+1}(\theta, \varphi, \xi)\|_{s_{n+1},r_{n+1},\mathcal{O}_{n+1}} \leq \epsilon_{n+1}.$$

And we have

$$\|DH^{(n)}\|_{s_{n+1},r_{n+1},\mathcal{O}_{n+1}} \leq \|D\widetilde{H}_0\|_{s_1,r_1,\mathcal{O}_1} \|D\widetilde{H}_1\|_{s_2,r_2,\mathcal{O}_2} \cdots \|D\widetilde{H}_n\|_{s_{n+1},r_{n+1},\mathcal{O}_{n+1}} \leq \prod_{i=0}^{n} (1 + 4\epsilon_i^{\frac{3}{4}}),$$

which implies

$$\|H^{(n+1)} - H^{(n)}\|_{s_{n+1},r_{n+1},\mathcal{O}_{n+1}} \leq \|DH^{(n)}\|_{s_n,r_n,\mathcal{O}_n} \|H_n - id\|_{s_{n+1},r_{n+1},\mathcal{O}_{n+1}} \leq 8\epsilon_n^{\frac{3}{4}}.$$

By the definition of $(\epsilon_n)_{\mathbb{N}}$, we know that for any $j \in \mathbb{Z}_+^{m+d}$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$, we have $8(\frac{4K_{n+1}^\alpha}{r_0})^{|j|}\epsilon_n^{\frac{3}{4}} \leq \epsilon_n^{\frac{1}{2}}$. By the Cauchy estimates, if we denote $x := (\theta, \varphi) \in \mathbb{T}^{m+d}$, we have

$$|\frac{\partial^{|j|}}{\partial x^j}(H^{(n+1)} - H^{(n)})| \leq \frac{1}{r_{n+1}^{|j|}} \|H^{(n+1)} - H^{(n)}\|_{s_{n+1},r_{n+1},\mathcal{O}_{n+1}} \leq 8(\frac{4K_{n+1}^\alpha}{r_0})^{|j|}\epsilon_n^{\frac{3}{4}} \leq \epsilon_n^{\frac{1}{2}}.$$

for any $n > N - 1$. This guarantees the limit $\lim_{n\to\infty} H^{(n)}$ belongs to C^∞.

Finally, let $\rho_\infty(\xi) = \lim_{n\to\infty} \rho_n(\xi)$, $m_\infty(\varphi, \xi) = \lim_{n\to\infty} g_n(\varphi, \xi)$ and $\Psi = (\lim_{n\to\infty} H^{(n)})^{-1}$, the proof of Proposition 3.1 is completed.

4.2. Proof of Proposition 3.2

In this section, we give the proof of Proposition 3.2. Since the proof process is very similar to [27], we only write the relevant lemma and the key points of proof that are different from Proposition 3.1. Readers can combine Proposition 3.2 with [27] for more detailed analysis.

Let N r_0, s_0, $\gamma > 0$, $M > d/2$ $\tau > d$. Suppose $K_0 > 0$ large enough such that

$$\frac{K_0^{\frac{M-\alpha}{16M}}}{\ln K_0} > c\tau N,$$

where $c > \frac{16\alpha(3\tau+2+\frac{d}{2})}{\tau N}$ and $\max\{\frac{M(2\tau_1-d-2)}{2M-d}, 1\} < \alpha < M$. ϵ_0 is sufficiently small. For given r_0, s_0, ϵ_0, we define some sequences depending on r_0, s_0, ϵ_0 for $n \geq 1$:

$$\begin{array}{ll} \Delta_n = \frac{s_0}{10\cdot 2^{n-1}}, & r_n = \frac{r_0}{4k_n^{\alpha}}, \\ s_n = s_{n-1} - \Delta_n, & \epsilon_n = \frac{\epsilon_{n-1}}{K_{n+1}^{2^{n+1}c\tau N}}, \\ \widetilde{\epsilon}_n = \sum_{i=0}^{n-1} \epsilon_i, & T^{(n-1)} = (\gamma^3/32\epsilon_{n-1}^{\frac{1}{2}})^{\frac{1}{3\tau+2}}, \end{array} \tag{4.18}$$

where $\max\{\frac{M\tau_1}{M-d-1}, 1\} < \alpha < M$.

4.2.1. One KAM step

In $(n-1)$-th step, vector field can be written as

$$X_{n-1} = \omega \cdot \partial\varphi + (\rho_f + g_{n-1}(\varphi) + f_{n-1}(\theta, \varphi)) \cdot \partial_\theta, \tag{4.19}$$

where

$$\begin{aligned} &||g_{n-1}(\varphi)||_{r_{n-1}} \leq 4\tilde{\epsilon}_{n-1}, \\ &||f_{n-1}(\theta, \varphi)||_{s_{n-1}, r_{n-1}} \leq \epsilon_{n-1}. \end{aligned}$$

Similar to Proposition 3.1, we can get the following Lemmas.

Lemma 4.5. *Denote $\bar{s}_n = s_{n-1} - \frac{\Delta_n}{3}$, $\bar{r}_n = \frac{r_0}{K_n^{\alpha}}$, then there exists $h_{n-1}(\varphi)$ with $||h_{n-1}(\varphi)||_{\bar{r}_n} \leq K_n^{\frac{M\tau_1+(d+1)\alpha}{M}}$, such that transformation $H_{n-1} : \theta = \bar{\theta} + h_{n-1}(\varphi)(\mod 2\pi)$ conjugates the vector field (4.19) into*

$$\overline{X}_{n-1} = \omega \cdot \partial_\varphi + (\rho_f + \bar{g}_{n-1}(\varphi) + \bar{f}_{n-1}(\bar{\theta}, \varphi)) \cdot \partial_{\bar{\theta}}, \tag{4.20}$$

where

$$\begin{aligned} &||g_{n-1}(\varphi)||_{\bar{r}_n} \leq \epsilon_{n-1}^{\frac{1}{2}}, \\ &||f_{n-1}(\theta, \varphi)||_{\bar{s}_n, \bar{r}_n} \leq \epsilon_{n-1}. \end{aligned}$$

Proof. Using transformation $\theta = \bar{\theta} + h_{n-1}(\varphi)(\mod 2\pi)$, system (4.19) becomes

$$\overline{X}_{n-1} = \omega \cdot \partial\varphi + \tilde{\bar{f}}_{n-1}(\bar{\theta}, \varphi) \cdot \partial_{\bar{\theta}},$$

where $\tilde{f}_{n-1}(\bar\theta,\varphi)=\rho_f+[g_{n-1}(\varphi)]+\mathcal{R}_{k_n}g_{n-1}(\varphi)+f_{n-1}(\bar\theta+h_{n-1}(\varphi),\varphi)$, if homological equation $\partial_\omega h_{n-1}(\varphi)=\mathcal{T}_{K_n}g_{n-1}(\varphi)-[g_{n-1}(\varphi)]$ is solvable. Then, we have

$$\|f_{n-1}(\bar\theta+h_{n-1}(\varphi),\varphi)\|_{\bar s_n,\bar r_n}\le\|f_{n-1}(\theta,\varphi)\|_{s_{n-1},r_{n-1}}\le\epsilon_{n-1},$$
$$\|\mathcal{R}_{k_n}g_{n-1}(\varphi)\|_{\bar r_n}\le\epsilon_0e^{-K_n(r_{n-1}-\bar r_n)}\le\epsilon_0e^{-K_n^{\frac{M-\alpha}{4M}}}\le\frac{1}{3}\epsilon_{n-1}^{\frac{1}{2}}.$$

By Lemma 2.2, we know $\rho(\omega,\tilde f_{n-1}(\bar\theta,\varphi))=\rho_f$. By Lemma 2.1, we know that

$$|[g_{n-1}(\varphi)]_\varphi|\le\|\mathcal{R}_{k_n}g_{n-1}(\varphi)+f_{n-1}(\bar\theta+h_{n-1}(\varphi),\varphi)\|_{\bar s_n,\bar r_n}\le\frac{2}{3}\epsilon_{n-1}^{\frac{1}{2}}.$$

Letting $\bar g_{n-1}(\varphi)=[g_{n-1}(\varphi)]_\varphi+\mathcal{R}_{k_n}g_{n-1}(\varphi)$ and $\bar f_{n-1}(\bar\theta,\varphi)=f_{n-1}(\bar\theta+h_{n-1}(\varphi),\varphi)$, we have

$$\|\bar f_{n-1}(\bar\theta,\varphi)\|_{\bar s_n,\bar r_n}=\|f_{n-1}(\bar\theta+h_{n-1}(\varphi),\varphi)\|_{\bar s_n,\bar r_n}\le\|f_{n-1}(\theta,\varphi)\|_{s_{n-1},r_{n-1},O_{n-1}}\le\epsilon_{n-1},$$
$$\|\bar g_{n-1}(\varphi)\|_{\bar s_n,\bar r_n}=\|[g_{n-1}(\varphi)]+\mathcal{R}_{k_n}g_{n-1}(\varphi)\|_{\bar r_n}\le\epsilon_0e^{-K_n(r_{n-1}-\bar r_n)}\le\epsilon_0e^{-K_n^{\frac{M-\alpha}{4M}}}\le\epsilon_{n-1}^{\frac{1}{2}}.$$

□

Lemma 4.6. *Denote $r_{n+}=\frac{r_0}{2K_n^\alpha}$, $s_{n+}=\bar s_n-\frac{\Delta}{3}$. Under the assumptions of Lemma 4.5, there exists $\overline{H}_{n-1}$ with estimates*

$$\|\overline{H}_{n-1}-id\|_{s_{n+},r_{n+}}\le4\epsilon_{n-1}^{\frac{3}{4}},\tag{4.21}$$
$$\|D(\overline{H}_{n-1}-id)\|_{s_{n+},r_{n+}}\le4\epsilon_{n-1}^{\frac{3}{4}},\tag{4.22}$$

such that $\overline{H}_{n-1}$ conjugates (4.20) to

$$X_{n-1,+}=\omega\cdot\partial_\varphi+(\rho_f+\bar g_{n-1,+}(\varphi)+\bar f_{n-1,+}(\bar\theta_+,\varphi))\cdot\partial_{\bar\theta_+}\tag{4.23}$$

with

$$\|\bar f_{n-1,+}(\bar\theta_+,\varphi)\|_{s_{n+},r_{n+}}=\le2\epsilon_{n-1}^{\frac{1}{2}},$$
$$\|\bar g_{n-1,+}(\varphi)\|_{r_{n+}}\le\epsilon_n,$$
$$\|\bar g_{n-1,+}(\varphi)-\bar g_{n-1}(\varphi)\|_{r_{n+}}\le4\epsilon_{n-1}.$$

Proof. Taking advantage of (ω,ρ_f) being Diophantine, we can obtain this Lemma immediately using the same process of Lemma 4.2. The detail proof is omitted here. □

Lemma 4.7. *Under the assumptions of Lemma 4.5 and Lemma 4.6, there exists $\widetilde{H}_{n-1}$ with estimates*

$$\|\widetilde{H}_{n-1}-id\|_{s_n,r_n}\le4\epsilon_{n-1}^{\frac{3}{4}},\tag{4.24}$$
$$\|D(\widetilde{H}_{n-1}-id)\|_{s_n,r_n}\le4\epsilon_{n-1}^{\frac{3}{4}},\tag{4.25}$$

such that $\widetilde{H}_{n-1}$ conjugates (4.20) to

$$X_n=\omega\cdot\partial_\varphi+(\rho_f+g_n(\varphi)+f_n(\theta_+,\varphi))\cdot\partial_{\theta_+}$$

with

$$\|f_n(\theta_+,\varphi)\|_{s_n,r_n}\le\epsilon_n,$$
$$\|g_n(\varphi)\|_{r_n}\le4\tilde\epsilon_n.$$

Proof. Applying the inverse transformation H_{n-1}^{-1}:

$$\bar{\theta}_+ = \theta_+ - h_{n-1}(\varphi),$$

the system (4.23) can be conjugated to

$$X_n = \omega \cdot \partial_\varphi + (\rho_f + g_n(\varphi) + f_n(\theta_+, \varphi)) \cdot \partial_{\theta_+}$$

where

$$g_n(\varphi) = \mathcal{T}_{K_n} g_{n-1}(\varphi) + \bar{g}_{n-1,+}(\varphi),\ f_n(\theta_+, \varphi) = \bar{f}_{n-1,+}(\theta_+ - h_{n-1}(\varphi), \varphi).$$

We have

$$\|g_n(\varphi) - g_{n-1}(\varphi)\|_{r_n, O_{n-1}} \leq \|\bar{g}_{n-1,+}(\varphi) - \bar{g}_{n-1}(\varphi)\|_{r_{n+}, O_{n-1}} \leq 4\epsilon_{n-1},$$

which implies $\|g_n(\varphi)\|_{r_n, O_{n-1}} \leq 4\tilde{\epsilon}_n$. Thus

$$\|f_n(\theta_+, \varphi)\|_{s_n, r_n} \leq \epsilon_n,$$
$$\|g_n(\varphi)\|_{r_n} \leq 4\tilde{\epsilon}_n.$$

Let $\widetilde{H}_{n-1} = H_{n-1} \circ \overline{H}_{n-1} \circ H_{n-1}^{-1}$. By (4.21) and (4.22), we get (4.24) and (4.25) hold. □

4.2.2. Iteration lemma

Lemmas 4.5, 4.6, and 4.7 can be summarized as the following iterative lemma. We do not give the proof because it can be seen in [27].

Lemma 4.8. *Suppose $\omega \in \mathcal{L}$, $\tilde{\rho}(\omega, \rho + f(\theta, \varphi)) = \rho_f \in DC_\omega(\gamma, \tau)$ and r_n, s_n, ϵ_n are defined in (4.18) for $n \geq 0$. Let ϵ_0 be sufficiently small. Then the following holds for all $n \geq 1$: If the system*

$$X_n = \omega \cdot \partial_\varphi + (\rho_f + g_n(\varphi) + f_n(\theta, \varphi)) \cdot \partial_\theta \tag{4.26}$$

satisfies

$$\|f_n(\theta, \varphi)\|_{s_n, r_n} \leq \epsilon_n,$$
$$\|g_n(\varphi)\|_{r_n} \leq 4\tilde{\epsilon}_n.$$

then there exists a change of variables $\widetilde{H}_n : \mathbb{T} \times \mathbb{T}^d \to \mathbb{T} \times \mathbb{T}^d$ with estimates

$$\|\widetilde{H}_n - id\|_{s_{n+1}, r_{n+1}} \leq 4\epsilon_n^{\frac{3}{4}},$$
$$\|D(\widetilde{H}_n - id)\|_{s_{n+1}, r_{n+1}} \leq 4\epsilon_n^{\frac{3}{4}},$$

such that it transforms the system (4.26) to

$$X_{n+1} = \omega \cdot \partial_\varphi + (\rho_f + g_{n+1}(\varphi) + f_{n+1}(\theta, \varphi)) \cdot \partial_\theta$$

with

$$\|f_{n+1}(\theta, \varphi)\|_{s_{n+1}, r_{n+1}} \leq \epsilon_{n+1},$$
$$\|g_{n+1}(\varphi)\|_{r_{n+1}} \leq 4\tilde{\epsilon}_{n+1}.$$

4.2.3. Convergence

Let ϵ_0 be sufficiently small and $r_0 = r$, $s_0 = s$. If $\tilde{\rho}(\omega, \rho + f) = \rho_f \in DC_\omega(\gamma, \tau)$ and $\|f\|_{s,r,O_\gamma} < \epsilon_0/2$, then the vector field

$$X = \omega \cdot \partial_\varphi + (\rho + f(\theta, \varphi)) \cdot \partial_\theta$$

can be written as

$$X = \omega \cdot \partial_\varphi + (\rho_f + \tilde{f}(\theta, \varphi)) \cdot \partial_\theta \tag{4.27}$$

where $\tilde{f}(\theta, \varphi) = \rho - \rho_f + f(\theta, \varphi)$. We can use Lemma 4.6 to system (4.27). Thus we can get $\widetilde{H}_0 \in H^{s_1, r_1, a}$, which conjugates (4.27) to

$$X_1 = \omega \cdot \partial_\varphi + (\rho + g_1(\varphi) + f_1(\theta, \varphi)) \cdot \partial_\theta$$

with

$$\|f_1(\theta, \varphi)\|_{s_1, r_1} \le \epsilon_1,$$
$$\|g_1(\varphi)\|_{r_1} \le 2\tilde{\epsilon}_1.$$

Then we apply Lemma 4.8 inductively, we can get $\widetilde{H}_i \in H^{s_{i+1}, r_{i+1}, a}$, $i = 1, \cdots, n$, such that $H^{(n)} = \widetilde{H}_0 \circ \widetilde{H}_1 \circ \cdots \circ \widetilde{H}_n$ conjugates (4.27) to

$$X_{n+1} = \omega \cdot \partial_\varphi + (\rho_f + g_{n+1}(\varphi) + f_{n+1}(\theta, \varphi)) \cdot \partial_\theta$$

with

$$\|f_{n+1}(\theta, \varphi)\|_{s_{n+1}, r_{n+1}} \le \epsilon_{n+1},$$
$$\|g_{n+1}(\varphi)\|_{r_{n+1}} \le 4\tilde{\epsilon}_{n+1}.$$

And we have

$$\|DH^{(n)}\|_{s_{n+1}, r_{n+1}} \le \|D\widetilde{H}_0\|_{s_1, r_1} \|D\widetilde{H}_1\|_{s_2, r_2} \cdots \|D\widetilde{H}_n\|_{s_{n+1}, r_{n+1}} \le \prod_{i=0}^{n} (1 + 4\epsilon_i^{\frac{3}{4}}),$$

which implies

$$\|H^{(n+1)} - H^{(n)}\|_{s_{n+1}, r_{n+1}} \le \|DH^{(n)}\|_{s_n, r_n} \|\widetilde{H}_n - id\|_{s_{n+1}, r_{n+1}} \le 8\epsilon_n^{\frac{3}{4}}.$$

By the definition of $(\epsilon_n)_{n \in \mathbb{N}}$, we know that for any $j \in \mathbb{Z}_+^{1+d}$, there exists $N \in \mathbb{N}$, such that for any $n \ge N$, we have $8(\frac{4K_{n+1}^\alpha}{r_0})^{|j|} \epsilon_n^{\frac{3}{4}} \le \epsilon_n^{\frac{1}{2}}$. By the Cauchy estimates, if we denote $x := (\theta, \varphi) \in \mathbb{T}^{1+d}$, we have

$$\left|\frac{\partial^{|j|}}{\partial x^j} (H^{(n+1)} - H^{(n)})\right| \le \frac{1}{r_{n+1}^{|j|}} \|H^{(n+1)} - H^{(n)}\|_{s_{n+1}, r_{n+1}} \le 8\left(\frac{4K_{n+1}^\alpha}{r_0}\right)^{|j|} \epsilon_n^{\frac{3}{4}} \le \epsilon_n^{\frac{1}{2}}.$$

for any $n > N - 1$. This guarantees the limit $\lim_{n\to\infty} H^{(n)}$ belongs to C^∞.

Finally, let $m(\varphi) = \lim_{n\to\infty} g_n(\varphi)$ and $\Psi = (\lim_{n\to\infty} H^{(n)})^{-1}$, the proof of Proposition 3.1 completed.

Author contributions

Xinyu Guan: Methodology, Writing-original draft; Nan Kang: Validation(equal), Writing-review & editing (equal). All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

References

1. V. I. Arnold, Small denominators I. On the mapping of a circle into itself, *Akad. Nauk. Math.*, **25** (1961), 21–86.
2. A. Avila, Global theory of one-frequency Schrödinger operators, *Acta Math.*, **215** (2015), 1–54. https://doi.org/10.1007/s11511-015-0128-7
3. A. Avila, Almost reducibility and absolute continuity, *arXiv preprint arXiv:1006.0704*, 2010.
4. A. Avila, B. Fayad, R. Krikorian, A KAM scheme for $SL(2,\mathbb{R})$ cocycles with frequencies, *Geom. Funct. Anal.,* **21** (2011), 1001–1009. https://doi.org/10.1007/s00039-011-0135-6
5. A. Avila, S. Jitomirskaya, The ten martini problem, *Ann. Math.,* **170** (2009), 303–342. https://doi.org/10.4007/annals.2009.170.303
6. A. Avila, R. Krikorian, Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles, *Ann. Math.*, **164** (2006), 911–940. https://doi.org/10.4007/annals.2006.164.911
7. D. Bambusi, Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, I, *Trans. Amer. Math. Soc.,* **370** (2018), 1823–1865. https://doi.org/10.1090/tran/7135
8. D. Bambusi, Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, II, *Comm. Math. Phys.,* **353** (2017), 353–378. https://doi.org/10.1007/s00220-016-2825-2
9. D. Bambusi, S. Graffi, Time quasi-periodic unbounded perturbations of Schrödinger operators and KAM methods, *Commun. Math. Phys.,* **219** (2001), 465–480. https://doi.org/10.1007/s002200100426

10. D. Bambusi, B. Grébert, A. Maspero, D. Robert, Reducibility of the quantum harmonic oscillator in d-dimensions with polynomial time-dependent perturbation, *Anal. PDE,* **11** (2018), 775–799. https://doi.org/10.2140/apde.2018.11.775

11. N. N. Bogoljubov, Ju. A. Mitropoliskii, A. M. Samoilenko, *Methods of accelerated convergence in nonlinear mechanics*, Springer, New York, 1976 (Russian original: Naukova Dumka, Kiev, 1960). https://doi.org/10.1007/978-3-642-61900-7

12. H. Cheng, W. Si, J. Si, Whiskered tori for forced beam equations with multi-dimensional liouvillean frequency, *J. Dynam. Differential Equations,* **32** (2020), 705–739. https://doi.org/10.1007/s10884-019-09754-1

13. M. Combescure, The quantum stability problem for time-periodic perturbations of the harmonic oscillator, *Ann. Inst. H. Poincaré Phys. Théor.,* **47** (1987), 63–83.

14. E. Dinaburg, Y. Sinai, The one-dimensional Schrödinger equation with a quasi-periodic potential, *Funct. Anal. Appl.*, **9** (1975), 279–289. https://doi.org/10.1007/BF01075873

15. L. H. Eliasson, Floquet solutions for the one-dimensional quasiperiodic Schrödinger equation, *Commun. Math. Phys.*, **146** (1992), 447–482. https://doi.org/10.1007/BF02097013

16. L. H. Eliasson, *Almost reducibility of linear quasi-periodic systems*, *Proceedings of symposia in pure mathematics*, **69** (2001), 679–706. https://doi.org/10.1090/pspum/069/1858550

17. L. H. Eliasson, B. Fayad, R. Krikorian, Around the stability of KAM-tori, *Duke Math. J.,* **164** (2015), 1733–1775. https://doi.org/10.1215/00127094-3120060

18. L. H. Eliasson, S. B. Kuksin, On reducibility of Schrödinger equations with quasiperiodic in time potentials, *Comm. Math. Phys.,* **286** (2009), 125–135. https://doi.org/10.1007/s00220-008-0683-2

19. B. Fayad, R. Krikorian, Rigidity results for quasiperiodic $SL(2,\mathbb{R})$-cocycles, *J. Mod. Dyn.,* **3** (2009), 479–510. https://doi.org/10.3934/jmd.2009.3.479

20. R. Feola, F. Giuliani, R. Montalto, M. Procesi, Reducibility of first order linear operators on tori via Moser's theorem, *J. Funct. Anal.*, **276** (2019), 932–970. https://doi.org/10.1016/j.jfa.2018.10.009

21. B. Grébert, E. Paturel, KAM for the Klein-Gordon equation on $\mathbb{S}^d$, *Boll. Unione Mat Ital.,* **9** (2016), 237–288. https://doi.org/10.1007/s40574-016-0072-2

22. H. Her, J. You, Full measure reducibility for generic one-parameter family of quasi-periodic linear systems, *J. Dynam. Differential Equations,* **20** (2008), 831–866. https://doi.org/10.1007/s10884-008-9113-6

23. M. Herman, Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, *Comment. Math. Helv.*, **58** (1983), 453–502. https://doi.org/10.1007/BF02564647

24. X. Hou, J. You, Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems, *Invent. Math.,* **190** (2012), 209–260. https://doi.org/10.1007/s00222-012-0379-2

25. À. Jorba, C. Simó, On the reducibility of linear differential equations with quasiperiodic coefficients, *J. Differential Equations,* **98** (1992), 111–124. https://doi.org/10.1016/0022-0396(92)90107-X

26. R. Krikorian, *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts,* Astérisque, 256 SMF, 1999.

27. R. Krikorian, J. Wang, J. You, Q. Zhou, Linearization of quasiperiodically forced circle flows beyond brjuno condition, *Commun. Math. Phys.,* **358** (2018), 81–100. https://doi.org/10.1007/s00220-017-3021-8

28. R. Kuksin, J. Pöschel, Invariant cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation, *Ann. Math.*, **143** (1996), 149–179. https://doi.org/10.2307/2118656

29. Z. Lou, J. Geng, Quasi-periodic response solutions in forced revesible systems with Liouvillean frequencies, *J. Differential Equations,* **263** (2017), 3894–3927. https://doi.org/10.1016/j.jde.2017.05.007

30. A. Maspero, D. Robert, On time dependent Schrödinger equations: global well-posedness and growth of Sobolev norms, *J. Funct. Anal.,* **273** (2017), 721–781. https://doi.org/10.1016/j.jfa.2017.02.029

31. R. Montalto, A Reducibility result for a class of Linear Wave Equations on $\mathbb{T}^d$, *International Mathematics Research Notices,* **2019** (2017), 1788–1862.

32. J. Moser, Convergent series expansions for quasiperiodic motions, *Math. Ann.,* **169** (1967), 136–176. https://doi.org/10.1007/BF01399536

33. W. Si, J. Si, Linearization of a quasi-periodically forced flow on $\mathbb{T}^m$ under Brjuno-Rüssmann non-resonant condition, *Appl. Anal,* **97** (2018), 2001–2024. https://doi.org/10.1080/00036811.2017.1350847

34. F. Wang, H. Cheng, J. Si, Response Solution to Ill-Posed Boussinesq Equation with Quasi-Periodic Forcing of Liouvillean Frequency, *J. Nonlinear Sci.*, **30** (2020), 657–710. https://doi.org/10.1007/s00332-019-09587-8

35. J. Wang, J. You, Boundedness of solutions for non-linear quasi-periodic differential equations with Liouvillean frequency, *J. Differential Equations,* **261** (2016), 1068–1098. https://doi.org/10.1016/j.jde.2016.03.038

36. J. Wang, J. You, Q. Zhou, Response solutions for quasi-periodically forced harmonic oscillators, *Trans. Amer. Math. Soc.,* **369** (2017), 4251–4274. https://doi.org/10.1090/tran/6800

37. X. Xu, W. Si, J. Si, Stoker's Problem for Quasi-periodically Forced Reversible Systems with Multidimensional Liouvillean Frequency, *SIAM. J. Appl. Dyn. Syst.*, **19** (2020), 2286–2321. https://doi.org/10.1137/19M1270033

38. X. Xu, J. You, Q. Zhou, Quasi-periodic solutions of NLS with Liouvillean Frequency, *Anal. PDE,* **14** (2021), 2327–2362. https://doi.org/10.2140/apde.2021.14.2327

39. J. You, Q. Zhou, Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications, *Comm. Math. Phys.*, **323** (2013), 975–1005. https://doi.org/10.1007/s00220-013-1800-4

40. J. You, Q. Zhou, Phase transition and semi-global reducibility, *Comm. Math. Phys.*, **330** (2014), 1095–1113. https://doi.org/10.1007/s00220-014-2012-2

41. J. You, Q. Zhou, Simple counter-examples to Kotani–Last conjecture via reducibility, *Int. Math. Res. Not.,* **19** (2015), 9450–9455. https://doi.org/10.1093/imrn/rnu240

42. J. You, S. Zhang, Q. Zhou, Point spectrum for quasi-periodic long range operators, *J. Spectr. Theory*, **4** (2014), 769–781. https://doi.org/10.4171/jst/85

43. D. Zhang, J. Xu, Reducibility of a class of nonlinear quasi-periodic systems with Liouvillean basic frequencies, *Ergod. Th. Dynam. Sys.*, **41** (2021), 1883–1920. https://doi.org/10.1017/etds.2020.23

44. Q. Zhou, J. Wang, Reducibility results for quasiperiodic cocycles with Liouvillean frequency, *J. Dynam. Differential Equations,* **24** (2012), 61–83. https://doi.org/10.1007/s10884-011-9235-0

A. Appendix

We give the conception of continued fraction expansion and CD-bridge. Let $\alpha \in (0,1)$ be irrational and int($\bullet$) denote the integer part of $\bullet$. Define that $a_0 = 0$, $\alpha_0 = \alpha$, and inductively for $k \geq 1$,

$$a_k = \mathrm{int}(\alpha_k^{-1}), \quad \alpha_k = \alpha_{k-1}^{-1} - a_k.$$

We also define that $p_0 = 0$, $p_1 = 1$, $q_0 = 1$, $q_1 = a_1$, and recursively,

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned}$$

Then (q_n) is the sequence of denominators of the best rational approximation for α. It satisfies

$$\|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}}, \quad \forall 1 \leq k < q_n$$

and

$$\frac{1}{q_n + q_{n+1}} < \|q_n\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}},$$

where $\|x\|_{\mathbb{T}}$ is defined by $\|x\|_{\mathbb{T}} = \inf_{p\in\mathbb{Z}} |x - p|$.

For each $\alpha \in \mathbb{R}/\mathbb{Q}$, in the sequel we will fix a particular subsequence (q_{n_k}) of the denominators of the best rational approximations for α, which for simplicity will be denote by (Q_k). Denote the sequences (q_{n_k+1}) and (p_{n_k}) by $(\overline{Q}_k)$ and (P_k) respectively. Now, we introduce the concept of a CD-bridge which first appeared in [4].

Definition A.1. *Let* $0 < \mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$. *We say that the pair of denominators* (q_l, q_n) *forms a* $CD(\mathcal{A},\mathcal{B},\mathcal{C})$ *bridge if*

- $q_{i+1} \leq q_i^{\mathcal{A}}, \ \forall i = l, \dots, n-1,$
- $q_l^{\mathcal{C}} \geq q_n \geq q_l^{\mathcal{B}}.$

Lemma A.1. *For each* $\mathcal{A} \geq 1$, *there exists a subsequence* (Q_k) *such that* $Q_0 = 1$, *and for each* $k \geq 0$, $Q_{k+1} \leq \overline{Q}_k^{\mathcal{A}^4}$, *and either* $\overline{Q}_k \geq Q_k^{\mathcal{A}}$, *or the pairs* $(\overline{Q}_{k-1}, Q_k)$ *and* (Q_k, Q_{k+1}) *are both* $CD(\mathcal{A},\mathcal{A},\mathcal{A}^3)$ *bridges.*

The proof can be seen in [4].