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Research article

On the central limit theorem for the elephant random walk with gradually increasing memory and random step size

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Abstract: In this paper, we investigate an extended version of the elephant random walk model. Unlike the traditional approach where step sizes remain constant, our model introduces a novel feature: step sizes are generated as a sequence of positive independent and identically distributed random variables, and the step of the walker at time $n+1$ depends only on the steps of the walker between times $1, ..., m_n$, where $(m_n)_{n\geqslant 1}$ is a sequence of positive integers growing to infinity as n goes to infinity. Our main results deal with the validity of the central limit theorem for this new variation of the standard ERW model introduced by Schütz and Trimper in 2004.

Keywords: elephant random walk; central limit theorem; asymptotic normality; phase transition; martingale theory
Mathematics Subject Classification: 60F05, 60F25, 60G42, 60G50

1. Introduction

The elephant random walk (ERW) represents a unique form of one-dimensional random walk along integers, notable for its retention of complete memory regarding its entire past trajectory. Schütz and Trimper [13] introduced the ERW as a means to investigate the memory effects inherent in a non-Markovian random walk. This model is inspired by the notion that elephants possess an exceptional memory, embodying the popular belief that elephants never forget their paths. It is well known that the asymptotic properties of the ERW depend on its memory parameter $p \in [0, 1]$. More precisely, the ERW exhibits three regimes: the diffusive regime for $0 \leqslant p < 3/4$, the critical regime for $p = 3/4$, and the superdiffusive regime of $3/4 < p \leqslant 1$. Many interesting limit theorems are already known for the ERW. In particular, $n^{-1}S_n \to 0$ a.s. for any $p \in]0, 1[$ and $n^{-1/2}S_n \to \mathcal{N}(0, (3-4p)^{-1})$ in distribution in the diffusive regime. In the critical regime, we have $(n \log n)^{-1/2}S_n \to \mathcal{N}(0, 1)$ in distribution and $n^{1-2p}S_n \to L$ almost surely and in $\mathbb{L}^4$ in the superdiffusive regime. The reader can refer to Baur and Bertoin [2], Bercu [3, 4], Bercu and Laulin [5], Kubota and Takei [10], Coletti, Gava, and Schütz [6], Laulin [11] and the references therein and to Ma et al. [12] and Dedecker et al. [7] for contributions to the rate of convergence in the central limit theorem of the ERW in the diffusive and critical regimes. Since the seminal paper by Schütz and Trimper [13], many variants of the ERW have been introduced in this literature. Aguech et al. [1] extended the results of Bercu [3] and Kubota and Takei [10] in the case where the memory of the elephant is increasing (see Gut and Stadtmuller [8]): At any step n, the elephant remembers only the steps at times $1, \dots, m_n$, where m_n is a non-decreasing integer sequence such that $m_n < n$ and $\lim_n m_n/n = \theta \in [0, 1]$ and $\lim_n m_n = +\infty$. In this paper, we combine the results in [7] and [1], and we assume that the size of the step of the elephant is random and with a non-decreasing memory m_n. More precisely, suppose we have an ascending sequence of integers denoted as m_n, where each m_n is less than or equal to n. Let $(U_n)_{n\geqslant 1}$ be a sequence of uniformly distributed random variables on the set $\{1, \dots, m_n\}$, and let $(V_n)_{n\geqslant 1}$ denote a sequence of random variables with values either -1 or 1 such that $\mathbb{P}(V_n = 1) = p$ for some fixed $p \in]0, 1[$. In [7], the authors assume that the size of the step of the elephant is a sequence $(Z_n)_{n\geqslant 1}$ of independent identically distributed random variables with finite variance $\sigma^2 = \mathbb{V}(Z_1)$, and they assume that the sequences of random variables $(U_n)_{n\geqslant 1}$, $(V_n)_{n\geqslant 1}$, and $(Z_n)_{n\geqslant 1}$ are independent. Denote by X_1 the first step of the elephant and assume that X_1 is a random variable such that $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = r$ for some fixed $r \in [0, 1]$. For any integer $n \geqslant 0$, we denote by S_n the position of the elephant at time n. Following [7], we have $S_0 = 0$, and for any integer $n \geqslant 1$,

$$S_n = \sum_{i=1}^{n} X_i Z_i,$$

where, for all $n \geq 1$,

$$X_{n+1} = V_n X_{U_n} = \begin{cases} X_{U_n}, & \text{with probability } p, \\ -X_{U_n}, & \text{with probability } 1-p, \end{cases}$$

and recall that U_n is uniformly distributed on $\{1, \dots, m_n\}$ for any $n \geqslant 1$. Without loss of generality, one can assume that $\mathbb{E}(Z_n) = 1$ for any $n \geqslant 1$, and consequently, the following decomposition will be usefull: For any $n \geqslant 1$,

$$S_n = T_n + H_n,$$

with

$$T_n = \sum_{k=1}^{n} X_i \quad \text{and} \quad H_n = \sum_{k=1}^{n} X_i (Z_i - 1).$$

Denote $a_1 = 1$, and for any $n \geq 2$,

$$a_n = \frac{\Gamma(n)\Gamma(2p)}{\Gamma(n+2p-1)} \quad \text{and} \quad v_n = \sum_{i=1}^{n} a_i^2,$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$, $s > 0$, is the Gamma function. For $m_n = n$, Dedecker et al. [7] proved the following result:

Theorem 1.1. *[7, Theorems 2.1 and 3.1] Assume that $p \in (0, 1]$ and $\mathbb{E}\left[Z^2\right] < +\infty$.*

(1) If $p \in (0, 3/4]$, then

$$\frac{a_n S_n - (2r-1)}{\sqrt{v_n + n a_n^2 \sigma^2}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, 1\right).$$

(2) If $p \in (3/4, 1]$*, then*

$$\frac{S_n}{n^{2p-1}} \xrightarrow[n\to\infty]{a.s} L,$$

where L is a non-denerate and non-Gaussian random variable.

Remark 1.1. *An important observation regarding the model investigated by Dedecker et al. [7] is its strong connection with the urn model. Specifically, the model operates as follows: Initially, an urn contains one white ball and one black ball. At each step n, a ball is randomly drawn from the urn. Subsequently, the drawn ball is replaced, with the addition of* $Y_n Z_n$ *balls of the same color and* $(1 - Y_n)Z_n$ *balls of the opposite color, where* Y_n *has a Bernoulli distribution with parameter p and is independent of* Z_n*. The addition matrix for this urn is random and is given by*

$$D_n = \begin{pmatrix} Y_n Z_n & (1 - Y_n) Z_n \\ (1 - Y_n) Z_n & Y_n Z_n \end{pmatrix}.$$

Some of the results of Dédecker et al. [7] and some extra results, can be obtained using this connection from [9], where, if we denote by W_n *and* B_n*, respectively, the number of white and black balls at time n, the position of the elephant at step n can be given, due to this connection, by* $S_n = W_n - B_n$*. This connection with the Pólya urn model is not possible if we consider gradually increasing memory random walks: Practically, it is not possible at step n to draw a ball only from the first* m_n *added balls.*

It is important to note that Theorem 1.1 holds in the particular case of $m_n = n$. Our aim in this work is to extend Theorem 1.1 to the case where $m_n \leqslant n$ such that $m_n/n \to \theta$ for some $\theta \in [0, 1]$. Additionally, for $p > 3/4$ (superdiffusive case), we are going to give the asymptotic distribution for the fluctuation of the elephant random walk around the random variable τL, where L is defined in Theorem 1.1 and τ in Theorem 2.1.

2. Results

First case: Diffusive regime $(0 < p < 3/4)$

Our first result concerns the case where $p \in (0, 3/4)$ and we give the asymptotic distribution of the elephant random walk.

Theorem 2.1. *Let* $\theta \in [0, 1]$ *such that* $\lim_{n\to\infty} m_n = +\infty$*, and* $m_n/n \to \theta$ *as n goes to infinity, and suppose that* $p \in (0, 3/4)$*. Assume that* $(Z_n)_{n\geqslant 1}$ *is iid positive with mean* 1 *and variance* σ^2*, and denote*

$$\tau = \theta + (1 - \theta)(2p - 1) \quad \textit{and} \quad \sigma_1^2 = \frac{\tau^2}{3 - 4p} + \theta(1 - \theta).$$

Then,

$$\frac{\sqrt{m_n}}{n} S_n \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \sigma_1^2 + \theta\sigma^2\right).$$

Remark 2.1. *If the step size is deterministic, then* $\sigma = 0$*, and we obtain the result in [1]. If* $\theta = 1$*, we obtain the result in [7].*

Proof of Theorem 2.1. Let

$$\bar{S}_n := \frac{\sqrt{m_n}}{n} S_n = \frac{\sqrt{m_n}}{n} H_n + \frac{\sqrt{m_n}}{n} T_n =: A_n + B_n.$$

By [1], we know that for $0 < p < 3/4$, we have

$$B_n = \frac{\sqrt{m_n}}{n} T_n \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\Big(0,\, \sigma_1^2\Big), \tag{2.1}$$

with

$$\sigma_1^2 = \theta(1-\theta) + \frac{\tau^2}{3-4p} \quad \text{and} \quad \tau^2 = \theta + (1-\theta)(2p-1).$$

For all $t \in \mathbb{R}$, denote $\varphi_n(t) = \mathbb{E}\Big[e^{it\bar{S}_n}\Big]$ and consider the σ-algebra $\mathcal{F} = \sigma\Big(X_i,\, U_i,\, V_i, i \in \mathbb{N}\Big)$. So, for any t in $\mathbb{R}$, we have

$$\varphi_n(t) = \mathbb{E}[\exp(it\{A_n + B_n\})] = \mathbb{E}[\mathbb{E}[\exp(itA_n)\,|\mathcal{F}]\exp(itB_n)]\,.$$

Since the random variables $(Z_i - 1)_i$ are iid centered with finite variance σ^2 and that $X_i^2 = 1$ a.s., we get

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\,(Z_i - 1) \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\Big(0,\, \sigma^2\Big). \tag{2.2}$$

In fact, if we denote

$$R_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\,(Z_i - 1) \quad \text{and} \quad \Phi_n(t) = \mathbb{E}\left[\exp(itR_n)\right],$$

then,

$$\begin{aligned}
\Phi_n(t) = \mathbb{E}\left[\mathbb{E}\left[\exp(itR_n)\mid\mathcal{F}\right]\right] &= \mathbb{E}\left[\prod_{k=1}^{n}\mathbb{E}\left[\exp\left(i\frac{t}{\sqrt{n}}X_k\,(Z_k-1)\right)\middle|\,\mathcal{F}\right]\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{n}\mathbb{E}\left[\exp\left(i\frac{t}{\sqrt{n}}X_k\,(Z_1-1)\right)\middle|\,\mathcal{F}\right]\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{n}\mathbb{E}\left[\left(1 + i\frac{t}{\sqrt{n}}X_k\,(Z_1-1) - \frac{t^2}{2n}\,(Z_1-1)^2 + o\left(\frac{1}{n}\right)\right)\middle|\,\mathcal{F}\right]\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{n}\left(1 - \frac{t^2\sigma^2}{2n} + o\left(\frac{1}{n}\right)\right)\right] \\
&= \left(1 - \frac{t^2\sigma^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \to \exp\left(-\frac{t^2\sigma^2}{2}\right),
\end{aligned}$$

where in the last three equations, the $o-$ term depends on the moment of order two of Z_1, which is finite by assumption.

Using (2.2), we obtain

$$A_n = \frac{\sqrt{m_n}}{\sqrt{n}}\frac{H_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\Big(0,\, \theta\sigma^2\Big).$$

As a first conclusion, we have

$$\lim_{n\to\infty}\mathbb{E}[\exp(itA_n)\,|\mathcal{F}] = \exp\left(-\frac{\theta\sigma^2 t^2}{2}\right) = \varphi\left(\sqrt{\theta}\sigma t\right).$$

where φ is the characteristic function of the standard normal law. On the other hand, for any $n \geqslant 1$ and any $t \in \mathbb{R}$,

$$\varphi_n(t) = \mathbb{E}\Big[\Big(\mathbb{E}[\exp(itA_n)\,|\mathcal{F}] - \varphi\left(\sqrt{\theta}\sigma t\right)\Big)\exp(itB_n)\Big] + \varphi\left(\sqrt{\theta}\sigma t\right)\mathbb{E}[\exp(itB_n)]\,.$$

Using (2.1) and noting that

$$\lim_{n\to+\infty}\left|\mathbb{E}\Big[\Big(\mathbb{E}[\exp(itA_n)\,|\mathcal{F}] - \varphi\left(\sqrt{\theta}\sigma t\right)\Big)\exp(itB_n)\Big]\right| \leq \lim_{n}\mathbb{E}\Big[\Big|\mathbb{E}[\exp(itA_n)\,|\mathcal{F}] - \varphi\left(\sqrt{\theta}\sigma t\right)\Big|\Big] = 0,$$

we derive [1]

$$\lim_{n\to+\infty}\varphi_n(t) = \varphi\left(\sqrt{\theta}\sigma t\right)\varphi(\sigma_1 t) = \exp\left(\frac{-(\theta\sigma^2+\sigma_1^2)t^2}{2}\right).$$

The proof of Theorem 2.1 is complete. □

Second case: Critical regime ($p = 3/4$)

Assume that $p = 3/4$ and consider the decompositon

$$S_n = S_{m_n} + D_{m_n,n}, \quad \text{where} \quad D_{m_n,n} = \sum_{k=m_n+1}^{n} X_k Z_k.$$

Theorem 2.2. *Let $\theta \in [0,1]$ such that $m_n/n \to \theta$ as n goes to infinity, and suppose that $p = 3/4$. If $(Z_n)_{n\geqslant 1}$ are iid positive with mean* 1 *and finite variance σ^2, then*

$$\frac{\sqrt{m_n}S_n}{n\sqrt{\ln m_n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\Big(0, \frac{(1+\theta)^2}{4}\Big).$$

Remark 2.2. *Notice that in this case, the step size does not influence the asymptotic behavior of the normalized position of the elephant. This is due to the fact that the normalized remainder term $D_{m_n,n}$ converges to* 0 *in probability.*

Proof of Theorem 2.2. From [7], we know that

$$\frac{S_{m_n}}{\sqrt{m_n\ln m_n}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\big(0,\ 1\big).$$

Moreover, given $\mathcal{F}_{m_n} := \sigma(X_i\,;\, i \leqslant m_n)$, for any $k \geqslant m_n + 1$, we have

$$\mathbb{P}\left[X_k = 1|\mathcal{F}_{m_n}\right] = \left(\frac{1}{2} + \frac{1}{4m_n}T_m\right) \text{ and } \mathbb{P}\left[X_k = -1|\mathcal{F}_{m_n}\right] = \left(\frac{1}{2} - \frac{1}{4m_n}T_m\right).$$

Note also that, given $\mathcal{F}_{m_n}$, the random variables $(X_k Z_k)_{m_n+1\leq k\leq n}$ are i.i.d. In the other part, we have the following decomposition:

$$\tilde{S}_n := \frac{\sqrt{m_n}S_n}{n\sqrt{\ln m_n}} = A_n + B_n,$$

where

$$A_n := \frac{m_n}{n} \frac{S_{m_n}}{\sqrt{m_n \ln m_n}} \quad \text{and} \quad B_n := \frac{\sqrt{m_n} D_{m_n, n}}{n \sqrt{\ln m_n}}.$$

For any $t \in \mathbb{R}$ and any $n \geqslant 1$, we denote $\phi_n(t) = \mathbb{E}\left[\exp\left(it\tilde{S}_n\right)\right]$ and $\mathcal{G}_n := \sigma(Z_1, \cdots, Z_n)$. Then, we have

$$\begin{aligned}
\phi_n(t) &= \mathbb{E}\left[\mathbb{E}\left[\exp(itA_n)\exp(itB_n)|\mathcal{F}_{m_n} \bigvee \mathcal{G}_{m_n}\right]\right] \\
&= \mathbb{E}\left[\exp(itA_n)\,\mathbb{E}\left[\exp(itB_n)|\mathcal{F}_{m_n} \bigvee \mathcal{G}_{m_n}\right]\right] \\
&= \mathbb{E}\left[\exp(itA_n)\left(\mathbb{E}\left[\exp\left(it\frac{\sqrt{m_n}}{n\sqrt{\ln m_n}}X_{m_n+1}Z_1\right)|\mathcal{F}_{m_n}\right]\right)^{(n-m_n)}\right].
\end{aligned}$$

Using the conditional distribution of X_{m_n+1} given $\mathcal{F}_{m_n}$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(it\frac{\sqrt{m_n}}{n\sqrt{\ln m_n}}X_{m_n+1}Z_1\right)|\mathcal{F}_{m_n}\right] \\
=&\left(\frac{1}{2}+\frac{1}{4m_n}T_{m_n}\right)\mathbb{E}\left[\exp\left(\frac{it\sqrt{m_n}}{n\sqrt{\ln m_n}}Z_1\right)\right]+\left(\frac{1}{2}-\frac{1}{4m_n}T_{m_n}\right)\mathbb{E}\left[\exp\left(-\frac{it\sqrt{m_n}}{n\sqrt{\ln m_n}}Z_1\right)\right] \\
=&\left(\frac{1}{2}+\frac{1}{4m_n}T_{m_n}\right)\left[1+\frac{it\sqrt{m_n}}{n\sqrt{\ln m_n}}\mu_1-\frac{t^2 m_n}{n^2\ln m_n}\mu_2+o\left(\frac{m_n^{3/2}}{n^3\ln m_n^{3/2}}\right)\right] \\
&+\left(\frac{1}{2}-\frac{1}{4m_n}T_{m_n}\right)\left[1-\frac{it\sqrt{m_n}}{n\sqrt{\ln m_n}}\mu_1-\frac{t^2 m_n}{n^2\ln m_n}\mu_2+o\left(\frac{m_n^{3/2}}{n^3\ln m_n^{3/2}}\right)\right] \\
=&1+\frac{it\mu_1}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}-\frac{t^2 m_n}{n^2\ln m_n}\mu_2+o\left(\frac{m_n^{3/2}}{n^3\ln m_n^{3/2}}\right)+\frac{T_{m_n}}{m_n}\times o\left(\frac{m_n^{3/2}}{n^3\ln m_n^{3/2}}\right),
\end{aligned}$$

and recall that $m_n^{-1}T_{m_n} \xrightarrow[n\to+\infty]{\text{a.s.}} 0$ (see [3]) and $|m_n^{-1}T_{m_n}| \leqslant 1$ a.s. Then, for n sufficiently large

$$\begin{aligned}
&\left(\mathbb{E}\left[\exp\left(it\frac{\sqrt{m_n}}{n\sqrt{\ln m_n}}X_{m_n+1}Z_1\right)|\mathcal{F}_{m_n}\right]\right)^{(n-m_n)} \\
=&\left(1+\frac{it\mu_1}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}-\frac{t^2 m_n}{n^2\ln m_n}\mu_2+\left(1+\frac{T_{m_n}}{m_n}\right)\times o\left(\frac{m_n^{3/2}}{n^3\ln m_n^{3/2}}\right)\right)^{(n-m_n)} \\
=&\exp\left[(n-m_n)\ln\left(1+\frac{it\mu_1}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}-\frac{t^2 m_n}{n^2\ln m_n}\mu_2+o\left(\frac{m_n^{3/2}}{n^3}\right)\right)\right] \\
=&\exp\left[\frac{it\mu_1(n-m_n)}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}-\frac{t^2 m_n(n-m_n)}{n^2\ln m_n}\mu_2+o\left(\frac{m_n^{3/2}}{n^2}\right)\right] \\
\approx&\exp\left[\frac{it\mu_1(n-m_n)}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}\right].
\end{aligned}$$

Then, the characteristic function $\phi_n(t)$ has an asymptotic expression

$$
\begin{aligned}
\phi_n(t) =&\mathbb{E}\left[\exp\left(itA_n\right)\exp\left(\frac{it\mu_1(n-m_n)}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}\right)\right]\\
=&\mathbb{E}\left[\exp\left(it\frac{m_n}{n}\left(\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}+\frac{H_{m_n}}{\sqrt{m_n\ln m_n}}\right)\right)\exp\left(\frac{it\mu_1(n-m_n)}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}\right)\right]\\
=&\mathbb{E}\left[\exp\left(it\frac{m_n+n}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}+it\frac{m_n}{n}\frac{H_{m_n}}{\sqrt{m_n\ln m_n}}\right)\right]\\
=&\mathbb{E}\left[\exp\left(it\frac{m_n+n}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}\right)\mathbb{E}\left[\exp\left(it\frac{m_n}{n}\frac{H_{m_n}}{\sqrt{m_n\ln m_n}}\right)|\mathcal{F}\right]\right].
\end{aligned}
$$

Since

$$\lim_{n\to\infty}\mathbb{E}\left[\exp\left(it\frac{m_n}{n}\frac{H_{m_n}}{\sqrt{m_n\ln m_n}}\right)|\mathcal{F}\right]=1,$$

we deduce that

$$\lim_{n\to\infty}\phi_n(t)=\lim_{n\to\infty}\mathbb{E}\left[\exp\left(it\frac{m_n+n}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}\right)\right].$$

Finally, by [3], we conclude

$$\lim_{n\to\infty}\mathbb{E}\left[\exp\left(it\frac{m_n+n}{2n}\frac{T_{m_n}}{\sqrt{m_n\ln m_n}}\right)\right]=\exp\left(-\frac{t^2}{2}\frac{(\theta+1)^2}{4}\right).$$

The proof of Theorem 2.2 is complete. □

Third case: Superdiffusive regime ($p > 3/4$)

In this section, we consider the case, where $p \in (3/4, 1)$, called the superdifusive regime. First, we give the almost sure convergence of S_n.

Theorem 2.3. *Let $\theta \in [0, 1]$ such that $m_n/n \to \theta$ as n goes to infinity, and suppose that $p \in (3/4, 1)$. If $(Z_n)_{n\geqslant 1}$ are iid positive with mean 1 and finite variance σ^2, then*

$$\frac{m_n^{2(1-p)}}{n}S_n \xrightarrow[n\to\infty]{a.s} (\theta+(2p-1)(1-\theta))\,L,$$

where L is a non-Gaussian random variable.

Proof of Theorem 2.3. Recall that

$$S_n = S_{m_n} + D_{m_n,n}, \quad \text{where} \quad D_{m_n,n} = \sum_{k=m_n+1}^{n} X_k Z_k,$$

and consequently,

$$\frac{m_n^{2(1-p)}}{n}S_n = \frac{m_n}{n}\frac{S_{m_n}}{m_n^{2p-1}} + \frac{m_n^{2(1-p)}}{n}D_{m_n,n}.$$

From [7], we know that

$$\frac{m_n}{n}\frac{S_{m_n}}{m_n^{2p-1}} \xrightarrow[n\to\infty]{\text{a.s}} \theta L.$$

On the other hand, according to $\mathcal{F}_{m_n}$, the sequence of random variables $(X_i Z_i)_{m_n+1\leq i\leq n}$ is i.i.d., and for all $i \geq m_n + 1$,

$$\mathbb{E}[X_i Z_i|\mathcal{F}_{m_n}] = \mathbb{E}[X_i|\mathcal{F}_{m_n}]\,\mathbb{E}[Z_i] = (2p-1)\,\frac{T_{m_n}}{m_n}.$$

Applying the strong law of large numbers condionally to $\mathcal{F}_m$, for n large, we have

$$\frac{D_{m_n,n}}{n-m_n} = \frac{1}{n-m_n}\sum_{k=m_n+1}^{n} X_k Z_k \approx (2p-1)\,\frac{T_{m_n}}{m_n} \quad \text{a.s.}$$

Then, almost surely, for n large

$$\frac{m_n^{2(1-p)}}{n} D_{m_n,n} = \frac{n-m_n}{n} m_n^{2(1-p)} \frac{1}{n-m_n} D_{m_n,n} \approx \frac{n-m_n}{n} m_n^{2(1-p)} \frac{T_{m_n}}{m_n} \approx (1-\theta)\,(2p-1)\,L. \qquad \square$$

The following result shows that the fluctuations of the elephant random walk are still gaussian around the random variable L as given in Theorem 2.3.

Theorem 2.4. *Let $\theta \in [0,1]$ such that $m_n/n \to \theta$ as n goes to infinity, and suppose that $p \in (3/4,\,1)$. If $(Z_n)_{n\geqslant 1}$ are iid positive with mean 1 and finite variance σ^2, then*

$$\sqrt{m_n^{4p-3}}\left(\frac{S_n m_n^{2(1-p)}}{n} - \tau L\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\Big(0,\ \lambda^2 + \theta\sigma^2\Big),$$

where

$$\lambda^2 = \frac{\tau^2}{4p-3} - \frac{\tau\left(\theta^{2p-1}-\theta\right)}{1-p} + \theta\,(1-\theta) \quad \text{and} \quad \tau = \theta + (1-\theta)\,(2p-1)\,.$$

Remark 2.3. *This result generalizes the result obtained in [7], it precises in addition the asymptotic distribution of the fluctuation, and coincides with the result of Aguech et al. [1], in the case where $\sigma = 0$.*

Proof of Theorem 2.4. We start again with the following decomposition:

$$S_n = \sum_{k=1}^{n} X_k + \sum_{k=1}^{n} X_k\,(Z_k - 1) = T_n + H_n.$$

First, using [1], if $\sqrt{m_n^{4p-3}}|n^{-1}m_n - \theta| \to 0$, then

$$\sqrt{m_n^{4p-3}}\left(\frac{S_n m_n^{2(1-p)}}{n} - \tau L\right) \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\Big(0,\ \lambda^2\Big).$$

Moreover, we have

$$\begin{aligned}\sqrt{m_n^{4p-3}}\left(\frac{S_n m_n^{2(1-p)}}{n} - \tau L\right) &= \sqrt{m_n^{4p-3}}\left(\frac{T_n m_n^{2(1-p)}}{n} - \tau L\right) + \frac{\sqrt{m_n}}{n}\sum_{k=1}^{n} X_k\,(Z_k-1)\\ &:= \Delta_n + \tilde{H}_n.\end{aligned}$$

Let, for all n, $\mathcal{H}_n = \sigma(X_1, \cdots, X_n, L)$. For all real t, let $\psi_Z(t) = \mathbb{E}[\exp(it[Z-1])]$ the characteristic function of $Z-1$ and $\Psi_n(t)$ is the characteristic function of $\sqrt{m_n^{4p-3}}\left(\frac{S_n m_n^{2(1-p)}}{n} - \tau L\right)$. So, we have

$$\begin{aligned}
\Psi_n(t) &= \mathbb{E}\Big[\exp(it\Delta_n)\exp\left(it\tilde{H}_n\right)\Big] \\
&= \mathbb{E}\Big[\exp(it\Delta_n)\,\mathbb{E}\Big[\exp\left(it\tilde{H}_n\right)|\mathcal{H}_n\Big]\Big] \\
&= \mathbb{E}\left[\exp(it\Delta_n)\prod_{k=1}^{n}\psi_Z\left(\frac{\sqrt{m}}{n}tX_k\right)\right].
\end{aligned}$$

If $\psi_{Z|\mathcal{H}_n}$ denotes the characteristic function of $Z-1$ conditionally to $\mathcal{H}_n$, then

$$\begin{aligned}
\psi_{Z|\mathcal{H}_n}\left(\frac{\sqrt{m_n}}{n}tX_k\right) &= \mathbb{E}\left[\exp\left(\frac{\sqrt{m_n}}{n}itX_k(Z_1-1)\right)|\mathcal{H}_n\right] \\
&= \mathbb{E}\left[\left(1 + it\frac{\sqrt{m_n}}{n}X_k(Z_1-1) - \frac{t^2}{2}\frac{m_n}{n^2}(Z_1-1)^2 + o\left(\frac{m_n^{3/2}}{n^3}\right)\right)|\mathcal{H}_n\right] \\
&\approx \left(1 - \frac{t^2}{2}\frac{m_n}{n^2}\sigma^2\right) \quad \text{(if } n \text{ is large)}.
\end{aligned}$$

Consequently, if n is large, then

$$\begin{aligned}
\Psi_n(t) &= \mathbb{E}\left[\exp(it\Delta_n)\left(1 - \frac{t^2}{2}\frac{m_n}{n^2}\sigma^2\right)^n\right] \\
&= \mathbb{E}\left[\exp(it\Delta_n)\exp\left(\sum_{k=1}^{n}\ln\left[1 - \frac{t^2}{2}\frac{m_n}{n^2}\sigma^2\right]\right)\right] \\
&\approx \mathbb{E}[\exp(it\Delta_n)]\exp\left(-\frac{t^2}{2}\frac{m_n}{n}\sigma^2\right) \\
&\to \exp\left(-\frac{t^2}{2}\left[\lambda^2 + \theta\sigma^2\right]\right).
\end{aligned}$$

The proof of Theorem 2.4 is complete. □

3. Conclusions and discussion

In this work, we place emphasis on the fact that the asymptotic normality for the elephant random walk with gradually increasing memory and random step size still holds in the three regimes of the model. Our results extend previous ones established by Dedecker et al. [7] and Aguech et al. [1]. Additionally, we observed that the connection with Polya urns is not feasible for gradually increasing memory. Finally, we argue that our approach can be used for deriving many others limit theorems for the elephant random walk with gradually increasing memory and random step size (law of the iterated logarithm, rate of convergence in the central limit theorem, invariance principle).

What about the asymptotic normality of the ERW model, remembering only its last m_n steps $n - m_n, \ldots, n-1$? This very interesting question will serve as the basis for a new research project. Another problem that we think is very interesting, and which will be one of our future projects, is to consider the same question but with an ERW model with a random step size. A more difficult problem is to establish the rate of convergence in the central limit theorems for the ERW model with restricted memory. In particular, it will be very interesting to understand the way that the restricted memory will influence the rate of convergence in the central limit theorem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

References

1. R. Aguech, M. El Elmachkouri, Gaussian fluctuations of the elephant random walk with gradually increasing memory, *J. Phys. A Math. Theor.*, **57** (2024), 065203. https://doi.org/10.1088/1751-8121/ad1c0d
2. E. Baur, J. Bertoin, Elephant random walks and their connection to Pólya-type urns, *Phys. Rev. E*, **94** (2016), 052134. https://doi.org/10.1103/PhysRevE.94.052134
3. B. Bercu, A martingale approach for the elephant random walk,, *J. Phys. A Math. Theor.*, **51** (2018), 015201. https://doi.org/10.1088/1751-8121/aa95a6
4. B. Bercu, On the elephant random walk with stops playing hide and seek with the Mittag-Leffler distribution, *J. Stat. Phys.*, **189** (2022), 12. https://doi.org/10.1007/s10955-022-02980-w
5. B. Bercu, L. Laulin, On the multi-dimensional elephant random walk, *J. Stat. Phys.*, **175** (2019), 1146–1163. https://doi.org/10.1007/s10955-019-02282-8
6. C. F. Coletti, R. Gava, G. M. Schutz., Central limit theorem and related results for the elephant random walk, *J. Math. Phys.*, **58** (2017), 053303. https://doi.org/10.1063/1.4983566
7. J. Dedecker, X. Q. Fan, H. J. Hu, F. Merlevede, Rates of convergence in the central limit theorem for the elephant random walk with random step sizes, *J. Stat. Phys.*, **190** (2023), 154. https://doi.org/10.1007/s10955-023-03168-6
8. A. Gut, U. Stadtmuller, The elephant random walk with gradually increasing memory, *Statist. Probab. Lett.*, **189** (2022), 109598. https://doi.org/10.1016/j.spl.2022.109598
9. S. Janson, Functional limit theorems for multitype branching processes and generalized Polya urns, *Stochastic Process. Appl.*, **110** (2004), 177–245. https://doi.org/10.1016/j.spa.2003.12.002

10. N. Kubota, M. Takei, Gaussian fluctuation for superdiffusive elephant random walks, *J. Stat. Phys.*, **177** (2019), 1157–1171. https://doi.org/10.1007/s10955-019-02414-0

11. L. Laulin, Introducing smooth amnesia to the memory of the elephant random walk, *Electron. Commun. Probab.*, **27** (2022), 1–12. https://doi.org/10.1214/22-ECP495

12. X. H. Ma, M. El Machkouri, X. Q. Fan, On Wasserstein-1 distance in the central limit theorem for elephant random walk, *J. Math. Phys.*, **63** (2022), 013301. https://doi.org/10.1063/5.0050312

13. G. Schutz, S. Trimper, Elephants can always remember: exact long-range memory effects in a non-Markovian random walk, *Phys. Rev. E*, **70** (2004), 045101. https://doi.org/10.1103/PhysRevE.70.045101