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Research article

The fourth power mean of the generalized quadratic Gauss sums associated with some Dirichlet characters

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Abstract: In this paper, the fourth power mean values of the generalized quadratic Gauss sums associated with the 3-order and 4-order Dirichlet characters are given by using the properties of the Dirichlet characters and Gauss sums.

Keywords: generalized quadratic Gauss sums; the fourth power mean; k-order Dirichlet character; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let $q \in \mathbf{Z}^+$ with $q \geq 2$. λ denotes a Dirichlet character modulo q. For any $m, n \in \mathbf{Z}$, the generalized quadratic Gauss sums $\mathcal{G}(m, n, \lambda; q)$ is defined as follows:

$$\mathcal{G}(m, n, \lambda; q) = \sum_{s=1}^{q} \lambda(s) \mathrm{e}_q\left(ms^2 + ns\right),$$

where $\mathrm{e}_q(x) = \exp(2\pi \mathrm{i} x/q)$ and i is the imaginary unit (i.e., $\mathrm{i}^2 = -1$).

If $m = 0$ and $n = 1$, then we call $\mathcal{G}(0, 1, \lambda; q)$ a classical Gauss sum and denote it as $\mathcal{G}(\lambda)$. Gauss sums have had an important effect on both cryptography and the analytic number theory. The analytic number theory and cryptography will greatly benefit from any significant advancements made in this area, so the study of the properties of $\mathcal{G}(m, n, \lambda; q)$ is a meaningful work. Up to now, many researchers have studied the calculation and estimation of the high-th power mean of $\mathcal{G}(m, n, \lambda; q)$, which can be roughly divided into three types. They are

$$\sum_{m=1}^{q-1}|\mathcal{G}(m,n,\lambda;q)|^k,\quad \sum_{\lambda \bmod q}|\mathcal{G}(m,n,\lambda;q)|^k,\quad \text{and}\quad \sum_{\lambda \bmod q}\sum_{m=1}^{q-1}|\mathcal{G}(m,n,\lambda;q)|^k.$$

For example, W. P. Zhang [13] proved the identities

$$\frac{1}{p-1}\sum_{\lambda \bmod p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2\right)\right|^4 = \begin{cases} 3p^2-6p-1+4\lambda_2(m) & \text{if } 4 \mid \phi(p), \\ 3p^2-6p-1 & \text{if } 4 \nmid \phi(p), \end{cases}$$

$$\frac{1}{p-1}\sum_{\lambda \bmod p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2\right)\right|^6 = 10p^3-25p^2-4p-1 \ \text{ if } 4 \nmid \phi(p),$$

where p is an odd prime and m is any integer with $(m,p)=1$, λ_2 denotes the Legendre's symbol modulo p, and $\phi(x)$ is the Euler function.

In 2020, N. Bag and R. Barman [2] proved that for odd prime p and any $m\in\mathbf{Z}$ with $(m,p)=1$, there are the asymptotic formulae

$$\sum_{\lambda \bmod p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2\right)\right|^6 = 10p^4+O\left(p^{\frac{7}{2}}\right)$$

and

$$\sum_{\lambda \bmod p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2\right)\right|^8 = 35p^5+O\left(p^{\frac{9}{2}}\right).$$

Over the next two years, N. Bag, A. Rojas-León, and W. P. Zhang [3, 4] proved not only the asymptotic formula for the 10-th power mean of $\mathcal{G}(m,0,\lambda;p)$, but also the asymptotic formula for the any $2k$-th power mean of $\mathcal{G}(m,0,\lambda;p)$. The results are as follows:

$$\sum_{\lambda \bmod p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2\right)\right|^{10} = 126p^6+O\left(p^{\frac{11}{2}}\right),$$

$$\sum_{\lambda \bmod p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2\right)\right|^{2k} = \binom{2k-1}{k}\cdot p^{k+1}+O\left(p^{\frac{2k+1}{2}}\right).$$

X. X. Li and Z. F. Xu [9] studied the fourth power mean of the generalized quadratic Gauss sums, and proved the following result: For an odd prime p and a character λ modulo p, there are the identities

$$\sum_{m=1}^{p}\left|\sum_{s=1}^{p-1}\lambda(s)\mathrm{e}_p\left(ms^2+s\right)\right|^4$$
$$= \begin{cases} p^3-3p^2+2p^2\lambda_2(-1)-p-8p\lambda_2(-1) & \text{if } \lambda=\lambda_0, \\ 2p^3-3p^2 & \text{if } \lambda(-1)=-1, \\ 2p^3-4p^2\lambda_2(-1)-3p^2-p\left|\sum_{t=1}^{p-1}\lambda(t+\bar{t})\right|^2 & \text{if } \lambda(-1)=1 \text{ and } \lambda\neq\lambda_0, \end{cases}$$

where $\bar{t}$ is the inverse of t modulo p (i.e., $\bar{t} \cdot t \equiv 1 \bmod p$), and λ_0 denotes principal characters mod p.

X. Y. Liu and W. P. Zhang [10] proved that for any odd prime p with $3 \nmid \phi(p)$, one has the identity

$$\frac{1}{p(p-1)} \sum_{\lambda \bmod p} \sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(ms^3 + s\right) \right|^6 = 6p^3 - 28p^2 + 39p + 5.$$

X. X. Lv and W. P. Zhang [11] obtained the identities, but there may be a bit of a miscalculation in this result, and the correct result should be

$$\frac{1}{p(p-1)} \sum_{\lambda \bmod p} \sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(ms^2 + s\right) \right|^6 = \begin{cases} 5p^3 - 27p^2 + 38p + 8 & \text{if } 4 \nmid \phi(p), \\ 5p^3 - 27p^2 + 38p + 20 & \text{if } 4 \mid \phi(p). \end{cases}$$

Some related work can also be found in [6, 12, 14, 16]. We will not list them all here. It is worth noting that for the power mean of $\mathcal{G}(m, 1, \lambda; p)$,

$$\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(ms^2 + s\right) \right|^{2k}, \ k \geq 2, \tag{1.1}$$

no one seems to have studied it so far; at least we have not seen any valid conclusions. Through research, we have found that even taking $k = 2$ in (1.1), it is difficult to get the exact results for general Dirichlet characters. Therefore, we settle for the second: select some special Dirichlet characters to get the exact result.

Throughout the article, we fix a few notations. We use $f = O(g)$ to denote $|f| \leq cg$ for some positive constant c. We use a λ_k to denote k-order Dirichlet character modulo p (i.e., $\lambda_k^k = \lambda_0$), where λ_0 denotes principal characters.

Theorem 1. *Let p be an odd prime with $3 \mid \phi(p)$. Then, for any 3-order character λ_3 modulo p with $\lambda_3(2) = 1$, we have the identities*

$$\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda_3(s) e_p\left(ms^2 + s\right) \right|^4 = \begin{cases} p \cdot \left(2p^2 - 7p - d^2\right) & \text{if } p \equiv 1 \bmod 12, \\ p \cdot \left(2p^2 - 11p + d^2\right) & \text{if } p \equiv 7 \bmod 12, \end{cases}$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Theorem 2. *Let p be an odd prime with $4 \mid \phi(p)$. Then, for any 4-order character λ_4 modulo p, we have the identity*

$$\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda_4(s) e_p\left(ms^2 + s\right) \right|^4 = 2p^3 - 7p^2 - 4p\alpha^2(p),$$

where $\alpha(p)$ is defined by $\alpha(p) = \sum_{s=1}^{\frac{p-1}{2}} \lambda_2(s + \overline{s})$.

From these two theorems, we may immediately get the following corollary:

Corollary 1. *Let p be an odd prime. Then, for any Dirichlet character λ with $\lambda \neq \lambda_0$, we have the asymptotic formula*

$$\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) e_p\left(ms^2 + s\right) \right|^4 = 2p^3 + O\left(p^2\right).$$

2. Several lemmas

The primary goal of this section is to introduce several necessary lemmas and their proofs for the paper's main results. In the proof of these lemmas, we use the properties of Gauss sums and the reduced (complete) residue system, as well as the properties and definitions of k-order character, odd character, and even character. References [1, 8, 15] have further details; we will not explain them here.

Lemma 1. *Let p be an odd prime. Then, for any Dirichlet character λ modulo p with $\lambda \neq \lambda_0$, we have the identity*

$$\mathcal{G}\left(\lambda^2\right) = \frac{\lambda^2(2) \cdot \mathcal{G}(\lambda) \cdot \mathcal{G}(\lambda\lambda_2)}{\mathcal{G}(\lambda_2)}.$$

Proof. We used Dirichlet character sum $\sum_{s=0}^{p-1} \lambda\left(s^2 - 1\right)$ as a bridge to analyze it from different perspectives. On one hand, there is

$$\begin{aligned}
\sum_{s=0}^{p-1} \lambda\left(s^2 - 1\right) &= \sum_{s=0}^{p-1} \lambda\left((s+1)^2 - 1\right) = \sum_{s=1}^{p-1} \lambda(s)\lambda(s+2) \\
&= \frac{1}{\mathcal{G}\left(\overline{\lambda}\right)} \sum_{t=1}^{p-1} \overline{\lambda}(t) \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(t(s+2)\right) = \frac{\mathcal{G}(\lambda)}{\mathcal{G}\left(\overline{\lambda}\right)} \sum_{t=1}^{p-1} \overline{\lambda}(t)\overline{\lambda}(t)\mathrm{e}_p(2t) \\
&= \frac{\mathcal{G}(\lambda)}{\mathcal{G}\left(\overline{\lambda}\right)} \sum_{t=1}^{p-1} \overline{\lambda}^2(t)\mathrm{e}_p(2t) = \frac{\lambda^2(2) \cdot \mathcal{G}(\lambda) \cdot \mathcal{G}\left(\overline{\lambda}^2\right)}{\mathcal{G}\left(\overline{\lambda}\right)}.
\end{aligned} \tag{2.1}$$

On the other hand, for any integer t with $(t, p) = 1$, the identity

$$\sum_{s=0}^{p-1} \mathrm{e}_p\left(ts^2\right) = 1 + \sum_{s=1}^{p-1} \left(1 + \lambda_2(s)\right) \mathrm{e}_p(ts) = \sum_{s=1}^{p-1} \lambda_2(s)\mathrm{e}_p(ts) = \lambda_2(t) \cdot \mathcal{G}(\lambda_2),$$

there is

$$\begin{aligned}
\sum_{s=0}^{p-1} \lambda\left(s^2 - 1\right) &= \frac{1}{\mathcal{G}\left(\overline{\lambda}\right)} \sum_{s=0}^{p-1} \sum_{t=1}^{p-1} \overline{\lambda}(t)\mathrm{e}_p\left(t(s^2 - 1)\right) \\
&= \frac{1}{\mathcal{G}\left(\overline{\lambda}\right)} \sum_{t=1}^{p-1} \overline{\lambda}(t)\mathrm{e}_p(-t) \sum_{s=0}^{p-1} \mathrm{e}_p\left(ts^2\right) = \frac{\mathcal{G}(\lambda_2)}{\mathcal{G}\left(\overline{\lambda}\right)} \sum_{t=1}^{p-1} \overline{\lambda}(t)\lambda_2(t)\mathrm{e}_p(-t) \\
&= \frac{\lambda_2(-1)\overline{\lambda}(-1)\mathcal{G}(\lambda_2) \cdot \mathcal{G}\left(\overline{\lambda}\lambda_2\right)}{\mathcal{G}\left(\overline{\lambda}\right)}.
\end{aligned} \tag{2.2}$$

Note that $\overline{\mathcal{G}(\lambda)} = \lambda(-1) \cdot \mathcal{G}\left(\overline{\lambda}\right)$, $\mathcal{G}^2(\lambda_2) = \lambda_2(-1) \cdot p$, $\mathcal{G}(\lambda) \cdot \overline{\mathcal{G}(\lambda)} = p$, from (2.1) and (2.2) we have the identities

$$\mathcal{G}\left(\overline{\lambda}^2\right) = \frac{\overline{\lambda}^2(2) \cdot \mathcal{G}\left(\overline{\lambda}\right) \cdot \mathcal{G}\left(\overline{\lambda}\lambda_2\right)}{\mathcal{G}(\lambda_2)} \quad \text{or} \quad \mathcal{G}\left(\lambda^2\right) = \frac{\lambda^2(2) \cdot \mathcal{G}(\lambda) \cdot \mathcal{G}(\lambda\lambda_2)}{\mathcal{G}(\lambda_2)}.$$

This proves Lemma 1. □

Lemma 2. *Let p be an odd prime, and let λ be any Dirichlet character modulo p with $\lambda \neq \lambda_0$. If λ is an odd character modulo p, that is, $\lambda(-1) = -1$, then we have*

$$\sum_{\substack{s=1 \\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) = 0.$$

If λ is an even character modulo p, that is, $\lambda(-1) = 1$, then we have

$$\sum_{\substack{s=1 \\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) = 6p + \frac{\lambda_2(-1)}{p^2} \cdot \left(\lambda(4) \cdot \mathcal{G}^4\left(\psi\right) \cdot \mathcal{G}^2\left(\overline{\lambda}\right) + \overline{\lambda}(4) \cdot \mathcal{G}^4\left(\overline{\psi}\right) \cdot \mathcal{G}^2(\lambda)\right),$$

where ψ is a Dirichlet character modulo p such that $\lambda = \psi^2$.

Proof. If λ is an odd character modulo p, this is $\lambda(-1) = -1$, then we have

$$\sum_{\substack{s=1 \\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) = \sum_{\substack{s=1 \\ (-s)^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(-st\overline{u}\right) = -\sum_{\substack{s=1 \\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right).$$

Hence

$$\sum_{\substack{s=1 \\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) = 0.$$

If λ is an even character modulo p, then there exists a Dirichlet character ψ such that $\lambda = \psi^2$, and we obtain

$$\begin{aligned}
\sum_{\substack{s=1 \\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) &= \sum_{\substack{s=1 \\ u^2\left(s^2-1\right)\equiv 1-t^2 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\right) \\
&= 4(p-1) + \sum_{s=2}^{p-2} \sum_{t=2}^{p-2} \lambda\left(st\right) \sum_{\substack{u=1 \\ u^2\left(s^2-1\right)\equiv 1-t^2 \bmod p}}^{p-1} 1 \\
&= 4(p-1) + \sum_{s=2}^{p-2} \sum_{t=2}^{p-2} \lambda\left(st\right)\left(1 + \lambda_2\left(\left(1-t^2\right)\overline{\left(s^2-1\right)}\right)\right) \\
&= 4(p-1) + \left(\sum_{s=1}^{p-1} \lambda(s) - 2\right)^2 + \lambda_2(-1) \cdot \left(\sum_{s=1}^{p-1} \lambda(s)\lambda_2\left(s^2-1\right)\right)^2 \\
&= 4p + \lambda_2(-1)\left(\sum_{s=1}^{p-1} \psi\left(s^2\right)\lambda_2\left(s^2-1\right)\right)^2
\end{aligned}$$

$$= \quad 4p+\lambda_2(-1)\left(\sum_{s=1}^{p-1}\psi(s)\lambda_2(s-1)\left(1+\lambda_2(s)\right)\right)^2$$

$$= \quad 4p+\lambda_2(-1)\left(\sum_{s=1}^{p-1}\psi(s)\lambda_2(s-1)+\sum_{s=1}^{p-1}\psi(s)\lambda_2(s)\lambda_2(s-1)\right)^2. \tag{2.3}$$

In Lemma 1, we have

$$\sum_{s=1}^{p-1}\psi(s)\lambda_2(s-1)=\frac{1}{\mathcal{G}(\lambda_2)}\sum_{t=1}^{p-1}\lambda_2(t)\sum_{s=1}^{p-1}\psi(s)\mathrm{e}_p\left(t(s-1)\right)$$

$$= \quad \frac{\mathcal{G}(\psi)}{\mathcal{G}(\lambda_2)}\sum_{t=1}^{p-1}\lambda_2(t)\overline{\psi}(t)\mathrm{e}_p\left(-t\right)=\frac{\lambda_2(-1)\psi(-1)\mathcal{G}(\psi)\,\mathcal{G}\left(\overline{\psi}\lambda_2\right)}{\mathcal{G}(\lambda_2)}$$

$$= \quad \frac{\lambda_2(-1)\cdot\lambda(2)\cdot\mathcal{G}^2(\psi)\cdot\mathcal{G}\left(\overline{\lambda}\right)}{p}. \tag{2.4}$$

Similarly,

$$\sum_{s=1}^{p-1}\psi(s)\lambda_2(s)\lambda_2(s-1) \quad = \quad \frac{1}{\mathcal{G}(\lambda_2)}\sum_{t=1}^{p-1}\lambda_2(t)\sum_{s=1}^{p-1}\psi(s)\lambda_2(s)\mathrm{e}_p\left(t(s-1)\right)$$

$$= \quad \frac{\overline{\lambda}(2)\cdot\mathcal{G}^2\left(\overline{\psi}\right)\cdot\mathcal{G}(\lambda)}{p}. \tag{2.5}$$

Now combine (2.3)–(2.5) to obtain identity

$$\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\sum_{\substack{u=1\\ s^2+t^2\equiv u^2+1 \bmod p}}^{p-1}\lambda\left(st\overline{u}\right)=6p+\frac{\lambda_2(-1)}{p^2}\cdot\left(\lambda(4)\cdot\mathcal{G}^4(\psi)\cdot\mathcal{G}^2\left(\overline{\lambda}\right)+\overline{\lambda}(4)\cdot\mathcal{G}^4\left(\overline{\psi}\right)\cdot\mathcal{G}^2(\lambda)\right).$$

This proves Lemma 2. □

Lemma 3. *Let p be an odd prime. Then, for any Dirichlet character λ modulo p with $\lambda\neq\lambda_0$, we have the identity*

$$\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\sum_{\substack{u=1\\ s^2+t^2\equiv u^2+1 \bmod p\\ s+t\equiv u+1 \bmod p}}^{p-1}\lambda\left(st\overline{u}\right)=2p-3.$$

Proof. Note that the conditions $s^2+t^2\equiv u^2+1 \bmod p$ and $s+t\equiv u+1 \bmod p$ equivalent to $s+t\equiv u+1 \bmod p$ and $st\equiv u \bmod p$, we obtain

$$\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\sum_{\substack{u=1\\ s^2+t^2\equiv u^2+1 \bmod p\\ s+t\equiv u+1 \bmod p}}^{p-1}\lambda\left(st\overline{u}\right)=\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\sum_{\substack{u=1\\ s+t\equiv u+1 \bmod p\\ st\equiv u \bmod p}}^{p-1}\lambda\left(st\overline{u}\right)=\sum_{s=1}^{p-1}\sum_{\substack{t=1\\ (s-1)(t-1)\equiv 0 \bmod p}}^{p-1}1=2(p-2)+1=2p-3.$$

This proves Lemma 3. □

Lemma 4. *Let p be an odd prime with $3 \mid \phi(p)$. Then, we have the identity*

$$\mathcal{G}^3(\lambda_3) + \mathcal{G}^3\left(\overline{\lambda_3}\right) = dp.$$

Proof. See This is consequence of [5](pp. 114). □

Lemma 5. *Let p be an odd prime with $4 \mid \phi(p)$. Then, we have the identity*

$$\mathcal{G}^2(\lambda_4) + \mathcal{G}^2\left(\overline{\lambda_4}\right) = 2\sqrt{p} \cdot \alpha(p).$$

Proof. See Lemma 2.2, Section 2 in [7](pp. 1253). □

3. Proofs of the theorems

Now we apply Lemmas 1–3 to complete the proof of our Theorem 1 and Theorem 2. In fact, note that the trigonometrical identities

$$\sum_{s=0}^{p-1} \mathrm{e}_p(ns) = \begin{cases} p & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n. \end{cases}$$

From the properties of the reduced residue system modulo p, we have

$$\begin{aligned}
\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(ms^2+s\right) \right|^4 &= p \cdot \sum_{\substack{s=1 \\ s^2+t^2 \equiv u^2+v^2 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \sum_{v=1}^{p-1} \lambda\left(st\overline{uv}\right) \mathrm{e}_p(s+t-u-v) \\
&= p \cdot \sum_{\substack{s=1 \\ s^2+t^2 \equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) \sum_{v=1}^{p-1} \mathrm{e}_p(v(s+t-u-1)) \\
&= p^2 \cdot \sum_{\substack{s=1 \\ s^2+t^2 \equiv u^2+1 \bmod p \\ s+t \equiv u+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right) - p \cdot \sum_{\substack{s=1 \\ s^2+t^2 \equiv u^2+1 \bmod p}}^{p-1} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \lambda\left(st\overline{u}\right). \qquad (3.1)
\end{aligned}$$

If $\lambda(-1) = -1$, then from (3.1), Lemmas 2 and 3, we have

$$\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(ms^2+s\right) \right|^4 = 2p^3 - 3p^2.$$

Similarly, if $\lambda(-1) = 1$, then there exists a Dirichlet character ψ such that $\lambda = \psi^2$, and we obtain

$$\begin{aligned}
&\sum_{m=0}^{p-1} \left| \sum_{s=1}^{p-1} \lambda(s) \mathrm{e}_p\left(ms^2+s\right) \right|^4 \\
= \;& p^2(2p-3) - 6p^2 - \frac{\lambda_2(-1)}{p} \cdot \left(\lambda(4) \cdot \mathcal{G}^4(\psi) \cdot \mathcal{G}^2\left(\overline{\lambda}\right) + \overline{\lambda}(4) \cdot \mathcal{G}^4\left(\overline{\psi}\right) \cdot \mathcal{G}^2(\lambda)\right)
\end{aligned}$$

$$= \quad 2p^3 - 9p^2 - \frac{\lambda_2(-1)}{p}\left(\lambda(4)\mathcal{G}^4(\psi)\,\mathcal{G}^2\left(\overline{\lambda}\right) + \overline{\lambda}(4)\mathcal{G}^4\left(\overline{\psi}\right)\mathcal{G}^2(\lambda)\right). \tag{3.2}$$

If $3 \mid \phi(p)$ and $\lambda_3(2) = 1$, then $\lambda_3(4) = 1$ and $\lambda_3 = \overline{\lambda_3}^2$. Taking $\lambda = \lambda_3$ and $\psi = \overline{\lambda_3}$, from (3.2) and identity

$$\mathcal{G}^3(\lambda_3) + \mathcal{G}^3\left(\overline{\lambda_3}\right) = dp,$$

we have

$$\begin{aligned}
&\sum_{m=0}^{p-1}\left|\sum_{s=1}^{p-1}\lambda_3(s)\mathrm{e}_p\left(ms^2+s\right)\right|^4 \\
= \quad & 2p^3 - 9p^2 - \frac{\lambda_2(-1)}{p}\left(\lambda_3(4)\mathcal{G}^4\left(\overline{\lambda_3}\right)\mathcal{G}^2\left(\overline{\lambda_3}\right) + \overline{\lambda_3}(4)\mathcal{G}^4(\lambda_3)\,\mathcal{G}^2(\lambda_3)\right) \\
= \quad & 2p^3 - 9p^2 - \frac{\lambda_2(-1)}{p}\left(\mathcal{G}^6\left(\overline{\lambda_3}\right) + \mathcal{G}^6(\lambda_3)\right) \\
= \quad & 2p^3 - 9p^2 - \frac{\lambda_2(-1)}{p}\left(\left(\mathcal{G}^3\left(\overline{\lambda_3}\right) + \mathcal{G}^3(\lambda_3)\right)^2 - 2p^3\right) \\
= \quad & \begin{cases} p\cdot\left(2p^2 - 7p - d^2\right) & \text{if } p \equiv 1 \bmod 12, \\ p\cdot\left(2p^2 - 11p + d^2\right) & \text{if } p \equiv 7 \bmod 12. \end{cases}
\end{aligned}$$

This proves Theorem 1.

If $4 \mid \phi(p)$, since $\lambda_2(4) = 1$, $\lambda_2(-1) = 1$, $\mathcal{G}(\lambda_2) = \sqrt{p}$ and $\lambda_2 = \lambda_4^2$. Taking $\lambda = \lambda_2$ in (3.2) and $\psi = \lambda_4$, from (3.2) and identity

$$\mathcal{G}^2(\lambda_4) + \mathcal{G}^2\left(\overline{\lambda_4}\right) = 2\sqrt{p}\cdot\alpha(p),$$

we have

$$\begin{aligned}
\sum_{m=0}^{p-1}\left|\sum_{s=1}^{p-1}\lambda_3(s)\mathrm{e}_p\left(ms^2+s\right)\right|^4 \quad &= \quad 2p^3 - 9p^2 - \left(\mathcal{G}^4\left(\overline{\lambda_4}\right) + \mathcal{G}^4(\lambda_4)\right) \\
&= \quad 2p^3 - 9p^2 - \left(\left(\mathcal{G}^2\left(\overline{\lambda_4}\right) + \mathcal{G}^2(\lambda_4)\right)^2 - 2p^2\right) \\
&= \quad 2p^3 - 7p^2 - 4p\alpha^2(p),
\end{aligned}$$

which implies Theorem 2 and Corollary 1.

4. Conclusions

Firstly, for any integer $k \geq 3$ in (1.1), it is difficult to obtain a corresponding result by using our methods, which is an open problem. Finally, our Theorem 1 only obtained a simple identity for the case where $\lambda_3(2) = 1$. What would happen if $\lambda_3(2) \neq 1$.

Author contributions

All authors have equally contributed to this work, and all authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. M. Apostol, *Introduction to Analytic Number Theory*, New York: Springer-Verlag, 1976.
2. N. Bag, R. Barman, Higher order moments of generalized quadratic Gauss sums weighted by L-functions, *Asian J. Math.*, **25** (2021), 413–430.
3. N. Bag, A. Rojas-León, W. P. Zhang, An explicit evaluation of 10-th power moment of quadratic Gauss sums and some applications, *Funct. Approx. Comment. Math.*, **66** (2022), 253–274. http://dx.doi.org/10.7169/facm/1995
4. N. Bag, A. Rojas-León, W. P. Zhang, On some conjectures on generalized quadratic Gauss sums and related problems, *Finite Fields Appl.*, **86** (2023), 102131. https://doi.org/10.1016/j.ffa.2022.102131
5. B. C. Berndt, R. J. Evans, The determination of Gauss sums, *Bull. Amer. Math. Soc.*, **5** (1981), 107–128.
6. L. Chen, X. Wang, A new fourth power mean of two-term exponential sums, *Open Math.*, **17** (2019), 407–414. https://doi.org/10.1515/math-2019-0034
7. Z. Y. Chen, W. P. Zhang, On the fourth-order linear recurrence formula related to classical Gauss sums, *Open Math.*, **15** (2017), 1251–1255. https://doi.org/10.1515/math-2017-0104
8. K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, New York: Springer-Verlag, 1982.
9. X. X. Li, X. F. Xu, The fourth power mean of the generalized two-term exponential sums and its upper and lower bound estimates, *J. Inequal. Appl.*, **2013** (2013), 504. https://doi.org/10.1186/1029-242X-2013-504
10. X. Y. Liu, W. P. Zhang, On the high-power mean of the generalized Gauss sums and Kloosterman sums, *Mathematics*, **7** (2019), 907. https://doi.org/10.3390/math7100907
11. X. X. Lv, W. P. Zhang, The generalized quadratic Gauss sums and its sixth power mean, *AIMS Math.*, **6** (2021), 11275–11285. http://dx.doi.org/10.3934/math.2021654

12. H. Zhang, W. P. Zhang, The fourth power mean of two-term exponential sums and its application, *Math. Rep.*, **19** (2017), 75–81.

13. W. P. Zhang, Moments of generalized quadratic Gauss sums weighted by L-functions, *J. Number Theory*, **92** (2002), 304–314. https://doi.org/10.1006/jnth.2001.2715

14. W. P. Zhang, J. Y. Hu, The number of solutions of the diagonal cubic congruence equation mod p, *Math. Rep.*, **20** (2018), 70–76.

15. W. P. Zhang, H. L. Li, *Elementary Number Theory*, Xi'an: Shaanxi Normal University Press, 2013.

16. W. P. Zhang, Y. Y. Meng, On the sixth power mean of the two-term exponential sums, *Acta Math. Sin. English Ser.*, **38** (2022), 510–518. https://doi.org/10.1007/s10114-022-0541-8