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*Research article*

# Existence and multiplicity of triple weak solutions for a nonlinear elliptic problem with fourth-order operator and Hardy potential

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**Abstract:** This study investigates the existence of triple weak solutions for a system of nonlinear elliptic equations with a fourth-order operator. The problem arises in the mathematical modeling of complex physical phenomena.

**Keywords:** critical points theorem; fourth order Leray-Lions operator; Hardy potential; variable exponent
**Mathematics Subject Classification:** 35J35, 35J60, 35G30

## 1. Introduction

Nonlinear partial differential equations serve as valuable mathematical tools for simulating complex physical phenomena across various scientific disciplines. The study of these equations is of great significance as it deepens our understanding of the underlying mathematical theory and provides insights into the behavior of real-world systems. This paper focuses on a specific problem: investigating the existence of triple weak solutions for a system of nonlinear elliptic equations featuring a fourth-order operator. The problem at hand revolves around mathematical modeling, where intricate physical processes with highly nonlinear dynamics are described by a system of equations. Examples of such phenomena can be found in fluid dynamics [1, 2], elasticity theory [3], image processing [4], and other fields where complex behavior arises from the interaction of multiple elements. Additionally, the inclusion of a Hardy potential in the Leray-Lions operator introduces further complexity and enriches its properties. The presence of singularities near the origin, induced by the Hardy potential, amplifies the intricacy of the operator's characteristics, making the behavior of solutions to be highly sensitive.

The main objective of this research is to establish the existence of solutions to this system of equations, which holds significant implications for both theoretical analysis and practical applications. Using a local minimum theorem and its variants Bonanno, Candito and D'Aguì [5] respectively and

Bonanno and Marano [6], we prove that the following coupled system admits one non-zero weak solutions and three weak distinct solutions respectively

$$\begin{cases} \sum_{i=1}^{n} (\Delta\,(a_i(x,\Delta u)) + \theta_i(x)\dfrac{|u|^{s_i-2}u}{|x|^{2s_i}}) = \lambda \sum_{i=1}^{n} f_i(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, for $i = 1,\cdots,n$, functions denoted by $f_i$ satisfy the condition that

$$(f) \qquad f_i(x,u) \le \xi_i(x) + c_i|u|^{q_i(x)-1}, \tag{1.2}$$

is a Carathéodory function such that $\xi_i \in L^1(\Omega)$ and $c_i$ is a positive constant. $\theta_i$ is a real function in $L^\infty(\Omega)$ with $ess\inf_{x\in\overline{\Omega}}\theta_i(x) > 0$. Here, $\Omega$ represents a bounded domain in $\mathbb{R}^N$ $(N \ge 2)$ with a $C^1$ boundary $\partial\Omega$. The constant $s_i$ is fixed, and $\lambda > 0$ is a parameter. The functions $q_i$ and $p_i$ belong to $C(\overline{\Omega})$ and satisfy the following inequalities

$$1 < s_i < \min_{x\in\overline{\Omega}} q_i(x) \le \max_{x\in\overline{\Omega}} q_i(x) < \frac{N}{2} < \tilde{p}^-.$$

In the given context, we have the following:

Let $\tilde{p}^- = \inf_{x\in\Omega} \tilde{p}(x)$ and $\tilde{p}(x) = \max_{1\le i\le n} p_i(x)$. The term $\Delta\,(a_i(x,\Delta u))$ represents the fourth-order Leray-Lions operator, which operates on the function $u$ and involves the second-order spatial derivative of $u$. The function $a_i$ is a Carathéodory function that satisfies additional requirements that are appropriate for the given context. Recently, Liu and Zhao [7] established the existence and multiplicity result for the following problem:

$$\begin{cases} \Delta(a(x,\Delta u)) + \frac{b(x)|u|^{h-2}u}{|x|^{2h}} = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded subset in $\mathbb{R}^N(N \ge 2)$ with the smooth boundary $\partial\Omega, \lambda > 0$ is a parameter, $0 < b(x) \in L^\infty(\Omega), 1 < h < \min\left\{p(x), \frac{N}{2}\right\}$, and $a : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying some required conditions. Two theorems about the existence of at least one and at least two nontrivial generalized solutions to their problem. In fact the authors established the existence of two solutions for a continuous spectrum; however, they used a condition of type (AR). We mention that our work is a generalization of the above problem because we consider a sum of finite Leray-Lions type operators with Hardy potentials; moreover, condition of type Amrosetti-Rabinowitz condition is not needed to establish that our main problem admits three weak solutions.

This paper appears to be one of the first to focus on investigating a coupled system involving a Leray-Lions operator with non-standard growth, a Hardy potential, and a coupled nonlinear source term. These additional features introduce further complexities and challenges in the analysis of this system. The paper is structured as follows. Section 2 introduces the Sobolev spaces with variable exponents and provides necessary background information. The proofs of the results are presented in Sections 3 and 4.

## 2. Background set up

Throughout this paper, the set is defined as follows:

$$C_+(\overline{\Omega}) := \{\beta \mid \beta \in C(\overline{\Omega}), \beta(x) > 1, \text{ for all } x \in \overline{\Omega}\}.$$

Additionally, we introduce the notations:

$$\beta^- := \inf_{x\in\overline{\Omega}} \beta(x) \quad \text{and} \quad \beta^+ := \sup_{x\in\overline{\Omega}} \beta(x).$$

In this study, we focus on a bounded regular domain $\Omega \subset \mathbb{R}^N$, where $N \geq 2$, and with a $C^1$ boundary. We make the assumption that the functions $p_i$ and $q_i$ belong to the set $C_+(\overline{\Omega})$ and satisfy the following conditions:

$$1 < s_i < \min_{x\in\overline{\Omega}} q_i(x) \leq \max_{x\in\overline{\Omega}} q_i(x) < \frac{N}{2} < \tilde{p}^-.$$

Moreover, we denote the Lebesgue space with variable exponents, as introduced in [8] by

$$L^{p_i(x)}(\Omega) = \left\{\Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p_i(x)} dx < \infty\right\}.$$

The Luxemburg norm of a function $u$ is given by:

$$|u|_{p_i(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p_i(x)} dx \leq 1\right\}.$$

For any function $u$ in the space $L^{p_i(x)}(\Omega)$ and $v$ in the conjugate space $L^{p_i'(x)}(\Omega)$ (where $L^{p_i'(x)}(\Omega)$ is the conjugate space of $L^{p_i(x)}(\Omega)$), there exists a Hölder-type inequality (see e.g., [9–12]), i.e.,

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_i^-} + \frac{1}{p_i'^-}\right)|u|_{p_i(x)}|v|_{p_i'(x)}. \tag{2.1}$$

Using the notation adapted in [13], for $\beta \in C_+(\overline{\Omega})$, put

$$[\zeta]^\beta := \max\{\zeta^{\beta^-}, \zeta^{\beta^+}\}, \quad [\zeta]_\beta := \min\{\zeta^{\beta^-}, \zeta^{\beta^+}\}.$$

A simple calculation shows that

(i) $[\zeta]^{\frac{1}{\beta}} = \max\left\{\zeta^{\frac{1}{\beta^-}}, \zeta^{\frac{1}{\beta^+}}\right\}$,

(ii) $[\zeta]_{\frac{1}{\beta}} = \min\left\{\zeta^{\frac{1}{\beta^-}}, \zeta^{\frac{1}{\beta^+}}\right\}$,

(iii) $[\zeta]_{\frac{1}{\beta}} = a \Longleftrightarrow \zeta = [a]^\beta, [\zeta]^{\frac{1}{\beta}} = a \Longleftrightarrow \zeta = [a]_\beta$,

(iv) $[\zeta]_\beta[\alpha]_\beta \leq [\zeta\alpha]_\beta \leq [\zeta\alpha]^\beta \leq [\zeta]^\beta[\alpha]^\beta$.

Now, let us recall the following proposition:

**Proposition 2.1.** *( [14]) For every $u$ in the function space $L^{p_i(x)}(\Omega)$, the following inequalities hold:*

$$[|u|_{p_i(x)}]_{p_i} \leq \int_\Omega |u(x)|^{p_i(x)} dx \leq [|u|_{p_i(x)}]^{p_i}.$$

Furthermore, we have the following proposition:

**Proposition 2.2.** *( [15]) If p and q are two functions in $C_+(\overline{\Omega})$ such that $q(x) \leq p(x)$ almost everywhere in $\Omega$, then the embedding $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ holds, and we have*

$$|u|_{q(x)} \leq c_q |u|_{p(x)},$$

*where $c_q$ is a positive constant.*

The Sobolev space with a variable exponent $W^{l,p(x)}(\Omega)$, where $l \in \{1, 2\}$ and $p \in \{p_i, i = 1, \ldots, n\}$, is defined as

$$W^{l,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq l\right\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index such that $|\alpha| = \sum_{i=1}^N \alpha_i$; also, $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}$.

The norm on the space $W^{l,p(x)}(\Omega)$ is given by

$$\|u\|_{l,p(x)} = \Sigma_{|\alpha| \leq l} |D^\alpha u|_{p(x)},$$

is a reflexive separable Banach space. Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, which has the norm $\|u\|_{1,p(x)} = |Du|_{p(x)}$. In the following, let

$$\tilde{X} := W_0^{1,\tilde{p}(x)}(\Omega) \cap W^{2,\tilde{p}(x)}(\Omega),$$

endowed with the norm

$$\|u\| = |\Delta u|_{\tilde{p}(x)} = \inf \left\{\mu > 0 : \int_\Omega \left|\frac{\Delta u}{\mu}\right|^{\tilde{p}(x)} dx \leq 1\right\}.$$

The modular on $\tilde{X}$ is the mapping $\rho_{\tilde{p}(x)} : \tilde{X} \to \mathbb{R}$ defined by $\rho_{\tilde{p}(x)}(u) = \int_\Omega |\Delta u|^{\tilde{p}(x)} dx$. This mapping meets the same characteristics as those defined in Proposition 2.3. To be more specific, we have the following:

**Proposition 2.3.** *For every $u \in L^{\tilde{p}(x)}(\Omega)$, one has*

*(1)* $|\Delta u|_{\tilde{p}(x)} < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_{\tilde{p}(x)}(u) < 1$ $(resp. = 1, > 1)$;
*(2)* $\left[|\Delta u|_{\tilde{p}(x)}\right]_{\tilde{p}} \leq \rho_{\tilde{p}(x)}(u) \leq \left[|\Delta u|_{\tilde{p}(x)}\right]^{\tilde{p}}$.

**Proposition 2.4** ( [16]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$, and let $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $w \in L^{q(x)}(\Omega)$, $w \neq 0$. Then*

$$[|w|_{p(x)q(x)}]_p \leq \||w|^{p(x)}|_{q(x)} \leq [|w|_{p(x)q(x)}]^p.$$

The space $\tilde{X}$ thus defined is a reflexive and separable Banach space. Remember that, the critical Sobolev exponent is defined as follows:

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - 2p(x)}, & p(x) < \dfrac{N}{2}, \\ +\infty, & p(x) \geq \dfrac{N}{2}. \end{cases}$$

As a consequence of Proposition 2.2, if $q(x) \leq p(x)$ almost everywhere on $\Omega$, we have the following embeddings

$$W_0^{1,\tilde{p}(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega) \quad \text{and} \quad W^{2,\tilde{p}(x)}(\Omega) \hookrightarrow W^{2,q(x)}(\Omega).$$

In particular, this implies

$$\tilde{X} \hookrightarrow W_0^{1,\tilde{p}^-}(\Omega) \cap W^{2,\tilde{p}^-}(\Omega).$$

where $\tilde{p}^- > \frac{N}{2}$. Since $\tilde{X} \hookrightarrow C^0(\overline{\Omega})$ is compact (see [11]), we obtain the inequality $|u|\infty \leq c_0|\Delta u|\tilde{p}(x)$, where $c_0$ is a positive constant.

Furthermore, for $1 \leq i \leq n$, the continuous embedding $\tilde{X} \hookrightarrow L^{\alpha_i(x)}(\Omega)$ holds for any $\alpha_i \in C_+(\overline{\Omega})$ such that $\alpha_i(x) \leq \tilde{p}(x)$ almost everywhere on $\Omega$. This leads to the inequality

$$|u|_{\alpha_i(x)} \leq c_{\alpha_i}|\Delta u|\tilde{p}(x), \tag{2.2}$$

where $c_{\alpha_i}$ is a positive constant.

The definition and statements required for the proofs presented in Section 3 are as follows:

**Definition 2.1.** *Consider two continuously Gâteaux differentiable functionals, $\Phi$ and $\Psi$, defined on a real Banach space $X$, and let $d \in \mathbb{R}$. The functional $I := \Phi - \Psi$ satisfies the Palais-Smale condition with an upper bound of $d$ if any sequence $\{u_k\}_{k\in\mathbb{N}} \in X$ that verifies the following conditions has a convergent subsequence:*

- *$I(u_k)$ is bounded,*
- $\lim\limits_{k\to+\infty} \|I'(u_k)\|_{X^*} = 0$,
- *$\Phi(u_k) < d$ for each $k \in \mathbb{N}$,*

*has a convergent subsequence. If $d = \infty$, we say that $I := \Phi - \Psi$ fulfill the Palais-Smale condition.*

In what follows, we recall the following local minimum theorem which plays a crucial role to prove our main result.

**Theorem 2.1.** *(Theorem 3.1 [5]) Let $X$ be a real Banach space, and let $\Phi$ and $\Psi$ be two continuously Gâteaux differentiable functionals defined on $X$. Suppose that the following conditions hold*

$$\inf_{x\in X} \Phi = \Phi(0) = \Psi(0) = 0.$$

*There exists a positive constant $d \in \mathbb{R}$ and $\overline{x} \in X$ with $0 < \Phi(\overline{x}) < d$ such that*

$$\frac{\sup\limits_{x\in\Phi^{-1}(]-\infty,d])} \Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$$

*and for any*

$$\lambda \in \Lambda := ]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup\limits_{x\in\Phi^{-1}(]-\infty,d])} \Psi(x)}[,$$

*$I_\lambda = \Phi - \lambda\Psi$ fulfill the $(PS)^{[d]}$-condition, so for any $\lambda \in \Lambda$, there is $x_\lambda \in \Phi^{-1}(]0, d])$ such that $I_\lambda(x_\lambda) \leq I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, d])$ and $I'_\lambda(x_\lambda) = 0$.*

The multiplicity result is attributed to the following theorem:

**Theorem 2.2.** *[6] Consider a reflexive real Banach space X, let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semi-continuous functional. Assume that the Gâteaux derivative of $\Phi$ has a continuous inverse on $X^*$. Furthermore, let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; assume that*

$$(a_0)\ \inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

*Suppose that there exist $d > 0$ and $\overline{x} \in X$, with $d < \Phi(\overline{x})$, such that*
*$(a_1)$ $\frac{\sup_{\Phi(x)<d}\Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,*
*$(a_2)$ for any $\lambda \in \Lambda_d := ]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup_{\Phi(x)\leq d}\Psi(x)}[$, $I_\lambda := \Phi - \lambda\Psi$ is coercive.*
*Then, for any $\lambda \in \Lambda_d$, $\Phi - \lambda\Psi$ has at least three distinct critical points in X.*

In what follows, let $D(x) := \sup\{D > 0 \mid B(x, D) \subseteq \Omega\}$, for all $x \in \Omega$, where $B$ is the ball centered at $x$ and of radius $D$. By the properties of the supremum, we can see easily that there exists $x^0 \in \Omega$ such that $B(x^0, R) \subseteq \Omega$, where $R = \sup_{x\in\Omega} D(x)$.

In the following we consider the following assumptions:

For $i = 1, \cdots, n$, let $A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function with the continuous derivative $a_i(x, \xi) = \partial_\xi A_i(x, \xi)$, satisfying

$(A0)$ $a_i(x, u + v) \leq c_i(a_i(x, u) + a_i(x, v))$, $\forall u, v \in W_0^{1,p_i(x)}(\Omega) \cap W^{2,p_i(x)}(\Omega)$, for some positive constant $c_i$, and $A_i$ satisfy the following conditions:

$(A1)$ $A_i(x, 0) = 0, A_i(x, \xi) = A_i(x, -\xi)$ for all $x \in \Omega, \xi \in \mathbb{R}$.

$(A2)$ $|a_i(x, \xi)| \leqslant c_{1_i}\left(\gamma_i(x) + |\xi|^{p_i(x)-1}\right)$ a.e. $(x, \xi) \in \Omega \times \mathbb{R}$, where $c_{1_i}$ is a positive constant, $\gamma_i(x) \in L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)$, $p_i \in C_+(\overline{\Omega})$.

$(A3)$ $|\xi|^{p_i(x)} \leq a_i(x, \xi) \cdot \xi \leq p_i(x)A_i(x, \xi)$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}$.

A typical example of $A_i$ and $a_i$ that can be chosen is as follows:

$$A_i(x, \xi) = \frac{1}{p_i(x)}|\xi|^{p_i(x)} \text{ and } a_i(x, \xi) = |\xi|^{p_i(x)-2}\xi.$$

**Remark 2.1.** *According to condition $(A2)$ and for $i = 1, \ldots, n$, the following inequality holds*

$$|A_i(x, t)| \leq C_i\left(\gamma_i(x)|t| + |t|^{p_i(x)}\right),$$

*for almost every $x \in \Omega$ and all $t \in \mathbb{R}$, where $C_i > 0$ is a constant.*

In this work, we will use the symbol $m$ to represent the value $\dfrac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$, where $\Gamma$ denotes the gamma function.

## 3. Existence

This section is dedicated to presenting some necessary results that are required to establish the existence and multiplicity of solutions. We begin by recalling the Hardy-Rellich inequality, which is stated in the following lemma [17].

**Lemma 3.1** ( [18]). *For $i = 1, \cdots, n$, $1 < s_i < N/2$ and $u \in W_0^{1,s_i}(\Omega) \cap W^{2,s_i}(\Omega)$, we have*

$$\int_\Omega \frac{|u(x)|^{s_i}}{|x|^{2s_i}} dx \leq \frac{k}{\mathcal{H}_{s_i}} \int_\Omega |\Delta u(x)|^{s_i} dx,$$

*where $\mathcal{H}_{s_i} := \left(\frac{N(s_i-1)(N-2s_i)}{s_i^2}\right)^{s_i}$.*

Now, let's review the definition of a weak solution to problem (1.1).

**Definition 3.1.** *We say that $u \in \tilde{X}\backslash\{0\}$ is a weak solution of problem* (1.1) *if $u = 0$ on $\partial\Omega$; then, the following integral equality is true:*

$$\sum_{i=1}^n \int_\Omega a_i(x, \Delta u) \Delta v dx + \sum_{i=1}^n \int_\Omega \theta_i(x) \frac{|u|^{s_i-2} uv}{|x|^{2s_i}} dx \\ - \lambda \sum_{i=1}^n \int_\Omega f_i(x, u) v dx = 0,$$

*for every $v \in \tilde{X}$.*

Let us define the functional $\Psi(u)$ as follows:

$$\Psi(u) := \sum_{i=1}^n \int_\Omega F_i(x, u) dx,$$

where $u \in \tilde{X}$. The Euler-Lagrange functional for problem (1.1) is then given under the condition that $I_\lambda : X \to \mathbb{R}$:

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \text{ for all } u \in \tilde{X},$$

where

$$\Phi(u) = \sum_{i=1}^n \int_\Omega A_i(x, \Delta u) dx + \sum_{i=1}^n \frac{1}{s_i} \int_\Omega \theta_i(x) \frac{|u(x)|^{s_i}}{|x|^{2s_i}} dx.$$

It is evident that condition $(a_0)$ of Theorem 2.2 holds. Furthermore, Remark 2.1 ensures that $\Phi$ is well-defined. Additionally, by employing (1.2), we have the following for all $u \in \tilde{X}$:

$$|F_i(x, u)| \leq \xi_i(x)|u| + \frac{c_i}{q_i(x)} |u|^{q_i(x)},$$

Therefore, we can write the following:

$$\Psi(u) \leq \sum_{i=1}^n \Big( |\xi_i(x)|_{L^1(\Omega)} |u|_\infty + \frac{c_i}{q_i^-} \int_\Omega (|u|^{q_i^+} + |u|^{q_i^-}) dx \Big).$$

Using Remark 2.2, we obtain

$$\Psi(u) \leq \sum_{i=1}^n \Big( |\xi_i(x)|_{L^1(\Omega)} |u|_\infty + \frac{c_i}{q_i^-} (c_{q_i^+}^{q_i^+} |\Delta u|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-} |\Delta u|_{\tilde{p}(x)}^{q_i^-}) |\Omega| \Big).$$

Hence, we conclude that $\Psi$ is well-defined. Furthermore, one has

$$\langle \Psi'(u), v\rangle := \Psi'(u)[v] = \sum_{i=1}^{n} \int_{\Omega} f_i(x,u)v dx,$$

for all $u, v \in \tilde{X}$, and it is compact. In fact, condition $(f)$ and the compact embedding $\tilde{X} \hookrightarrow L^{q_i(x)}(\Omega), 1 < q_i(x) < p^*(x)$ implies the compactness of $\Psi'(u)$. In fact, let $(u_k)_k \subset \tilde{X}$ be a sequence such that $u_k \rightharpoonup u$. Noting that the embedding $\tilde{X} \hookrightarrow L^{q_i(x)}(\Omega), 1 < q_i(x) < p^*(x)$ is compact, there is a subsequence, still denoted by $(u_k)_k$, such that $u_k \to u,$ strongly in $L^{q_i(x)}(\Omega)$. We claim that the Nemytskii operator $N_{f_i}(u)(x) = f_i(x, u(x))$ is continuous since $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $(f)$; thus, $N_{f_i}(u_k) \to N_{f_i}(u)$ in $L^{\frac{q_i(x)}{q_i(x)-1}}(\Omega)$. In view of the Hölder's inequality mentioned in (2.1) and the compact embedding $\tilde{X} \hookrightarrow L^{q_i(x)}(\Omega), 1 < q_i(x) < p^*(x)$, for all $v \in \tilde{X}$, one has

$$\begin{aligned}
|\Psi'(u_k)(v) - \Psi'(u)(v)| &= |\sum_{i=1}^{n}\Big(\int_{\Omega} f_i(x,u_k)v dx - \int_{\Omega} f_i(x,u)v dx\Big)| \\
&\le \sum_{i=1}^{n}\int_{\Omega}\Big(|(f_i(x,u_k) - f_i(x,u))v|\Big)dx \\
&\le \sum_{i=1}^{n}\Big(2|N_{f_i}(u_k) - N_{f_i}(u)|_{\frac{q_i(x)}{q_i(x)-1}} |v|_{q_i(x)}\Big) \\
&\le \sum_{i=1}^{n}\Big(2k_i|N_{f_i}(u_k) - N_{f_i}(u)|_{\frac{q_i(x)}{q_i(x)-1}} |\Delta v|_{\tilde{p}(x)}\Big),
\end{aligned}$$

where $c_{q_i}$ is the embedding constant of the embedding $\tilde{X} \hookrightarrow L^{q_i(x)}(\Omega), 1 < q_i(x) < p^*(x)$. Thus $\Psi'(u_k) \to \Psi'(u)$ in $\tilde{X}^*$, i.e., $\Psi'$ is completely continuous, thus $\Psi'$ is compact.

Moreover, by using Proposition 2.3 and the hypothesis $(A3)$ for $u \in \tilde{X}$ with $\|u\| \ge 1$, one has

$$\Phi(u) \ge \int_{\Omega} \frac{1}{\tilde{p}(x)}|\Delta u|^{\tilde{p}(x)}dx \ge \frac{1}{\tilde{p}^+}\rho_{\tilde{p}(x)}(u) \ge \frac{1}{\tilde{p}^+}|\Delta u|^{\tilde{p}^-}_{\tilde{p}(x)} = \|u\|^{\tilde{p}^-}, \tag{3.1}$$

since $\tilde{p}^- > 1$, we deduce that $\Phi$ is coercive. On the other hand $\Phi$ is sequentially weakly lower semicontinuous as sum of sequentially weakly lower semicontinuous functionals and of class $C^1$ on $\tilde{X}$ for the same reason, for more details one can see [7] and note that

$$\Phi'(u))[v] = \sum_{i=1}^{n}\int_{\Omega}\big(a_i(x,\Delta u)\cdot \Delta v + \sum_{i=1}^{n}\theta_i(x)\frac{|u(x)|^{s_i-2}uv}{|x|^{2s_i}}\big)dx,$$

for any $v \in \tilde{X}$. Moreover, we have the following proposition:

**Proposition 3.1.** $\Phi' : \tilde{X} \to \tilde{X}^*$ *is uniformly monotonic and admits a continuous inverse in* $\tilde{X}^*$ .

*Proof.* By using the assumption on $\theta_i$, $i \in \{1, \cdots, n\}$, one has

$$\int_{\Omega}\frac{\theta_i(x)}{|x|^{2s_i}}(|u|^{s_i-2}u - |v|^{s_i-2}v)(u-v)dx \ge \frac{ess\inf_{x\in\overline{\Omega}}\theta_i(x)}{(diam(\Omega))^{2s_i}}\int_{\Omega}(|u|^{s_i-2}u - |v|^{s_i-2}v)(u-v)dx. \tag{3.2}$$

Now, let $U_{\beta_i} = \{x \in \Omega : \beta_i(x) \geq 2\}$ and $V_{\beta_i} = \{x \in \Omega : 1 < \beta_i(x) < 2\}$; by using the elementary inequality [19], for $i \in \{1, \cdots, n\}$ and $\beta_i > 1$, we get that there exists a positive constant $C_{\beta_i}$; such that if $\beta_i \geq 2$, then

$$\left\langle |x|^{\beta_i - 2} x - |y|^{\beta_i - 2} y, x - y \right\rangle \geq C_{\beta_i} |x - y|^{\beta}, \text{ for } \beta_i \geq 2 \tag{3.3}$$

and if $1 < \beta_i < 2$, then

$$\left\langle |x|^{\beta_i - 2} x - |y|^{\beta_i - 2} y, x - y \right\rangle \geq C_{\beta_i} \frac{|x - y|^2}{(|x| + |y|)^{2 - \beta_i}}, \text{ for } 1 < \beta_i < 2, \tag{3.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$. Due to $(A0)$ and by assumptions (A1) and (A3), we have

$$\begin{aligned}
\langle \Phi'(u) - \Phi'(v), u - v \rangle &= \sum_{i=1}^{n} \int_{\Omega} (a_i(x, \Delta u) - a(x, \Delta v)) \cdot (\Delta u - \Delta v) dx, \\
&+ \sum_{i=1}^{n} \int_{\Omega} \frac{\theta_i(x)}{|x|^{2s_i}} (|u|^{s_i - 2} u - |v|^{s_i - 2} v)(u - v) dx, \\
&= \sum_{i=1}^{n} \int_{\Omega} (a_i(x, \Delta u) + a(x, -\Delta v)) \cdot (\Delta u - \Delta v) dx, \\
&+ \sum_{i=1}^{n} \int_{\Omega} \frac{\theta_i(x)}{|x|^{2s_i}} (|u|^{s_i - 2} u - |v|^{s_i - 2} v)(u - v) dx, \\
&\geq \sum_{i=1}^{n} \frac{1}{c_i} \int_{\Omega} a(x, \Delta u - \Delta v) \cdot (\Delta u - \Delta v) dx, \\
&\geq \sum_{i=1}^{n} \frac{1}{c_i} \int_{\Omega} |\Delta u - \Delta v|^{p_i(x)} dx, \\
&\geq \min_{1 \leq i \leq n} \frac{1}{c_i} \sum_{i=1}^{n} \int_{\Omega} |\Delta(u - v)|^{p_i(x)} dx,
\end{aligned}$$

To end our proof let $\check{p} = \inf_{x \in \Omega} (\min_{1 \leq i \leq n} (p_i(x))$, since $|\Delta(u - v)|^{p_i(x)} \in L^1(\Omega)$, we shall distinguish two cases.
**First case:** Suppose that $|\Delta(u - v)| \geq 1$, for all $1 \leq i \leq n$, which yields,

$$\begin{aligned}
\langle \Phi'(u) - \Phi'(v), u - v \rangle &\geq \min_{1 \leq i \leq n} \frac{1}{c_i} \int_{\Omega} \sum_{i=1}^{n} |\Delta(u - v)|^{\check{p}} dx \\
&\geq \min_{1 \leq i \leq n} \frac{1}{c_i} \|u - v\|^{\check{p}}.
\end{aligned} \tag{3.5}$$

**Second case:** Suppose that there exists $1 \leq i_0 \leq n$, such that, $|\Delta(u_{i_0} - v_{i_0})| < 1$; thus,

$$\begin{aligned}
\langle \Phi'(u) - \Phi'(v), u - v \rangle &\geq \min_{1 \leq i \leq n} \frac{1}{c_i} \int_{\Omega} |\Delta(u_{i_0} - v_{i_0})|^{p_{i_0(x)}} dx \\
&\geq \min_{1 \leq i \leq n} \frac{1}{c_i} \int_{\Omega} |\Delta(u_{i_0} - v_{i_0})|^{p_{i_0}^-} dx
\end{aligned}$$

$$\geq \min_{1\leq i\leq n} \frac{1}{c_i} \|u_{i_0} - v_{i_0}\|^{p^-_{i_0}}. \tag{3.6}$$

Now, by adding (3.5) and (3.6), we can deduce that $\Phi'$ is uniformly monotonic Theorem 26(A)(d) of [20] ends the proof. □

**Proposition 3.2.** *$\Phi'$ satisfies the condition $(S)_+$, which means that, if $u_k \rightharpoonup u$ and $\overline{\lim}_{k\to+\infty}\langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle \leq 0$, then $u_k \to u$ (strongly).*

*Proof.* Since $\Phi'$ is uniformly monotone, so due to [20, Example 27.2(b)], $\Phi'$ satisfy the condition $(S)_+$. □

**Remark 3.1.** *Under assumptions $(A2)$ and $(A3)$, one has*

$$\frac{1}{\tilde{p}^+}[|\Delta u|_{\tilde{p}(x)}]_{\tilde{p}} \leq \Phi(u) \leq \hat{K} \sum_{i=1}^{n} \Big( |\Delta u|_{\tilde{p}(x)} + [|\Delta u|_{\tilde{p}(x)}]^{p_i} + |\Delta u|^{s_i}_{\tilde{p}(x)} \Big),$$

*where*

$$\hat{K} = \max_{1\leq i\leq n} \left\{ C_i c_{p_i}, C_i c_{p_i}^{p_i} \|\gamma_i\|_{\frac{p_i(x)}{p_i(x)-1}}, c_{s_i}^{s_i} k \frac{|\theta_i|_\infty}{s_i \mathcal{H}_{s_i}} \right\}.$$

*Proof.* By using assumptions $(A2)$, $(A3)$, Proposition 2.3, Lemma 3.1 and finally Proposition 2.2 we have

$$\begin{aligned}
\frac{1}{\tilde{p}^+}[|\Delta u|_{\tilde{p}(x)}]_{\tilde{p}} &\leq \frac{1}{\tilde{p}^+} \int_\Omega |\Delta u|^{\tilde{p}(x)}\, dx, \\
&\leq \Phi\,(u) \\
&= \sum_{i=1}^{n} \int_\Omega A_i\,(x, \Delta u)\, dx + \sum_{i=1}^{n} \int_\Omega \theta_i(x) \frac{|u|^{s_i}}{s_i |x|^{2s_i}} dx, \\
&\leq \sum_{i=1}^{n} \Big( C_i \int_\Omega \gamma_i(x) |\Delta u| dx + C_i \int_\Omega |\Delta u|^{p_i(x)} dx + \frac{1}{s_i} \int_\Omega \theta_i(x) \frac{|u(x)|^{s_i}}{|x|^{2s_i}} dx \Big), \\
&\leq \sum_{i=1}^{n} \Big( C_i |\gamma_i(x)|_{\frac{p_i(x)}{p_i(x)-1}} |\Delta u|_{p_i(x)} + C_i [|\Delta u|_{p_i(x)}]^{p_i} + \frac{k}{s_i \mathcal{H}_{s_i}} |\theta_i|_\infty |\Delta u|^{s_i}_{p_i(x)} \Big), \\
&\leq \sum_{i=1}^{n} \Big( C_i c_{p_i} |\gamma_i(x)|_{\frac{p_i(x)}{p_i(x)-1}} |\Delta u|_{\tilde{p}(x)} + C_i c_{p_i}^{p_i} [|\Delta u|_{\tilde{p}(x)}]^{p_i} + \frac{c_{s_i}^{s_i} k}{s_i \mathcal{H}_{s_i}} |\theta_i|_\infty |\Delta u|^{s_i}_{\tilde{p}(x)} \Big), \\
&\leq \hat{K} \sum_{i=1}^{n} \Big( |\Delta u|_{\tilde{p}(x)} + [|\Delta u|_{\tilde{p}(x)}]^{p_i} + |\Delta u|^{s_i}_{\tilde{p}(x)} \Big).
\end{aligned}$$

where

$$\hat{K} = \max_{1\leq i\leq n} \left\{ C_i c_{p_i}, C_i c_{p_i}^{p_i} \|\gamma_i\|_{\frac{p_i(x)}{p_i(x)-1}}, c_{s_i}^{s_i} k \frac{|\theta_i|_\infty}{s_i \mathcal{H}_{s_i}} \right\},$$

this ends the proof. □

**Remark 3.2.** *For $u \in \tilde{X}\backslash\{0\}$. If $I'_\lambda(u) = 0$, we have*

$$\sum_{i=1}^{n} \int_\Omega a_i\left(x, \Delta u\right) \Delta v dx + \sum_{i=1}^{n} \int_\Omega \theta_i(x) \frac{|u|^{s_i-2} uv}{|x|^{2s_i}} dx - \lambda \sum_{i=1}^{n} \int_\Omega f_i(x,u) v dx = 0,$$

*for any $v \in \tilde{X}\backslash\{0\}$, which assures that the critical points of $I_\lambda$ are exactly weak solutions of problem* (1.1).

**Lemma 3.2.** *$I_\lambda$ fulfill the Palais-Smale condition for any $\lambda > 0$.*

*Proof.* Let $\{u_k\} \subseteq \tilde{X}$ be a Palais-Smale sequence; so, one has

$$\sup_k I_\lambda\left(u_k\right) < +\infty \quad \text{and} \quad \lim_{k\to+\infty} \left\| I'_\lambda\left(u_k\right)\right\|_{\tilde{X}^*} \longrightarrow 0. \tag{3.7}$$

Let us show that $\{u_k\} \subseteq X$ contains a convergent subsequence. By the Hölder inequality, Proposition 2.4 and Remark 2.2, we have

$$\begin{aligned}
\langle \Psi'(u), u\rangle &= \sum_{i=1}^{n} \int_\Omega f_i(x,u) u dx \\
&\leq \sum_{i=1}^{n} \Big( |\xi_i(x)|_{L^1(\Omega)} |u|_\infty + \frac{c_i}{q_i^-} ( c_{q_i^+}^{q_i^+} |\Delta u|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-} |\Delta u|_{\tilde{p}(x)}^{q_i^-}) |\Omega| \Big) \\
&\leq \sum_{i=1}^{n} \Big( c_0 |\xi_i(x)|_{L^1(\Omega)} |\Delta u|_{\tilde{p}(x)} + \frac{c_i}{q_i^-} ( c_{q_i^+}^{q_i^+} |\Delta u|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-} |\Delta u|_{\tilde{p}(x)}^{q_i^-}) |\Omega| \Big).
\end{aligned}$$

So, for $k$ large enough, by assumption $(A3)$ and Proposition 2.3, one has

$$\begin{aligned}
\left\langle I'_\lambda\left(u_k\right), u_k\right\rangle &= \left\langle \Phi'_\lambda\left(u_k\right), u_k\right\rangle - \lambda \left\langle \Psi'_\lambda\left(u_k\right), u_k\right\rangle \\
&\geq \left[|\Delta u_k|_{\tilde{p}(x)}\right]_{\tilde{p}} - \lambda \sum_{i=1}^{n} \Big( c_0 |\xi_i(x)|_{L^1(\Omega)} |\Delta u_k|_{\tilde{p}(x)} + \frac{c_i}{q_i^-} ( c_{q_i^+}^{q_i^+} |\Delta u_k|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-} |\Delta u_k|_{\tilde{p}(x)}^{q_i^-}) |\Omega| \Big).
\end{aligned}$$

Moreover, by using (3.7), we have

$$\left[|\Delta u_k|_{\tilde{p}(x)}\right]_{\tilde{p}} \leq \sum_{i=1}^{n} \Big( c_0 |\xi_i(x)|_{L^1(\Omega)} |\Delta u_k|_{\tilde{p}(x)} + \frac{c_i}{q_i^-} ( c_{q_i^+}^{q_i^+} |\Delta u_k|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-} |\Delta u_k|_{\tilde{p}(x)}^{q_i^-}) |\Omega| \Big).$$

Let us assume that $\lim_{k\to+\infty} |\Delta u_k|_{\tilde{p}(x)} = +\infty$ and divide by $|\Delta u_k|_{\tilde{p}(x)}^{q_i^+}$ since $q_i^+ < \tilde{p}^-$, for all $1 \leq i \leq n$, we obtain an absurdity, then $\{u_k\}$ is bounded, since $\tilde{X}$ is a reflexive separable Banach space, then, passing to a subsequence if necessary, we can assume that $u_k \rightharpoonup u$. By Proposition 3.2, $u_k \to u$ (strongly) in $X$ and so $I_\lambda$ satisfies the Palais-Smale condition. □

Our existence result is as follows:

**Theorem 3.1.** *For $i = 1, \cdots, n$, let $a_i : \Omega \times \mathbb{R} \to R$ be a potential which satisfies the hypotheses $(A0)-(A3)$ and let $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfies condition $(f)$ and exists such that*

$$\operatorname*{ess\,inf}_{x\in\Omega} F_i(x,t) := \begin{cases} \operatorname{ess\,inf}_{x\in\Omega} \displaystyle\int_0^t f_{i_0}(x,s)ds > 0, & \text{for some}, i_0 \in \{1,\cdots,n\}, \\ \operatorname{ess\,inf}_{x\in\Omega} \displaystyle\int_0^t f_i(x,s)ds \geq 0, & \text{for } i \neq i_0, \end{cases} \tag{3.8}$$

*for all $t \in [0,h]$, where $h$ is a non-negative constant.*

*Suppose that there exist $d, \delta > 0$ such that*

$$\hat{K} \sum_{i=1}^{n} \left( \left[ \frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2} \right]^{p_i} + \left( \frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2} \right)^{s_i} + \left( \frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2} \right) \right) m \left( R^N - \left(\frac{R}{2}\right)^N \right) < d, \tag{3.9}$$

*then, for any $\lambda \in ]A_\delta, B_d[$, with*

$$A_\delta := \frac{\left(2^N - 1\right) \hat{K} \displaystyle\sum_{i=1}^{n} \left( \left[ \frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2} \right]^{p_i} + \left( \frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2} \right)^{s_i} + \left( \frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2} \right) \right)}{\displaystyle\sum_{i=1}^{n} \operatorname*{ess\,inf}_{x\in\Omega} F_i(x,h)}$$

*and*

$$B_d := \frac{d}{\displaystyle\sum_{i=1}^{n} \left( c_0 |\xi_i(x)|_{L^1(\Omega)} \left[\tilde{p}^+ d\right]^{\frac{1}{\tilde{p}}} + \frac{c_i}{q_i^-} ( c_{q_i^+}^{q_i^+} \left( \left[\tilde{p}^+ d\right]^{\frac{1}{\tilde{p}}} \right)^{q_i^+} + c_{q_i^-}^{q_i^-} \left( \left[\tilde{p}^+ d\right]^{\frac{1}{\tilde{p}}} \right)^{q_i^-} ) |\Omega| \right)},$$

*Problem (1.1) has at least one nontrivial weak solution.*

*Proof.* We try to prove our existence result by using Theorem 2.1. For this purpose, we have to show that all conditions of Theorem 2.1 are met. To begin and for a given $\lambda > 0$, we mention that, given from Lemma 3.2, the functional $I_\lambda$ satisfies the $(PS)^{[d]}$ condition. Let $d > 0, \delta > 0$ be as in (3.9) and let $w \in X$ defined by

$$w(x) := \begin{cases} 0, & x \in \Omega \backslash B\left(x^0, R\right), \\ \delta, & x \in B\left(x^0, \frac{R}{2}\right), \\ \frac{\delta}{R^2 - \left(\frac{R}{2}\right)^2} \left( R^2 - \displaystyle\sum_{k=1}^{N} \left(x_k - x_k^0\right)^2 \right), & x \in B\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right), \end{cases}$$

where $x = (x_1, \ldots, x_N) \in \Omega$. Then,

$$\sum_{k=1}^{N} \frac{\partial^2 w}{\partial x_k^2}(x) = \begin{cases} 0, & x \in \left(\Omega \backslash B\left(x^0, R\right)\right) \cup B\left(x^0, \frac{R}{2}\right) \\ -\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}, & x \in B\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right). \end{cases}$$

So, by applying Remark 3.1, one has

$$\begin{aligned}
&\frac{1}{\tilde{p}^+}\Big[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\Big]_{\tilde{p}} m\left(R^N-\left(\frac{R}{2}\right)^N\right)\\
&\quad< \Phi(w)\\
&\quad\le \hat{K}\sum_{i=1}^{n}\left(\Big[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\Big]^{p_i}+\left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)^{s_i}+\left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)\right) m\left(R^N-\left(\frac{R}{2}\right)^N\right).
\end{aligned}$$

So, $\Phi(w) < d$. On the other hand, one has

$$\Psi(w) \ge \sum_{i=1}^{n}\int_{B\left(x^0,\frac{R}{2}\right)} F_i(x,w)dx \ge m\Big(\frac{R}{2}\Big)^N \sum_{i=1}^{n}\operatorname*{ess\,inf}_{x\in\Omega} F_i(x,h), \tag{3.10}$$

then, we deduce that

$$\frac{\Psi(w)}{\Phi(w)} > \frac{\displaystyle\sum_{i=1}^{n}\operatorname*{ess\,inf}_{x\in\Omega} F_i(x,h)}{(2^N-1)\,\hat{K}\displaystyle\sum_{i=1}^{n}\left(\Big[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\Big]^{p_i}+\left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)^{s_i}+\left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)\right)}.$$

Using Remark 2.3, for any $u \in \Phi^{-1}((-\infty, d])$, we have

$$\frac{1}{\tilde{p}^+}\Big[|\Delta u|_{\tilde{p}(x)}\Big]_{\tilde{p}} \le \Phi(u) \le d.$$

Therefore

$$|\Delta u|_{\tilde{p}(x)} \le \Big[\tilde{p}^+\Phi(u)\Big]^{\frac{1}{\tilde{p}}} \le \Big[\tilde{p}^+ d\Big]^{\frac{1}{\tilde{p}}}.$$

Hence, from Proposition 2.4 and Remark 3.1, we deduce that

$$\begin{aligned}
\Psi(u) \;&\le\; \sum_{i=1}^{n}\Big(|\xi_i(x)|_{L^1(\Omega)}|u|_\infty + \frac{c_i}{q_i^-}\big(c_{q_i^+}^{q_i^+}|\Delta u|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-}|\Delta u|_{\tilde{p}(x)}^{q_i^-}\big)|\Omega|\Big),\\
&\le\; \sum_{i=1}^{n}\Big(c_0|\xi_i(x)|_{L^1(\Omega)}|\Delta u|_{\tilde{p}(x)} + \frac{c_i}{q_i^-}\big(c_{q_i^+}^{q_i^+}|\Delta u|_{\tilde{p}(x)}^{q_i^+} + c_{q_i^-}^{q_i^-}|\Delta u|_{\tilde{p}(x)}^{q_i^-}\big)|\Omega|\Big).
\end{aligned}$$

So

$$\sup_{\Phi(u)\le d}\Psi(u) \;\le\; \sum_{i=1}^{n}\Big(c_0|\xi_i(x)|_{L^1(\Omega)}\Big[\tilde{p}^+ d\Big]^{\frac{1}{\tilde{p}}} + \frac{c_i}{q_i^-}\big(c_{q_i^+}^{q_i^+}\Big(\Big[\tilde{p}^+ d\Big]^{\frac{1}{\tilde{p}}}\Big)^{q_i^+} + c_{q_i^-}^{q_i^-}\Big(\Big[\tilde{p}^+ d\Big]^{\frac{1}{\tilde{p}}}\Big)^{q_i^-}\big)|\Omega|\Big).$$

As a result, the criteria of Theorem 2.1 are confirmed. So, for any

$$\lambda \in ]A_\delta, B_d[\subseteq \Big]\frac{\Phi(w)}{\Psi(w)}, \frac{d}{\sup_{u\in\Phi^{-1}(]-\infty,d])}\Psi(u)}\Big[,$$

$I_\lambda$ admits at least one non-zero critical point, which is the problem's weak solution. □

## 4. Triple solutions

**Theorem 4.1.** *For any $\lambda \in ]A_\delta, B_d[$, $A_\delta$ and $B_d$ are those of Theorem 3.1, i.e.,*

$$A_\delta := \frac{\left(2^N-1\right)\hat{K}\sum_{i=1}^{n}\left(\left[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right]^{p_i}+\left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)^{s_i}+\left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)\right)}{\sum_{i=1}^{n}\operatorname*{ess\,inf}_{x\in\Omega}F_i(x,h)}$$

*and*

$$B_d := \frac{d}{\sum_{i=1}^{n}\left(c_0|\xi_i(x)|_{L^1(\Omega)}\left[\tilde{p}^+d\right]^{\frac{1}{\tilde{p}}}+\frac{c_i}{q_i^-}\left(c_{q_i^+}^{q_i^+}\left(\left[\tilde{p}^+d\right]^{\frac{1}{\tilde{p}}}\right)^{q_i^+}+c_{q_i^-}^{q_i^-}\left(\left[\tilde{p}^+d\right]^{\frac{1}{\tilde{p}}}\right)^{q_i^-}\right)|\Omega|\right)},$$

*Problem (1.1) admits at least three weak solutions.*

*Proof.* Note that $\Phi$ and $\Psi$ satisfy the regularity assumptions of Theorem 2.2; let us verify conditions $(i)$ and $(ii)$ of this theorem. For this purpose, let

$$\frac{1}{\tilde{p}^+}\left[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right]_{\tilde{p}}m\left(R^N-\left(\frac{R}{2}\right)^N\right)=d$$

and let $w \in X$ be as mentioned above, that is

$$w(x) := \begin{cases} 0 & x\in\Omega\backslash B\left(x^0,R\right), \\ \delta & x\in B\left(x^0,\frac{R}{2}\right), \\ \frac{\delta}{R^2-\left(\frac{R}{2}\right)^2}\left(R^2-\sum_{k=1}^{N}\left(x_k-x_k^0\right)^2\right) & x\in B\left(x^0,R\right)\backslash B\left(x^0,\frac{R}{2}\right). \end{cases}$$

So, by applying assumption $(A3)$ and Remark 3.1, one has

$$\Phi(w) \geq \sum_{i=1}^{n}\int_\Omega A_i(x,\Delta w)dx > \frac{1}{\tilde{p}^+}\left[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right]_{\tilde{p}}m\left(R^N-\left(\frac{R}{2}\right)^N\right)=d.$$

Therefore, the assumption $(i)$ of Theorem 2.2 holds. Let us show that $I_\lambda$ is coercive for any $\lambda > 0$. By using (3.11), one has

$$\Psi(u) \leq \sum_{i=1}^{n}\left(c_0|\xi_i(x)|_{L^1(\Omega)}|\Delta u|_{\tilde{p}(x)}+\frac{c_i}{q_i^-}\left(c_{q_i^+}^{q_i^+}|\Delta u|_{\tilde{p}(x)}^{q_i^+}+c_{q_i^-}^{q_i^-}|\Delta u|_{\tilde{p}(x)}^{q_i^-}\right)|\Omega|\right).$$

Then, from Remark 3.1, $\frac{1}{\tilde{p}^+}\left[|\Delta u|_{\tilde{p}(x)}\right]_{\tilde{p}} \leq \Phi(u)$. So,

$$I_\lambda(u) \geq \frac{1}{\tilde{p}^+}\left[|\Delta u|_{\tilde{p}(x)}\right]_{\tilde{p}}-\sum_{i=1}^{n}\left(c_0|\xi_i(x)|_{L^1(\Omega)}|\Delta u|_{\tilde{p}(x)}+\frac{c_i}{q_i^-}\left(c_{q_i^+}^{q_i^+}|\Delta u|_{\tilde{p}(x)}^{q_i^+}+c_{q_i^-}^{q_i^-}|\Delta u|_{\tilde{p}(x)}^{q_i^-}\right)|\Omega|\right),$$

where $\tilde{p}^+ = \sup_{x\in\Omega}\tilde{p}(x)$; by using $\tilde{p}^- > q_i^+ > 1$, for all $1 \leq i \leq n$, we deduce that $I_\lambda$ is coercive; consequently condition (ii) is satisfied, which assures that all assumptions of Theorem 4.1 are satisfied. So, for any $\lambda \in ]A_\delta, B_d[$, $I_\lambda$ has at least three distinct critical points which represents the weak solutions of problem (1.1). □

## 5. Conclusions

The main objective of this paper was to establish the existence of solutions to the coupled system of Eq (1.1), which has significant implications for both theoretical analysis and practical applications. Using a local minimum theorem and its variants, we are able to prove the existence of one non-zero weak solution and three distinct weak solutions. This is an important result, as it demonstrates the solvability of this system of equations under the stated assumptions. The findings of this work can contribute to the understanding of such systems and their potential applications in various fields.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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