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Research article

Covering properties of $C_p(Y|X)$

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Abstract: Let X be an infinite Tychonoff space, and Y be a topological subspace of X. In this paper, we study some covering properties of the subspace $C_p(Y|X)$ of $C_p(Y)$ consisting of those functions $f \in C(Y)$ which admit a continuous extension to X equipped with the relative topology of $C_p(Y)$. Among other results, we show that (i) $C_p(Y|X)$ has a fundamental bounded resolution if and only if Y is countable; when X is realcompact and Y is closed in X, we have (ii) if $C_p(Y|X)$ admits a resolution of convex compact sets that swallows the local null sequences in $C_p(Y|X)$, then Y is countable and discrete; (iii) if $C_p(Y|X)$ admits a compact resolution that swallows the compact sets, then Y is also countable and discrete, and, as a corollary, we deduce that $C_p(Y|X)$ admits a compact resolution that swallows the compact sets if and only if $C_p(Y|X)$ is a Polish space. We also prove that (iv) for a metrizable space X, $C_p(X)$ is a quasi-(LB)-space if and only if X is σ-compact, and hence for a subspace Y of X, the space $C_p(Y|X)$ is a quasi-(LB)-space. We include some examples and observations that answer natural questions raised in this paper.

Keywords: Lindelöf Σ-space; Polish space; P-space; bounded resolution; bornological space; Banach disk; quasi-(LB)-space
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1. Introduction

Unless otherwise stated, X will stand for an infinite Tychonoff space. We assume that all linear spaces are over the field $\mathbb{R}$ of real numbers and all locally convex spaces are Hausdorff. We denote by $C_p(X)$ the ring $C(X)$ of real-valued continuous functions on X endowed with the pointwise topology τ_p, and by $C_k(X)$ the space $C(X)$ equipped with the compact-open topology τ_k. The subspace of $C(X)$ of uniformly bounded functions is represented by $C^b(X)$. The topological dual of $C_p(X)$ is denoted by

$L(X)$, or by $L_p(X)$ when equipped with the weak* topology $\sigma(L(X), C(X))$. If we provide $L(X)$ with the strong topology $\beta(L(X), C(X))$, we shall write $L_\beta(X)$, and shall refer to $L_\beta(X)$ as the strong dual of $C_p(X)$. A family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of a set X is called a resolution for X if it covers X and verifies that $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$. If the sets A_α are compact, we speak of a compact resolution. If E is a locally convex space and $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a resolution for E consisting of bounded sets (bounded in the locally convex sense [5, 1.4.5 Definition]), we say that $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution. A bounded resolution for E that swallows the bounded sets of E will be referred to as a fundamental bounded resolution. Recall that a Banach disk B in a locally convex space E is an absolutely convex bounded set such that E_B, the linear span of B equipped with the Minkowski functional of B as a norm, is a Banach space. A locally convex space E is said to be a quasi-(LB)-space (in the sense of Valdivia) if it admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ consisting of Banach disks [33].

If Y is a topological subspace of X, the locally convex space $C_p(Y|X)$ is defined in [2, 0.4] as the range (the image) $R(T)$ of $C_p(X)$ in $C_p(Y)$ under the restriction map $T : C(X) \to C(Y)$, given by $f \mapsto f|_Y$, equipped with the relative topology of $C_p(Y)$. Note that T is a linear and continuous map. Clearly, $C(Y|X) = R(T)$ consists of those $f \in C(Y)$ that admit a continuous extension to the whole of X. Observe that $C_p(Y|X)$ is always a dense linear subspace of $C_p(Y)$, so the topological dual of $C_p(Y|X)$ coincides algebraically with $L(Y)$. However, since the weak* topology $\sigma(L(Y), C(Y))$ on $L(Y)$ is stronger than the weak* topology $\sigma(L(Y), C(Y|X))$, we shall denote by $L_p(Y|X)$ the space $L(Y)$ equipped with this second topology to distinguish it from $L_p(Y)$, which as we know represents the space $L(Y)$ with the topology $\sigma(L(Y), C(Y))$. In what follows we shall denote by $M(X)$ the bidual of $C_p(X)$, i.e., the dual of $L_\beta(X)$. Likewise, to be consistent with our notation, we shall represent by $M(Y|X)$ the bidual of $C_p(Y|X)$.

If Y is dense in X, then the restriction map $T : C_p(X) \to C_p(Y)$ is continuous and one-to-one, i.e., it is a so-called condensation map from $C_p(X)$ onto $C_p(Y|X)$. In this case, $C_p(Y|X)$ is linearly homeomorphic to the space $\left(C(X), \tau_p(Y)\right)$, where $\tau_p(Y)$ denotes the locally convex topology on $C(X)$ of the pointwise convergence on Y. Moreover, the adjoint map $T^* : L_p(Y|X) \to L_p(X)$ carries $L_p(Y|X)$ onto a dense linear subspace of $L_p(X)$. If Y is closed in X, then $C_p(Y|X)$ is an open subspace of $C_p(Y)$ [2, 0.4.1 Proposition]; consequently, in this case the restriction map T is in addition a quotient map. If Y is C-embedded in X (which happens in particular if Y is a compact set in X or if X is normal and Y closed), then T is onto and hence $C_p(Y|X) = C_p(Y)$.

We denote by $\delta(Y)$ the canonical copy of Y consisting of delta measures, i.e., $\delta(Y) = \left\{\delta_y : y \in Y\right\}$. The set $\delta(Y)$ looks different when regarded as a subset of $L_p(Y)$ or a subset of $L_p(Y|X)$, since in the former case each δ_y acts as a continuous linear functional on $C_p(Y)$, and in the latter each δ_y acts as a continuous linear functional on $C_p(Y|X)$. Of course, in the first case $\delta(Y)$ is the standard homeomorphic copy of Y in $L_p(Y)$. When a function $f \in C(Y|X)$ is regarded as a linear functional on $w \in L_p(Y|X)$, we shall write $\langle f, w\rangle$. So, if $w = \sum_{i=1}^n a_i \delta_{y_i} \in L(Y)$ with $a_i \in \mathbb{R}$ and $y_i \in Y$ for $1 \leq i \leq n$, then we have $\langle f, w\rangle := \sum_{i=1}^n a_i f(y_i)$.

Note that $C(Y|X) \subseteq C(Y) \subseteq \mathbb{R}^Y$, which implies that $M(Y|X) \subseteq M(Y) \subseteq \mathbb{R}^Y$. However, the following property holds.

Theorem 1.1. *[18, Corollary 7] The strong duals of $C_p(Y|X)$ and $C_p(Y)$ coincide, as do their biduals $M(Y|X)$ and $M(Y)$.*

In other words, if we denote by $L_\beta(Y|X)$ the dual space $L(Y)$ provided with the strong topology

$\beta(L(Y), C(Y|X))$ of the dual pair $\langle L(Y), C(Y|X)\rangle$, it turns out that $L_\beta(Y|X) = L_\beta(Y)$. Hence, $M(Y|X) = M(Y)$ and, consequently, the weak* bidual of $C_p(Y)$, i.e., the dual space $M(Y)$ equipped with the relative product topology of $\mathbb{R}^Y$ coincides with the weak* bidual of $C_p(Y|X)$. So, the complete picture is

$$C(Y|X) \subseteq C(Y) \subseteq M(Y|X) = M(Y) \subseteq \mathbb{R}^Y.$$

This result, along with the next three, will be used along the paper.

Theorem 1.2. *[13, Theorem 28] The weak* bidual of $C_p(X)$ admits a bounded resolution if and only if $|X| = \aleph_0$.*

Recall that the envelope $\mathcal{E} = \{A(\alpha \mid n) : \alpha \in \Sigma, n \in \mathbb{N}\}$ of a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of a locally convex space E with $\Sigma \subseteq \mathbb{N}^\mathbb{N}$, where

$$A(\alpha \mid n) = \bigcup \left\{A_\beta : \beta \in \Sigma, \beta(i) = \alpha(i), 1 \leq i \leq n\right\}$$

is called limited if for each $\alpha \in \Sigma$ and each absolutely convex neighborhood of the origin U in E there is $n \in \mathbb{N}$ with $A(\alpha \mid n) \subseteq nU$. A covering $\{A_\alpha : \alpha \in \Sigma\}$ with $\Sigma \subseteq \mathbb{N}^\mathbb{N}$ of a set X is called a Σ-covering of X.

Theorem 1.3. *[10, Lemma 2] Let E be a locally convex space. If E admits a Σ-covering $\{A_\alpha : \alpha \in \Sigma\}$ with $\Sigma \subseteq \mathbb{N}^\mathbb{N}$ of limited envelope, then there exists a Lindelöf Σ-space Z such that $E' \subseteq Z \subseteq \mathbb{R}^E$.*

In the following result, υX stands for the Hewitt realcompactification of X, as usual.

Theorem 1.4. *[10, Theorem 3] The space $C_p(X)$ admits a Σ-covering of limited envelope if and only if υX is a Lindelöf Σ-space.*

It is worth mentioning that each bounded resolution for E is always a Σ-covering of E of limited envelope [10, Proposition 12]. So, in particular, any compact resolution for E is a Σ-covering of E of limited envelope.

Although $C_p(Y|X)$ is a continuous linear image of $C_p(X)$ and a dense linear subspace of $C_p(Y)$, the topological properties of X and Y induced by certain covering properties of $C_p(Y|X)$ are not always the same as the topological properties of X and Y induced by analogous covering properties of $C_p(X)$ or $C_p(Y)$. For instance, if $C_p(X)$ is σ-countably compact, then X is finite [31]. But if $C_p(Y|X)$ is σ-countably compact and Y is dense in X, then X is pseudocompact and Y is a P-space [2, I.2.2 Theorem]. More generally, if $C_p(X)$ is σ-bounded relatively sequentially complete, then X is finite [17, Corollary 3.2]; but, if $C_p(Y|X)$ is σ-bounded relatively sequentially complete and Y is dense in X, then X is pseudocompact and Y is a P-space [17, Theorem 3.3].

Motivated by the research carried out in [17, 18] on countable coverings of $C_p(Y|X)$ and in [13, 14] on uncountable coverings of $C_p(X)$, we extend our study of the locally convex space $C_p(Y|X)$ for (*i*) fundamental bounded resolutions, (*ii*) compact resolutions swallowing the compact sets, (*iii*) resolutions consisting of Banach disks, and (*iv*) resolutions made up of convex compact sets swallowing the local null sequences. We refer the reader to [2, 24, 32] for topological and locally convex notions not defined here.

2. Two covering properties of $C_p(Y|X)$

First, we prove that if $C_p(Y|X)$ has a fundamental resolution consisting of bounded sets, Y must necessarily be countable, which extends [15, Theorem 3.3 (i)] as well as [16, Theorem 1.9], the latter because $C_p(X|\beta X) = C_p^b(X)$. For the definitions and properties of quasi-Suslin and web-compact spaces that we use below, see [32, I.4.2] and [24, Chapter 4], respectively.

Theorem 2.1. *Let $Y \subseteq X$. $C_p(Y|X)$ admits a fundamental bounded resolution if and only if Y is countable.*

Proof. Assume that $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental resolution for $C_p(Y|X)$ consisting of absolutely convex bounded sets. Since each A_α is absolutely convex, the bipolar theorem ensures that $A_\alpha^{00} = \overline{A_\alpha}$, closure in $\mathbb{R}^Y$. Consequently, the fact that $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows all bounded sets in $C_p(Y|X)$ means that

$$M(Y|X) = \{\overline{A_\alpha} : \alpha \in \mathbb{N}^\mathbb{N}\}.$$

Since $M(Y|X)$ coincides with $M(Y)$, according to Theorem 1.1, we see that $M(Y)$ admits a bounded resolution, namely $\{\overline{A_\alpha} : \alpha \in \mathbb{N}^\mathbb{N}\}$, where each $\overline{A_\alpha}$ is a compact set in $\mathbb{R}^Y$. In this case, Theorem 1.2 tells us that $|Y| = \aleph_0$.

Conversely, if Y is countable then $C_p(Y)$ is metrizable, so does $C_p(Y|X)$. But, as is well-known, every metrizable locally convex space E admits a fundamental bounded resolution, namely the family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with

$$A_\alpha = \bigcap_{i=1}^{\infty} \alpha(i)\, U_i,$$

where $\{U_i : i \in \mathbb{N}\}$ is a decreasing base of absolutely convex neighborhoods of the origin in E. So, $C_p(Y|X)$ has a fundamental bounded resolution. □

Corollary 2.1. *Let X be a P-space and let Y be a dense subspace of X. Then $C_p(Y|X)$ admits a bounded resolution if and only if Y is countable and discrete.*

Proof. If X is a P-space and Y is dense in X, according to [18, Theorem 18] the space $C_p(Y|X)$ is sequentially complete, and hence locally complete, i.e., such that every bounded set is contained in a Banach disk [26, 39.2]. So, if $C_p(Y|X)$ admits a bounded resolution, then $C_p(Y|X)$ is a quasi-(LB)-space. Hence, by [33, Proposition 22] or [24, Theorem 3.5], there is a resolution for $C_p(Y|X)$ consisting of Banach disks that swallows all Banach disks in $C_p(Y|X)$. Thus, the local completeness of $C_p(Y|X)$ guarantees that $C_p(Y|X)$ has a fundamental bounded resolution, which allows Theorem 2.1 to ensure that Y is countable. As every countable P-space is discrete, we are done. Conversely, if Y is countable, then $C_p(Y|X)$, as a linear subspace of $\mathbb{R}^Y$, is metrizable. Hence, $C_p(Y|X)$ has a bounded resolution. □

Example 2.1. *Let X be the one-point Lindelöfication of a discrete space Y with $|Y| \geq \aleph_1$. Since Y is uncountable, the previous corollary prevents $C_p(Y|X)$ to admit a bounded resolution.*

Example 2.2. *If Y is a dense P-space of X and $C_p(Y|X)$ admits a bounded resolution, even a countable one consisting of sequentially complete bounded sets, Y need not be countable or discrete. In fact, if Y is a nondiscrete P-space and $X := \beta Y$, we claim that $C_p(Y|X)$ admits a countable resolution consisting of sequentially complete bounded sets.*

Proof. Clearly, $C_p(Y|X) = C_p^b(Y)$, where $C_p^b(Y)$ is the dense linear subspace of $C_p(Y)$ consisting of all bounded functions with relative pointwise topology. If B stands for the closed unit ball of the Banach space $\left(C^b(Y), \|\cdot\|_\infty\right)$, then $C(Y|X) = C^b(Y) = \bigcup_{n=1}^{\infty} nB$. So, the family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $A_\alpha = \alpha(1) \cdot B$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ is a countable bounded resolution for $C_p(Y|X)$. Since Y is a P-space, $C_p(Y)$ is sequentially complete [6]. Hence, if $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_p(Y|X)$ contained in B, there exists $f \in C(Y)$ such that $f_n \to f$ in $C_p(Y)$. But, as $|f_n(y)| \leq 1$ for all $(n, y) \in \mathbb{N} \times Y$, we have that $|f(y)| \leq 1$ for all $y \in Y$. Thus, $f \in B$, which shows that B is sequentially complete in $C_p(Y|X)$. □

Theorem 2.2. *Let Y be a topological subspace of X. If $C_p(Y|X)$ admits a compact resolution that swallows the compact sets, then the compact sets in Y are finite.*

Proof. First note that the topologies $\sigma(L(Y), C(Y))$ and $\sigma(L(Y), C(Y|X))$ coincide on each compact subset K of Y, or rather on $\delta(K)$, regarded as a subset of $L(Y)$. In other words, that each compact set K in Y is embedded both in $L_p(Y)$ and in $L_p(Y|X)$.

In fact, if a net $\{y_d : d \in D\}$ in K and a point $y \in K$ verify that $\left\langle f, \delta_{y_d}\right\rangle \to \left\langle f, \delta_y\right\rangle$ for every $f \in C(Y)$, it is obvious that $\left\langle f, \delta_{y_d}\right\rangle \to \left\langle f, \delta_y\right\rangle$ for every $f \in C(Y|X)$. Thus, $\delta_{y_d} \to \delta_y$ under $\sigma(L(Y), C(Y|X))$. Conversely, if a net $\{y_d : d \in D\}$ in K and a point $y \in K$ verify that $\left\langle f, \delta_{y_d}\right\rangle \to \left\langle f, \delta_y\right\rangle$ for every $f \in C(Y|X)$, we claim that $\left\langle g, \delta_{y_d}\right\rangle \to \left\langle g, \delta_y\right\rangle$ for every $g \in C(Y)$. Indeed, if $g \in C(Y)$, then $g|_K \in C(K)$ and there is $h \in C(X)$ with $h|_K = g|_K$. Since $h|_Y \in C(Y|X)$, by assumption $\left\langle h|_Y, \delta_{y_d}\right\rangle \to \left\langle h|_Y, \delta_y\right\rangle$. Thus, $\left\langle h|_K, \delta_{y_d}\right\rangle \to \left\langle h|_K, \delta_y\right\rangle$, i.e., $\left\langle g, \delta_{y_d}\right\rangle \to \left\langle g, \delta_y\right\rangle$. Consequently, $\delta_{y_d} \to \delta_y$ under $\sigma(L(Y), C(Y))$.

Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution for $C_p(Y|X)$ consisting of compact sets that swallows the compact sets in $C_p(Y|X)$. To prove that each compact set in Y is finite, we adapt the first part of the argument of [30, 3.7 Theorem] (see also [24, Theorem 9.14]).

Assume by contradiction that there exists an infinite compact set K in Y. As $C_p(Y|X)$ has a compact resolution, it is a quasi-Suslin space in the sense of Valdivia by [7, Proposition 1], hence it is web-compact in the sense of Orihuela (cf. [28]). Thus, the space $C_p\left(C_p(Y|X)\right)$ is angelic by [28, Theorem 3].

Since K is a compact subspace of Y, it follows from the first part of this proof that K is embedded in $C_p\left(C_p(Y|X)\right)$. Hence, K is an infinite Fréchet-Urysohn compact subset of the angelic space $C_p\left(C_p(Y|X)\right)$ and consequently there exists a non-trivial sequence $\{x_n\}_{n=1}^{\infty}$ converging to some $x \in K$. If we set $S := \{x_n : n \in \mathbb{N}\} \cup \{x\}$, then S is a compact countable set, and hence a metrizable compact space. Thus, there is a linear extender map $\varphi : C_p(S) \to C_p(Y)$, i.e., such that $\varphi(f|_S) = f$ for every $f \in C(Y)$, which embeds $C_p(S)$ as a closed linear subspace of $C_p(Y)$, [4, Proposition 4.1]. But, if $f := \varphi(g)$ with $g \in C(S)$, then $f \in C(Y|X)$. Indeed, if $h \in C(X)$ extends g to the whole of X, then

$$h|_Y = \varphi(h|_S) = \varphi(g) = f,$$

which means that $f \in C(Y|X)$. Thus, φ can be regarded as a map from $C_p(S)$ into $C_p(Y|X)$. Therefore, $C_p(S)$ embeds in $C_p(Y|X)$ as a closed linear subspace of $C_p(Y|X)$. So, the metrizable space $C_p(S)$ contains a resolution of compact sets, namely the family $\{A_\alpha \cap C(S) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, that swallows the compact sets in $C_p(S)$. By Christensen's theorem [8, Theorem 3.3] (see also [12, Theorem 94]) $C_p(S)$ must be a Polish space. Thus, [2, I.3.3 Corollary] guarantees that the compact set S is discrete, and hence finite, a contradiction. □

For the next lemma, recall that, according to [19, 15.14 Corollary], a Tychonoff space X is realcompact if and only if its homeomorphic copy $\delta(X)$ in $L_p(X)$ is a complete set.

Theorem 2.3. *Assume that X is realcompact and Y is a closed subspace of X. If $C_p(Y|X)$ has a Σ-covering of limited envelope, then Y is a Lindelöf Σ-space.*

Proof. We need the following auxiliary result.

Claim 2.1. *Under our assumptions, the copy $\delta(Y)$ of Y in $L_p(Y|X)$ is a complete set.*

Proof of the claim. Let $\{\delta_{y_d} : d \in D\}$ be a Cauchy net in the canonical copy $\delta(Y)$ of Y in $L_p(Y|X)$, i.e., such that for each $f \in C(Y|X)$ and $\epsilon > 0$ there is $d(f,\epsilon) \in D$ with

$$|f(y_r) - f(y_s)| = \left|\left\langle f, \delta_{y_r} - \delta_{y_s}\right\rangle\right| < \epsilon$$

for $r, s \geq d(f,\epsilon)$. As for each $f \in C(X)$, one has that $f|_Y \in C(Y|X)$, and clearly $\{\delta_{y_d} : d \in D\}$, regarded as a net in $\delta(X)$, is a Cauchy net in $L_p(X)$. Since X is realcompact there is $z \in X$ such that $\delta_{y_d} \to \delta_z$ in $L_p(X)$, i.e., such that $\left\langle f, \delta_{y_d}\right\rangle \to \langle f, \delta_z\rangle$ for every $f \in C(X)$. But, as Y is closed in X, the canonical copy $\delta(Y)$ of Y in $L_p(X)$ is closed in $\delta(X)$, which implies that $\delta_z \in \delta(Y)$. This means that $z \in Y$, and thus we have in particular that $\left\langle f|_Y, \delta_{y_d}\right\rangle \to \langle f|_Y, \delta_z\rangle$ for every $f \in C(X)$. From this, it follows that $\left\langle g, \delta_{y_d}\right\rangle \to \langle g, \delta_z\rangle$ for every $g \in C(Y|X)$, which shows that $\delta(Y)$ is a complete set in $L_p(Y|X)$.

Now, if $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a Σ-covering of limited envelope for $C_p(Y|X)$, Theorem 1.3 provides a Lindelöf Σ-space Z such that

$$L_p(Y|X) \subseteq Z \subseteq \mathbb{R}^{C(Y|X)}.$$

By the claim, $\delta(Y)$ is a complete subspace of $L_p(Y|X)$, and consequently, a closed set of the complete locally convex space $\mathbb{R}^{C(Y|X)}$. Thus, $\delta(Y)$, as a closed topological subspace of Z, is a Lindelöf Σ-space. □

Observe that if $\{\delta_{y_d} : d \in D\}$ is a Cauchy net in the copy $\delta(Y)$ of Y in $L_p(Y|X)$, then $\{\delta_{y_d} : d \in D\}$ need not be a Cauchy net in $L_p(Y)$. So, if we require only Y to be realcompact (regardless of whether X is realcompact or not), the argument of the previous claim does not work.

Corollary 2.2. *Let X be realcompact. If $C_p(X)$ has a bounded resolution, then X has countable extent.*

Proof. If X has uncountable extent, there exists an uncountable closed discrete subspace Y, i.e., consisting of relative isolated points. If $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution for $C_p(X)$, the restriction map $T : C_p(X) \to C_p(Y|X)$ given by $T(f) = f|_Y$ maps continuously $C_p(X)$ onto $C_p(Y|X)$. Thus, the family $\{T(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution for $C_p(Y|X)$, and hence a Σ-covering of limited envelope. According to Theorem 2.3, the space Y must be Lindelöf, and hence countable, a contradiction. □

Corollary 2.3. *If $\mathbb{M}$ denotes the Michael line and $\mathbb{P}$ the subspace of the irrational numbers, the following properties hold.*

(1) $C_p(\mathbb{M})$ does not admit a bounded resolution.
(2) $C_p(\mathbb{P}|\mathbb{M})$ does not admit a bounded resolution.

Proof. Recall that in the Michael line the set $\mathbb{P}$ of irrationals is a discrete open subspace. As the topology of $\mathbb{M}$ is stronger than that of the real line $\mathbb{R}$ and each subspace of $\mathbb{R}$ is realcompact, it turns out that $\mathbb{M}$ is realcompact by virtue of [19, 8.18 Corollary].

(1) Since there exists an uncountable Euclidean closed set Y in $\mathbb{R}$ consisting entirely of irrational numbers, this set Y is closed and discrete in $\mathbb{M}$, so $\mathbb{M}$ has uncountable extent and (the contrapositive statement of) Corollary 2.2 applies.

(2) If Y is again an uncountable closed set in $\mathbb{M}$ consisting of irrational numbers, the restriction map $S : C_p(\mathbb{P}|\mathbb{M}) \to C_p(Y|\mathbb{M})$ defined as usual by $S(f) = f|_Y$ is a continuous linear map. It is also surjective, for if $g \in C(Y|\mathbb{M})$, there exists $f \in C(\mathbb{M})$ such that $f|_Y = g$. So, clearly $f|_{\mathbb{P}} \in C(\mathbb{P}|\mathbb{M})$ and $S(f|_{\mathbb{P}}) = f|_Y = g$. If $C_p(\mathbb{P}|\mathbb{M})$ had a bounded resolution, say $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, then $\{S(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ would be a bounded resolution for $C_p(Y|\mathbb{M})$. So, according to the proof of Corollary 2.2, the subspace Y would be countable, a contradiction. □

Example 2.3. *The converse of Corollary 2.2 does not hold. Let $\mathbb{S}$ be the Sorgenfrey line. Since $\mathbb{S}$ is a Lindelöf space, it is realcompact and has countable extent. However, $C_p(\mathbb{S})$ does not admit a bounded resolution. Otherwise, by Theorem 1.4, the Sorgenfrey line would be a Lindelöf Σ-space, which is not true since the product of two Lindelöf Σ-spaces is a Lindelöf Σ-space, but $\mathbb{S} \times \mathbb{S}$ is not Lindelöf.*

Corollary 2.4. *Let X be realcompact and let Y be a closed subspace which is a P-space. Then $C_p(Y|X)$ admits a bounded resolution if and only if Y is countable and discrete.*

Proof. Since Y is a P-space, each compact set in Y is finite. If, in addition, $C_p(Y|X)$ admits a Σ-covering of limited envelope, then Theorem 2.3 ensures that Y is a Lindelöf Σ-space. But, every Lindelöf Σ-space with finite compact sets is countable (by [2, IV.6.15 Proposition]), and every countable P-space is discrete. The converse is clear. □

Hence, if X is a realcompact P-space and Y a closed subspace, then $C_p(Y|X)$ admits a bounded resolution if and only if Y is countable and discrete.

Theorem 2.4. *Let Y be a closed subspace of a realcompact space X. Then $C_p(Y|X)$ admits a compact resolution that swallows the compact sets if and only if Y is countable and discrete.*

Proof. According to Theorem 2.2, each compact set K in Y is finite. Moreover, as a compact resolution is a Σ-covering of limited envelope, Theorem 2.3 tells us that Y is a Lindelöf Σ-space. But, as we know, a Lindelöf Σ-space with finite compact sets is countable, so Y is countable and, consequently, both spaces $C_p(Y|X)$ and $C_p(Y)$ are metrizable.

Therefore we have a metrizable topological space $C_p(Y|X)$ with a compact resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ that swallows the compact sets in $C_p(Y|X)$. So, again by Christensen's theorem, $C_p(Y|X)$ is a Polish space, and hence a Čech-complete space [9, 4.3.26 Theorem]. Since $C_p(Y|X)$ is a Čech-complete dense subspace of $C_p(Y)$, [2, I.3.1 Theorem] asserts that Y is (countable and) discrete. □

Example 2.4. *In the previous theorem, the requirement that X be realcompact is not necessary. In fact, if X is a non-realcompact P-space [19, Problem 9L] and Y is a countable subspace of X, then Y is C-embedded in X, so $C_p(Y|X) = C_p(Y) = \mathbb{R}^Y$ since Y, a countable subspace, is (closed and) discrete. If $Y = \{y_n : n \in \mathbb{N}\}$, then $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $A_\alpha = \{f \in \mathbb{R}^{\mathbb{N}} : |f(y_n)| \leq \alpha(n)\}$ is a compact resolution for $C_p(Y|X)$ that swallows the compact sets.*

Remark 2.1. *If X is an arbitrary Tychonoff space and $Y = X$, using the well-known fact that if $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution for $C_p(X)$ that swallows the compact sets of $C_p(X)$ and $S : C_p(\upsilon X) \to C_p(X)$ is the restriction map $f \mapsto f|_X$, then $\{S^{-1}(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution for $C_p(\upsilon X)$ that swallows the compact sets in $C_p(\upsilon X)$, and thus, we can deduce [30, 3.7 Theorem] from Theorem 2.4. So, if $Y = X$, we may exempt the space X from being realcompact in the statement of Theorem 2.4.*

Corollary 2.5. *Let Y be a closed subspace of a realcompact space X. The following are equivalent*

(1) $C_p(Y|X)$ admits a compact resolution that swallows the compact sets.
(2) $C_p(Y|X)$ is a Polish space.

Proof. If $C_p(Y|X)$ has a compact resolution that swallows the compact sets, Y is countable and discrete by Theorem 2.4. Consequently, $C_p(Y|X)$, as a dense linear subspace of $C_p(Y) = \mathbb{R}^Y$, is a metrizable space with a compact resolution swallowing the compact sets. Hence, a Polish space by Christensen's theorem. Conversely, if $C_p(Y|X)$ is a Polish space, it clearly has a compact resolution swallowing the compact sets. □

Remark 2.2. *Let Y be a countable dense subspace of X. If $C_p(Y|X)$ admits a compact resolution that swallows the compact sets, then Y is discrete. In fact, if Y is countable, then $C_p(Y|X)$ is metrizable and we may argue as in the final paragraph of the proof of Theorem 2.4.*

Example 2.5. *If $C_p(Y|X)$ admits a bounded resolution that swallows the compact sets in $C_p(Y|X)$, then Y need not be discrete. If $\mathbb{M}$ denotes the Michael line, the set $\mathbb{Q}$ of rational numbers is a closed subspace of $\mathbb{M}$. Since, $\mathbb{R}^{\mathbb{Q}} = \mathbb{R}^{\mathbb{N}}$ clearly has a compact resolution that swallows the compact sets in $\mathbb{R}^{\mathbb{Q}}$, we see that $C_p(\mathbb{Q}|\mathbb{M}) = C_p(\mathbb{Q})$ has a bounded resolution that swallows the compact sets in $C_p(\mathbb{Q}|\mathbb{M})$, but $\mathbb{Q}$ is not discrete.*

3. Resolutions for $C_p(Y|X)$ consisting of Banach disks

Let us mention that, originally, the definition of quasi-(LB)-spaces emerged as an appropriate range class for the closed graph theorem when strictly barrelled spaces are located at the domain class [33, Corollary 1.5]. Valdivia's class of quasi-(LB)-spaces has proven to be useful both in functional analysis and in topology (see for instance [24, Chapter 3]). Recently, some lifting properties involving quasi-(LB)-spaces at the range class in the closed graph theorem has been relaxed by replacing the strictly barrelled spaces of the domain class by locally convex spaces with a sequential web (see [22, Theorem 1] for details).

Lemma 3.1. *The spaces $C_p(X)$ and $C_k(X)$ have the same Banach disks.*

Proof. If B is a Banach disk in $C_k(X)$, it is clear that B is a Banach disk in $C_p(X)$. Conversely, we claim that if Q is a Banach disk in $C_p(X)$, then Q is a bounded set in $C_k(X)$. Indeed, if U is a basic τ_k-closed absolutely convex neighborhood of the origin in $C_k(X)$, there is a compact set K in X and $\epsilon > 0$ such that

$$U = \{f \in C(X) : \sup_{x \in K} |f(x)| \le \epsilon\}.$$

Clearly, U is a τ_p-closed set, for if $\{f_d : d \in D\}$ is a net in U such that $f_d \to f$ in $C_p(X)$, then $|f(x)| \le \epsilon$ for every $x \in K$. As the norm topology of E_Q is stronger than the pointwise topology, $U \cap E_Q$ is also a closed absolutely convex set for the norm topology of E_Q. Since $E = \bigcup_{n=1}^{\infty} nU$, the Baire category theorem provides some $m \in \mathbb{N}$ such that $Q \subseteq mU$, which shows that Q is a bounded set in $C_k(X)$, as stated. As Q is a τ_k-bounded set such that E_Q is a Banach space, it turns out that Q is a Banach disk in $C_k(X)$. □

Theorem 3.1. *Let X be metrizable. Then $C_p(X)$ is a quasi-(LB)-space if and only if X is σ-compact.*

Proof. If $C_p(X)$ is a quasi-(LB)-space, according to Lemma 3.1 the space $C_k(X)$ is also a quasi-(LB)-space. As X is a $k_{\mathbb{R}}$-space, $C_k(X)$ is complete, and hence locally complete, that is, such that the τ_k-closed absolutely convex cover of each bounded set in $C_k(X)$ is a Banach disk or, according to [20, Theorem 2.1], such that each convergent sequence in $C_k(X)$ is equicontinuous. Since by [33, Proposition 22] the space $C_k(X)$ admits a resolution consisting of Banach disks that swallows the Banach disks in $C_k(X)$, it follows that $C_k(X)$ has a fundamental bounded resolution. Consequently, by [11, Proposition 3], the space X must be σ-compact.

Conversely, if X is metrizable and σ-compact, again by [11, Proposition 3] the space $C_k(X)$ has a (fundamental) bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since the local completeness of $C_k(X)$ guarantees that $B_\alpha := \overline{\operatorname{abx}(A_\alpha)}^{\tau_k}$, i.e. the τ_k-closed absolutely convex cover of A_α, is a Banach disk for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, and $C_k(X)$ is clearly a quasi-(LB)-space. Thus, $C_p(X)$ is also a quasi-(LB)-space. □

Corollary 3.1. *Let Y be a topological subspace of a metrizable space X. If X is σ-compact, then $C_p(Y|X)$ is a quasi-(LB)-space.*

Proof. If X is metrizable and σ-compact, $C_p(X)$ is a quasi-(LB)-space by Theorem 3.1. If T is the restriction map from $C_p(X)$ onto $C_p(Y|X)$, the image $T(B)$ of each Banach disk of $C_p(X)$ is a Banach disk of $C_p(Y|X)$. Thus, $C_p(Y|X)$ is a quasi-(LB)-space. □

Example 3.1. *Metrizability cannot be dropped in the 'if' part of Theorem 3.1. If $p \in \beta\mathbb{N} \setminus \mathbb{N}$, equip $X = \mathbb{N} \cup \{p\}$ with the relative topology of $\beta\mathbb{N}$. Since X is countable, it is σ-compact. However, $C_p(X)$ is not a quasi-(LB)-space. In fact, since $C_p(X)$ is a locally convex Baire space (see [27, Example 7.1]), i.e., such that it cannot be covered by countably many rare, balanced sets [29], if $C_p(X)$ were a quasi-(LB)-space then $C_p(X)$ would be a Fréchet space by virtue of [24, Corollary 3.12]. As $C_p(X)$ is a dense linear subspace of $\mathbb{R}^X$, this would imply that $C_p(X) = \mathbb{R}^X$. Consequently, the space X should be discrete, which is not true.*

4. Resolutions for $C_p(Y|X)$ consisting of convex compact sets

Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ in a locally convex space E is local null or Mackey convergent to zero [25, 28.3] if there is a bounded, closed, absolutely convex set B in E (a closed disk) such that $x_n \to \mathbf{0}$ in the normed space $E_B := \operatorname{span}(B)$ equipped with the Minkowski functional of B as a norm. Each local null sequence in E is a null sequence. If E is metrizable, each null sequence is local null [25, 28.3.(1) c)]. A linear form u defined on a bornological space E (see [5, 3.6.2 Definition] or [25, 28]) is continuous if and only if $u(x_n) \to 0$ for each local null sequence $\{x_n\}_{n=1}^{\infty}$ in E, [25, 28.3.(4)]. Recall that, according to the Buchwalter-Schmets theorem [6], the space $C_p(X)$ is bornological if and only if X is realcompact.

Theorem 4.1. *Let Y be a closed subspace of a realcompact space X. If $C_p(Y|X)$ admits a resolution of convex compact sets that swallows the local null sequences in $C_p(Y|X)$, then Y is countable and discrete.*

Proof. As in the proof of [13, Theorem 12], we may assume without loss of generality that $C_p(Y|X)$ admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of absolutely convex compact sets swallowing the local null sequences in $C_p(Y|X)$.

We adapt as possible the argument of the proof of [13, Theorem 12] to the present setting. Let $\mathcal{M}$ be the family of all local null sequences in $C_p(Y|X)$. Since the family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows the members of $\mathcal{M}$, the Mackey* topology $\mu(L(Y), C(Y|X))$ of $L(Y)$ is stronger than the topology τ_{c_0} on $L(Y)$ of the uniform convergence on the local null sequences of $C_p(Y|X)$. As it is clear that $\sigma(L(Y), C(Y|X)) \leq \tau_{c_0}$, we conclude that $(L(Y), \tau_{c_0})' = C(Y|X)$. Moreover, as X is realcompact, $C_p(X)$ is bornological, and since Y is a closed subspace of X, the restriction map from $C_p(X)$ onto $C_p(Y|X)$ is a quotient map [2, 0.4.1 (2) Proposition], which ensures that $C_p(Y|X)$ is also a bornological space [25, 28.4.(2)]. Consequently, the τ_{c_0}-dual $(L(Y), \tau_{c_0})$ of $C_p(Y|X)$ is complete by [25, 28.5.(1)]. In fact, it is even $\mu(L(Y), C(Y|X))$-complete by [25, 18.4.(4)].

Now we claim that every compact set in Y is finite. Indeed, if K is a compact set in Y, as we have seen in the proof of Theorem 2.2, the canonical copy $\delta(K)$ of K in $L_p(Y|X)$ is embedded in $L_p(Y|X)$, i.e., it is a $\sigma(L(Y), C(Y|X))$-compact set in $L(Y)$. So, the completeness of $(L(Y), \tau_{c_0})$, together with Krein's theorem and the fact that τ_{c_0} is a locally convex topology of the dual pair $\langle L(Y), C(Y|X)\rangle$, ensures that the weak* closure Q in $L_p(Y|X)$ of the absolutely convex hull of $\delta(K)$ is a compact set in $L_p(Y|X)$, and hence a $\beta(L(Y), C(Y|X))$-bounded set. On the other hand, as $C_p(Y)$ is a quasibarrelled space [23, 11.7.3 Corollary], the $\beta(L(Y), C(Y))$-bounded sets in $L(Y)$ are finite-dimensional. But, according to Theorem 1.1, we have that $\beta(L(Y), C(Y)) = \beta(L(Y), C(Y|X))$. So, every $\beta(L(Y), C(Y|X))$-bounded set in $L(Y)$ is finite-dimensional. Thus, in particular, Q is finite-dimensional. This means that $\delta(K)$, as a linearly independent system of vectors in $L(Y)$ contained in Q, must be finite. Hence, the compact set K is finite, as stated.

Since each compact resolution is a Σ-covering of limited envelope, Theorem 2.3 tells us that Y is a Lindelöf Σ-space. So Y is countable by [2, IV.6.15 Proposition]. Hence, $C_p(Y|X)$ is a metrizable space, which ensures that, in $C_p(Y|X)$, local null sequences and null sequences are the same. Moreover, if M is a compact set in the metrizable space $C_p(Y|X)$, then M lies in the closed absolutely convex cover of a null sequence $\{f_n\}_{n=1}^{\infty}$, [25, 21.10.(3)]. So, if $\{f_n\}_{n=1}^{\infty} \subseteq A_\gamma$, the fact that A_γ is a closed absolutely convex set guarantees that $M \subseteq A_\gamma$. Thus, $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution for $C_p(Y|X)$ that swallows the compact sets of $C_p(Y|X)$. So, we apply Theorem 2.4 to get that Y is (countable and) discrete. □

Remark 4.1. *If $Y = X$, one can get rid of the realcompactness from X by working with $C_p(\upsilon X)$ instead of with $C_p(X)$. So, Theorem 4.1 essentially contains [13, Theorem 12].*

5. Bounded resolutions for $C_p(Y|X)$ swallowing Cauchy sequences

Let us examine the existence of a bounded resolution for $C_p(Y)$ that swallows the Cauchy sequences in $C_p(Y|X)$. We denote by $\mathbb{R}^{(X)}$ the linear subspace of $\mathbb{R}^X$ consisting of functions with finite support, i.e., those which vanish off a finite set in X.

Theorem 5.1. *Assume that X is metrizable and Y hemicompact with $Y \subseteq X$. Then $C_p(Y|X)$ has a bounded resolution that swallows the Cauchy sequences in $C_p(Y|X)$ if and only if Y is countable.*

Proof. We may assume there is a bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ in $C_p(Y|X)$ consisting of absolutely convex sets swallowing the Cauchy sequences in $C_p(Y|X)$. Given $e_y \in \mathbb{R}^{(Y)}$ defined by $e_y(y) = 1$ and $e_y(z) = 0$ if $z \neq y$, we extend e_y to the whole of X by setting $\widehat{e_y}(z) = e_y(z)$ if $z \in Y$ and $\widehat{e_y}(x) = 0$ if $x \notin Y$. Since X is first countable, as in the proof of [13, Theorem 33] we find a sequence $\left\{f_{y,n}\right\}_{n=1}^{\infty}$ in $C(X)$ such that $f_{y,n} \to \widehat{e_y}$ in $\mathbb{R}^X$. As $f_{y,n}\big|_Y \to e_y$ in $C_p(Y)$, there is $\gamma \in \mathbb{N}^\mathbb{N}$ with $f_{y,n}\big|_Y \in A_\gamma$ for all $n \in \mathbb{N}$, so $e_y \in \overline{A_\gamma}$. Now, given $(n_1, \alpha) \in \mathbb{N} \times \mathbb{N}^\mathbb{N}$, let $\beta \in \mathbb{N}^\mathbb{N}$ be such that $\beta(1) = n_1$ and $\beta(i+1) = \alpha(i)$ for each $i \in \mathbb{N}$ and define $B_\beta := n_1\overline{A_\alpha}$, closure in $\mathbb{R}^Y$. Since $\left\{e_y : y \in Y\right\}$ is a Hamel basis of $\mathbb{R}^{(Y)}$, it follows that $\{B_\beta : \beta \in \mathbb{N}^\mathbb{N}\}$ is a bounded family in $\mathbb{R}^Y$ such that $\mathbb{R}^{(Y)} \subseteq \bigcup\{B_\beta : \beta \in \mathbb{N}^\mathbb{N}\}$.

On the other hand, note that $C_k(Y)$ is a metrizable space due to Arens' theorem [1] (see also [34, Theorem 13.2.4]). As $C(Y|X)$ is clearly a dense linear subspace of $C_k(Y)$, each $g \in C_k(Y)$ is the limit of a sequence $\{g_n\}_{n=1}^{\infty}$ in $C(Y|X)$ under the compact-open topology τ_k, and hence under the pointwise topology τ_p. Thus, there exists $\delta \in \mathbb{N}^\mathbb{N}$ such that $g_n \in A_\delta$ for every $n \in \mathbb{N}$ so that $g \in \overline{A_\delta}$, which shows that $C(Y) \subseteq \bigcup\{B_\beta : \beta \in \mathbb{N}^\mathbb{N}\}$. Thus, we conclude that $C(Y) + \mathbb{R}^{(Y)} \subseteq \bigcup\{B_\beta : \beta \in \mathbb{N}^\mathbb{N}\}$, which means that the linear subspace $C(Y) + \mathbb{R}^{(Y)}$ of $M(Y)$ admits a bounded resolution under the weak* topology of $M(Y)$. According to [13, Remarks 20, 27], Y must be countable.

For the converse, note that if Y is countable, $\mathbb{R}^Y$ has a compact resolution $\{Q_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ that swallows the compact sets in $\mathbb{R}^Y$. Hence, $\{Q_\alpha \cap C(Y|X) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution for $C_p(Y|X)$ such that if $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_p(Y|X)$ and $h \in \mathbb{R}^Y$ verifies that $h_n \to h$ in $\mathbb{R}^Y$, there is $\gamma \in \mathbb{N}^\mathbb{N}$ with $\{h, h_n : n \in \mathbb{N}\} \in Q_\gamma$. Hence, $\{h_n : n \in \mathbb{N}\} \in Q_\gamma \cap C(Y|X)$. □

Proposition 5.1. *Let $C_p(Y)$ be Fréchet-Urysohn. If $C_p(Y|X)$ has a bounded resolution that swallows the Cauchy sequences in $C_p(Y|X)$, then $C_p(Y)$ has a bounded resolution.*

Proof. Let $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a bounded resolution for $C_p(Y|X)$ that swallows the Cauchy sequences in $C_p(Y|X)$. As $C(Y) \subseteq M(Y) = M(Y|X)$, the latter equality because of Theorem 1.1, given $g \in C(Y)$ there is a bounded set Q in $C_p(Y|X)$ such that $g \in \overline{Q}$, closure in $C_p(Y)$. But, since $C_p(Y)$ is Fréchet-Urysohn, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in $Q \subseteq C(Y|X)$ such that $g_n \to g$ in $C_p(Y)$. As $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_p(Y|X)$, there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $g_n \in A_\gamma$ for every $n \in \mathbb{N}$. Consequently, we have $g \in \overline{A_\gamma}$, closure in $C_p(Y)$. Since each bounded set in $C_p(Y|X)$ is a bounded set in $C_p(Y)$ and the closure of a bounded set is bounded, it follows that the family $\{\overline{A_\alpha} : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution for $C_p(Y)$. □

Corollary 5.1. *Assume that Y is a Lindelöf P-space, with $Y \subseteq X$. Then $C_p(Y|X)$ has a bounded resolution that swallows the Cauchy sequences in $C_p(Y|X)$ if and only if Y is countable and discrete.*

Proof. Since Y is a Lindelöf P-space, it turns out that $C_p(Y)$ is Fréchet-Urysohn [3, 10.2 Theorem]. Consequently, $C_p(Y)$ has a bounded resolution by the preceding proposition. So, according to [15, Proposition 3.6], the subspace Y must be countable and discrete. The proof of the converse is analogous to that of Theorem 5.1. □

6. $\Sigma(Y)$ and $\mathbb{R}^{(Y)}$ as dense subspaces of proper $C_p(Y|X)$ spaces

If F is a linear subspace of $C(X)$, let $Y = \{x \in X : \delta_x \notin F^\perp\}$ so that $\delta(Y) = \delta(X) \setminus F^\perp$. Since $F^\perp$, the annihilator of F in $L(X)$, is a closed subspace of $L_p(X)$, Y is an open subset of X. Define $T : F \to C_p(Y|X)$ by the rule $Tf = f|_Y$.

Theorem 6.1. *T is a linear homeomorphism from F onto its range, which embeds F isomorphically into $C_p(Y|X)$.*

Proof. Let us first check that T is one-to-one. So, if $f \in \ker T \Leftrightarrow f|_Y = \mathbf{0}$, we have to show that $f = \mathbf{0}$. Equivalently, if $f \in F$ is such that $f \neq \mathbf{0}$, we must prove that $Tf \neq \mathbf{0}$. But, if $f \neq \mathbf{0}$, there is $u \in L(X)$ with $\langle f, u\rangle \neq 0$. If $u = v + w$ with $v = \sum_{i=1}^{p} a_i \delta_{x_i}$, where $\delta_{x_i} \in F^\perp$ or, equivalently, $x_i \in X \setminus Y$ for $1 \leq i \leq p$, and $w = \sum_{i=1}^{q} a_i \delta_{y_i}$ with $\delta_{y_i} \notin F^\perp$ or, equivalently, $y_i \in Y$ for $1 \leq i \leq q$, then

$$0 \neq \langle f, u\rangle = \langle f, w\rangle = \langle f|_Y, w\rangle = \langle Tf, w\rangle,$$

since $w \in L(Y)$. As $L(Y)$ is the dual of $C_p(Y|X)$, this shows that $Tf \neq \mathbf{0}$, as desired.

Let us check that $T^{-1} : \mathrm{Im}T \to F$ is continuous when $\mathrm{Im}T$ is provided with the relative topology of $C_p(Y|X)$ and F is equipped with the relative topology of $C_p(X)$. Indeed, if $\{f_d : d \in D\}$ is a net in F and $f \in F$ are such that $Tf_d \to Tf$ in $\mathrm{Im}T$, it is clear that $f_d(y) \to f(y)$ for every $y \in Y$. Now, if $x \in X$, either $\delta_x \in F^\perp$ or $\delta_x \notin F^\perp$. In the first case, it is obvious that $f_d(x) = 0 \to 0 = f(x)$ since $f_d, f \in F$. In the second case, $x \in Y$, so $f_d(x) \to f(x)$. This shows that $f_d \to f$ in F pointwise on X. Since T is continuous, one-to-one, and a topological homomorphism, it turns out that T is a linear homeomorphism that embeds isomorphically F into $C_p(Y|X)$. □

Theorem 6.2. *Let Z be a closed set in X. If $F = \{f \in C(X) : f(z) = 0\ \forall z \in Z\}$, then F is linearly homeomorphic to a dense subspace of $C_p(Y|X)$.*

Proof. Since it is clear that $\delta_z \in F^\perp$ for each $z \in Z$, we have $Z \subseteq X \setminus Y$. But, if $x \in X \setminus Y$ and $x \notin Z$, there is $g \in C(X)$ with $g(x) = 1$ and $g(z) = 0$ for all $z \in Z$, which means that $g \in F$. But $\langle g, \delta_x\rangle = 1$, so $\delta_x \notin F^\perp$, i.e., $x \in Y$, a contradiction. Therefore, $X \setminus Y = Z$. Thus, if $T : F \to C_p(Y|X)$ is the map considered in the previous theorem, then

$$T(F) = \{g \in C(Y) : \exists f \in C(X),\ f|_Y = g,\ f|_{X\setminus Y} = 0\}.$$

By Theorem 6.1 the subspace $T(F)$ of $C_p(Y|X)$ is isomorphic to F. If Δ is a finite subset of Y and $h \in C(Y|X)$, the fact that Z is closed allows us to find $f \in C(X)$ such that $f(y) = h(y)$ for each $y \in \Delta$ and $f(z) = 0$ for every $z \in Z$, i.e., a function $f \in C(X)$ such that $g := f|_Y$ lies in $T(F)$ and verifies that $g(y) = h(y)$ for $y \in \Delta$. This shows that $T(F)$ is dense in $C_p(Y|X)$. □

Example 6.1. *If $X = D \cup \{\xi\}$ is the one-point Lindelöfication of the discrete space D, the subspace $\Sigma(D)$ of $\mathbb{R}^D$ consisting of all countably supported functions on D is linearly homeomorphic to a dense subspace of $C_p(D|X)$. In particular, $C_p(D|X)$ is a Baire space.*

Proof. If $F := \{f \in C(X) : f(\xi) = 0\}$, the restriction map $S : F \to \Sigma(D)$ given by $Sf = f|_D$ is a linear homeomorphism from F onto $\Sigma(D)$. If we put $Y := \{x \in X : \delta_x \notin F^\perp\}$, Theorem 6.2 asserts that the subspace F of $C_p(X)$ is isomorphic to a dense subspace of $C_p(Y|X)$. Consequently, $\Sigma(D)$ is isomorphic to a dense subspace of $C_p(Y|X)$.

But, clearly $D = Y$, since, if $x \in D$, there is $f \in C(X)$ with $f(x) = 1$ and $f(\xi) = 0$, such that $f \in F$ and $\langle f, \delta_x \rangle = 1$, i.e., $\delta_x \notin F^{\perp}$. Thus, $x \in Y$. Whereas if $x \in Y$, then $\delta_x \notin F^{\perp}$, which means that $x \neq \xi$, so $x \in D$. The second statement follows from the fact that $\Sigma(D)$ is a Baire space.

□

Remark 6.1. *If X is uncountable, then $\Sigma(X)$ does not admit a bounded resolution. This is because $\Sigma(X)$ is locally complete. Thus, Valdivia's theorem [24, Theorem 3.5] ensures that $\Sigma(X)$ admits a bounded resolution if and only if it is a quasi-(LB)-space. Since $\Sigma(X)$ is a Baire space, $\Sigma(X)$ is necessarily a Fréchet space [24, Corollary 3.12], which implies that $\Sigma(X) = \mathbb{R}^X$ with X countable. Note that this implies that if $X = D \cup \{\xi\}$ is the one-point Lindelöfication of the discrete space D with $|D| \geq \aleph_1$, then $C_p(D|X)$ does not have a bounded resolution (cf. Example 2.1).*

Example 6.2. *If $X = D \cup \{\xi\}$ is the one-point compactification of the discrete space D, the subspace $\mathbb{R}^{(D)}$ of $\mathbb{R}^D$ consisting of all finitely supported functions on D is linearly homeomorphic to a dense subspace of the space $C_p(D|X)$.*

Proof. If $F := \{f \in C(X) : f(\xi) = 0\}$, the linear map $S : F \to \mathbb{R}^{(D)}$ defined by $Sf = f|_D$ is a linear homeomorphism from the closed one-codimensional linear subspace F of $C_p(X)$ onto $\mathbb{R}^{(D)}$. Likewise, the set $Y := \{x \in X : \delta_x \notin F^{\perp}\}$ coincides with D since $\delta(X) \cap F^{\perp} = \left\{\delta_\xi\right\}$. By Theorem 6.2, F is linearly homeomorphic to a dense subspace of $C_p(Y|X)$. So, $\mathbb{R}^{(D)}$ is linearly isomorphic to a dense linear subspace of $C_p(D|X)$. □

Remark 6.2. *$\mathbb{R}^{(X)}$ has a fundamental bounded resolution if and only if X is countable.*

Proof. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a fundamental resolution for $\mathbb{R}^{(X)}$ consisting of absolutely convex bounded sets. Thus, the bipolar theorem ensures that $A_\alpha^{00} = \overline{A_\alpha}$, closure in $\mathbb{R}^X$. Hence, the fact that $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows all compact sets in $\mathbb{R}^{(X)}$ means that the bidual E of $\mathbb{R}^{(X)}$ is given by $E = \{\overline{A_\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. As each $\overline{A_\alpha}$ is an absolutely convex compact set in $\mathbb{R}^X$, and thus a Banach disk in E, it turns out that E is a quasi-(LB)-space. On the other hand, since each $f \in \Sigma(X)$ is the limit of a sequence in $\mathbb{R}^{(X)}$, we have that $\Sigma(X) \subseteq E$. As $\Sigma(X)$ is a dense Baire subspace of $\mathbb{R}^X$, it follows that E is a Baire space. Since each locally convex space which is both a quasi-(LB)-space and a Baire space is a Fréchet space, necessarily $E = \mathbb{R}^X$ with X countable. The converse is obvious. □

As a consequence, if $X = Y \cup \{\xi\}$ is the one-point compactification of the discrete space Y, then $C_p(Y|X)$ admits a fundamental bounded resolution if and only if Y is countable, which is a particular case of Theorem 2.1.

Author contributions

Juan C. Ferrando: conceptualization, research, methodology, formal analysis, writing-original draft, review & editing, validation; Manuel López-Pellicer: formal analysis, validation, writing review; Santiago Moll-López: validation, writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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