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Research article

On generalised framed surfaces in the Euclidean space

Masatomo Takahashi[1] **and Haiou Yu**[2,*]

[1] Department of Mathematical Science, Muroran Institute of Technology, Muroran 0508585, Japan
[2] Department of Mathematics, Jilin University of Finance and Economic, Changchun 130117, China

* **Correspondence:** Email: yuhaiou@jlufe.edu.cn.

Abstract: We have introduced framed surfaces as smooth surfaces with singular points. The framed surface is a surface with a moving frame based on the unit normal vector of the surface. Thus, the notion of framed surfaces (respectively, framed base surfaces) is locally equivalent to the notion of Legendre surfaces (respectively, frontals). A more general notion of singular surfaces, called generalised framed surfaces, is introduced in this paper. The notion of generalised framed surfaces includes not only the notion of framed surfaces, but also the notion of one-parameter families of framed curves. It also includes surfaces with corank one singularities. We investigate the properties of generalised framed surfaces.

Keywords: generalised framed surface; framed surface; parallel surface; singular point; basic invariants
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1. Introduction

We investigate differential geometric invariants of surfaces with singular points, that is, singular surfaces. The geometry of singular surfaces in the Euclidean space is a classical object (cf. [1–5, 9, 11, 18–21]). For regular surfaces, the Gauss curvature and mean curvature are important invariants up to congruence. However, if we consider a deformation of a regular surface (for instance, parallel surfaces or caustics), it may have singular points. One of the idea is to consider the fronts or frontals as smooth surfaces with singular points (cf. [1, 2, 10, 13, 16, 19]). The other idea is to consider one-parameter families of framed curves as smooth surfaces with singular points (cf. [7, 15]). We generalise the consideration to treat the smooth surfaces with singular points. A more general notion of singular surfaces, called generalised framed surfaces, is introduced in this paper. The notion of generalised framed surfaces includes not only the notion of framed surfaces, but also the notion of one-parameter families of framed curves. It also includes surfaces with corank one singularities (cf. [12, 14]).

We have introduced framed surfaces as surfaces with singular points in [6]. The framed surface is a surface with a moving frame based on the unit normal vector of the surface. Thus, the notion of framed surfaces (respectively, framed base surfaces) is locally equivalent to the notion of Legendre surfaces (respectively, frontals). In fact, if f is a frontal, then the Jacobi ideal J_f of f is generated by one element [10]. On the other hand, we have also introduced one-parameter families of framed curves as surfaces with singular points in [8, 15]. The relation between framed surfaces and one-parameter families of framed curves was investigated in [7]. In §2, we review the theories of framed surfaces and one-parameter families of framed curves. In §3, we introduce the basic invariants of generalised framed surfaces and give the existence and uniqueness theorems for the basic invariants of generalised framed surfaces. The properties of the generalised framed surfaces are investigated. We give conditions for a surface to become a generalised framed base surface (Theorem 3.10) and for a generalised framed surface to become a framed base surface (Theorem 3.11). In §4 and §5, we focus on surfaces with corank one singularities and corank two singularities, respectively. We prove that surfaces with corank one singularities can always be considered generalised framed surfaces at least locally (Theorem 4.1). Moreover, we find that a part of surfaces with corank two singularities can be considered as generalised framed surfaces. The conditions for special cases of surfaces with corank two singularities to become generalised framed surfaces and framed surfaces are given. As an application, we investigate two types of parallel surfaces of generalised framed surfaces and give concrete examples to illustrate our results in §6.

All maps and manifolds considered in this paper are differentiable of class C^∞ unless stated otherwise.

2. Preliminaries

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the canonical basis of $\mathbb{R}^3$. Let U be a simply connected domain in $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | |\boldsymbol{a}| = 1\}$. We denote a 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ.

2.1. Framed surfaces

We quickly review the theory of framed surfaces in Euclidean 3-space; in detail, see [6, 7]. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a smooth mapping.

Definition 2.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = \boldsymbol{x}_v(u, v) \cdot \boldsymbol{n}(u, v) = 0$ for all $(u, v) \in U$, where $\boldsymbol{x}_u(u, v) = (\partial \boldsymbol{x}/\partial u)(u, v)$ and $\boldsymbol{x}_v(u, v) = (\partial \boldsymbol{x}/\partial v)(u, v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

By definition, the framed base surface is a frontal. For the definition and properties of frontals see [1, 2]. On the other hand, the frontal is a framed base surface at least locally.

We denote $\boldsymbol{t}(u, v) = \boldsymbol{n}(u, v) \times \boldsymbol{s}(u, v)$. Then $\{\boldsymbol{n}(u, v), \boldsymbol{s}(u, v), \boldsymbol{t}(u, v)\}$ is a moving frame along $\boldsymbol{x}(u, v)$, and we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix},$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions. We call these functions *basic invariants* of the framed surface. We denote the above matrices by $\mathcal{G}, \mathcal{F}_1$ and $\mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that (u, v) is a singular point of $\boldsymbol{x}$ if and only if $\det \mathcal{G}(u, v) = 0$.

Since the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should satisfy some conditions. Note that there are fundamental theorems for framed surfaces, namely, the existence and uniqueness theorems for the basic invariants of framed surfaces (cf. [6]).

2.2. One-parameter families of framed curves

We also review the theory of one-parameter families of framed curves in the Euclidean 3-space, in detail, see [7, 15]. Let $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a smooth mapping.

Definition 2.2. We say that $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a *one-parameter family of framed curves with respect to u* (respectively, *with respect to v*) if $(\gamma(\cdot, v), \nu_1(\cdot, v), \nu_2(\cdot, v))$ is a framed curve for each v (respectively, $(\gamma(u, \cdot), \nu_1(u, \cdot), \nu_2(u, \cdot))$ is a framed curve for each u), that is, $\gamma_u(u, v) \cdot \nu_1(u, v) = \gamma_u(u, v) \cdot \nu_2(u, v) = 0$ (respectively, $\gamma_v(u, v) \cdot \nu_1(u, v) = \gamma_v(u, v) \cdot \nu_2(u, v) = 0$) for all $(u, v) \in U$. We say that γ is a *one-parameter family of framed base curves with respect to u* (respectively, *with respect to v*) if there exists $(\nu_1, \nu_2) : U \to \Delta$ such that (γ, ν_1, ν_2) is a one-parameter family of framed curves with respect to u (respectively, *with respect to v*).

We denote $\mu(u, v) = \nu_1(u, v) \times \nu_2(u, v)$. Then $\{\nu_1(u, v), \nu_2(u, v), \mu(u, v)\}$ is a moving frame along $\gamma(u, v)$ and we have the Frenet-Serret type formula.

$$\begin{aligned}
\begin{pmatrix} \nu_{1u}(u, v) \\ \nu_{2u}(u, v) \\ \mu_u(u, v) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(u, v) & m(u, v) \\ -\ell(u, v) & 0 & n(u, v) \\ -m(u, v) & -n(u, v) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(u, v) \\ \nu_2(u, v) \\ \mu(u, v) \end{pmatrix}, \\
\begin{pmatrix} \nu_{1v}(u, v) \\ \nu_{2v}(u, v) \\ \mu_v(u, v) \end{pmatrix} &= \begin{pmatrix} 0 & L(u, v) & M(u, v) \\ -L(u, v) & 0 & N(u, v) \\ -M(u, v) & -N(u, v) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(u, v) \\ \nu_2(u, v) \\ \mu(u, v) \end{pmatrix}, \\
\gamma_u(u, v) &= r(u, v)\mu(u, v), \\
\gamma_v(u, v) &= P(u, v)\nu_1(u, v) + Q(u, v)\nu_2(u, v) + R(u, v)\mu(u, v).
\end{aligned}$$

We call the mapping $(\ell, m, n, r, L, M, N, P, Q, R)$ the *curvature of the one-parameter family of framed curves with respect to u* of (γ, ν_1, ν_2).

Since the integrability conditions $\gamma_{uv}(u,v) = \gamma_{vu}(u,v)$, $\nu_{1uv}(u,v) = \nu_{1vu}(u,v)$, $\nu_{2uv}(u,v) = \nu_{2vu}(u,v)$ and $\mu_{uv}(u,v) = \mu_{vu}(u,v)$, the basic invariants should satisfy some conditions. Note that there are fundamental theorems for one-parameter families of framed curves, namely, the existence and uniqueness theorems for curvatures of one-parameter families of framed curves (cf. [7]).

3. Generalised framed surfaces

We give a definition of a generalisation of framed surfaces and one-parameter families of framed curves. Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a smooth mapping. We denote $\nu = \boldsymbol{x}_u \times \boldsymbol{x}_v$.

Definition 3.1. We say that $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a *generalised framed surface* if there exist smooth functions $\alpha, \beta : U \to \mathbb{R}$ such that $\nu(u,v) = \alpha(u,v)\nu_1(u,v) + \beta(u,v)\nu_2(u,v)$ for all $(u,v) \in U$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *generalised framed base surface* if there exists $(\nu_1, \nu_2) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface.

Remark 3.2. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Then $\nu(u,v) = \boldsymbol{x}_u(u,v) \times \boldsymbol{x}_v(u,v) = (a_1(u,v)b_2(u,v) - a_2(u,v)b_1(u,v))\boldsymbol{n}(u,v)$. If we take $\alpha(u,v) = a_1(u,v)b_2(u,v) - a_2(u,v)b_1(u,v)$ and $\beta(u,v) = 0$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is also a generalised framed surface.

Remark 3.3. Let $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to u with curvature $(\ell, m, n, r, L, M, N, P, Q, R)$. Then $\nu(u,v) = \gamma_u(u,v) \times \gamma_v(u,v) = -r(u,v)Q(u,v)\nu_1(u,v) + r(u,v)P(u,v)\nu_2(u,v)$. If we take $\alpha(u,v) = -r(u,v)Q(u,v)$ and $\beta(u,v) = r(u,v)P(u,v)$, then (γ, ν_1, ν_2) is also a generalised framed surface.

We denote $\nu_3(u,v) = \nu_1(u,v) \times \nu_2(u,v)$. Then $\{\nu_1(u,v), \nu_2(u,v), \nu_3(u,v)\}$ is a moving frame along $\boldsymbol{x}(u,v)$, and we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

$$\begin{pmatrix} \nu_{1u} \\ \nu_{2u} \\ \nu_{3u} \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad \begin{pmatrix} \nu_{1v} \\ \nu_{2v} \\ \nu_{3v} \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

where $a_i, b_i, c_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions with $a_1b_2 - a_2b_1 = 0$. We call the functions *basic invariants* of the generalised framed surface. We denote the above matrices by $\mathcal{G}, \mathcal{F}_1$ and $\mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$. By definition, we have

$$\alpha(u,v) = \det \begin{pmatrix} b_1(u,v) & c_1(u,v) \\ b_2(u,v) & c_2(u,v) \end{pmatrix}, \ \beta(u,v) = -\det \begin{pmatrix} a_1(u,v) & c_1(u,v) \\ a_2(u,v) & c_2(u,v) \end{pmatrix}.$$

Since the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2u} - \mathcal{F}_{1v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should satisfy the following conditions:

$$\begin{cases} a_{1v} - b_1e_2 - c_1f_2 = a_{2u} - b_2e_1 - c_2f_1, \\ b_{1v} + a_1e_2 - c_1g_2 = b_{2u} + a_2e_1 - c_2g_1, \\ c_{1v} + a_1f_2 + b_1g_2 = c_{2u} + a_2f_1 + b_2g_1, \end{cases} \tag{3.1}$$

$$\begin{cases} e_{1v} - f_1 g_2 = e_{2u} - f_2 g_1, \\ f_{1v} - e_2 g_1 = f_{2u} - e_1 g_2, \\ g_{1v} - e_1 f_2 = g_{2u} - e_2 f_1. \end{cases} \tag{3.2}$$

We give fundamental theorems for generalised framed surfaces, that is, the existence and uniqueness theorems for the basic invariants of generalised framed surfaces.

Theorem 3.4 (Existence Theorem for generalised framed surfaces)**.** *Let* $(a_i, b_i, c_i, e_i, f_i, g_i) : I \to \mathbb{R}^{12}, i = 1, 2$ *be a smooth mapping satisfying* $a_1 b_2 - a_2 b_1 = 0$*, satisfying the integrability conditions* (3.1) *and* (3.2)*. Then there exists a generalised framed surface* $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ *whose associated basic invariants are* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$*.*

Proof. Since the integrability conditions (3.1) and (3.2) exist, there exists a smooth mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ and an orthonormal frame $\{\nu_1, \nu_2, \nu_3\}$ such that the condition holds. Therefore, there exists a generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \Delta$ whose associated basic invariants are $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. □

Definition 3.5. Let $(\boldsymbol{x}, \nu_1, \nu_2), (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ be generalised framed surfaces. We say that $(\boldsymbol{x}, \nu_1, \nu_2)$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as generalised framed surfaces* if there exists a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\boldsymbol{x}}(u, v) = A(\boldsymbol{x}(u, v)) + \boldsymbol{a}$, $\widetilde{\nu}_1(u, v) = A(\nu_1(u, v))$ and $\widetilde{\nu}_2(u, v) = A(\nu_2(u, v))$ for all $(u, v) \in U$.

Theorem 3.6 (Uniqueness Theorem for generalised framed surfaces)**.** *Let* $(\boldsymbol{x}, \nu_1, \nu_2), (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ *be generalised framed surfaces with basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$*, respectively. Then* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *and* $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ *are congruent as generalised framed surfaces if and only if the basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *and* $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$ *coincide.*

In order to prove the uniqueness theorem, we prepare the following two lemmas.

Lemma 3.7. *Let* $(\boldsymbol{x}, \nu_1, \nu_2), (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ *be generalised framed surfaces with basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$*, respectively. If* $(\boldsymbol{x}, \nu_1, \nu_2)$ *and* $(\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *are congruent as generalised framed surfaces, then* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2) = (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$*.*

Proof. By Definition 3.1 and a direct calculation, we obtain the lemma. □

Lemma 3.8. *Let* $(\boldsymbol{x}, \nu_1, \nu_2), (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ *be generalised framed surfaces with basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$*, respectively. If* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2) = (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$ *and* $(\boldsymbol{x}, \nu_1, \nu_2)(u_0, v_0) = (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0, v_0)$ *for some point* $(u_0, v_0) \in U$*, then* $(\boldsymbol{x}, \nu_1, \nu_2) = (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)$*.*

Proof. Define a smooth function $f : U \to \mathbb{R}$ by

$$f(u, v) = \nu_1(u, v) \cdot \widetilde{\nu}_1(u, v) + \nu_2(u, v) \cdot \widetilde{\nu}_2(u, v) + \nu_3(u, v) \cdot \widetilde{\nu}_3(u, v).$$

By the definition of the basic invariants, we have

$$\begin{aligned} f_u =& (e_1 - \widetilde{e}_1)\widetilde{\nu}_1 \cdot \nu_2 + (\widetilde{e}_1 - e_1)\nu_1 \cdot \widetilde{\nu}_2 + (f_1 - \widetilde{f}_1)\widetilde{\nu}_1 \cdot \nu_3 + (\widetilde{f}_1 - f_1)\nu_1 \cdot \widetilde{\nu}_3 \\ &+ (g_1 - \widetilde{g}_1)\widetilde{\nu}_2 \cdot \nu_3 + (\widetilde{g}_1 - g_1)\nu_2 \cdot \widetilde{\nu}_3. \end{aligned}$$

By the assumption $(\mathcal{F}_1, \mathcal{F}_2) = (\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$, we have $f_u(u,v) = 0$ for all $(u,v) \in U$. Similarly, we also have $f_v(u,v) = 0$ for all $(u,v) \in U$. Moreover, by the assumption $(\nu_1, \nu_2)(u_0, v_0) = (\widetilde{\nu}_1, \widetilde{\nu}_2)(u_0, v_0)$, we have $f(u_0, v_0) = 3$. It concludes that $f(u,v) = 3$ for all $(u,v) \in U$. Hence, we have $\nu_1 \cdot \widetilde{\nu}_1 = 1,\ \nu_2 \cdot \widetilde{\nu}_2 = 1,\ \nu_3 \cdot \widetilde{\nu}_3 = 1$. It follows that $\nu_1 = \widetilde{\nu}_1, \nu_2 = \widetilde{\nu}_2, \nu_3 = \widetilde{\nu}_3$. Next, we show $\boldsymbol{x} = \widetilde{\boldsymbol{x}}$. By the assumption $\mathcal{G} = \widetilde{\mathcal{G}}$, we have $\boldsymbol{x}_u = a_1\nu_1 + b_1\nu_2 + c_1\nu_3 = \widetilde{a}_1\widetilde{\nu}_1 + \widetilde{b}_1\widetilde{\nu}_2 + \widetilde{c}_1\widetilde{\nu}_3 = \widetilde{\boldsymbol{x}}_u$ and $\boldsymbol{x}_v = a_2\nu_1 + b_2\nu_2 + c_2\nu_3 = \widetilde{a}_2\widetilde{\nu}_1 + \widetilde{b}_2\widetilde{\nu}_2 + \widetilde{c}_2\widetilde{\nu}_3 = \widetilde{\boldsymbol{x}}_v$. Then, we have $(\boldsymbol{x} - \widetilde{\boldsymbol{x}})_u = (\boldsymbol{x} - \widetilde{\boldsymbol{x}})_v = 0$. Since $\boldsymbol{x}(u_0, v_0) = \widetilde{\boldsymbol{x}}(u_0, v_0)$, we have $\boldsymbol{x}(u,v) = \widetilde{\boldsymbol{x}}(u,v)$ for all $(u,v) \in U$. Therefore, we have $(\boldsymbol{x}, \nu_1, \nu_2) = (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)$. □

Proof of the Uniqueness Theorem. The necessary part of the theorem is Lemma 3.7. We prove the sufficient part of the theorem. For fix a point $(u_0, v_0) \in U$, there exist $A \in SO(3)$ and $\boldsymbol{a} \in \mathbb{R}^3$ such that $(\boldsymbol{x}, \nu_1, \nu_2)(u_0, v_0) = (A\widetilde{\boldsymbol{x}} + \boldsymbol{a}, A\widetilde{\nu}_1, A\widetilde{\nu}_2)(u_0, v_0)$. By Lemmas 3.7 and 3.8, we have $(\boldsymbol{x}, \nu_1, \nu_2) = (A\widetilde{\boldsymbol{x}} + \boldsymbol{a}, A\widetilde{\nu}_1, A\widetilde{\nu}_2)$, that is, $(\boldsymbol{x}, \nu_1, \nu_2)$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as generalised framed surfaces. □

By relations among basic invariants of generalised framed surfaces, basic invariants of framed surfaces, and curvatures of one-parameter families of framed curves, we have the following:

Proposition 3.9. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.*

(1) *If $a_1(u,v) = a_2(u,v) = 0$ (respectively, $b_1(u,v) = b_2(u,v) = 0$) for all $(u,v) \in U$, then $(\boldsymbol{x}, \nu_1, \nu_2)$ (respectively, $(\boldsymbol{x}, \nu_2, \nu_1)$) is a framed surface.*

(2) *If $a_1(u,v) = b_1(u,v) = 0$ (respectively, $a_2(u,v) = b_2(u,v) = 0$) for all $(u,v) \in U$, then $(\boldsymbol{x}, \nu_1, \nu_2)$ (respectively, $(\boldsymbol{x}, \nu_2, \nu_1)$) is a one-parameter family of framed curves with respect to u (respectively, with respect to v).*

We give a condition for a surface to become a generalised framed base surface.

Theorem 3.10. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping. We denote $\nu = \boldsymbol{x}_u \times \boldsymbol{x}_v = p_1\boldsymbol{e}_1 + p_2\boldsymbol{e}_2 + p_3\boldsymbol{e}_3$, where $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the canonical basis. Then $\boldsymbol{x}$ is a generalised framed base surface at least locally if and only if the functions p_1, p_2 and p_3 are linearly dependent.*

Proof. If $\boldsymbol{x}$ is a generalised framed base surface at least locally, then there exist $(\nu_1, \nu_2) : U \to \Delta$ and $\alpha, \beta : U \to \mathbb{R}$, such that $\nu = \alpha\nu_1 + \beta\nu_2$. If we denote $\nu_3 = \nu_1 \times \nu_2$, then there exists a rotation $A(u,v) \in SO(3)$, such that ${}^T(\nu_1, \nu_2, \nu_3) = A^T(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, where T is the transpose of the matrix. Therefore,

$$\nu = (p_1, p_2, p_3)\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix} = (\alpha, \beta, 0)\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = (\alpha, \beta, 0)A\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}.$$

It follows that $(\alpha, \beta, 0)A = (p_1, p_2, p_3)$. Since the orthogonal transformation does not change the linearly relation of the set of functions, we have that the functions p_1, p_2 and p_3 are linearly dependent by $\alpha, \beta, 0$.

Conversely, if p_1, p_2, p_3 are linearly dependent, then there exist functions $k_1, k_2, k_3 : U \to \mathbb{R}$ with $(k_1, k_2, k_3) \neq (0,0,0)$ such that $k_1p_1 + k_2p_2 + k_3p_3 = 0$. Without loss of generality, we may assume $k_1 \neq 0$, at least locally. Then we have

$$\nu = p_1\boldsymbol{e}_1 + p_2\boldsymbol{e}_2 + p_3\boldsymbol{e}_3 = p_2\Big(\boldsymbol{e}_2 - \frac{k_2}{k_1}\boldsymbol{e}_1\Big) + p_3\Big(\boldsymbol{e}_3 - \frac{k_3}{k_1}\boldsymbol{e}_1\Big).$$

If we take

$$\nu_1 = \frac{k_2\boldsymbol{e}_1 - k_1\boldsymbol{e}_2}{\sqrt{k_1^2 + k_2^2}},\ \nu_2 = \frac{k_1k_3\boldsymbol{e}_1 + k_2k_3\boldsymbol{e}_2 - (k_1^2 + k_2^2)\boldsymbol{e}_3}{\sqrt{(k_1^2 + k_2^2)(k_1^2 + k_2^2 + k_3^2)}}$$

and

$$\alpha = \frac{-(k_1^2 + k_2^2)p_2 - k_2k_3p_3}{k_1\sqrt{k_1^2 + k_2^2}},\ \beta = -\sqrt{\frac{k_1^2 + k_2^2 + k_3^2}{k_1^2 + k_2^2}}p_3,$$

then $\nu = \alpha\nu_1 + \beta\nu_2$. It follows that $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface, and hence $\boldsymbol{x}$ is a generalised framed base surface, at least locally. □

We also give a condition for a generalised framed surface to become a framed base surface.

Theorem 3.11. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with $\nu = \alpha\nu_1 + \beta\nu_2$.*

(1) *If $\boldsymbol{x}$ is a framed base surface, then the functions α and β are linearly dependent.*

(2) *Suppose that the set of regular points of $\boldsymbol{x}$ is dense in U. If the functions α and β are linearly dependent, then $\boldsymbol{x}$ is a framed base surface, at least locally.*

Proof. (1) If $\boldsymbol{x}$ is a framed base surface, then there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface. Therefore, there exists a smooth function $\ell : U \to \mathbb{R}$ such that $\nu = \alpha\nu_1 + \beta\nu_2 = \ell\boldsymbol{n}$. By $\alpha^2 + \beta^2 = \ell^2$, there exists a smooth function $\theta : U \to \mathbb{R}$ such that $\alpha = \ell\cos\theta, \beta = \ell\sin\theta$. Since $\alpha\sin\theta - \beta\cos\theta = 0$, the functions α and β are linearly dependent.

(2) If α and β are linearly dependent, then there exist functions $k_1, k_2 : U \to \mathbb{R}$ with $(k_1, k_2) \neq (0, 0)$ such that $k_1\alpha + k_2\beta = 0$. Without loss of generality, we may assume $k_1 \neq 0$ at least locally. Since the set of regular points of $\boldsymbol{x}$ is dense in U, we have

$$\nu = \alpha\nu_1 + \beta\nu_2 = \beta\Big(-\frac{k_2}{k_1}\nu_1 + \nu_2\Big).$$

It follows that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface, where

$$\boldsymbol{n} = \frac{-k_2\nu_1 + k_1\nu_2}{\sqrt{k_1^2 + k_2^2}},\ \boldsymbol{s} = \frac{k_1\nu_1 + k_2\nu_2}{\sqrt{k_1^2 + k_2^2}}$$

and hence $\boldsymbol{x}$ is a framed base surface at least locally. □

Corollary 3.12. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Suppose that the set of regular points of $\boldsymbol{x}$ is dense in U.*

(1) *If $\boldsymbol{a}(u, v) = (a_1, a_2)(u, v)$ and $\boldsymbol{b}(u, v) = (b_1, b_2)(u, v)$ are linearly dependent, then $\boldsymbol{x}$ is a framed base surface, at least locally.*

(2) *If* $\mathrm{rank}(\boldsymbol{a}, \boldsymbol{b}) = 1$ *at $p \in U$, then $\boldsymbol{x}$ is a framed base surface around p.*

Proof. (1) By assumption, there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ with $(k_1, k_2) \neq (0, 0)$ such that $k_1\boldsymbol{a} + k_2\boldsymbol{b} = 0$. Since

$$k_2\alpha - k_1\beta = k_2\det(\boldsymbol{b}, \boldsymbol{c}) + k_1\det(\boldsymbol{a}, \boldsymbol{c}) = \det(k_1\boldsymbol{a} + k_2\boldsymbol{b}, \boldsymbol{c}) = 0,$$

α and β are linearly dependent. By Theorem 3.11 (2), $\boldsymbol{x}$ is a framed base surface, at least locally.

(2) By assumption, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent around p. Therefore, we have the result by (1). □

Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with $\nu = \alpha\nu_1 + \beta\nu_2$ and basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We consider other frames by using rotation and reflection.

$$\begin{pmatrix}\widetilde{\nu}_1(u,v)\\ \widetilde{\nu}_2(u,v)\end{pmatrix} = \begin{pmatrix}\cos\theta(u,v) & -\sin\theta(u,v)\\ \sin\theta(u,v) & \cos\theta(u,v)\end{pmatrix}\begin{pmatrix}\nu_1(u,v)\\ \nu_2(u,v)\end{pmatrix},$$
$$\begin{pmatrix}\overline{\nu}_1(u,v)\\ \overline{\nu}_2(u,v)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\nu_1(u,v)\\ \nu_2(u,v)\end{pmatrix}.$$

Proposition 3.13. *Under the above notations, we have the following:*

(1) $(\boldsymbol{x}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ *is also a generalised framed surface with*

$$(\widetilde{\alpha}(u,v), \widetilde{\beta}(u,v)) = (\alpha(u,v), \beta(u,v))\begin{pmatrix}\cos\theta(u,v) & \sin\theta(u,v)\\ -\sin\theta(u,v) & \cos\theta(u,v)\end{pmatrix}$$

and the basic invariants

$$\begin{pmatrix}\widetilde{a}_1(u,v) & \widetilde{b}_1(u,v)\\ \widetilde{a}_2(u,v) & \widetilde{b}_2(u,v)\end{pmatrix} = \begin{pmatrix}a_1(u,v) & b_1(u,v)\\ a_2(u,v) & b_2(u,v)\end{pmatrix}\begin{pmatrix}\cos\theta(u,v) & \sin\theta(u,v)\\ -\sin\theta(u,v) & \cos\theta(u,v)\end{pmatrix},$$
$$\begin{pmatrix}\widetilde{c}_1(u,v)\\ \widetilde{c}_2(u,v)\end{pmatrix} = \begin{pmatrix}c_1(u,v)\\ c_2(u,v)\end{pmatrix}, \begin{pmatrix}\widetilde{e}_1(u,v)\\ \widetilde{e}_2(u,v)\end{pmatrix} = \begin{pmatrix}e_1(u,v) - \theta_u(u,v)\\ e_2(u,v) - \theta_v(u,v)\end{pmatrix},$$
$$\begin{pmatrix}\widetilde{f}_1(u,v) & \widetilde{g}_1(u,v)\\ \widetilde{f}_2(u,v) & \widetilde{g}_2(u,v)\end{pmatrix} = \begin{pmatrix}f_1(u,v) & g_1(u,v)\\ f_2(u,v) & g_2(u,v)\end{pmatrix}\begin{pmatrix}\cos\theta(u,v) & \sin\theta(u,v)\\ -\sin\theta(u,v) & \cos\theta(u,v)\end{pmatrix}.$$

(2) $(\boldsymbol{x}, \overline{\nu}_1, \overline{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ *is also a generalised framed surface with* $(\overline{\alpha}(u,v), \overline{\beta}(u,v)) = (\beta(u,v), \alpha(u,v))$ *and the basic invariants*

$$\begin{pmatrix}\overline{a}_1(u,v) & \overline{b}_1(u,v) & \overline{c}_1(u,v)\\ \overline{a}_2(u,v) & \overline{b}_2(u,v) & \overline{c}_2(u,v)\end{pmatrix} = \begin{pmatrix}b_1(u,v) & a_1(u,v) & -c_1(u,v)\\ b_2(u,v) & a_2(u,v) & -c_2(u,v)\end{pmatrix},$$
$$\begin{pmatrix}\overline{e}_1(u,v) & \overline{f}_1(u,v) & \overline{g}_1(u,v)\\ \overline{e}_2(u,v) & \overline{f}_2(u,v) & \overline{g}_2(u,v)\end{pmatrix} = \begin{pmatrix}-e_1(u,v) & -g_1(u,v) & -f_1(u,v)\\ -e_2(u,v) & -g_2(u,v) & -f_2(u,v)\end{pmatrix}.$$

Proof. (1) Since

$$\begin{aligned}\nu &= \alpha\nu_1 + \beta\nu_2\\ &= \alpha(\cos\theta\widetilde{\nu}_1 + \sin\theta\widetilde{\nu}_2) + \beta(-\sin\theta\widetilde{\nu}_1 + \cos\theta\widetilde{\nu}_2)\\ &= (\alpha\cos\theta - \beta\sin\theta)\widetilde{\nu}_1 + (\alpha\sin\theta + \beta\cos\theta)\widetilde{\nu}_2,\end{aligned}$$

we have $\widetilde{\alpha} = \alpha\cos\theta - \beta\sin\theta$, $\widetilde{\beta} = \alpha\sin\theta + \beta\cos\theta$. Moreover, $\widetilde{\nu}_1 \cdot \widetilde{\nu}_2 = (\cos\theta\nu_1 - \sin\theta\nu_2) \cdot (\sin\theta\nu_1 + \cos\theta\nu_2) = 0$. Therefore, $(\boldsymbol{x}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ is also a generalised framed surface. Since

$$\widetilde{\nu}_3 = \widetilde{\nu}_1 \times \widetilde{\nu}_2 = (\cos\theta\nu_1 - \sin\theta\nu_2) \times (\sin\theta\nu_1 + \cos\theta\nu_2) = \nu_3,$$

we have the basic invariants

$$\begin{aligned}
\widetilde{a}_1 &= \boldsymbol{x}_u \cdot \widetilde{\nu}_1 = \boldsymbol{x}_u \cdot (\cos\theta\nu_1 - \sin\theta\nu_2) = a_1\cos\theta - b_1\sin\theta,\\
\widetilde{a}_2 &= \boldsymbol{x}_v \cdot \widetilde{\nu}_1 = \boldsymbol{x}_v \cdot (\cos\theta\nu_1 - \sin\theta\nu_2) = a_2\cos\theta - b_2\sin\theta,\\
\widetilde{b}_1 &= \boldsymbol{x}_u \cdot \widetilde{\nu}_2 = \boldsymbol{x}_u \cdot (\sin\theta\nu_1 + \cos\theta\nu_2) = a_1\sin\theta + b_1\cos\theta,\\
\widetilde{b}_2 &= \boldsymbol{x}_v \cdot \widetilde{\nu}_2 = \boldsymbol{x}_v \cdot (\sin\theta\nu_1 + \cos\theta\nu_2) = a_2\sin\theta + b_2\cos\theta,\\
\widetilde{c}_1 &= \boldsymbol{x}_u \cdot \widetilde{\nu}_3 = c_1,\\
\widetilde{c}_2 &= \boldsymbol{x}_v \cdot \widetilde{\nu}_3 = c_2,\\
\widetilde{e}_1 &= \widetilde{\nu}_{1u} \cdot \widetilde{\nu}_2 = \big((e_1 - \theta_u)(\sin\theta\nu_1 + \cos\theta\nu_2) + (f_1\cos\theta - g_1\sin\theta)\nu_3\big) \cdot (\sin\theta\nu_1 + \cos\theta\nu_2)\\
&= e_1 - \theta_u,\\
\widetilde{e}_2 &= \widetilde{\nu}_{1v} \cdot \widetilde{\nu}_2 = \big((e_2 - \theta_v)(\sin\theta\nu_1 + \cos\theta\nu_2) + (f_2\cos\theta - g_2\sin\theta)\nu_3\big) \cdot (\sin\theta\nu_1 + \cos\theta\nu_2)\\
&= e_2 - \theta_v,\\
\widetilde{f}_1 &= \widetilde{\nu}_{1u} \cdot \widetilde{\nu}_3 = \big((e_1 - \theta_u)(\sin\theta\nu_1 + \cos\theta\nu_2) + (f_1\cos\theta - g_1\sin\theta)\nu_3\big) \cdot \nu_3 = f_1\cos\theta - g_1\sin\theta,\\
\widetilde{f}_2 &= \widetilde{\nu}_{1v} \cdot \widetilde{\nu}_3 = \big((e_2 - \theta_v)(\sin\theta\nu_1 + \cos\theta\nu_2) + (f_2\cos\theta - g_2\sin\theta)\nu_3\big) \cdot \nu_3 = f_2\cos\theta - g_2\sin\theta,\\
\widetilde{g}_1 &= \widetilde{\nu}_{2u} \cdot \widetilde{\nu}_3 = \big((\theta_u - e_1)(\cos\theta\nu_1 - \sin\theta\nu_2) + (f_1\sin\theta + g_1\cos\theta)\nu_3\big) \cdot \nu_3 = f_1\sin\theta + g_1\cos\theta,\\
\widetilde{g}_2 &= \widetilde{\nu}_{2v} \cdot \widetilde{\nu}_3 = \big((\theta_v - e_2)(\cos\theta\nu_1 - \sin\theta\nu_2) + (f_2\sin\theta + g_2\cos\theta)\nu_3\big) \cdot \nu_3 = f_2\sin\theta + g_2\cos\theta.
\end{aligned}$$

(2) Since $\nu = \alpha\nu_1 + \beta\nu_2 = \beta\overline{\nu}_1 + \alpha\overline{\nu}_2$, we have $\overline{\alpha} = \beta$, $\overline{\beta} = \alpha$. Moreover, $\overline{\nu}_1 \cdot \overline{\nu}_2 = \nu_2 \cdot \nu_1 = 0$. Therefore, $(\boldsymbol{x}, \overline{\nu}_1, \overline{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ is also a generalised framed surface. Since $\overline{\nu}_3 = \overline{\nu}_1 \times \overline{\nu}_2 = \nu_2 \times \nu_1 = -\nu_3$, we have the basic invariants

$$\begin{aligned}
\overline{a}_1 &= \boldsymbol{x}_u \cdot \overline{\nu}_1 = \boldsymbol{x}_u \cdot \nu_2 = b_1,\\
\overline{a}_2 &= \boldsymbol{x}_v \cdot \overline{\nu}_1 = \boldsymbol{x}_v \cdot \nu_2 = b_2,\\
\overline{b}_1 &= \boldsymbol{x}_u \cdot \overline{\nu}_2 = \boldsymbol{x}_u \cdot \nu_1 = a_1,\\
\overline{b}_2 &= \boldsymbol{x}_v \cdot \overline{\nu}_2 = \boldsymbol{x}_v \cdot \nu_1 = a_2,\\
\overline{c}_1 &= \boldsymbol{x}_u \cdot \overline{\nu}_3 = \boldsymbol{x}_u \cdot (-\nu_3) = -c_1,\\
\overline{c}_2 &= \boldsymbol{x}_v \cdot \overline{\nu}_3 = \boldsymbol{x}_v \cdot (-\nu_3) = -c_2,\\
\overline{e}_1 &= \overline{\nu}_{1u} \cdot \overline{\nu}_2 = \nu_{2u} \cdot \nu_1 = -e_1,\\
\overline{e}_2 &= \overline{\nu}_{1v} \cdot \overline{\nu}_2 = \nu_{2v} \cdot \nu_1 = -e_2,\\
\overline{f}_1 &= \overline{\nu}_{1u} \cdot \overline{\nu}_3 = \nu_{2u} \cdot (-\nu_3) = -g_1,\\
\overline{f}_2 &= \overline{\nu}_{1v} \cdot \overline{\nu}_3 = \nu_{2v} \cdot (-\nu_3) = -g_2,\\
\overline{g}_1 &= \overline{\nu}_{2u} \cdot \overline{\nu}_3 = \nu_{1u} \cdot (-\nu_3) = -f_1,\\
\overline{g}_2 &= \overline{\nu}_{2v} \cdot \overline{\nu}_3 = \nu_{1v} \cdot (-\nu_3) = -f_2.
\end{aligned}$$

□

Next, we consider a parameter change in the domain U and a diffeomorphism in the target space $\mathbb{R}^3$.

Proposition 3.14. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Let $\phi : V \to U, (p, q) \mapsto \phi(p, q) = (u(p, q), v(p, q))$ be a parameter change, that is, a* *diffeomorphism of the domain. Then* $(\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) = (\boldsymbol{x}, \nu_1, \nu_2) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ *is a generalised framed surface with*

$$(\widetilde{\alpha}(p,q), \widetilde{\beta}(p,q)) = \left(\alpha(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q), \beta(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\right)$$

and the basic invariants

$$\begin{aligned}
\begin{pmatrix} \widetilde{a}_1 & \widetilde{b}_1 & \widetilde{c}_1 \\ \widetilde{a}_2 & \widetilde{b}_2 & \widetilde{c}_2 \end{pmatrix}(p,q) &= \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}(\phi(p,q)), \\
\begin{pmatrix} \widetilde{e}_1 & \widetilde{f}_1 & \widetilde{g}_1 \\ \widetilde{e}_2 & \widetilde{f}_2 & \widetilde{g}_2 \end{pmatrix}(p,q) &= \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix}(\phi(p,q)).
\end{aligned}$$

Proof. Since

$$\begin{aligned}
\widetilde{\nu}(p,q) &= \widetilde{\boldsymbol{x}}_p(p,q) \times \widetilde{\boldsymbol{x}}_q(p,q) \\
&= \boldsymbol{x}_u(\phi(p,q))u_p(p,q) \times \boldsymbol{x}_v(\phi(p,q))v_q(p,q) - \boldsymbol{x}_u(\phi(p,q))u_q(p,q) \times \boldsymbol{x}_v(\phi(p,q))v_p(p,q) \\
&= \nu(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q) \\
&= (\alpha\nu_1 + \beta\nu_2)(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q) \\
&= \alpha(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\nu_1(\phi(p,q)) + \beta(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\nu_2(\phi(p,q)) \\
&= \alpha(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\widetilde{\nu}_1(p,q) + \beta(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\widetilde{\nu}_2(p,q),
\end{aligned}$$

we have

$$\widetilde{\alpha}(p,q) = \alpha(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q),\ \widetilde{\beta}(p,q) = \beta(\phi(p,q))\det\begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q).$$

Moreover, $\widetilde{\nu}_1(p,q) \cdot \widetilde{\nu}_2(p,q) = \nu_1(\phi(p,q)) \cdot \nu_2(\phi(p,q)) = 0$. Therefore, $(\boldsymbol{x}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ is also a generalised framed surface. By the chain rule, we have

$$\begin{aligned}
\widetilde{\boldsymbol{x}}_p(p,q) &= \boldsymbol{x}_u(\phi(p,q))u_p(p,q) + \boldsymbol{x}_v(\phi(p,q))v_p(p,q) \\
&= \{a_1(\phi(p,q))\nu_1(\phi(p,q)) + b_1(\phi(p,q))\nu_2(\phi(p,q)) + c_1(\phi(p,q))\nu_3(\phi(p,q))\}u_p(p,q) \\
&\quad + \{a_2(\phi(p,q))\nu_1(\phi(p,q)) + b_2(\phi(p,q))\nu_2(\phi(p,q)) + c_2(\phi(p,q))\nu_3(\phi(p,q))\}v_p(p,q) \\
&= \{a_1(\phi(p,q))u_p(p,q) + a_2(\phi(p,q))v_p(p,q)\}\widetilde{\nu}_1(p,q) \\
&\quad + \{b_1(\phi(p,q))u_p(p,q) + b_2(\phi(p,q))v_p(p,q)\}\widetilde{\nu}_2(p,q) \\
&\quad + \{c_1(\phi(p,q))u_p(p,q) + c_2(\phi(p,q))v_p(p,q)\}\widetilde{\nu}_3(p,q), \\
\widetilde{\boldsymbol{x}}_q(p,q) &= \boldsymbol{x}_u(\phi(p,q))u_q(p,q) + \boldsymbol{x}_v(\phi(p,q))v_q(p,q) \\
&= \{a_1(\phi(p,q))\nu_1(\phi(p,q)) + b_1(\phi(p,q))\nu_2(\phi(p,q)) + c_1(\phi(p,q))\nu_3(\phi(p,q))\}u_q(p,q) \\
&\quad + \{a_2(\phi(p,q))\nu_1(\phi(p,q)) + b_2(\phi(p,q))\nu_2(\phi(p,q)) + c_2(\phi(p,q))\nu_3(\phi(p,q))\}v_q(p,q)
\end{aligned}$$

$$\begin{aligned}&= \{a_1(\phi(p,q))u_q(p,q) + a_2(\phi(p,q))v_q(p,q)\}\widetilde{\nu}_1(p,q)\\&\quad + \{b_1(\phi(p,q))u_q(p,q) + b_2(\phi(p,q))v_q(p,q)\}\widetilde{\nu}_2(p,q)\\&\quad + \{c_1(\phi(p,q))u_q(p,q) + c_2(\phi(p,q))v_q(p,q)\}\widetilde{\nu}_3(p,q).\end{aligned}$$

It follows that we have the first equation of the basic invariants. The second equation of the basic invariants can be proved similarly to the above by using the chain rule. □

Proposition 3.15. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface and $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Then there exists a smooth mapping $(\nu_1^\Phi, \nu_2^\Phi) : U \to \Delta$ such that $(\Phi \circ \boldsymbol{x}, \nu_1^\Phi, \nu_2^\Phi) : U \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface.*

Proof. We denote the Jacobi matrix of Φ at x by $D_\Phi(x)$, that is,

$$D_\Phi(x) = \begin{pmatrix} \dfrac{\partial \Phi_1}{\partial x_1}(x) & \dfrac{\partial \Phi_2}{\partial x_1}(x) & \dfrac{\partial \Phi_3}{\partial x_1}(x) \\ \dfrac{\partial \Phi_1}{\partial x_2}(x) & \dfrac{\partial \Phi_2}{\partial x_2}(x) & \dfrac{\partial \Phi_3}{\partial x_2}(x) \\ \dfrac{\partial \Phi_1}{\partial x_3}(x) & \dfrac{\partial \Phi_2}{\partial x_3}(x) & \dfrac{\partial \Phi_3}{\partial x_3}(x) \end{pmatrix}.$$

Since Φ is a diffeomorphism, $D_\Phi(x) \in GL(3, R)$. We define a mapping $(\nu_1^\Phi, \nu_2^\Phi) : U \to \Delta$ by

$$(\nu_1^\Phi, \nu_2^\Phi)(u,v) = \left(\frac{A\nu_1(u,v)}{|A\nu_1(u,v)|}, \frac{(A\nu_1(u,v) \cdot A\nu_1(u,v))A\nu_2(u,v) - (A\nu_1(u,v) \cdot A\nu_2(u,v))A\nu_1(u,v)}{|A\nu_1(u,v)||A\nu_1(u,v) \times A\nu_2(u,v)|} \right),$$

where $A =^T ((D_\Phi)^{-1} \circ \boldsymbol{x})$. Then we show that $(\Phi \circ \boldsymbol{x}, \nu_1^\Phi, \nu_2^\Phi) : U \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface. In fact,

$$\begin{aligned}\nu^\Phi &= (d/du)(\Phi \circ \boldsymbol{x}) \times (d/dv)(\Phi \circ \boldsymbol{x}) = (D_\Phi \circ \boldsymbol{x})\boldsymbol{x}_u \times (D_\Phi \circ \boldsymbol{x})\boldsymbol{x}_v\\&= (\det D_\Phi \circ \boldsymbol{x})A\boldsymbol{x}_u \times \boldsymbol{x}_v = (\det D_\Phi \circ \boldsymbol{x})A\nu = (\det D_\Phi \circ \boldsymbol{x})A(\alpha\nu_1 + \beta\nu_2)\\&= (\det D_\Phi \circ \boldsymbol{x})\frac{\alpha(A\nu_1 \cdot A\nu_1) + \beta(A\nu_1 \cdot A\nu_2)}{|A\nu_1|}\frac{A\nu_1}{|A\nu_1|}\\&\quad + (\det D_\Phi \circ \boldsymbol{x})\frac{\beta|A\nu_1 \times A\nu_2|}{|A\nu_1|}\frac{(A\nu_1 \cdot A\nu_1)A\nu_2 - (A\nu_1 \cdot A\nu_2)A\nu_1}{|A\nu_1||A\nu_1 \times A\nu_2|}\\&= (\det D_\Phi \circ \boldsymbol{x})\frac{\alpha(A\nu_1 \cdot A\nu_1) + \beta(A\nu_1 \cdot A\nu_2)}{|A\nu_1|}\nu_1^\Phi + (\det D_\Phi \circ \boldsymbol{x})\frac{\beta|A\nu_1 \times A\nu_2|}{|A\nu_1|}\nu_2^\Phi.\end{aligned}$$

Thus, $\nu^\Phi = \alpha^\Phi \nu_1^\Phi + \beta^\Phi \nu_2^\Phi$, where

$$\begin{aligned}\alpha^\Phi &= (\det D_\Phi \circ \boldsymbol{x})\frac{\alpha(A\nu_1 \cdot A\nu_1) + \beta(A\nu_1 \cdot A\nu_2)}{|A\nu_1|},\\\beta^\Phi &= (\det D_\Phi \circ \boldsymbol{x})\frac{\beta|A\nu_1 \times A\nu_2|}{|A\nu_1|}.\end{aligned}$$

Moreover,

$$\nu_1^\Phi \cdot \nu_2^\Phi = \frac{A\nu_1}{|A\nu_1|} \cdot \frac{(A\nu_1 \cdot A\nu_1)A\nu_2 - (A\nu_1 \cdot A\nu_2)A\nu_1}{|A\nu_1||A\nu_1 \times A\nu_2|} = 0.$$

Therefore, $(\Phi \circ \boldsymbol{x}, \nu_1^\Phi, \nu_2^\Phi) : U \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface. □

4. Corank one singularities

Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping. Suppose that corank$(d\boldsymbol{x}) = 1$ at a point $p \in U$. By using a parameter change of U, we may assume that $\boldsymbol{x}$ is given by $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$ at least locally, where $f, g : U \to \mathbb{R}$ are smooth functions. Then corank one singularities are always generalised framed base surfaces, at least locally.

Theorem 4.1. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$. Then $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface, where*

$$\begin{aligned}
\nu_1(u, v) &= \frac{(f_u(u, v), -1, 0)}{\sqrt{1 + f_u(u, v)^2}},\\
\nu_2(u, v) &= \frac{(g_u(u, v), f_u(u, v)g_u(u, v), -f_u(u, v)^2 - 1)}{\sqrt{1 + f_u(u, v)^2}\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}}
\end{aligned}$$

with

$$\begin{aligned}
\alpha(u, v) &= \frac{(1 + f_u(u, v)^2)g_v(u, v) - f_u(u, v)f_v(u, v)g_u(u, v)}{\sqrt{1 + f_u(u, v)^2}},\\
\beta(u, v) &= -\frac{f_v(u, v)\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}}{\sqrt{1 + f_u(u, v)^2}}
\end{aligned}$$

and the basic invariants

$$\begin{aligned}
a_1(u, v) &= 0,\\
b_1(u, v) &= 0,\\
c_1(u, v) &= \sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2},\\
a_2(u, v) &= -\frac{f_v(u, v)}{\sqrt{1 + f_u(u, v)^2}},\\
b_2(u, v) &= \frac{f_u(u, v)f_v(u, v)g_u(u, v) - (f_u(u, v)^2 + 1)g_v(u, v)}{\sqrt{1 + f_u(u, v)^2}\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}},\\
c_2(u, v) &= \frac{f_u(u, v)f_v(u, v) + g_u(u, v)g_v(u, v)}{\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}},\\
e_1(u, v) &= \frac{f_{uu}(u, v)g_u(u, v)}{(1 + f_u(u, v)^2)\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}},\\
f_1(u, v) &= \frac{f_{uu}(u, v)}{\sqrt{1 + f_u(u, v)^2}\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}},\\
g_1(u, v) &= \frac{-f_u(u, v)f_{uu}(u, v)g_u(u, v) + (f_u(u, v)^2 + 1)g_{uu}(u, v)}{\sqrt{1 + f_u(u, v)^2}(1 + f_u(u, v)^2 + g_u(u, v)^2)},\\
e_2(u, v) &= \frac{f_{uv}(u, v)g_u(u, v)}{(1 + f_u(u, v)^2)\sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}},
\end{aligned}$$

$$f_2(u,v) = \frac{f_{uv}(u,v)}{\sqrt{1+f_u(u,v)^2}\sqrt{1+f_u(u,v)^2+g_u(u,v)^2}},$$
$$g_2(u,v) = \frac{-f_u(u,v)f_{uv}(u,v)g_u(u,v)+(f_u(u,v)^2+1)g_{uv}(u,v)}{\sqrt{1+f_u(u,v)^2}(1+f_u(u,v)^2+g_u(u,v)^2)}.$$

Proof. Since $\boldsymbol{x}_u(u,v) = (1, f_u(u,v), g_u(u,v))$, $\boldsymbol{x}_v(u,v) = (0, f_v(u,v), g_v(u,v))$, we have

$$\begin{aligned} \nu(u,v) &= \boldsymbol{x}_u(u,v) \times \boldsymbol{x}_v(u,v) \\ &= (f_u(u,v)g_v(u,v) - f_v(u,v)g_u(u,v), -g_v(u,v), f_v(u,v)). \end{aligned}$$

By $\boldsymbol{x}_u(u,v) \neq 0$ and $\boldsymbol{x}_u(u,v) \cdot \nu(u,v) = 0$ for all $(u,v) \in U$, we have that the components of $\nu(u,v)$ are linearly dependent. Therefore, $\boldsymbol{x}$ is a generalised framed base surface by Theorem 3.10. By direct calculation, we have the basic invariants. □

Example 4.2 (Cross cap)**.** Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$,

$$\boldsymbol{x}(u,v) = (u, v^2, uv),\ \nu_1(u,v) = (0,-1,0),\ \nu_2(u,v) = \frac{1}{\sqrt{1+v^2}}(v, 0, -1).$$

Note that $\boldsymbol{x}$ at 0 is a cross cap singular point (cf. [21]). Then $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ. By $f(u,v) = v^2$, $g(u,v) = uv$ in Theorem 4.1, we have $\alpha(u,v) = u$, $\beta(u,v) = -2v\sqrt{1+v^2}$ and the basic invariants

$$\begin{pmatrix} a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_2(u,v) & b_2(u,v) & c_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{1+v^2} \\ -2v & \dfrac{-u}{\sqrt{1+v^2}} & \dfrac{uv}{\sqrt{1+v^2}} \end{pmatrix},$$
$$\begin{pmatrix} e_1(u,v) & f_1(u,v) & g_1(u,v) \\ e_2(u,v) & f_2(u,v) & g_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{1+v^2} \end{pmatrix}.$$

Example 4.3 ($S_1^\pm$ singular point)**.** Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$,

$$\boldsymbol{x}(u,v) = (u, v^2, u^2v \pm v^3),\ \nu_1(u,v) = (0,-1,0),\ \nu_2(u,v) = \frac{1}{\sqrt{1+4u^2v^2}}(2uv, 0, 1).$$

Note that $\boldsymbol{x}$ at 0 is a $S_1^\pm$ singular point (cf. [14,17]). Then $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ. By $f(u,v) = v^2$, $g(u,v) = u^2v \pm v^3$ in Theorem 4.1, we have $\alpha(u,v) = u^2 \pm 3v^2$, $\beta(u,v) = -2v\sqrt{1+4u^2v^2}$ and the basic invariants

$$\begin{pmatrix} a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_2(u,v) & b_2(u,v) & c_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{1+4u^2v^2} \\ -2v & \dfrac{-(u^2 \pm 3v^2)}{\sqrt{1+4u^2v^2}} & \dfrac{2uv(u^2 \pm 3v^2)}{\sqrt{1+4u^2v^2}} \end{pmatrix},$$
$$\begin{pmatrix} e_1(u,v) & f_1(u,v) & g_1(u,v) \\ e_2(u,v) & f_2(u,v) & g_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dfrac{2v}{1+4u^2v^2} \\ 0 & 0 & \dfrac{2u}{1+4u^2v^2} \end{pmatrix}.$$

Remark 4.4. The $\mathcal{A}$-simple singularities of a map from a 2-dimensional manifold to a 3-dimensional one are also of corank one; see [14]. We can treat them as generalised framed surfaces.

We say that $\boldsymbol{x}$ at p is a cross cap singular point (respectively, $S_1^\pm$ singular point) if $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent (that is, right-left equivalent) to $(u, v) \mapsto (u, v^2, uv)$ (respectively, $(u, v) \mapsto (u, v^2, v(u^2 \pm v^2))$). By using the criteria of cross cap and $S_1^\pm$ singular points, we have the following:

Proposition 4.5. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface which is given by the form of Theorem 4.1. Suppose that p is a singular point of $\boldsymbol{x}$; that is, $a_2(p) = b_2(p) = c_2(p) = 0$. Then we have the following:*

(1) $\boldsymbol{x}$ at p is a cross cap singular point if and only if $-f_2 b_{2v} + g_2 a_{2v} \neq 0$ at p.

(2) $\boldsymbol{x}$ at p is a S_1^+ singular point if and only if $a_{2u} b_{2v} - b_{2u} a_{2v} = 0$, $(a_{2v}, b_{2v}) \neq (0, 0)$ and $H < 0$ at p, where

$$H = \Big(a_{2uu} b_{2v} - b_{2uu} a_{2v} + 2a_{2u} b_{2uv} - 2b_{2u} a_{2uv} + 2(a_{2u}^2 + b_{2u}^2)e_2\Big)$$
$$\Big(a_{2v} b_{2vv} - a_{2vv} b_{2v} + 2(a_{2v}^2 + b_{2v}^2)e_2\Big) - (a_{2u} b_{2vv} - a_{2vv} b_{2u} + 2(a_{2u} a_{2v} + b_{2u} b_{2v})e_2)^2 .$$

(3) $\boldsymbol{x}$ at p is a S_1^- singular point if and only if $a_{2u} b_{2v} - b_{2u} a_{2v} = 0$ and $H > 0$ at p.

Proof. Since $f_v(p) = g_v(p) = 0$, we have $a_2(p) = b_2(p) = c_2(p) = 0$. Note that $c_1(p) \neq 0$.

(1) By $\boldsymbol{x}_u = c_1 \nu_3,\ \boldsymbol{x}_v = a_2 \nu_1 + b_2 \nu_2 + c_2 \nu_3$,

$$\boldsymbol{x}_{uv} = -c_1 f_2 \nu_1 - c_1 g_2 \nu_2 + c_{1v} \nu_3,$$
$$\boldsymbol{x}_{vv} = (a_{2v} - b_2 e_2 - c_2 f_2)\nu_1 + (b_{2v} + a_2 e_2 - c_2 g_2)\nu_2 + (c_{2v} + a_2 f_2 + b_2 g_2)\nu_3,$$

we have $\det(\boldsymbol{x}_u, \boldsymbol{x}_{uv}, \boldsymbol{x}_{vv})(p) = c_1^2(p)(-f_2(p) b_{2v}(p) + g_2(p) a_{2v}(p))$. By the criterion of the cross cap singular point in [21], we have the result.

(2) We denote $\varphi = \det(\boldsymbol{x}_u, \boldsymbol{x}_v, \boldsymbol{x}_{vv})$. By the calculation of (1) and

$$\boldsymbol{x}_{vv} = (a_{2v} - b_2 e_2 - c_2 f_2)\nu_1 + (a_2 e_2 + b_{2v} - c_2 g_2)\nu_2 + (a_2 f_2 + b_2 g_2 + c_{2v})\nu_3,$$

we have $\varphi = c_1(a_2(a_2 e_2 + b_{2v} - c_2 g_2) - b_2(a_{2v} - b_2 e_2 - c_2 f_2))$. It follows that

$$\begin{aligned}
\varphi_u = {} & c_{1u}(a_2(a_2 e_2 + b_{2v} - c_2 g_2) - b_2(a_{2v} - b_2 e_2 - c_2 f_2)) \\
& + c_1(a_{2u}(a_2 e_2 + b_{2v} - c_2 g_2) + a_2(a_2 e_2 + b_{2v} - c_2 g_2)_u \\
& - b_{2u}(a_{2v} - b_2 e_2 - c_2 f_2) - b_2(a_{2v} - b_2 e_2 - c_2 f_2)_u), \\
\varphi_v = {} & c_{1v}(a_2(a_2 e_2 + b_{2v} - c_2 g_2) - b_2(a_{2v} - b_2 e_2 - c_2 f_2)) \\
& + c_1(a_{2v}(a_2 e_2 + b_{2v} - c_2 g_2) + a_2(a_2 e_2 + b_{2v} - c_2 g_2)_v \\
& - b_{2v}(a_{2v} - b_2 e_2 - c_2 f_2) - b_2(a_{2v} - b_2 e_2 - c_2 f_2)_v).
\end{aligned}$$

Therefore, $\varphi_u(p) = c_1(p)(a_{2u}(p) b_{2v}(p) - b_{2u}(p) a_{2v}(p))$ and $\varphi_v(p) = 0$. By the integrability condition (3.1) of the generalised framed surface, $-c_1(p) f_2(p) = a_{2u}(p), -c_1(p) g_2(p) = b_{2u}(p)$, and $c_{1v}(p) = c_{2u}(p)$. Hence $d\varphi(p) = 0$ if and only if $a_{2u}(p) b_{2v}(p) - b_{2u}(p) a_{2v}(p) = 0$ (equivalently, $f_2(p) b_{2v}(p) - g_2(p) a_{2v}(p) = 0$). By a direct calculation, we have

$$\varphi_{uu} = c_1(a_{2uu} b_{2v} - b_{2uu} a_{2v} + 2a_{2u} b_{2uv} - 2b_{2u} a_{2uv} + 2(a_{2u}^2 + b_{2u}^2)e_2),$$

$$\varphi_{uv} = c_1(a_{2u}b_{2vv} - a_{2vv}b_{2u} + 2(a_{2u}a_{2v} + b_{2u}b_{2v})e_2),$$
$$\varphi_{vv} = c_1(a_{2v}b_{2vv} - a_{2vv}b_{2v} + 2(a_{2v}^2 + b_{2v}^2)e_2)$$

at p. By the criterion of the $S_1^\pm$ singular point in [17], $\boldsymbol{x}$ at p is a S_1^+ singular point if and only if $d\varphi(p) = 0$, $\boldsymbol{x}_u$ and $\boldsymbol{x}_{vv}$ are linearly independent at p, and $H = \varphi_{uu}(p)\varphi_{vv}(p) - \varphi_{uv}^2(p) < 0$. We have the result.

(3) $\boldsymbol{x}$ at p is a S_1^- singular point if and only if $d\varphi(p) = 0$ and $H = \varphi_{uu}(p)\varphi_{vv}(p) - \varphi_{uv}^2(p) > 0$. We have the result. □

Remark 4.6. By the integrability condition (3.1), the condition $-f_2b_{2v} + g_2a_{2v} \neq 0$ at p in Proposition 4.5 (1) is equivalent to the condition $a_{2u}b_{2v} - b_{2u}a_{2v} \neq 0$ at p. That is, $d\varphi(p) \neq 0$ (cf. [17]).

5. Corank two singularities

Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping. Suppose that $\mathrm{corank}(d\boldsymbol{x}) = 2$ at a point $p \in U$. We consider one of the components of $\boldsymbol{x}(u, v)$ to be 2-jet; that is, by using parameter change and up to sign, $\boldsymbol{x}(u, v)$ is given by

$$(i)\ \left(\frac{1}{2}(u^2 + v^2), f(u, v), g(u, v)\right),$$
$$(ii)\ \left(\frac{1}{2}(u^2 - v^2), f(u, v), g(u, v)\right),$$
$$(iii)\ \left(\frac{1}{2}u^2, f(u, v), g(u, v)\right),$$

where $f, g : U \to \mathbb{R}$ are smooth functions. By direct calculation, $\nu(u, v)$ is given by

$$(i)\ (f_u(u, v)g_v(u, v) - f_v(u, v)g_u(u, v), -(ug_v(u, v) - vg_u(u, v)), uf_v(u, v) - vf_u(u, v)),$$
$$(ii)\ (f_u(u, v)g_v(u, v) - f_v(u, v)g_u(u, v), -(ug_v(u, v) + vg_u(u, v)), uf_v(u, v) + vf_u(u, v)),$$
$$(iii)\ (f_u(u, v)g_v(u, v) - f_v(u, v)g_u(u, v), -ug_v(u, v), uf_v(u, v)),$$

respectively. By Theorem 3.10, $\boldsymbol{x}$ is a generalised framed base surface at least locally if and only if the components of $\nu(u, v)$ are linearly dependent.

As special cases, we consider two of the components of $\boldsymbol{x}(u, v)$ to be 2-jet.

Proposition 5.1. *Let* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *be given by*

$$\boldsymbol{x}(u, v) = \left(\frac{1}{2}(u^2 + v^2), \frac{1}{2}(u^2 - v^2), g(u, v)\right)$$

and $j^2g(0) = 0$. *Then we have the following:*

(1) $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a generalised framed base surface germ if and only if there exists a function* $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh$ *or* $g_v = vh$.

(2) *Suppose that* $\boldsymbol{x}$ *is a generalised framed base surface germ. Then* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a framed base surface germ if and only if there exist functions* $h_1, h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh_1$ *and* $g_v = vh_2$.

Proof. (1) We show the sufficient part of the proposition. Since

$$\boldsymbol{x}_u(u,v) = (u, u, g_u(u,v)),\ \boldsymbol{x}_v(u,v) = (v, -v, g_v(u,v)),$$

we have $\nu(u,v) = (ug_v(u,v)+vg_u(u,v), vg_u(u,v)-ug_v(u,v), -2uv)$. If there exists a function $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $g_u = uh$ or $g_v = vh$, then we have

$$(ug_v(u,v) + vg_u(u,v)) + (vg_u(u,v) - ug_v(u,v)) + h(u,v)(-2uv) = 0$$

or

$$(ug_v(u,v) + vg_u(u,v)) - (vg_u(u,v) - ug_v(u,v)) + h(u,v)(-2uv) = 0$$

for all $(u,v) \in (\mathbb{R}^2, 0)$. It follows that the components of $\nu(u,v)$ are linearly dependent. By Theorem 3.10, $\boldsymbol{x}$ is a generalised framed base surface germ. In fact, we can take

$$\nu_1(u,v) = \frac{1}{\sqrt{2}}(1,-1,0),\ \nu_2(u,v) = \frac{(h(u,v), h(u,v), -2)}{\sqrt{2}\sqrt{h(u,v)^2+2}}$$

with

$$\alpha(u,v) = \sqrt{2}(-(vg_u(u,v) - ug_v(u,v)) + uvh(u,v)),\ \beta(u,v) = \sqrt{2}uv\sqrt{h(u,v)^2+2} \tag{5.1}$$

$i = 1,2$. Therefore, $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface germ.

Conversely, if $\boldsymbol{x}$ is a generalised framed surface germ, we have $ug_v + vg_u, vg_u - ug_v$ and $-2uv$ are linearly dependent. Then there exist functions $k_1, k_2, k_3 : (\mathbb{R}^2, 0) \to \mathbb{R}$ with $(k_1, k_2, k_3) \neq 0$ at 0, such that $k_1(ug_v + vg_u) + k_2(vg_u - ug_v) + k_3(-2uv) = 0$. By $j^2 g(0) = 0$, we have $k_1(0) \neq 0$ or $k_2(0) \neq 0$. Without loss of generality, we assume $k_1(0) \neq 0$. Then we have

$$v\left(1 + \frac{k_2(u,v)}{k_1(u,v)}\right) g_u(u,v) + u\left(1 - \frac{k_2(u,v)}{k_1(u,v)}\right) g_v(u,v) - 2\frac{k_3(u,v)}{k_1(u,v)} uv = 0$$

for all $(u,v) \in (\mathbb{R}^2, 0)$. It follows that $v\,(1 + k_2(0,v)/k_1(0,v))\, g_u(0,v) = 0$ for all $v \in (\mathbb{R}, 0)$ and $u\,(1 - k_2(u,0)/k_1(u,0))\, g_v(u,0) = 0$ for all $u \in (\mathbb{R}, 0)$. If $1 + k_2(0,0)/k_1(0,0) \neq 0$, we have $vg_u(0,v) = 0$. By the continuous condition, $g_u(0,v) = 0$ for all $v \in (\mathbb{R}, 0)$. Thus, there exists a function h, such that $g_u(u,v) = uh(u,v)$. If $1 - k_2(0,0)/k_1(0,0) \neq 0$, we have $ug_v(u,0) = 0$. By the continuous condition, $g_v(u,0) = 0$ for all $u \in (\mathbb{R}, 0)$. Thus, there exists a function $h : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that $g_v(u,v) = vh(u,v)$.

(2) We show the sufficient part of the proposition. If there exist functions $h_1, h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $g_u = uh_1$ and $g_v = vh_2$, then $vg_u - ug_v = uv(h_1 - h_2)$. It follows that α and β are linearly dependent from Eq (5.1). Since the set of regular points of $\boldsymbol{x}$ is dense, according to Theorem 3.11 (2), $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ is a framed base surface germ.

Conversely, since $\boldsymbol{x}$ is a generalised framed base surface germ, we have $g_u = uh_1$ or $g_v = vh_1$. Without loss of generality, we assume $g_u = uh_1$. By Theorem 3.11 (1), α and β are linearly dependent. Then there exists a function $k : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that $vg_u(u,v) - ug_v(u,v) = k(u,v)uv$. It follows that

$$ug_v(u,v) = uvh_1(u,v) - k(u,v)uv = uv(h_1(u,v) - k(u,v)).$$

Thus, $ug_v(u,0) = 0$ for all $u \in (\mathbb{R}, 0)$. Then there exists a function $h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $g_v(u,v) = vh_2(u,v)$ for all $(u,v) \in (\mathbb{R}^2, 0)$. □

Proposition 5.2. *Let* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *be given by*

$$\boldsymbol{x}(u, v) = \left(\frac{1}{2}u^2, \frac{1}{2}v^2, g(u, v)\right)$$

and $j^2 g(0) = 0$. *Then we have the following:*

(1) $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a generalised framed base surface germ if and only if there exists a function* $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh$ *or* $g_v = vh$.

(2) *Suppose that* $\boldsymbol{x}$ *is a generalised framed base surface germ. Then* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a framed base surface germ if and only if there exist functions* $h_1, h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh_1$ *and* $g_v = vh_2$.

Proof. (1) We show the sufficient part of the proposition. Since

$$\boldsymbol{x}_u(u, v) = (u, 0, g_u(u, v)),\ \boldsymbol{x}_v(u, v) = (0, v, g_v(u, v)),$$

we have $\nu(u, v) = (-vg_u(u, v), -ug_v(u, v), uv)$. If there exists a function $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $g_u = uh$, we have $-vg_u(u, v) + h(u, v)uv = 0$. It follows that the components of $\nu(u, v)$ are linearly dependent. By Theorem 3.10, $\boldsymbol{x}$ is a generalised framed base surface germ. In fact, we can take

$$\nu_1(u, v) = (0, -1, 0),\ \nu_2(u, v) = \frac{(h(u, v), 0, -1)}{\sqrt{h(u, v)^2 + 1}}$$

with

$$\alpha(u, v) = -ug_v(u, v),\ \beta(u, v) = -uv\sqrt{h(u, v)^2 + 1}. \tag{5.2}$$

Therefore, $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface germ.

If there exists a function $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $g_v = vh$, we have $-ug_v(u, v) + h(u, v)uv = 0$. It follows that the components of $\nu(u, v)$ are linearly dependent. By Theorem 3.10, $\boldsymbol{x}$ is a generalised framed base surface germ. In fact, we can take

$$\nu_1(u, v) = (1, 0, 0),\ \nu_2(u, v) = \frac{(0, h(u, v), -1)}{\sqrt{h(u, v)^2 + 1}}$$

with

$$\alpha(u, v) = -vg_u(u, v) - h(u, v)uv,\ \beta(u, v) = -uv\sqrt{h(u, v)^2 + 1}. \tag{5.3}$$

Therefore, $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface germ.

Conversely, if $\boldsymbol{x}$ is a generalised framed surface germ, we have $-vg_u, -ug_v$ and uv are linearly dependent. Then there exist functions $(k_1, k_2, k_3) \neq 0$ at 0, such that $k_1(-vg_u) + k_2(-ug_v) + k_3uv = 0$. By $j^2 g(0) = 0$, we have $k_1(0) \neq 0$ or $k_2(0) \neq 0$. If $k_1(0) \neq 0$, we have

$$-vg_u(u, v) - \frac{k_2(u, v)}{k_1(u, v)}ug_v(u, v) + \frac{k_3(u, v)}{k_1(u, v)}uv = 0$$

for all $(u, v) \in (\mathbb{R}^2, 0)$. It follows that $vg_u(0, v) = 0$ for all $v \in (\mathbb{R}, 0)$. Thus, there exists a function $h : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that $g_u(u, v) = uh(u, v)$. If $k_2(0) \neq 0$, we have

$$-\frac{k_1(u,v)}{k_2(u,v)} v g_u(u,v) - u g_v(u,v) + \frac{k_3(u,v)}{k_2(u,v)} uv = 0$$

for all $(u, v) \in (\mathbb{R}^2, 0)$. It follows that $ug_v(u, 0) = 0$ for all $u \in (\mathbb{R}, 0)$. Thus, there exists a function $h : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that $g_v(u, v) = vh(u, v)$.

(2) We show the sufficient part of the proposition. If there exist functions $h_1, h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $g_u = uh_1$ and $g_v = vh_2$, then $\alpha(u, v) = -uvh_2(u, v)$ or $\alpha(u, v) = -uv(h_1(u, v) + h_2(u, v))$. It follows that α and β are linearly dependent by Eqs (5.2) and (5.3). Since the set of regular points of $\boldsymbol{x}$ is dense, according to Theorem 3.11 (2), $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ is a framed base surface germ.

Conversely, since $\boldsymbol{x}$ is a generalised framed base surface germ, we have $g_u = uh_1$ or $g_v = vh_1$. Without loss of generality, we assume $g_u = uh_1$. By Theorem 3.11 (1), α and β are linearly dependent. Then there exists a function $k : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that $ug_v(u, v) = k(u, v)uv$ by Eq (5.2). Thus, $ug_v(u, 0) = 0$ for all $u \in (\mathbb{R}, 0)$. Then there exists a function $h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that $g_v(u, v) = vh_2(u, v)$ for all $(u, v) \in (\mathbb{R}^2, 0)$. □

Example 5.3. Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}(u,v) = \left(\frac{1}{2}u^2, \frac{1}{2}v^2, u^{k+2}v\right),\ \nu_1(u,v) = (0,-1,0),\ \nu_2(u,v) = \frac{((k+2)u^k v, 0, -1)}{\sqrt{(k+2)^2u^{2k}v^2+1}},$$

where k is a natural number. Note that 0 is a corank two singular point of $\boldsymbol{x}$. By $g(u, v) = u^{k+2}v$ in Proposition 5.2 (1), $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ with

$$\alpha(u,v) = -u^{k+3},\ \beta(u,v) = -uv\sqrt{(k+2)^2u^{2k}v^2+1}$$

and the basic invariants

$$\begin{pmatrix} a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_2(u,v) & b_2(u,v) & c_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & u\sqrt{(k+2)^2u^{2k}v^2+1} \\ -v & \dfrac{-u^{k+2}}{\sqrt{(k+2)^2u^{2k}v^2+1}} & \dfrac{(k+2)u^{2k+2}v}{\sqrt{(k+2)^2u^{2k}v^2+1}} \end{pmatrix},$$

$$\begin{pmatrix} e_1(u,v) & f_1(u,v) & g_1(u,v) \\ e_2(u,v) & f_2(u,v) & g_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dfrac{k(k+2)u^{k-1}v}{(k+2)^2u^{2k}v^2+1} \\ 0 & 0 & \dfrac{(k+2)u^k}{(k+2)^2u^{2k}v^2+1} \end{pmatrix}.$$

By Proposition 5.2 (2), $\boldsymbol{x}$ is not a framed base surface germ.

6. Parallel surfaces of generalised framed surface

Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with $\nu = \alpha\nu_1 + \beta\nu_2$ and basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We consider parallel surfaces of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$.

Definition 6.1. We say that $\boldsymbol{x}^\lambda : U \to \mathbb{R}^3$, $\boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\nu$ is a *parallel surface* of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$, where λ is a non-zero constant.

Remark 6.2. If there exist functions $\widetilde{\alpha}, \widetilde{\beta}, \ell : U \to \mathbb{R}$, such that $\alpha = \ell\widetilde{\alpha}, \beta = \ell\widetilde{\beta}$, then we take $\nu = \ell(\widetilde{\alpha}\nu_1 + \widetilde{\beta}\nu_2) = \ell\widetilde{\nu}$. In this case, we consider the parallel surface as $\boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\widetilde{\nu}$. Moreover, if the set of regular points of $\boldsymbol{x}$ is dense in U and, α and β are linearly dependent, then there exists a function $\widetilde{\ell}$ such that $\nu = \widetilde{\ell}\boldsymbol{n}$, where $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface (cf. Theorem 3.11). In this case, the parallel surface is given by $\boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\boldsymbol{n}$.

Since $\boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\nu$, we have

$$\boldsymbol{x}^\lambda_u = (a_1 + \lambda(\alpha_u - \beta e_1))\nu_1 + (b_1 + \lambda(\beta_u + \alpha e_1))\nu_2 + (c_1 + \lambda(\alpha f_1 + \beta g_1))\nu_3, \tag{6.1}$$

$$\boldsymbol{x}^\lambda_v = (a_2 + \lambda(\alpha_v - \beta e_2))\nu_1 + (b_2 + \lambda(\beta_v + \alpha e_2))\nu_2 + (c_2 + \lambda(\alpha f_2 + \beta g_2))\nu_3. \tag{6.2}$$

It follows that

$$\begin{aligned}
\nu^\lambda = &\Big(\alpha + \lambda(\alpha(b_1 f_2 - b_2 f_1) + \beta(b_1 g_2 - b_2 g_1) + (c_2\beta_u - c_1\beta_v) + \alpha(e_1 c_2 - e_2 c_1)) \\
&+ \lambda^2(\alpha\beta(e_1 g_2 - e_2 g_1) + \alpha^2(e_1 f_2 - e_2 f_1) + \alpha(f_2\beta_u - f_1\beta_v) + \beta(g_2\beta_u - g_1\beta_v))\Big)\nu_1 \\
&+ \Big(\beta + \lambda(\alpha(a_2 f_1 - a_1 f_2) + \beta(a_2 g_1 - a_1 g_2) + (c_1\alpha_v - c_2\alpha_u) + \beta(e_1 c_2 - e_2 c_1)) \\
&+ \lambda^2(\alpha\beta(e_1 f_2 - e_2 f_1) + \alpha^2(e_1 g_2 - e_2 g_1) + \alpha(f_1\alpha_v - f_2\alpha_u) + \beta(g_1\alpha_v - g_2\alpha_u))\Big)\nu_2 \\
&+ \lambda\Big(\alpha(a_1 e_2 - a_2 e_1) + \beta(b_1 e_2 - b_2 e_1) + (a_1\beta_v - a_2\beta_u) + (b_2\alpha_u - b_1\alpha_v) \\
&+ \lambda(\alpha(e_2\alpha_u - e_1\alpha_v) + \beta(e_2\beta_u - e_1\beta_v) + (\alpha_u\beta_v - \alpha_v\beta_u))\Big)\nu_3.
\end{aligned}$$

By Theorem 3.10, $\boldsymbol{x}^\lambda$ is a generalised framed base surface at least locally if and only if the components of ν^λ are linearly dependent.

By Remark 6.2, if $k : U \to \mathbb{R}$ is a non-zero function, then $\alpha = \ell\widetilde{\alpha} = (\ell/k)k\widetilde{\alpha}, \beta = \ell\widetilde{\beta} = (\ell/k)k\widetilde{\beta}$. Therefore, $\widetilde{\nu}$ is not unique, that is, $\boldsymbol{x}^\lambda[k] = \boldsymbol{x} + \lambda k\widetilde{\nu}$.

Proposition 6.3. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, p) \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with $k\widetilde{\nu} = k\widetilde{\alpha}\nu_1 + k\widetilde{\beta}\nu_2$ and basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Suppose that $(\widetilde{\alpha}, \widetilde{\beta})(p) = 0$. Then we have the following:*

(1) *If $(a_1, a_2)(p) = (b_1, b_2)(p) = 0$ and $(\widetilde{\alpha}_u, \widetilde{\alpha}_v, \widetilde{\beta}_u, \widetilde{\beta}_v)(p) \neq 0$, then $\boldsymbol{x}^\lambda[k]$ is also a generalised framed base surface around p.*

(2) *If $(c_1, c_2)(p) \neq 0$, then $\boldsymbol{x}^\lambda[k]$ is also a generalised framed base surface around p.*

Proof. Since $\boldsymbol{x}^\lambda[k] = \boldsymbol{x} + \lambda k\widetilde{\nu}$, we have

$$\begin{aligned}
\boldsymbol{x}^\lambda[k]_u &= (a_1 + \lambda(k_u\widetilde{\alpha} + k\widetilde{\alpha}_u - k\widetilde{\beta}e_1))\nu_1 + (b_1 + \lambda(k_u\widetilde{\beta} + k\widetilde{\beta}_u + k\widetilde{\alpha}e_1))\nu_2 \\
&\quad + (c_1 + \lambda(k\widetilde{\alpha}f_1 + k\widetilde{\beta}g_1))\nu_3, && (6.3)\\
\boldsymbol{x}^\lambda[k]_v &= (a_2 + \lambda(k_v\widetilde{\alpha} + k\widetilde{\alpha}_v - k\widetilde{\beta}e_2))\nu_1 + (b_2 + \lambda(k_v\widetilde{\beta} + k\widetilde{\beta}_v + k\widetilde{\alpha}e_2))\nu_2 \\
&\quad + (c_2 + \lambda(k\widetilde{\alpha}f_2 + k\widetilde{\beta}g_2))\nu_3. && (6.4)
\end{aligned}$$

If $(\widetilde{\alpha}, \widetilde{\beta})(p) = 0$, $(a_1, a_2)(p) = (b_1, b_2)(p) = 0$ and $(\widetilde{\alpha}_u, \widetilde{\alpha}_v, \widetilde{\beta}_u, \widetilde{\beta}_v)(p) \neq 0$, or $(\widetilde{\alpha}, \widetilde{\beta})(p) = 0$ and $(c_1, c_2)(p) \neq 0$, then $(\boldsymbol{x}^\lambda[k]_u, \boldsymbol{x}^\lambda[k]_v)(p) \neq 0$ by the Eqs (6.3) and (6.4). It follows that the components of $\nu^\lambda[k]$ are linearly dependent by $\boldsymbol{x}^\lambda[k]_u(u, v) \cdot \nu^\lambda[k](u, v) = 0$ and $\boldsymbol{x}^\lambda[k]_v(u, v) \cdot \nu^\lambda[k](u, v) = 0$ for all $(u, v) \in (\mathbb{R}^2, p)$. By Theorem 3.10, $\boldsymbol{x}^\lambda[k]$ is a generalised framed base surface around p. □

Proposition 6.4. *Let* $(\boldsymbol{x},\nu_1,\nu_2):(\mathbb{R}^2,0)\to\mathbb{R}^3\times\Delta$ *be a generalised framed surface germ, which is given by the form of Theorem 4.1. Suppose that* $f_u(0)=f_v(0)=g_u(0)=g_v(0)=0$*. Then* $(\boldsymbol{x}^\lambda,\nu_1^\lambda,\nu_2^\lambda):(\mathbb{R}^2,0)\to\mathbb{R}^3\times\Delta$ *is also a generalised framed surface germ, where*

$$\nu_1^\lambda=\frac{(f_u-\lambda g_{uv},-1-\lambda(f_ug_v-f_vg_u)_u,0)}{\sqrt{(1+\lambda(f_ug_v-f_vg_u)_u)^2+(f_u-\lambda g_{uv})^2}},$$

$$\nu_2^\lambda=\frac{\left(\dfrac{(g_u+\lambda f_{uv})}{1+\lambda(f_ug_v-f_vg_u)_u},\dfrac{(f_u-\lambda g_{uv})(g_u+\lambda f_{uv})}{(1+\lambda(f_ug_v-f_vg_u)_u)^2},-\dfrac{(f_u-\lambda g_{uv})^2}{(1+\lambda(f_ug_v-f_vg_u)_u)^2}-1\right)}{\sqrt{1+\dfrac{(f_u-\lambda g_{uv})^2}{(1+\lambda(f_ug_v-f_vg_u)_u)^2}}\sqrt{1+\dfrac{(f_u-\lambda g_{uv})^2}{(1+\lambda(f_ug_v-f_vg_u)_u)^2}+\dfrac{(g_u+\lambda f_{uv})^2}{(1+\lambda(f_ug_v-f_vg_u)_u)^2}}}$$

with

$$\begin{aligned}\alpha^\lambda=&\frac{1}{\sqrt{(1+\lambda(f_ug_v-f_vg_u)_u)^2((1+\lambda(f_ug_v-f_vg_u)_u)^2+(f_u-\lambda g_{uv})^2)}}\\&(-\lambda(f_ug_v-f_vg_u)_v(g_u+\lambda f_{uv})(1+\lambda(f_ug_v-f_vg_u)_u)-(f_v-\lambda g_{vv})(g_u+\lambda f_{uv})(f_u-\lambda g_{uv})\\&+(g_v+\lambda f_{vv})((1+\lambda(f_ug_v-f_vg_u)_u)^2+(f_u-\lambda g_{uv})^2),\\\beta^\lambda=&\frac{\sqrt{(1+\lambda(f_ug_v-f_vg_u)_u)^2+(f_u-\lambda g_{uv})^2+(g_u+\lambda f_{uv})^2}}{\sqrt{(1+\lambda(f_ug_v-f_vg_u)_u)^2+(f_u-\lambda g_{uv})^2}}\\&(-(f_v-\lambda g_{vv})(1+\lambda(f_ug_v-f_vg_u)_u)+\lambda(f_u-\lambda g_{uv})(f_ug_v-f_vg_u)_v).\end{aligned}$$

Proof. Since $\boldsymbol{x}^\lambda=(u+\lambda(f_ug_v-f_vg_u),f-\lambda g_v,g+\lambda f_v)$, we have

$$\begin{aligned}\boldsymbol{x}_u^\lambda&=(1+\lambda(f_ug_v-f_vg_u)_u,f_u-\lambda g_{uv},g_u+\lambda f_{uv}),\\\boldsymbol{x}_v^\lambda&=(\lambda(f_ug_v-f_vg_u)_v,f_v-\lambda g_{vv},g_v+\lambda f_{vv}).\end{aligned}$$

Then

$$\begin{aligned}\nu^\lambda=((&f_u-\lambda g_{uv})(g_v+\lambda f_{vv})-(g_u+\lambda f_{uv})(f_v-\lambda g_{vv}),\\&\lambda(f_ug_v-f_vg_u)_v(g_u+\lambda f_{uv})-(1+\lambda(f_ug_v-f_vg_u)_u)(g_v+\lambda f_{vv}),\\&(1+\lambda(f_ug_v-f_vg_u)_u)(f_v-\lambda g_{vv})-\lambda(f_ug_v-f_vg_u)_v(f_u-\lambda g_{uv})).\end{aligned}$$

Since $f_u(0)=f_v(0)=g_u(0)=g_v(0)=0$, we have $\boldsymbol{x}_u^\lambda(0)=(1,-\lambda g_{uv}(0),\lambda f_{uv}(0))$. It follows that the components of ν^λ are linearly dependent by $\boldsymbol{x}_u^\lambda(u,v)\cdot\nu^\lambda(u,v)=0$ for all $(u,v)\in(\mathbb{R}^2,0)$. By Theorem 3.10, $\boldsymbol{x}^\lambda$ is a generalised framed base surface germ. □

Corollary 6.5. *Let* $(\boldsymbol{x},\nu_1,\nu_2):(\mathbb{R}^2,0)\to\mathbb{R}^3\times\Delta$ *be a generalised framed surface germ, which is given by the form of Theorem 4.1. Suppose that the set of regular points of* $\boldsymbol{x}^\lambda$ *is dense in* U*. Under the same assumptions in Proposition 6.4, we have the following:*

(1) *If* $g_{vv}(0)\neq 0$*, then* $\boldsymbol{x}^\lambda$ *is a framed base surface germ.*

(2) *If* $g_{vv}(0)=0$ *and* $f_{vv}(0)\neq 0$*, then* $\boldsymbol{x}^\lambda$ *is a framed base surface germ.*

Proof. By Proposition 6.4, we have

$$\alpha^\lambda(0) = \frac{1}{\sqrt{1+\lambda^2 g_{uv}(0)^2}}\Big(-\lambda(1+\lambda^2 g_{uv}(0)^2)f_{vv}(0) - \lambda^3 g_{uv}(0) f_{uv}(0) g_{vv}(0)\Big),$$
$$\beta^\lambda(0) = \frac{\sqrt{1+\lambda^2 f_{uv}(0)^2 + \lambda^2 g_{uv}(0)^2}}{\sqrt{1+\lambda^2 g_{uv}(0)^2}}\lambda g_{vv}(0).$$

If $g_{vv}(0) \neq 0$, then $\beta^\lambda(0) \neq 0$. It follows that α^λ and β^λ are linearly dependent around 0. Also, if $g_{vv}(0) = 0$ and $f_{vv}(0) \neq 0$, then $\alpha^\lambda(0) \neq 0$. It follows that α^λ and β^λ are linearly dependent around 0. By Theorem 3.11 (2), $\boldsymbol{x}^\lambda$ is a framed base surface germ. □

We define the other type of parallel surfaces of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$.

Definition 6.6. We say that $\boldsymbol{x}^\lambda[\theta] : U \to \mathbb{R}^3$, $\boldsymbol{x}^\lambda[\theta] = \boldsymbol{x} + \lambda(\cos\theta\nu_1 + \sin\theta\nu_2)$ is a *θ-parallel surface* of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$, where λ is a non-zero constant and θ is a constant.

Remark 6.7. Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. If $a_1 = a_2 = 0$ (respectively, $b_1 = b_2 = 0$), then $(\boldsymbol{x}, \nu_1, \nu_2)$ (respectively, $(\boldsymbol{x}, \nu_2, \nu_1)$) is a framed surface by Proposition 3.9. That is $\boldsymbol{n} = \nu_1$ (respectively, $\boldsymbol{n} = \nu_2$). It follows that $\boldsymbol{x}^\lambda[0] = \boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\boldsymbol{n}$ (respectively, $\boldsymbol{x}^\lambda[\pi/2] = \boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\boldsymbol{n}$).

Since $\boldsymbol{x}^\lambda[\theta] = \boldsymbol{x} + \lambda(\cos\theta\nu_1 + \sin\theta\nu_2)$, we have

$$\boldsymbol{x}^\lambda[\theta]_u = (a_1 - \lambda e_1\sin\theta)\nu_1 + (b_1 + \lambda e_1\cos\theta)\nu_2 + (c_1 + \lambda f_1\cos\theta + \lambda g_1\sin\theta)\nu_3,$$
$$\boldsymbol{x}^\lambda[\theta]_v = (a_2 - \lambda e_2\sin\theta)\nu_1 + (b_2 + \lambda e_2\cos\theta)\nu_2 + (c_2 + \lambda f_2\cos\theta + \lambda g_2\sin\theta)\nu_3.$$

It follows that

$$\begin{aligned}\nu^\lambda[\theta] = \Big(&\alpha + \lambda((e_1c_2 - e_2c_1)\cos\theta + (b_1f_2 - b_2f_1)\cos\theta + (b_1g_2 - b_2g_1)\sin\theta)\\ &+ \lambda^2((e_1f_2 - e_2f_1)\cos^2\theta + (e_1g_2 - e_2g_1)\cos\theta\sin\theta)\Big)\nu_1\\ &\Big(\beta + \lambda((f_1a_2 - f_2a_1)\cos\theta + (g_1a_2 - g_2a_1)\sin\theta + (e_1c_2 - e_2c_1)\sin\theta)\\ &+ \lambda^2((e_1f_2 - e_2f_1)\sin\theta\cos\theta + (e_1g_2 - e_2g_1)\sin^2\theta)\Big)\nu_2\\ &+ \lambda\Big((a_1e_2 - a_2e_1)\cos\theta + (b_1e_2 - b_2e_1)\sin\theta\Big)\nu_3.\end{aligned}$$

By Theorem 3.10, $\boldsymbol{x}^\lambda[\theta]$ is a generalised framed base surface at least locally if and only if the components of $\nu^\lambda[\theta]$ are linearly dependent.

Proposition 6.8. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface germ, which is given by the form of Theorem 4.1. Suppose that $f_u(0) = f_v(0) = g_u(0) = g_v(0) = 0$. Then we have the following:*

(1) *If $f_{uu}(0)\cos\theta + g_{uu}(0)\sin\theta = 0$, then $\boldsymbol{x}^\lambda[\theta]$ is a generalised framed base surface germ.*

(2) *If $f_{uv}(0)\cos\theta + g_{uv}(0)\sin\theta \neq 0$, then $\boldsymbol{x}^\lambda[\theta]$ is a generalised framed base surface germ.*

Proof. Since $\boldsymbol{x}^\lambda[\theta] = \boldsymbol{x} + \lambda(\cos\theta\nu_1 + \sin\theta\nu_2)$, we have

$$\begin{aligned}
\boldsymbol{x}^\lambda[\theta]_u =& -\lambda\frac{f_{uu}g_u\sin\theta}{(1+f_u^2)\sqrt{1+f_u^2+g_u^2}}\nu_1+\lambda\frac{f_{uu}g_u\cos\theta}{(1+f_u^2)\sqrt{1+f_u^2+g_u^2}}\nu_2\\
&+\Big(\sqrt{1+f_u^2+g_u^2}+\lambda\frac{f_{uu}\cos\theta}{\sqrt{1+f_u^2}\sqrt{1+f_u^2+g_u^2}}+\lambda\frac{(-f_uf_{uu}g_u+(1+f_u^2)g_{uu})\sin\theta}{\sqrt{1+f_u^2}(1+f_u^2+g_u^2)}\Big)\nu_3,\\
\boldsymbol{x}^\lambda[\theta]_v =& \Big(-\frac{f_v}{\sqrt{1+f_u^2}}-\lambda\frac{f_{uv}g_u\sin\theta}{(1+f_u^2)\sqrt{1+f_u^2+g_u^2}}\Big)\nu_1\\
&+\Big(\frac{f_uf_vg_u-(1+f_u^2)g_v}{\sqrt{1+f_u^2}\sqrt{1+f_u^2+g_u^2}}+\lambda\frac{f_{uv}g_u\cos\theta}{(1+f_u^2)\sqrt{1+f_u^2+g_u^2}}\Big)\nu_2\\
&+\Big(\frac{f_uf_v+g_ug_v}{\sqrt{1+f_u^2+g_u^2}}+\lambda\frac{f_{uv}\cos\theta}{\sqrt{1+f_u^2}\sqrt{1+f_u^2+g_u^2}}+\lambda\frac{(-f_uf_{uv}g_u+(1+f_u^2)g_{uv})\sin\theta}{\sqrt{1+f_u^2}(1+f_u^2+g_u^2)}\Big)\nu_3.
\end{aligned}$$

It follows that

$$\begin{aligned}
\boldsymbol{x}^\lambda[\theta]_u(0) &= (1+\lambda f_{uu}(0)\cos\theta+\lambda g_{uu}(0)\sin\theta)\nu_3(0),\\
\boldsymbol{x}^\lambda[\theta]_v(0) &= \lambda(f_{uv}(0)\cos\theta+g_{uv}(0)\sin\theta)\nu_3(0).
\end{aligned}$$

If $f_{uu}(0)\cos\theta+g_{uu}(0)\sin\theta=0$ or $f_{uv}(0)\cos\theta+g_{uv}(0)\sin\theta\neq 0$, then 0 is a corank one singular point of $\boldsymbol{x}^\lambda[\theta]$. By Proposition 3.14 and Theorem 4.1, $\boldsymbol{x}^\lambda[\theta]$ is a generalised framed base surface germ. □

We consider special cases where ν_2 or ν_1 is a constant. If ν_2 (respectively, ν_1) is a constant, then we denote $\boldsymbol{x}_1^\lambda=\boldsymbol{x}^\lambda[0]=\boldsymbol{x}+\lambda\nu_1$ (respectively, $\boldsymbol{x}_2^\lambda=\boldsymbol{x}^\lambda[\pi/2]=\boldsymbol{x}+\lambda\nu_2$).

Proposition 6.9. *Let $(\boldsymbol{x},\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ be a generalised framed surface with $\nu=\alpha\nu_1+\beta\nu_2$ and basic invariants $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$.*

(1) *If ν_2 is a constant, then $(\boldsymbol{x}_1^\lambda,\nu_1,\nu_2)$ is also a generalised framed surface with $\widetilde{\alpha}^\lambda=\alpha+\lambda(b_1f_2-b_2f_1)$, $\widetilde{\beta}^\lambda=\beta+\lambda(a_2f_1-a_1f_2)$ and the basic invariants*

$$\begin{pmatrix}\widetilde{a}_1^\lambda & \widetilde{b}_1^\lambda & \widetilde{c}_1^\lambda\\ \widetilde{a}_2^\lambda & \widetilde{b}_2^\lambda & \widetilde{c}_2^\lambda\end{pmatrix}=\begin{pmatrix}a_1 & b_1 & c_1+\lambda f_1\\ a_2 & b_2 & c_2+\lambda f_2\end{pmatrix},\ \begin{pmatrix}\widetilde{e}_1^\lambda & \widetilde{f}_1^\lambda & \widetilde{g}_1^\lambda\\ \widetilde{e}_2^\lambda & \widetilde{f}_2^\lambda & \widetilde{g}_2^\lambda\end{pmatrix}=\begin{pmatrix}0 & f_1 & 0\\ 0 & f_2 & 0\end{pmatrix}.$$

(2) *If ν_1 is a constant, then $(\boldsymbol{x}_2^\lambda,\nu_1,\nu_2)$ is also a generalised framed surface with $\widetilde{\alpha}^\lambda=\alpha+\lambda(b_1g_2-b_2g_1)$, $\widetilde{\beta}^\lambda=\beta+\lambda(a_2g_1-a_1g_2)$ and the basic invariants*

$$\begin{pmatrix}\widetilde{a}_1^\lambda & \widetilde{b}_1^\lambda & \widetilde{c}_1^\lambda\\ \widetilde{a}_2^\lambda & \widetilde{b}_2^\lambda & \widetilde{c}_2^\lambda\end{pmatrix}=\begin{pmatrix}a_1 & b_1 & c_1+\lambda g_1\\ a_2 & b_2 & c_2+\lambda g_2\end{pmatrix},\ \begin{pmatrix}\widetilde{e}_1^\lambda & \widetilde{f}_1^\lambda & \widetilde{g}_1^\lambda\\ \widetilde{e}_2^\lambda & \widetilde{f}_2^\lambda & \widetilde{g}_2^\lambda\end{pmatrix}=\begin{pmatrix}0 & 0 & g_1\\ 0 & 0 & g_2\end{pmatrix}.$$

Proof. (1) Since ν_2 is a constant and $\boldsymbol{x}_1^\lambda=\boldsymbol{x}+\lambda\nu_1$, we have

$$\boldsymbol{x}_{1u}^\lambda=a_1\nu_1+b_1\nu_2+(c_1+\lambda f_1)\nu_3,\ \boldsymbol{x}_{1v}^\lambda=a_2\nu_1+b_2\nu_2+(c_2+\lambda f_2)\nu_3.$$

By a direct calculation, we have

$$\widetilde{\nu}^\lambda=\boldsymbol{x}_{1u}^\lambda\times\boldsymbol{x}_{1v}^\lambda=(\alpha+\lambda(b_1f_2-b_2f_1))\nu_1+(\beta+\lambda(a_2f_1-a_1f_2))\nu_2.$$

It follows that $(\boldsymbol{x}_1^\lambda, \nu_1, \nu_2)$ is also a generalised framed surface with $\widetilde{\alpha}^\lambda = \alpha + \lambda(b_1 f_2 - b_2 f_1)$, $\widetilde{\beta}^\lambda = \beta + \lambda(a_2 f_1 - a_1 f_2)$. By a direct calculation, we have the basic invariants.

(2) Since ν_1 is a constant and $\boldsymbol{x}_2^\lambda = \boldsymbol{x} + \lambda\nu_2$, we have

$$\boldsymbol{x}_{2u}^\lambda = a_1\nu_1 + b_1\nu_2 + (c_1 + \lambda g_1)\nu_3,\ \boldsymbol{x}_{2v}^\lambda = a_2\nu_1 + b_2\nu_2 + (c_2 + \lambda g_2)\nu_3.$$

By a direct calculation, we have

$$\widetilde{\nu}^\lambda = \boldsymbol{x}_{2u}^\lambda \times \boldsymbol{x}_{2v}^\lambda = (\alpha + \lambda(b_1 g_2 - b_2 g_1))\nu_1 + (\beta + \lambda(a_2 g_1 - a_1 g_2))\nu_2.$$

It follows that $(\boldsymbol{x}_2^\lambda, \nu_1, \nu_2)$ is also a generalised framed surface with $\widetilde{\alpha}^\lambda = \alpha + \lambda(b_1 g_2 - b_2 g_1)$, $\widetilde{\beta}^\lambda = \beta + \lambda(a_2 g_1 - a_1 g_2)$. By a direct calculation, we have the basic invariants. □

We give a relation between parallel surfaces $\boldsymbol{x}^\lambda$ and θ-parallel surfaces $\boldsymbol{x}^\lambda[\theta]$.

Proposition 6.10. *Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface, and $\nu = \alpha\nu_1 + \beta\nu_2$.*

(1) *If $\boldsymbol{x}^\lambda = \boldsymbol{x}^\lambda[\theta]$, then $\boldsymbol{x}$ is a regular surface.*

(2) *Suppose that the set of regular points of $\boldsymbol{x}$ is dense in U and $\alpha = \ell\widetilde{\alpha}, \beta = \ell\widetilde{\beta}$. If $\boldsymbol{x}^\lambda = \boldsymbol{x}^\lambda[\theta]$, where $\boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\widetilde{\nu}$, then $\boldsymbol{x}$ is a framed base surface at least locally.*

Proof. (1) By assumption, $\alpha = \cos\theta$ and $\beta = \sin\theta$. Since $(\alpha, \beta) \neq (0, 0)$, $\boldsymbol{x}$ is a regular surface and hence a framed base surface.

(2) By assumption, $\widetilde{\alpha} = \cos\theta$ and $\widetilde{\beta} = \sin\theta$. Then α and β are linearly dependent. By Theorem 3.11 (2), $\boldsymbol{x}$ is a framed base surface at least locally. □

By Proposition 6.10 (2), if $\ell(u_0, v_0) = 0$, then (u_0, v_0) is a singular point of $\boldsymbol{x}$. The relation between two parallel surfaces $\boldsymbol{x}^\lambda$ and $\boldsymbol{x}^\lambda[\theta]$ measures not only regular surfaces, but also framed base surfaces.

Example 6.11 (Cross cap)**.** Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$,

$$\boldsymbol{x}(u, v) = (u, v^2, uv),\ \nu_1(u, v) = (0, -1, 0),\ \nu_2(u, v) = \frac{1}{\sqrt{1 + v^2}}(v, 0, -1).$$

Then $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ with $\alpha(u, v) = u, \beta(u, v) = -2v\sqrt{1 + v^2}$, see Example 4.2.

We consider the parallel surfaces of $(\boldsymbol{x}, \nu_1, \nu_2)$. Let $(\boldsymbol{x}^\lambda, \nu_1^\lambda, \nu_2^\lambda) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}^\lambda(u, v) = (u - 2\lambda v^2, v^2 - \lambda u, uv + 2\lambda v),$$
$$\nu_1^\lambda(u, v) = \frac{(-\lambda, -1, 0)}{\sqrt{1 + \lambda^2}},\ \nu_2^\lambda(u, v) = \frac{(v, -\lambda v, -1 - \lambda^2)}{\sqrt{(1 + \lambda^2)(1 + v^2 + \lambda^2)}}.$$

Then $(\boldsymbol{x}^\lambda, \nu_1^\lambda, \nu_2^\lambda)$ is a generalised framed surface germ with

$$\alpha^\lambda(u, v) = \frac{6\lambda v^2 + (1 + \lambda^2)(u + 2\lambda)}{\sqrt{1 + \lambda^2}},\ \beta^\lambda(u, v) = \frac{2(2\lambda^2 - 1)v\sqrt{1 + v^2 + \lambda^2}}{\sqrt{1 + \lambda^2}}.$$

In fact, $(\boldsymbol{x}^\lambda, \boldsymbol{n}, \boldsymbol{s}) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ is a framed surface germ, where

$$\boldsymbol{n} = \frac{\alpha^\lambda\nu_1^\lambda + \beta^\lambda\nu_2^\lambda}{\sqrt{\alpha^{\lambda 2} + \beta^{\lambda 2}}},\ \boldsymbol{s} = \frac{-\beta^\lambda\nu_1^\lambda + \alpha^\lambda\nu_2^\lambda}{\sqrt{\alpha^{\lambda 2} + \beta^{\lambda 2}}}.$$

The curvature of $(\boldsymbol{x}^\lambda, \boldsymbol{n}, \boldsymbol{s})$ is given by

$$J^{F^\lambda}(u,v) = \frac{\sqrt{(6\lambda v^2 + (u+2\lambda)(\lambda^2+1))^2 + 4(2\lambda^2-1)^2 v^2(1+v^2+\lambda^2)}}{\sqrt{1+\lambda^2}},$$
$$K^{F^\lambda}(u,v) = -\frac{4(2\lambda^2-1)^2(1+\lambda^2)^{\frac{3}{2}} v^2}{((6\lambda v^2 + (u+2\lambda)(\lambda^2+1))^2 + 4(2\lambda^2-1)^2 v^2(1+v^2+\lambda^2))^{\frac{3}{2}}},$$
$$H^{F^\lambda}(u,v) = -\frac{2(2\lambda^2-1)((1+\lambda^2)((1+\lambda^2)(u+2\lambda) + 2v^2(u-\lambda)) + (-4\lambda+u)\sqrt{1+\lambda^2}v^2)}{(6\lambda v^2 + (u+2\lambda)(\lambda^2+1))^2 + 4(2\lambda^2-1)^2 v^2(1+v^2+\lambda^2)}.$$

Moreover, $\boldsymbol{x}^\lambda$ is a regular surface germ, since $J^{F^\lambda}(0) = \sqrt{4\lambda^2(1+\lambda^2)} \neq 0$.

Next, we consider θ-parallel surface of $(\boldsymbol{x}, \nu_1, \nu_2)$. Since ν_1 is a constant, we consider

$$\boldsymbol{x}_2^\lambda(u,v) = \boldsymbol{x}(u,v) + \lambda\nu_2(u,v) = \left(u + \lambda\frac{v}{\sqrt{1+v^2}}, v^2, uv - \lambda\frac{1}{\sqrt{1+v^2}}\right).$$

By a direct calculation, we have

$$\widetilde{\nu}^\lambda(u,v) = u\nu_1(u,v) - 2v\sqrt{1+v^2}\nu_2(u,v).$$

It follows that $(\boldsymbol{x}_2^\lambda, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface germ with

$$\widetilde{\alpha}^\lambda(u,v) = u,\ \widetilde{\beta}^\lambda(u,v) = -2v\sqrt{1+v^2}$$

and the basic invariants

$$\begin{pmatrix} \widetilde{a}_1^\lambda(u,v) & \widetilde{b}_1^\lambda(u,v) & \widetilde{c}_1^\lambda(u,v) \\ \widetilde{a}_2^\lambda(u,v) & \widetilde{b}_2^\lambda(u,v) & \widetilde{c}_2^\lambda(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{1+4u^2v^2} + \lambda\frac{2v}{1+4u^2v^2} \\ -2v & \frac{-(u^2 \pm 3v^2)}{\sqrt{1+4u^2v^2}} & \frac{2uv(u^2 \pm 3v^2)}{\sqrt{1+4u^2v^2}} + \lambda\frac{2u}{1+4u^2v^2} \end{pmatrix},$$
$$\begin{pmatrix} \widetilde{e}_1^\lambda(u,v) & \widetilde{f}_1^\lambda(u,v) & \widetilde{g}_1^\lambda(u,v) \\ \widetilde{e}_2^\lambda(u,v) & \widetilde{f}_2^\lambda(u,v) & \widetilde{g}_2^\lambda(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{2v}{1+4u^2v^2} \\ 0 & 0 & \frac{2u}{1+4u^2v^2} \end{pmatrix}.$$

Moreover, we can see that the functions $\widetilde{\alpha}^\lambda$ and $\widetilde{\beta}^\lambda$ are not linearly dependent. By Theorem 3.11 (1), $\boldsymbol{x}_2^\lambda$ is not a framed base surface germ.

Example 6.12 (H_k singular point)**.** Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}(u,v) = (u, uv + v^{3k-1}, v^3),\ \nu_1(u,v) = \frac{(-v, 1, 0)}{\sqrt{1+v^2}},\ \nu_2(u,v) = (0,0,1),$$

where k is a natural number with $k \geq 2$. Note that 0 is a H_k singular point of $\boldsymbol{x}$ (cf. [14]). Then $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ with

$$\alpha(u,v) = -3v^2\sqrt{1+v^2},\ \beta(u,v) = u + (3k-1)v^{3k-2}$$

and the basic invariants

$$\begin{pmatrix} a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_2(u,v) & b_2(u,v) & c_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{1+v^2} \\ \dfrac{u+(3k-1)v^{3k-2}}{\sqrt{1+v^2}} & 3v^2 & \dfrac{v(u+(3k-1)v^{3k-2})}{\sqrt{1+v^2}} \end{pmatrix},$$

$$\begin{pmatrix} e_1(u,v) & f_1(u,v) & g_1(u,v) \\ e_2(u,v) & f_2(u,v) & g_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{1}{1+v^2} & 0 \end{pmatrix}.$$

We consider the parallel surface of $(\boldsymbol{x}, \nu_1, \nu_2)$. Let $(\boldsymbol{x}^\lambda, \nu_1^\lambda, \nu_2^\lambda) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}^\lambda(u,v) = (u + 3\lambda v^3, uv + v^{3k-1} - 3\lambda v^2, v^3 + \lambda(u + (3k-1)v^{3k-2})),$$
$$\nu_1^\lambda(u,v) = \frac{(v,-1,0)}{\sqrt{1+v^2}},\ \nu_2^\lambda(u,v) = \frac{(\lambda, \lambda v, -(1+v^2))}{\sqrt{1+v^2}\sqrt{1+\lambda^2+v^2}}.$$

Note that 0 is a corank one singular point of $\boldsymbol{x}^\lambda$. Then $(\boldsymbol{x}^\lambda, \nu_1^\lambda, \nu_2^\lambda)$ is a generalised framed surface germ with

$$\alpha^\lambda(u,v) = \frac{-\lambda uv - 3(\lambda^2-1)v^2 + 3v^4 + \lambda(3k-1)(3k-2)v^{3k-3} + \lambda(3k-1)(3k-3)v^{3k-1}}{\sqrt{1+v^2}},$$
$$\beta^\lambda(u,v) = \frac{\sqrt{1+v^2+\lambda^2}}{\sqrt{1+v^2}}(-u + 6\lambda v + 9\lambda v^3 - (3k-1)v^{3k-2})$$

and the basic invariants

$$a_1{}^\lambda(u,v) = 0,\ b_1{}^\lambda(u,v) = 0,\ c_1{}^\lambda(u,v) = \sqrt{1+\lambda^2+v^2},$$
$$a_2{}^\lambda(u,v) = \frac{-u + 6\lambda v + 9\lambda v^3 - (3k-1)v^{3k-2}}{\sqrt{1+v^2}},$$
$$b_2{}^\lambda(u,v) = \frac{\lambda uv + 3(\lambda^2-1)v^2 - 3v^4 - \lambda(3k-1)(3k-2)v^{3k-3} - \lambda(3k-1)(3k-3)v^{3k-1}}{\sqrt{1+v^2}\sqrt{1+\lambda^2+v^2}},$$
$$c_2{}^\lambda(u,v) = \frac{uv + 6\lambda v^2 + \lambda^2(3k-1)(3k-2)v^{3k-3} + (3k-1)v^{3k-1}}{\sqrt{1+\lambda^2+v^2}},$$
$$e_1{}^\lambda(u,v) = 0,\ f_1{}^\lambda(u,v) = 0,\ g_1{}^\lambda(u,v) = 0,$$
$$e_2{}^\lambda(u,v) = \frac{\lambda}{(1+v^2)\sqrt{1+\lambda^2+v^2}},\ f_2{}^\lambda(u,v) = \frac{1}{1+v^2},\ g_2{}^\lambda(u,v) = \frac{-\lambda v}{\sqrt{1+v^2}(1+\lambda^2+v^2)}.$$

Moreover, we can see that the functions α^λ and β^λ are not linearly dependent. By Theorem 3.11 (1), $\boldsymbol{x}^\lambda$ is not a framed base surface germ.

Next, we consider θ-parallel surface of $(\boldsymbol{x}, \nu_1, \nu_2)$. Since ν_2 is a constant, we consider

$$\boldsymbol{x}_1^\lambda(u,v) = \boldsymbol{x}(u,v) + \lambda\nu_1(u,v) = \left(u - \frac{\lambda v}{\sqrt{1+v^2}}, uv + v^{3k-1} + \frac{\lambda}{\sqrt{1+v^2}}, v^3\right).$$

By a direct calculation, we have

$$\widetilde{\nu}^\lambda(u,v) = -3v^2\sqrt{1+v^2}\nu_1(u,v) + (u + (3k-1)v^{3k-2})\nu_2(u,v).$$

It follows that $(\boldsymbol{x}_1^\lambda, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ is a generalised framed surface germ with

$$\widetilde{\alpha}^\lambda(u, v) = -3v^2\sqrt{1+v^2},\ \widetilde{\beta}^\lambda(u, v) = u + (3k-1)v^{3k-2}$$

and the basic invariants

$$\begin{pmatrix} \widetilde{a}_1^\lambda(u,v) & \widetilde{b}_1^\lambda(u,v) & \widetilde{c}_1^\lambda(u,v) \\ \widetilde{a}_2^\lambda(u,v) & \widetilde{b}_2^\lambda(u,v) & \widetilde{c}_2^\lambda(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{1+v^2} \\ \dfrac{u+(3k-1)v^{3k-2}}{\sqrt{1+v^2}} & 3v^2 & \dfrac{v(u+(3k-1)v^{3k-2})}{\sqrt{1+v^2}} - \dfrac{\lambda}{1+v^2} \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{e}_1^\lambda(u,v) & \widetilde{f}_1^\lambda(u,v) & \widetilde{g}_1^\lambda(u,v) \\ \widetilde{e}_2^\lambda(u,v) & \widetilde{f}_2^\lambda(u,v) & \widetilde{g}_2^\lambda(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{1}{1+v^2} & 0 \end{pmatrix}.$$

Moreover, we can see that the functions $\widetilde{\alpha}^\lambda$ and $\widetilde{\beta}^\lambda$ are not linearly dependent. By Theorem 3.11 (1), $\boldsymbol{x}_1^\lambda$ is not a framed base surface germ.

Author contributions

Writing-original draft preparation, M.T. and H.Y.; writing-review and editing, M.T. and H.Y.; All authors equally contributed to this work. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. V. I. Arnol'd, *Singularities of caustics and wave fronts*, Mathematics and Its Applications, Vol. 62, Kluwer Academic Publishers, 1990. https://doi.org/10.1007/978-94-011-3330-2
2. V. I. Arnol'd, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of differentiable maps*, Vol. I, Birkhäuser Boston, 1986.

3. E. Abbena, S. Salamon, A. Gray, *Modern differential geometry of curves and surfaces with mathematica,* 3 Eds., Studies in Advanced Mathematics, Chapman and Hall/CRC, Boca Raton, FL, 2006. https://doi.org/10.1201/9781315276038
4. R. L. Bishop, There is more than one way to frame a curve, *Am. Math. Mon.*, **82** (1975), 246–251. https://doi.org/10.1080/00029890.1975.11993807
5. J. W. Bruce, P. J. Giblin, *Curves and singularities*, 2 Eds., Cambridge University Press, Cambridge, 1992. https://doi.org/10.1017/CBO9781139172615
6. T. Fukunaga, M. Takahashi, Framed surfaces in the Euclidean space, *Bull. Braz. Math. Soc. (N.S.)*, **50** (2019), 37–65. https://doi.org/10.1007/s00574-018-0090-z
7. T. Fukunaga, M. Takahashi, Framed surfaces and one-parameter families of framed curves in Euclidean 3-space, *J. Singul.*, **21** (2020), 30–49. https://doi.org/10.5427/jsing.2020.21c
8. S. Honda, M. Takahashi, Framed curves in the Euclidean space, *Adv. Geom.*, **16** (2016), 265–276. https://doi.org/10.1515/advgeom-2015-0035
9. G. Ishikawa, *Singularities of curves and surfaces in various geometric problems*, CAS Lecture Notes 10, Exact Sciences, 2015.
10. G. Ishikawa, Recognition problem of frontal singularities, *J. Singul.*, **21** (2020), 149–166. https://doi.org/10.5427/jsing.2020.21i
11. S. Izumiya, M. C. Romero-Fuster, M. A. S. Ruas, F. Tari, *Differential geometry from a singularity theory viewpoint*, World Scientific Pub. Co Inc., 2015. https://doi.org/10.1142/9108
12. L. F. Martins, J. J. Nuño-Ballesteros, Contact properties of surfaces in $\mathbb{R}^3$ with corank 1 singularities, *Tohoku Math. J.*, **67** (2015), 105–124. https://doi.org/10.2748/tmj/1429549581
13. C. Muñoz-Cabello, J. J. Nuño-Ballesteros, R. Oset Sinha, Deformations of corank 1 frontals, *Proc. R. Soc. Edinburgh Sect. A*, (2023), 1–31. https://doi.org/10.1017/prm.2023.55
14. D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, *Proc. London Math. Soc.*, **s3-50** (1985), 333–369. https://doi.org/10.1112/plms/s3-50.2.333
15. D. Pei, M. Takahashi, H. Yu, Envelopes of one-parameter families of framed curves in the Euclidean space, *J. Geom.*, **110** (2019), 1–31. https://doi.org/10.1007/s00022-019-0503-1
16. K. Saji, M. Umehara, K. Yamada, The geometry of fronts, *Ann. Math.*, **169** (2009), 491–529.
17. K. Saji, Criteria for cuspidal S_k singularities and their applications, *J. Gökova Geom. Topol.*, **4** (2010), 67–81.
18. M. Takahashi, Envelopes of Legendre curves in the unit tangent bundle over the Euclidean plane, *Results Math.*, **71** (2017), 1473–1489. https://doi.org/10.1007/s00025-016-0619-7
19. K. Teramoto, Parallel and dual surfaces of cuspidal edges, *Differ. Geom. Appl.*, **44** (2016), 52–62. https://doi.org/10.1016/j.difgeo.2015.10.005
20. J. M. West, *The differential geometry of the cross cap*, Ph. D. thesis, The University of Liverpool, 1995.
21. H. Whitney, The singularities of a smooth n-manifold in $(2n-1)$-space, *Ann. Math.*, **45** (1944), 247–293. https://doi.org/10.2307/1969266

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