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Research article

Novel fractional inequalities measured by Prabhakar fuzzy fractional operators pertaining to fuzzy convexities and preinvexities

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Abstract: In this article, we implemented the idea of a fuzzy interval-valued function with the well-known generalized fuzzy fractional operators, associated with different types of convexities and preinvexities. We developed the Prabhakar fuzzy fractional operators using the fuzzy interval-valued function. We presented the novel extensions of Hermite-Hadamard fuzzy-type and trapezoidal fuzzy-type inequalities, based on the h-Godunova-Levin convex and h-Godunova preinvex fuzzy interval-valued functions.

Keywords: fuzzy fractional integral; fuzzy interval-valued function; preinvex function; fuzzy convexity; Hermite-Hadamard inequality
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1. Introduction

Convexities and preinvexities significantly contribute to advancing the optimization theory in both pure and applied mathematics. The increasing demand for fractional operators stems from the extensions and generalizations of fractional inequalities. Modifications applied to fractional operators enhance their kernels, and multi-dimensional series of special functions play a pivotal role in deriving new kernel versions, thereby improving the precision and applicability of fractional operators. Over the past two decades, extensive research has been conducted on extending generalized fractional operators, utilizing multi-index special functions as kernels. Researchers have diligently explored the use of these generalized operators to illustrate inequalities, garnering considerable attention in the field of fractional order analysis [1–5]. The interconnection between control optimization and the theory of inequalities is closely linked to the presence of (non)convex functions within their structures. This relationship has been thoroughly examined by researchers [6–8]. The classical theory of convexities facilitates numerous generalizations and extensions. One such essential generalization is the concept of an invex function, introduced by Hanson in [9]. This concept serves as a foundational extension of convex functions.

Furthermore, the fundamental characteristics of preinvex mappings and their applications in the realms of optimization and variational inequalities have been extensively explored. Noor [10,11], along with the works of Weir and Mond [12], has addressed these preinvex mappings, solving a multitude of problems related to equilibria. The application of preinvex mappings has proven to be instrumental in addressing challenges within optimization and variational inequalities, contributing to the resolution of complex equilibrium-related issues.

Diverse strategies are employed to tackle issues related to community specifications, decision-making, and optimizations, with their significance growing daily. The conventional crisp sets, which provide membership values of 0 or 1, pose a challenge when an object exhibits a property to varying degrees. To address this limitation, Zadeh introduced the concept of the fuzzy set theory, designed to handle ambiguous or uncertain information by allowing membership values in the range $[0, 1]$. The degree of membership within this interval reflects the level of assurance or conviction associated with the corresponding output value.

The most commonly concept of the fuzzy number theory is based on the characteristics function, but in fuzzy domain, the characteristics function is known as membership function that depends on the unit interval $[0, 1]$. This means that each subset of $\mathbb{R}$, and any real number, are exponential cases of the fuzzy number theory. In decision-making processes entailing unclear or imprecise information, fuzzy interval-valued functions prove invaluable. They enable the representation and analysis of hazy preferences or ambiguous results. Pioneering works by Moore in 1966, Kulish and Miranker [13] delved into the idea of interval analysis, uncovering its applications in arithmetic. Furthermore, a multitude of studies in [14] have explored fuzzy-set theory and system development across various fields, making noteworthy contributions to solving a wide array of problems in applied and pure mathematics. These contributions extend to disciplines such as computer science, operational research, management sciences, control optimization theory, decision sciences, and artificial intelligence.

So far, numerous researchers have explored the diverse properties and applications of variational inequalities within a set of functions exhibiting generalized convexity [15, 16]. Additionally, in the realm of studying fractional inequalities using fuzzy interval-valued functions (FIV-functions), which involve fractional integrals with the integrand as an FIV-function, several noteworthy works have been addressed by researchers. For instance, Flores et al. [17] extended the Gronwall inequality to encompass interval-valued functions (IV-functions). Furthermore, Chalco et al. [18] applied generalized Hukuhara derivatives to establish Ostrowski-type inequalities for FIV-functions and delved into their numerical applications. Zhao et al. [19, 20] directed their attention to IV-functions for the modification of well-known inequalities such as Jensen's, Chebyshev, and Hermite-Hadamard type inequalities. Following this line of research, Zhang et al. [21] derived Jensen's inequalities through the utilization of fuzzy set-valued functions.

The primary motivation of this study stems from the essential and widely applicable nature of two classes of functions characterized by convexity and preinvexity properties within the realm of optimization. In the context of fuzzy optimization, we aim to articulate optimality conditions under such functions and explore the derivation of various variational inequalities of fuzzy type. FIV-functions emerge as a pivotal fuzzy notion, offering a means to establish innovative applied fractional fuzzy inequalities through their combination with fractional operators. The incorporation of FIV-functions into these inequalities provides a unique avenue for investigating uncertainties in prediction scenarios, departing from the semantic norms of standard non-fuzzy inequalities. We delve into the utilization of FIV-functions to formulate fuzzy integral inequalities, where uncertainties manifest in the integrand rather than the measures of integration. To achieve this, we draw on a specific group of h-convex and h-preinvex FIV-functions of the Godunova-Levin type. This strategic utilization facilitates the derivation of notable fuzzy-based integral inequalities. In essence, the research endeavors to bridge the conceptual gap between convexity, preinvexity, and fuzzy optimization, offering a novel perspective on uncertainties in prediction situations through the lens of FIV-functions and fractional fuzzy inequalities.

In the following section, certain essential concepts are revisited to serve as a reminder. Section 3 explores Prabhakar fuzzy-based Hermite-Hadamard (H-H) type inequalities utilizing h-Godunova-Levin FIV-functions endowed with convexity. The convexity property is formulated explicitly within this section. In Section 4, the focus shifts to Prabhakar fuzzy-based trapezoidal-type inequalities, this time incorporating h-Godunova-Levin FIV-functions with the preinvexity property. The study culminates in Section 5, presenting concluding remarks and insights drawn from the findings.

2. Preliminaries

In the present section, we focus on some basic definitions and properties, which are helpful to understand the contents of the paper. Throughout the paper, $K \subseteq \mathbb{R}$ is a convex set.

Definition 2.1. *[22] The function $\varkappa : K \to \mathbb{R}$ is said to be convex if*

$$y(tu + (1-t)v) \leq ty(u) + (1-t)y(v),$$

where $t \in [0,1]$ and $u, v \in K$ are arbitrary.

By considering the convex function defined above, the Hermite-Hadamard (H-H) type inequality

$$y(\frac{u+v}{2}) \leq \frac{1}{v-u}\int_v^u y(u)du \leq \frac{y(u)+y(v)}{2}$$

was discussed many remarkable results in [23], where $u, v \in K \subseteq \mathbb{R}$ and $u < v$.

Definition 2.2. *[24] The preinvex function $y : K \to \mathbb{R}$, where $u, v \in K$ (Here, $K \subset \mathbb{R}$ is an invex set w.r.t. the bi-function ξ on $K \times K$) and $\lambda \in [0,1]$, is defined as*

$$y(v + \lambda\xi(u,v)) \leq \lambda y(u) + (1-\lambda)y(v).$$

Definition 2.3. *[25] Consider the functions $h : (0,1) \to \mathbb{R}$ and $y : K \to \mathbb{R}^{\geq 0}$. Then, the h-Godunova-Levin function y satisfies*

$$y(tu + (1-t)v) \leq \frac{y(u)}{h(t)} + \frac{y(v)}{h(1-t)},$$

for all $u, v \in K$ and $t \in (0,1)$.

Definition 2.4. *[25] Consider the functions $h : (0,1) \to \mathbb{R}$ and $y : K \to \mathbb{R}$. Then the h-Godunova-Levin preinvex function y w.r.t. ξ satisfies*

$$y(u + t\xi(v,u)) \leq \frac{y(u)}{h(1-t)} + \frac{y(v)}{h(t)},$$

so that $u, v \in K, t \in (0,1)$ are arbitrary.

Definition 2.5. *[26] Let $y \in L^1[u,v]$. The Riemann-Liouville left and right integrals $\mathbb{I}^b_{u^+}y$ and $\mathbb{I}^b_{v^-}y$ of order $b > 0$ are defined as follows*

$$\mathbb{I}^b_{u^+}y(r) = \frac{1}{\Gamma(b)}\int_u^r (r-t)^{b-1}y(t)dt, \quad r > u,$$

and

$$\mathbb{I}^b_{v^-}y(r) = \frac{1}{\Gamma(b)}\int_r^v (t-r)^{b-1}y(t)dt, \quad r < v.$$

By considering $F \subseteq \mathbb{R}$ as the fuzzy interval, let $\hbar_1, \hbar_2 \in F$. The partial order relation $\preccurlyeq$ is given by

$$\hbar_1 \preccurlyeq \hbar_2 \quad \Leftrightarrow \quad [\hbar_1]^\ell \leq_I [\hbar_2]^\ell, \quad \forall \ell \in [0,1],$$

where $[\hbar]^\ell$ is called the ℓ-level set of $\hbar$ defined by

$$[\hbar]^\ell = \{r \in \mathbb{R} : \ \ell \leq \hbar(r)\}.$$

Proposition 2.6. *[27] For $\hbar_1, \hbar_2 \in F$, $c \in \mathbb{R}$ and $\ell \in [0,1]$, we have*

$$\begin{aligned} [\hbar_1 \widetilde{+} \hbar_2]^\ell &= [\hbar_1]^\ell + [\hbar_2]^\ell, \\ \left[\hbar_1 \widetilde{\times} \hbar_2\right]^\ell &= [\hbar_1]^\ell \times [\hbar_2]^\ell, \\ [c \cdot \hbar_1]^\ell &= c[\hbar_1]^\ell, \\ [c \widetilde{+} \hbar_1]^\ell &= c + [\hbar_1]^\ell. \end{aligned}$$

Let $\hbar_1 = \hbar_2 \widetilde{+} H$, where $H \in F$. In this case, the Hukuhara difference (H-difference) of $\hbar_1$ and $\hbar_2$ is represented by the symbol $\hbar_1 \widetilde{-} \hbar_2$ and we have

$$(H)^*(\ell) = (\hbar_1 \widetilde{-} \hbar_2)^*(\ell) = \hbar_1^*(\ell) - \hbar_2^*(\ell), \quad (H)_*(\ell) = (\hbar_1 \widetilde{-} \hbar_2)_*(\ell) = \hbar_{1*}(\ell) - \hbar_{2*}(\ell).$$

The left fuzzy-based Riemann-Liouville integral of the function y is defined as

$$\begin{aligned} [\mathbb{I}^b_{u^+}y(x)]^\ell &= \frac{1}{\Gamma(b)}\int_u^x (x-t)^{b-1}y_\ell(t)dt \\ &= \frac{1}{\Gamma(b)}\int_u^x (x-t)^{b-1}[y_*(t,\ell), y^*(t,\ell)]dt, \quad x > u, \end{aligned}$$

where

$$[\mathbb{I}^b_{u^+}y_*(x,\ell)] = \frac{1}{\Gamma(b)}\int_u^x (x-t)^{b-1}[y_*(t,\ell)]dt, \quad x > u,$$

and

$$[\mathbb{I}^b_{u^+}y^*(x,\ell)] = \frac{1}{\Gamma(b)}\int_{u'}^x (x-t)^{b-1}[y^*(t,\ell)]dt, \quad x > u'.$$

We denote the family of all bounded closed fuzzy intervals of $\mathbb{R}$ by K_{bc}.

Definition 2.7. *[28] For all* $\ell \in [0,1]$*, the FIV-function* $y : K \subseteq \mathbb{R} \to K_{bc}$ *is formulated by* $y_\ell(x) = \big[y_*(x,\ell), y^*(x,\ell)\big]$, $\forall\, x \in K$ *in which the left and right functions* $y^*(x,\ell), y_*(x,\ell) : K \to \mathbb{R}$ *are named as the upper and lower functions of* y*.*

Note that the FIV-function $y : K \subseteq \mathbb{R} \to K_{bc}$ is continuous at $x \in K$ if for all $\ell \in [0,1]$, both $y_*(x,\ell)$ and $y^*(x,\ell)$ are continuous at $x \in K$.

Definition 2.8. *[29] Assume that* K_{bc} *is the aforesaid class of intervals and* $H \in K_{bc}$*. Then*

$$\Lambda = [\Lambda_*, \Lambda^*] = \{\alpha \in \mathbb{R} \,|\, \Lambda_* \preccurlyeq \alpha \preccurlyeq \Lambda^*\}, (\Lambda_*,\ \ \Lambda^* \in \mathbb{R}).$$

If $\Lambda_* = \Lambda^*$*, then we say that* Λ *is degenerate.*

In this research, we consider all the fuzzy intervals as the non-degenerate intervals. If $\Lambda_* \geq 0$, then $[\Lambda_*, \Lambda^*]$ is named as a positive interval, and all of them can be specified by the symbol K_{bc}^+ so that

$$K_{bc}^+ = \Big\{[\Lambda_*, \Lambda^*] : [\Lambda_*, \Lambda^*] \in K_{bc} \text{ and } \Lambda_* \geq 0\Big\}.$$

Let $\sigma \in \mathbb{R}$. Then, we define the scalar multiplication as [29]

$$\sigma \cdot \Lambda = \begin{cases} [\sigma\Lambda_*, \sigma\Lambda^*] & \text{if } \ \sigma \geq 0, \\ [\sigma\Lambda^*, \sigma\Lambda_*] & \text{if } \ \sigma \leq 0. \end{cases}$$

Also, the Minkowski algebraic difference and addition actions $\Lambda_1 - \Lambda_2$ and $\Lambda_1 + \Lambda_2$, where $\Lambda_1, \Lambda_2 \in K_{bc}$, are defined as

$$[\Lambda_{1*}, \Lambda_1^*] - [\Lambda_{2*}, \Lambda_2^*] = [\Lambda_{1*} - \Lambda_{2*}, \Lambda_1^* - \Lambda_2^*],$$
$$[\Lambda_{1*}, \Lambda_1^*] + [\Lambda_{2*}, \Lambda_2^*] = [\Lambda_{1*} + \Lambda_{2*}, \Lambda_1^* + \Lambda_2^*].$$

Moreover, the inclusion " $\subseteq$ " means that $\Lambda_1 \subseteq \Lambda_2$ if and only if $[\Lambda_{1*}, \Lambda_1^*] \subseteq [\Lambda_{2*}, \Lambda_2^*]$ iff $\Lambda_{1*} \preccurlyeq \Lambda_{2*}, \Lambda_1^* \preccurlyeq \Lambda_2^*$ [29].

The fuzzy order relation $\preccurlyeq_1$ is defined on K_{bc} as

$$[\Lambda_{1*}, \Lambda_1^*] \preccurlyeq_1 [\Lambda_{2*}, \Lambda_2^*] \quad \Leftrightarrow \quad \Lambda_{1*} \preccurlyeq \Lambda_{2*}, \Lambda_1^* \preccurlyeq \Lambda_2^*,$$

for all $[\Lambda_{1*}, \Lambda_1^*], [\Lambda_{2*}, \Lambda_2^*] \in K_{bc}$ [30].

Definition 2.9. *[31] The FIV-function* $y : [u,v] \to K_{bc}$ *is convex on* $[u,v]$ *if*

$$y\Big(tu + (1-t)v\Big) \preccurlyeq ty(u)\widetilde{+}(1-t)y(v),$$

so that $u, v \in [u,v], t \in [0,1]$*,* $y(v) \succcurlyeq 0$ *for all* $v \in [u,v]$*.*

Definition 2.10. *[32] For* $u, v \in K_{bc}$*,* $\lambda \in [0,1]$*, if* $\xi : [\alpha,\beta] \to \mathbb{R}$ *is the bi-function, then the invex set* $K_{bc} \subset \mathbb{R}$ *is defined as follows:*

$$v + \lambda\xi(u,v) \in K_{bc}. \tag{2.1}$$

Definition 2.11. *[32] The FIV-function* $y : [\alpha,\beta] \to K_{bc}$ *is preinvex w.r.t.* ξ *on invex interval* $[\alpha,\beta]$ *if*

$$y(v + \lambda\xi(u,v)) \preccurlyeq \lambda y(u)\widetilde{+}(1-\lambda)y(v),$$

so that $u, v \in [\alpha,\beta], t \in [0,1]$*, and* $y(u) \succcurlyeq 0, \forall\, v \in [\alpha,\beta]$*.*

Definition 2.12. *[29] Let P be a partition on $[\alpha,\beta]$ as $P = \{u = x_1 < x_2 < x_3 < x_4 < x_5 < \cdots < x_m = v\}$. The subintervals including the point p, having the maximum length, are termed as the mesh of partition P given by*

$$Mesh(p) = \max\{x_j - x_{j-1} : j = 1, 2, \ldots, m\}.$$

The collection of all possible partitions for $[u, v]$ is denoted by $P\{\delta, [\alpha,\beta]\}$ and for $p \in P(\delta, [\alpha,\beta])$, we have $Mesh(p) < \delta$. For every point b_j belonging to the subinterval $[x_{j-1}, x_j]$, where $1 \leq j \leq m$, we define the Riemann sum of the function y corresponding to $p \in P\{\delta, [\alpha,\beta]\}$ as

$$S(y, p, \delta) = \sum_{j=1}^{m} y(b_j)(x_j - x_{j-1}),$$

so that $y : [\alpha,\beta] \to K_{bc}$.

Definition 2.13. *[33] Assume $y : [\alpha,\beta] \to K_{bc}$. We say that y is Riemann-integrable ($\mathcal{R}$-integrable) on $[\alpha,\beta]$ if*

$$d(S(y, P, \delta), B) < \delta,$$

where $B \in \mathbb{R}$ and $\delta > 0$. For each choice of $b_j \in [y_{j-1}, y_j]$, $1 \leq j \leq m$, B is called the $\mathcal{R}$-integral of y w.r.t. $[\alpha,\beta]$, and is denoted by $B = (\mathcal{R}) \int_u^v y(t)dt$.

Definition 2.14. *The integral form of gamma function is defined as given below*

$$\Gamma(y) = \int_0^\infty k^{y-1} e^{-k} dk, \tag{2.2}$$

for $\mathbb{R}(k) > 0$

Definition 2.15. *The Pochammer's symbol is defined as follows*

$$(y)_k = \begin{cases} 1, & for \quad k = 0, y \neq 0, \\ y(y+1)...(y+k-1), & for \quad k \geq 1. \end{cases} \tag{2.3}$$

For $k \in \mathbb{N}$ and $y \in \mathbb{C}$:

$$(y)_k = \frac{\Gamma(y+k)}{\Gamma(y)}, \tag{2.4}$$

where Γ is the gamma function.

Theorem 2.16. *[34] Let $y : [\alpha,\beta] \subseteq \mathbb{R} \to K_{bc}$ be an FIV-function such that $y(x) = [y_*, y^*]$. Then y is $\mathcal{R}$-integrable on $[\alpha,\beta]$ iff y_* and y^* are $\mathcal{R}$-integrable on $[\alpha,\beta]$, and*

$$(\mathcal{R}) \int_u^v y(x)dx = \left[(\mathcal{R}) \int_u^v y_*(x), (\mathcal{R}) \int_u^v y^*(x)dx\right].$$

Definition 2.17. *[35] The Prabhakar fractional linear operator on a space L of functions by integral and employs fractional operator integration $I^\mu : L \to L$ to prove results for $a < x$, $\alpha,\beta,\rho,\lambda \in \mathbb{C}$ and $\Re(\alpha) > 0, \Re(\rho) > 0$ as follows:*

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x) = \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta} \lambda (x-t)^\alpha f(t)dt \quad (\Re(\beta)) > 0, \tag{2.5}$$

and operator for $a > x$

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x) = \int_x^a (t-x)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(t-x)^{\alpha} f(t)dt \quad (\Re(\beta)) > 0, \tag{2.6}$$

for any real number $\alpha \geq 0$ *where the function*

$$E^{\rho}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\rho)_n x^n}{\Gamma(\alpha n + \beta) n!} \quad \Re(\alpha) > 0, \tag{2.7}$$

is an entire function of order $(\Re(\alpha))^{-1}$.

Definition 2.18. *The left- and right-sided Prabhakar fuzzy fractional integral operators pertaining left, right end point functions, are defined as given below*

$$\begin{aligned} \left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^{l} &= \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha} f(l,t)dt \\ &= \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha} [f^{*}(l,t), f_{*}(l,t)]dt, \end{aligned} \tag{2.8}$$

where

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f^{*}(x,l) = \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha} f^{*}(l,t)dt \quad (x > a), \tag{2.9}$$

$$\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f_{*}(x,l) = \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha} f_{*}(l,t)dt \quad (x > a). \tag{2.10}$$

And

$$\begin{aligned} \left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^{l} = \left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^{l} &= \int_x^a (t-x)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(t-x)^{\alpha} f(l,t)dt \\ &= \int_x^a (t-x)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(t-x)^{\alpha} [f^{*}(l,t), f_{*}(l,t)]dt. \end{aligned} \tag{2.11}$$

Remark 2.19. *Prabhakar fuzzy fractional integral operators are represented with the difference* $\xi(x,a)$ *in short notation as follows:*

$$\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^{l} = \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda\Big(\frac{x-t}{\xi(x,a)}\Big)^{\alpha} f(l,t)dt = \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',a^{+}} f(x,l), \tag{2.12}$$

$$\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda} f(x)\right]^{l} = \int_x^a (t-x)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda\Big(\frac{t-x}{\xi(x,a)}\Big)^{\alpha} f(l,t)dt = \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',a^{-}} f(x,l). \tag{2.13}$$

3. On the (H-H)-type inequalities using the Prabhakar FIV-functions endowed with *h*-Godunova-Levin convexity

In the present section, we introduce a novel category of Prabhakar FIV-functions characterized by the convexity property aligned with the *h*-Godunova-Levin type. Through the application of these Prabkakar FIV-functions, we obtain (H-H)-type inequalities and delve into their applications, illustrated through various examples and cases.

Definition 3.1. *The FIV-function* $y : [\alpha, \beta] \to K_{bc}$ *is said to be the Godunova-Levin FIV-function if*

$$y(tu + (1-t)v) \preccurlyeq \frac{y(u)}{t} \tilde{+} \frac{y(v)}{1-t},$$

is satisfied for all $u, v \in [u, v], t \in (0, 1)$.

Definition 3.2. *Consider* $h : (0, 1) \to \mathbb{R}$. *The FIV-function* $y : [\alpha, \beta] \to K_{bc}$ *is named as a convex FIV-function of the h-Godunova-Levin type if*

$$y(tu + (1-t)v) \preccurlyeq \frac{y(u)}{h(t)} \tilde{+} \frac{y(v)}{h(1-t)},$$

is satisfied for all $u, v \in [u, v], t \in (0, 1)$.

Definition 3.3. *Consider* $h : (0, 1) \to \mathbb{R}$. *The FIV-function* $y : [\alpha, \beta] \to K_{bc}$ *is named as a preinvex FIV-function of the h-Godunova-Levin type w.r.t.* ξ *if*

$$y(u + t\xi(v, u)) \preccurlyeq \frac{y(u)}{h(1-t)} \tilde{+} \frac{y(v)}{h(t)},$$

is satisfied for all $u, v \in [u, v], t \in (0, 1)$.

Now, we are going to prove the (H-H)-type inequality for convex FIV-functions of the h-Godunova-Levin type.

Theorem 3.4. *Consider* $h : (0, 1) \to \mathbb{R}$ *and* $y \in L_1[u, v]$ *such that* $h(w) \neq 0$ *and* $y : [u, v] \to K_{bc}$ *is a convex FIV-function of the h-Godunova-Levin type with* $0 < u < v$. *Then the fuzzy fractional integral inequalities*

$$\begin{aligned} \frac{h(\frac{1}{2})}{2} y\Big(\frac{u+v}{2}\Big) E^{\rho}_{\alpha,\beta+1}\lambda &\preccurlyeq \frac{1}{2(v-u)^{\beta}} [\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+} y(v) \tilde{+} \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-} y(u)] \\ &\preccurlyeq \frac{y(u) \tilde{+} y(v)}{2} \int_0^1 \Big[\frac{1}{h(w)} + \frac{1}{h(1-w)}\Big] w^{\beta-1} E^{\rho}_{\alpha,\beta} \lambda w^{\alpha} dw, \end{aligned} \tag{3.1}$$

hold.

Proof. In view of the h-Godunova-Levin convexity for the FIV-function y on the closed interval $[u, v]$, for each $x, y \in [u, v]$, we have

$$y(\delta x + (1-\delta) y) \leq \frac{y(x)}{h(\delta)} \tilde{+} \frac{y(y)}{h(1-\delta)}, \quad \delta \in (0, 1).$$

Put $x = wu + (1-w)v, y = (1-w)u + wv$ and $\delta = \frac{1}{2}$. Hence

$$h\Big(\frac{1}{2}\Big) y\Big(\frac{u+v}{2}\Big) \leq [y(wu + (1-w)v) \tilde{+} y((1-w)u + wv)].$$

For each $\ell \in [0, 1]$ and for the lower FIV-function y_*, we can write

$$h\Big(\frac{1}{2}\Big) y_*\Big(\frac{u+v}{2}, \ell\Big) \leq [y_*(wu + (1-w)v, \ell) + y_*((1-w)u + wv, \ell)]. \tag{3.2}$$

Multiplying $w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}$ by Eq (3.2) and integrating w.r.t. w on $[0,1]$, we obtain

$$\begin{aligned} & h\Big(\frac{1}{2}\Big)y_*\Big(\frac{u+v}{2},\ell\Big)\int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw \\ \leq \; & \int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y_*(wu+(1-w)v,\ell)dw+\int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y_*((1-w)u+wv,\ell)dw. \end{aligned} \tag{3.3}$$

Now, by substituting $r = wu + (1-w)v$ and $t = (1-w)u + wv$, it becomes

$$\begin{aligned} & h\Big(\frac{1}{2}\Big)y_*\Big(\frac{u+v}{2},\ell\Big)E^{\rho}_{\alpha,\beta+1}\lambda \\ \leq \; & \int_u^v \left(\frac{v-r}{v-u}\right)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda\left(\frac{v-r}{v-u}\right)^{\alpha} y_*(r,\ell)\frac{dr}{v-u}+\int_u^v \left(\frac{t-u}{v-u}\right)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda\left(\frac{t-u}{v-u}\right)^{\alpha} y_*(t,\ell)\frac{dt}{v-u} \\ \leq \; & \frac{1}{(v-u)^{\beta}}\left[\int_u^v (v-r)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda\left(\frac{v-r}{v-u}\right)^{\alpha} y_*(r,\ell)dr+\int_u^v (t-u)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda\left(\frac{t-u}{v-u}\right)^{\alpha} y_*(t,\ell)dt\right] \end{aligned} \tag{3.4}$$

and so,

$$\frac{h(\frac{1}{2})}{2}y_*\Big(\frac{u+v}{2},\ell\Big)E^{\rho}_{\alpha,\beta+1}\lambda \leq \frac{1}{2(v-u)^{\beta}}\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y_*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y_*(u,\ell)\right]. \tag{3.5}$$

Similarly, we continue the calculation for upper FIV-function y^*, and we obtain

$$\frac{h(\frac{1}{2})}{2}y^*\Big(\frac{u+v}{2},\ell\Big)E^{\rho}_{\alpha,\beta+1}\lambda \leq \frac{1}{2(v-u)^{\beta}}\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y^*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y^*(u,\ell)\right]. \tag{3.6}$$

From inequalities (3.5) and (3.6), the inequality

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2}\Big[y_*\Big(\frac{u+v}{2},\ell\Big),y^*\Big(\frac{u+v}{2},\ell\Big)\Big]E^{\rho}_{\alpha,\beta+1}\lambda \\ \leq & \frac{1}{2(v-u)^{\beta}}\Big[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y_*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y_*(u,\ell)),\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y^*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y^*(u,\ell)\Big] \end{aligned}$$

is derived; that is,

$$\frac{h(\frac{1}{2})}{2}y\Big(\frac{u+v}{2}\Big)E^{\rho}_{\alpha,\beta+1}\lambda \preccurlyeq \frac{1}{2(v-u)^{\beta}}\Big[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y(v)\tilde{+}\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y(u)\Big]. \tag{3.7}$$

Now, we again consider the h-Godunova-Levin convexity for the FIV-function y to prove the second part of the inequality (3.1). We have

$$y\Big(wu+(1-w)v\Big) \leq \frac{y(u)}{h(w)}\tilde{+}\frac{y(v)}{h(1-w)}, \tag{3.8}$$

and

$$y\Big((1-w)u+wv\Big) \leq \frac{y(u)}{h(1-w)}\tilde{+}\frac{y(v)}{h(w)}. \tag{3.9}$$

By adding (3.8) and (3.9), we get

$$y\Big(wu+(1-w)v\Big)+y\Big((1-w)u+wv\Big)\le\Big(y(u)\widetilde{+}y(v)\Big)\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big].$$

For each $\ell\in[0,1]$ and for the lower FIV-function y_*, we have

$$y_*\Big(wu+(1-w)v,\ell\Big)+y_*\Big((1-w)u+wv,\ell\Big)\le\Big(y_*(u,\ell)+y_*(v),\ell\Big)\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]. \tag{3.10}$$

By multiplying $w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}$ by both sides of (3.10) and integrating on $[0,1]$ w.r.t. w, we get

$$\begin{aligned}&\int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y_*\Big(wu+(1-w)v,\ell\Big)dw+\int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y_*\Big((1-w)u+wv,\ell\Big)dw\\ \le\;&\Big(y_*(u,\ell)+y_*(v,\ell)\Big)\int_0^1\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw.\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}&\frac{1}{(v-u)^{\beta}}[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y_*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y_*(u,\ell)]\\ \le\;&[y_*(u,\ell)+y_*(v,\ell)]\int_0^1\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw.\end{aligned} \tag{3.11}$$

Similarly, we continue our computation for the upper FIV-function y^*, and we obtain

$$\begin{aligned}&\frac{1}{(v-u)^{\beta}}[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y^*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y^*(u,\ell)]\\ \le\;&[y^*(u,\ell)+y^*(v,\ell)]\int_0^1\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw.\end{aligned} \tag{3.12}$$

Combining inequalities (3.11) and (3.12), the inequality

$$\begin{aligned}&\frac{1}{(v-u)^{\beta}}\Big[\{\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y_*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda,'v^-}y_*(u,\ell)\},\{\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y^*(v,\ell)+\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y^*(u,\ell)\}\Big]\\ \le&[[y_*(u,\ell)+y_*(v,\ell)],[y^*(u,\ell)+y^*(v,\ell)]]\int_0^1\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw,\end{aligned}$$

is derived; that is,

$$\frac{[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda,u^+}y(v)\widetilde{+}\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda,v^-}y(u)]}{2(v-u)^{\beta}}\preccurlyeq\frac{y(u)\widetilde{+}y(v)}{2}\int_0^1\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw. \tag{3.13}$$

From (3.7) and (3.13), we get

$$\begin{aligned}\frac{h(\frac{1}{2})}{2}y\Big(\frac{u+v}{2}\Big)E^{\rho}_{\alpha,\beta+1}\lambda&\preccurlyeq\frac{1}{2(v-u)^{\beta}}[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+}y(v)\widetilde{+}\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-}y(u)]\\ &\preccurlyeq\frac{y(u)\widetilde{+}y(v)}{2}\int_0^1\Big[\frac{1}{h(w)}+\frac{1}{h(1-w)}\Big]w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}dw,\end{aligned} \tag{3.14}$$

which is our desired result. □

Some special cases of the conclusion of the above theorem can be stated in the framework of several examples.

Example 3.5. *In Theorem 3.4, put $h(w) = w^s$. Then the (H-H)-type inequality*

$$\frac{(\frac{1}{2})^s}{2} y\Big(\frac{u+v}{2}\Big) E^{\rho}_{\alpha,\beta+1}\lambda \leqslant \frac{1}{2(v-u)^{\beta}} [\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+} y(v) \widetilde{+} \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-} y(u)]$$
$$\leqslant \frac{y(u)\widetilde{+}y(v)}{2} \int_0^1 \left[\frac{1}{(w)^s} + \frac{1}{(1-w)^s}\right] w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} dw,$$

is derived for the FIV-function with convexity of the s-Godunova-Levin type.

Example 3.6. *In Theorem 3.4, put $h(w) = 1$. Then*

$$\frac{1}{2} y\Big(\frac{u+v}{2}\Big)\lambda \leqslant \frac{[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+} y(v) \widetilde{+} \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-} y(u)]}{2(v-u)^{\beta} E^{\rho}_{\alpha,\beta+1}} \leqslant \frac{y(u)\widetilde{+}y(v)}{2}. \tag{3.15}$$

Example 3.7. *In Theorem 3.4, put $h(w) = \frac{1}{w}$. Then the (H-H)-type inequality*

$$y\Big(\frac{u+v}{2}\Big) \leqslant \frac{\left[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+} y(v) \widetilde{+} \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-} y(u)\right]}{2(v-u)^{\beta} E^{\rho}_{\alpha,\beta+1}\lambda} \leqslant \frac{y(u)\widetilde{+}y(v)}{2}, \tag{3.16}$$

is derived for the convex FIV-function.

Example 3.8. *In Theorem 3.4, put $h(w) = w$. Then the (H-H)-type inequality*

$$\frac{h(\frac{1}{4})}{2} y\Big(\frac{u+v}{2}\Big) E^{\rho}_{\alpha,\beta+1}\lambda \leqslant \frac{1[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+} y(v) \widetilde{+} \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-} y(u)]}{2(v-u)^{\beta}} \leqslant \frac{y(u)\widetilde{+}y(v)}{2} \int_0^1 \frac{w^{\beta-2}}{1-w} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} dw, \tag{3.17}$$

is derived for the FIV-function with convexity of the Godunova-Levin type.

Example 3.9. *(1) By choosing the value $h(w) = \frac{1}{w^s}$ in Theorem 3.4, the (H-H)-type inequality*

$$y\Big(\frac{u+v}{2}\Big) \frac{E^{\rho}_{\alpha,\beta+1}\lambda}{2^{1-s}} \leqslant \frac{[\mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',u^+} y(v) \widetilde{+} \mathfrak{E}^{\alpha,\beta}_{\rho,\lambda',v^-} y(u)]}{2(v-u)^{\beta}} \leqslant \frac{y(u)\widetilde{+}y(v)}{2} \int_0^1 \frac{[w^s + (1-w)^s]}{w^{1-\beta}} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} dw,$$

is derived for the FIV-function with s-convexity.

(2) If we take the value $\lambda = 0, h(w) = \frac{1}{w^s}$ in Theorem 3.4, then

$$2^{s-1} y\Big(\frac{u+v}{2}\Big) \leqslant \frac{\Gamma(\beta+1)}{2(v-u)^{\beta}} \Big[\mathbb{I}^{\beta}_{u^+} y(v) \widetilde{+} \mathbb{I}^{\beta}_{v^-} y(u)\Big] \leqslant \frac{y(u)\widetilde{+}y(v)}{2} \beta \int_0^1 [w^s + (1-w)^s] w^{\beta-1} dw,$$

is derived for the FIV-function with s-convexity.

(3) If we take the value $\lambda = 0, h(w) = \frac{1}{w^s}, \beta = 1$ in Theorem 3.4, then

$$2^{s-1} y\Big(\frac{u+v}{2}\Big) \leqslant \frac{1}{v-u} \int_u^v y(w) dw \leqslant \frac{y(u)\widetilde{+}y(v)}{s+1}.$$

4. On the trapezoidal-type inequalities using Prabhakar FIV-functions characterized by the preinvexity property of the h-Godunova-Levin type

In this context, we formulate a lemma specific to Prabhakar FIV-functions featuring h-Godunova-Levin preinvexity, providing valuable support in establishing our key findings. Indeed, within this section, we deduce fuzzy-based trapezoidal-type inequalities and explore their applications through illustrated examples and cases.

Lemma 4.1. *Let* $y : K = [u, u + \xi(v, u)] \to \mathbb{R}$ *be a differentiable function, and* K *be an invex set w.r.t.* $\xi : K \times K \to \mathbb{R}$ *so that* $\xi(v, u) > 0$ *for* $u, v \in K$. *Then*

$$\frac{y(u)\widetilde{+}y(u+\xi(v,u))}{2}E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y(u,\ell)\widetilde{+}\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y(v,\ell)\right]}{2\xi^{\beta-1}(v,u)} = \frac{\xi(u,v)}{2}I.$$

Proof. Consider the integral

$$I = \int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y'(u+w\xi(v,u))dw - \int_0^1 (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}y'(u+w\xi(v,u))dw.$$

For every $\ell \in [0, 1]$, we have

$$I = \int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y'_*(u+w\xi(v,u),l)dw - \int_0^1 (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}y'_*(u+w\xi(v,u),l)dw.$$

Now we solve the integrals of the above equation by considering it as I_1 and I_2. Simply, for I_1, we get

$$\begin{aligned} I_1 &= \frac{y_*(u+\xi(v,u),\ell)}{\xi(v,u)}E^{\rho}_{\alpha,\beta}\lambda - \sum_{n=0}^{\infty}\frac{\lambda^n(\rho)_n(\alpha n+\beta-1)}{n!\Gamma(\alpha n+\beta)\xi(v,u)}\int_0^1 w^{\alpha n+\beta-2}y_*(u+w\xi(v,u),\ell)dw \\ &= \frac{y_*(u+\xi(v,u),\ell)}{\xi(v,u)}E^{\rho}_{\alpha,\beta}\lambda - \frac{1}{\xi(v,u)}\int_u^v\left(\frac{t-u}{\xi(v,u)}\right)^{\beta-2}E^{\rho}_{\alpha,\beta-1}\lambda\left(\frac{t-u}{\xi(v,u)}\right)^{\alpha}y_*(t,\ell)dt, \end{aligned}$$

and so,

$$I_1 = \frac{y_*(u+\xi(v,u),\ell)}{\xi(v,u)}E^{\rho}_{\alpha,\beta}\lambda - \frac{\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y_*(u,\ell)}{\xi^{\beta}(v,u)}.$$

Using the similar inference to that of I_2, we have

$$I_2 = \frac{y_*(u,\ell)}{\xi(v,u)}E^{\rho}_{\alpha,\beta}\lambda - \frac{\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y_*(u,\ell)}{\xi^{\beta}(v,u)}.$$

Consequently,

$$\begin{aligned} &\left[\frac{y_*(u+\xi(v,u),\ell)}{\xi(v,u)} + \frac{y_*(u,\ell)}{\xi(v,u)}\right]E^{\rho}_{\alpha,\beta}\lambda - \frac{1}{\xi^{\beta}(v,u)}\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y_*(u,\ell) + \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y_*(v,\ell)\right] \\ &[y_*(u,\ell) + y_*(u+\xi(v,u),\ell)]\frac{E^{\rho}_{\alpha,\beta}\lambda}{\xi(u,\ell)} - \frac{1}{\xi^{\beta}(v,u)}\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y_*(u,\ell) + \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y_*(v,\ell)\right] \end{aligned}$$

$$= \int_0^1 w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} y'_*(u + w\xi(v,u), l)dw - \int_0^1 (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha} y'_*(u + w\xi(v,u), l)dw. \quad (4.1)$$

Similarly, we continue our computations for the upper FIV-function $y^{'*}$, and we get

$$[y^*(u,\ell) + y^*(u+\xi(v,u),\ell)] \frac{E^{\rho}_{\alpha,\beta}\lambda}{\xi(u,\ell)} - \frac{1}{\xi^{\beta}(v,u)} \left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y^*(u,\ell) + \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y^*(v,\ell)\right]$$
$$= \int_0^1 w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} y'^*(u + w\xi(v,u), l)dw - \int_0^1 (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha} y'^*(u + w\xi(v,u), l)dw. \quad (4.2)$$

By combining Eqs (4.1) and (4.2), then we have

$$[y_*(u,\ell) + y_*(u+\xi(v,u),\ell), y^*(u,\ell) + y^*(u+\xi(v,u),\ell)] \frac{E^{\rho}_{\alpha,\beta}\lambda}{\xi(u,\ell)}$$
$$-\frac{1}{\xi^{\beta}(v,u)} \left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y_*(u,\ell) + \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y_*(v,\ell), \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y^*(u,\ell) + \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y^*(v,\ell)\right]$$
$$= \int_0^1 w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} (y'^*(u + w\xi(v,u), l), y'_*(u + w\xi(v,u), l))dw \quad (4.3)$$
$$- \int_0^1 (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha} (y'^*(u + w\xi(v,u), l), y'_*(u + w\xi(v,u), l))dw.$$

That is,

$$[y(u)\widetilde{+}y(u+\xi(v,u))] \frac{E^{\rho}_{\alpha,\beta}\lambda}{\xi(u,\ell)} - \frac{1}{\xi^{\beta}(v,u)} \left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y(u,\ell) \widetilde{+} \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y(v,\ell)\right] = I.$$

This completes the proof. □

By Lemma 4.1, we can state the following theorem.

Theorem 4.2. *Consider the function $y : K = [u, u+\xi(v,u)] \to (0,\infty)$. Assume that the differentiable FIV-function $|y'|$ has the preinvexity property of the h-Godunova-Levin type on K. Then*

$$\left|\frac{y(u)\widetilde{+}y(u+\xi(v,u))}{2} E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y(u,\ell) \widetilde{+} \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y(v,\ell)\right]}{2\xi^{\beta-1}(v,u)}\right|$$
$$\preccurlyeq \frac{\xi(v,u)}{2}[|y'(u)|\widetilde{+}|y'(v)|] \int_0^1 \left|w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| \frac{1}{h(w)} dw.$$

Proof. Using Lemma 4.1, we obtain that for every $\ell \in [0,1]$

$$\left|\frac{y(u)\widetilde{+}y(u+\xi(v,u))}{2} E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y(u,\ell) \widetilde{+} \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y(v,\ell)\right]}{2\xi^{\beta-1}(v,u)}\right|$$
$$= \left|\frac{\xi(u,v)}{2} \int_0^1 w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} y'(u + w\xi(v,u))dw - \int_0^1 (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha} y'(u + w\xi(v,u))dw\right|$$
$$= \left|\frac{\xi(u,v)}{2} \int_0^1 w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} y'(u + w\xi(v,u))dw - \int_0^1 (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha} y'(u + w\xi(v,u), l)dw\right|$$

$$\leq \frac{\xi(u,v)}{2}\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y'(u+w\xi(v,u))\right|dw - \int_0^1 \left|(1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}y'(u+w\xi(v,u),l)\right|dw$$
$$= \frac{\xi(u,v)}{2}\left|\sum_{n=0}^{\infty}\frac{(\rho)_n\lambda^n}{\Gamma(\alpha n+\beta)n!}\right|\int_0^1 \left|w^{\alpha n+\beta-1}-(1-w)^{\alpha n+\beta-1}\right||y'(u+w\xi(v,u),l)|\,dw.$$

The h-Godunova-Levin preinvexity property of the FIV-function $\left|y'_*(u+w\xi(v,u),\ell)\right|$ gives

$$\text{L.H.S.} \leq \frac{\xi(u,v)}{2}\left|\sum_{n=0}^{\infty}\frac{(\rho)_n\lambda^n}{\Gamma(\alpha n+\beta)n!}\right|\int_0^1 \left|w^{\alpha n+\beta-1}-(1-w)^{\alpha n+\beta-1}\right|\left|\frac{y'_*(u,\ell)}{h(w)}+\frac{y'_*(v,\ell)}{h(1-w)}\right|dw$$
$$\leq \frac{\xi(u,v)}{2}\left|\sum_{n=0}^{\infty}\frac{(\rho)_n\lambda^n}{\Gamma(\alpha n+\beta)n!}\right|\int_0^1 \left|w^{\alpha n+\beta-1}-(1-w)^{\alpha n+\beta-1}\right|\left[\frac{|y'_*(u,\ell)|}{h(w)}+\frac{|y'_*(v,\ell)|}{h(1-w)}\right]dw. \qquad (4.4)$$

Similarly, we repeat the computations to solve it this time for the upper FIV-function y'^*. We have

$$\text{L.H.S.} \leq \frac{\xi(u,v)}{2}\left|\sum_{n=0}^{\infty}\frac{(\rho)_n\lambda^n}{\Gamma(\alpha n+\beta)n!}\right|\int_0^1 \left|w^{\alpha n+\beta-1}-(1-w)^{\alpha n+\beta-1}\right|\left[\frac{|y'^*(u,\ell)|}{h(w)}+\frac{|y'^*(v,\ell)|}{h(1-w)}\right]dw. \qquad (4.5)$$

From inequalities (4.4) and (4.5), we get

$$\text{L.H.S.}$$
$$\leqslant_I \frac{\xi(v,u)}{2}[|y'_*(u,\ell)+y'_*(v,\ell)|,|y'^*(u,\ell)+y'^*(v,\ell)|]\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}-(1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|\frac{1}{h(w)}dw$$
$$\leqslant \frac{\xi(v,u)}{2}[|y'(u)|\widetilde{+}|y'(v)|]\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}-(1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|\frac{1}{h(w)}dw,$$

as required. □

Some special cases of the conclusion of the above theorem can be stated in the framework of several examples.

Example 4.3. *In Theorem 4.2, put $\xi(v,u)=v-u$. Then*

$$\left|\frac{y(u)\widetilde{+}y(v)}{2}E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y(u,\ell)\widetilde{+}\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y(v,\ell)\right]}{2(v-u)^{\beta-1}}\right|$$
$$\leqslant \frac{(v-u)}{2}[|y'(u)|\widetilde{+}|y'(v)|]\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}-(1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|\frac{1}{h(w)}dw.$$

Theorem 4.4. *Consider the function $y: K=[u,u+\xi(v,u)]\to(0,\infty)$. Assume that the differentiable FIV-function $|y'|^q$ has the preinvexity property of the h-Godunova-Levin type on K so that $q=\frac{p}{p-1}$ with $p>1$. Then*

$$\left|\frac{y(u)\widetilde{+}y(u+\xi(v,u))}{2}E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y(u,\ell)\widetilde{+}\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y(v,\ell)\right]}{2\xi^{\beta-1}(v,u)}\right|$$
$$\leqslant \frac{\xi(v,u)}{2}(|y'(u)|^q\widetilde{+}|y'(v)|^q)^{\frac{1}{q}}\left(\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}-(1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|^p dw\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{h(w)}dw\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 4.1, and Hölder's integral inequality for every $\ell \in [0,1]$, we obtain

$$
\begin{aligned}
& \left| \frac{y(u) \widetilde{+} y(u+\xi(v,u))}{2} E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y(u,\ell) \widetilde{+} \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y(v,\ell) \right]}{2\xi^{\beta-1}(v,u)} \right| \\
= & \left| \frac{\xi(u,v)}{2} \int_0^1 w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} y'(u+w\xi(v,u))dw - \int_0^1 (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda (1-w)^{\alpha} y'(u+w\xi(v,u))dw \right| \\
\leq & \frac{\xi(u,v)}{2} \int_0^1 \left| w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda (1-w)^{\alpha} \right| |y'(u+w\xi(v,u))| \, dw \\
\leq & \frac{\xi(v,u)}{2} \left(\int_0^1 \left| w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda (1-w)^{\alpha} \right|^p dw \right)^{\frac{1}{p}} \left(\int_0^1 \left| y'(u+w\xi(v,u)) \right|^q dw \right)^{\frac{1}{q}} \\
\leq & \frac{\xi(v,u)}{2} \left(\int_0^1 \left| w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda (1-w)^{\alpha} \right|^p dw \right)^{\frac{1}{p}} \left(\int_0^1 \left| y'_*(u+w\xi(v,u)), \ell \right|^q dw \right)^{\frac{1}{q}},
\end{aligned}
\tag{4.6}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The h-Godunova-Levin preinvexity property of the FIV-function $\left| y'_*(u+w\xi(v,u)), \ell \right|^q$ gives

$$
\int_0^1 |y'_*(u+w\xi(v,u)), \ell|^q dw \leq \int_0^1 \left(\frac{|y'_*(u), \ell|^q}{h(w)} + \frac{|y'_*(v), \ell|^q}{h(1-w)} \right) dw. \tag{4.7}
$$

Similarly, we solve it for the upper FIV-function $\left| y'^* \right|^q$, and we get

$$
\int_0^1 |y'^*(u+w\xi(v,u)), \ell|^q dw \leq \int_0^1 \left(\frac{|y'^*(u), \ell|^q}{h(w)} + \frac{|y'^*(v), \ell|^q}{h(1-w)} \right) dw. \tag{4.8}
$$

By combining (4.7) and (4.8), we have

$$
\begin{aligned}
\text{L.H.S.} & \leq_I \left(|y'_*(u,\ell) + y'_*(v,\ell)|^q, |y'^*(u,\ell) + y'^*(v,\ell)|^q \right) \int_0^1 \frac{1}{h(w)} dw \\
& \preccurlyeq \left(|y'(u)|^q \widetilde{+} y'(v)|^q \right) \int_0^1 \frac{1}{h(w)} dw.
\end{aligned}
\tag{4.9}
$$

Using (4.9) in (4.6), we get the required result. □

Theorem 4.5. *The H-H inequality related to the hypotheses of Theorem 4.4 is as follows*

$$
\begin{aligned}
& \left| \frac{y(u) \widetilde{+} y(u+\xi(v,u))}{2} E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-} y(u,\ell) \widetilde{+} \mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+} y(v,\ell) \right]}{2\xi^{\beta-1}(v,u)} \right| \\
\preccurlyeq & \frac{\xi(v,u)}{2^{\frac{1}{q}}} \left(|y'(u)|^q \widetilde{+} |y'(v)|^q \right)^{\frac{1}{q}} \left[E^{\rho}_{\alpha,\beta+1}\lambda - \left(\frac{1}{2} \right)^{\beta-1} E^{\rho}_{\alpha,\beta+1}\lambda \left(\frac{1}{2} \right)^{\alpha} \right]^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \frac{\left| w^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1} E^{\rho}_{\alpha,\beta}\lambda (1-w)^{\alpha} \right|}{h(w)} dw \right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Using Lemma 4.1, and thanks to the power-mean inequality, the last inequality becomes for every $\ell \in [0,1]$

$$\begin{aligned}
&\left|\frac{y(u)\widetilde{+}y(u+\xi(v,u))}{2}E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y(u,\ell)\widetilde{+}\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y(v,\ell)\right]}{2\xi^{\beta-1}(v,u)}\right| \\
= \ & \left|\frac{\xi(u,v)}{2}\int_0^1 w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha}y'(u+w\xi(v,u))dw - \int_0^1 (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}y'(u+w\xi(v,u))dw\right| \\
\leq \ & \frac{\xi(u,v)}{2}\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| |y'(u+w\xi(v,u))|\, dw, \qquad (4.10)
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{y(u)\widetilde{+}y(u+\xi(v,u))}{2}E^{\rho}_{\alpha,\beta}\lambda - \frac{\left[\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(u+\xi(v,u))^-}y(u,\ell)\widetilde{+}\mathfrak{E}^{\alpha,\beta-1}_{\rho,\lambda',(v+\xi(u,v))^+}y(v,\ell)\right]}{2\xi^{\beta-1}(v,u)}\right| \\
\leq \ & \frac{\xi(v,u)}{2}\left(\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| dw\right)^{1-\frac{1}{q}} \qquad (4.11) \\
&\left(\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| \left|y'(u+w\xi(v,u))\right|^q dw\right)^{\frac{1}{q}}.
\end{aligned}$$

The h-Godunova-Levin preinvexity of the FIV-function $\left|y^{'}(u+w\xi(v,u))\right|^q$, we can obtain

$$\begin{aligned}
&\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| \left|y^{'}_{*}(u+w\xi(v,u),\ell)\right|^q dw \\
\leq \ & \int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| \left(\frac{|y^{'}_{*}(u),\ell|^q}{h(w)} + \frac{|y^{'}_{*}(v,\ell)|^q}{h(1-w)}\right)dw \\
\leq \ & \int_0^1 \frac{\left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|}{h(w)}(|y^{'}_{*}(u,\ell|^q + |y^{'}_{*}(v,\ell)|^q)dw. \qquad (4.12)
\end{aligned}$$

Similarly, for the upper FIV-function $|y^{'*}|^q$, we get

$$\begin{aligned}
&\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| \left|y^{'*}(u+w\xi(v,u),\ell)\right|^q dw \\
\leq \ & \int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| \left(\frac{|y^{'*}(u,\ell)|^q}{h(w)} + \frac{|y^{'*}(v,\ell)|^q}{h(1-w)}\right)dw \\
\leq \ & \int_0^1 \frac{\left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|}{h(w)}(|y^{'*}(u,\ell)|^q + |y^{'*}(v,\ell)|^q)dw. \qquad (4.13)
\end{aligned}$$

By combining (4.12) and (4.13), we have

$$\text{L.H.S.} \preccurlyeq \int_0^1 \frac{\left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right|}{h(w)}(|y^{'}(u)|^q\widetilde{+}|y^{'}(v)|^q)dw. \qquad (4.14)$$

On the other hand, using the equation

$$\int_0^1 \left|w^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda w^{\alpha} - (1-w)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(1-w)^{\alpha}\right| dw = 2\left[E^{\rho}_{\alpha,\beta+1}\lambda - \left(\frac{1}{2}\right)^{\beta-1} E^{\rho}_{\alpha,\beta+1}\lambda\left(\frac{1}{2}\right)^{\alpha}\right],$$

and (4.14) and (4.12), we get the required result. □

5. Conclusions

In this research, we introduced a novel classification of convex FIV-functions and preinvex FIV-functions, employing an arbitrary auxiliary h-function of the Godunova-Levin type. Utilizing this category, we derived updated versions of Hermite-Hadamard (H-H) and trapezoidal-type inequalities by implementation of newly described fuzzy fractional operators (Prabhakar fuzzy-based operators). The application of these findings was demonstrated through various examples, highlighting diverse cases. We conclude that extending Prabhakar (H-H) and Prabhakar trapezoidal-type fractional inequalities to encompass different types of fuzzy-based convexities and preinvexities under Prabhakar FIV-functions is both feasible and applicable. Looking ahead, we plan to explore this concept further within the domain of generalized convex FIV-functions, with a focus on applications in fuzzy-interval-valued non-linear programming. We anticipate that this exploration will contribute valuable insights to the broader field of fuzzy-based optimization theory and provide a foundation for other researchers in fulfilling their roles within this realm. Moreover, we conclude that such type of fractional operators, extensions, and generalizations of inequalities and its refinments could be discussed for fuzzy interval valued functions. The validated results are expected to open new avenues for future research in this evolving area. These outcomes indicate a promising route for further exploration in the sphere of integral inequalities.

Author contributions

Iqra Nayab: Conceptualization, Formal Analysis, Software, Software, Writing-Original Draft; Shahid Mubeen: Conceptualization, Formal Analysis, Writing-Review & Editing; Rana Safdar Ali: Conceptualization, Formal Analysis, Investigation, Methodology, Writing-Original Draft; Faisal Zahoor: Formal Analysis, Investigation, Software; Muath Awadalla: Investigation, Writing-Original Draft; Abd Elmotaleb A. M. A. Elamin: Investigation, Methodology, Writing-Review & Editing. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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