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Research article

Time-inhomogeneous Hawkes processes and its financial applications

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Abstract: We consider time-inhomogeneous Hawkes processes with an exponential kernel, and we analyze some properties of the model. Time-inhomogeneity for the Hawkes process is indispensable for short rate models or for other calibration purposes, while financial applications for the time-homogeneous case already well known. Distributional properties for such a model generate computational tractability for a financial application. In this paper, moments and the Laplace transform of time-inhomogeneous Hawkes processes are obtained from the distributional properties of the underlying processes. As an applications to finance, we investigate the pricing formula for zero-coupon bonds when short-term interest rates are governed by the time-inhomogeneous Hawkes process. Numerical illustrations are also provided. As an illustrative example, we apply the derived moments and Laplace transform of time-inhomogeneous Hawkes processes to the pricing of zero-coupon bonds within a financial context. By considering the short-term interest rate as driven by inhomogeneous Hawkes processes, we develop explicit formulae for valuing zero-coupon bonds. This application is particularly relevant for modeling interest rate dynamics in real-world scenarios, allowing for a more nuanced understanding of pricing dynamics. Through numerical illustrations, we demonstrate the computational tractability of our approach, showcasing its practical utility for financial practitioners and providing insights into the intricate interplay between time-inhomogeneous Hawkes processes and bond pricing in dynamic markets.

Keywords: time-inhomogeneous Hawkes processes; self-exciting processes; zero coupon bond; moments; transforms
Mathematics Subject Classification: 60G42, 60G55, 60H10

1. Introduction

In this paper, we study time-inhomogeneous Hawkes processes having an exponential kernel. Hawkes process is typically a self-exciting simple point process with clustering effect where the jump rate varies depending on its entire past history, as introduced by Hawkes [12]. Let N be a simple point process on $\mathbb{R}$ and $\mathcal{F}_t^{-\infty} := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t])$ increases with the family of $\sigma-$algebras. Non-negative $\mathcal{F}_t^{-\infty}$- incrementally measurable process λ_t

$$E\left[N(a,b]|\mathcal{F}_a^{-\infty}\right] = E\left[\int_a^b \lambda_s ds|\mathcal{F}_a^{-\infty}\right] \tag{1.1}$$

a. s. for every interval $(a, b]$ is called the $\mathcal{F}_t^{-\infty}$- intensity of N. We use the notation $N_t := N(0, t]$ to represent the number of points in the interval $(0, t]$.

A general Hawkes process is a simple point process N that allows $\mathcal{F}_t^{-\infty}$-intensity

$$\lambda_t := \phi\left(\int_{-\infty}^t h(t-s)N(ds)\right), \tag{1.2}$$

where $\phi(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and left continuous, and $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and we always assume that $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. Here $\int_{-\infty}^t h(t-s)N(ds)$ represents $\int_{(-\infty,t)} h(t-s)N(ds)$. We always assume that $N(-\infty, 0] = 0$, i.e., the history of the Hawkes process is empty. In the literatures, $h(\cdot)$ and $\phi(\cdot)$ are usually called exciting functions and rate functions respectively. The Hawkes process is determined to be linear or nonlinear depending on the intensity function $\phi(\cdot)$. In general, the Hawkes process is a non-Markov process. This is because the future evolution of the self-exciting simple point process is ruled by the timing of past events. However, in the special case where the function of intensity is exponential, it is a Markov process. There are some crucial differences between a Hawkes process and a standard Poisson process, with important properties for the intensity process. A Poisson process is stationary with independent increments. In contrast, the Hawkes process has dependent increments and clustering effects. Unlike the usual Poisson process, for which the intensity is a positive constant, the intensity of the Hawkes process increases when it observes the arrivals of points and decays when no point arrives. Moreover, Kim-Kim [17–20] has studied several important results of the degenerate Poisson process, which is a specialized form of the Poisson process where events inter-arrival times remain constant. This implies a regular arrival of events at fixed intervals, modeled mathematically by the exponential distribution for inter-event times. In contrast, the Hawkes process extends the Poisson process by allowing past events to influence the arrival probability of future events, introducing interaction. Past events can trigger new events, leading to variable inter-arrival times. By introducing the degenerate Poisson process, we emphasize its role as a case of no interaction, serving as a special instance within the broader framework of the Hawkes process. This juxtaposition highlights the notion that when there is no interaction, the arrival intervals behave as a degenerate Poisson process within the Hawkes model. Thus, understanding the degenerate Poisson process provides valuable insights into how the Hawkes process can encompass a scenarios with and without interaction, enriching comprehension of the model's dynamics.

The Hawkes process has many applications in neuroscience, seismology, DNA modeling, finance, and many other fields. It has extraordinary properties, which are called self-exciting and clustering properties and they are very appealing to some financial applications. In particular, the self-exciting, and clustering properties of the Hawkes process are important for correlated fundamental modeling and evaluating credit derivatives valuation in finance; for example, see Errais et al. [5] and Dassios and Zhao [3]. The structural properties of the Hawkes process have been explored and illustrated in [4] for the case of time homogeneity, and covers many related applications. However, there are many situations where time-dependent parameters are needed, such as model calibration. The defining feature of Hawkes process is the exponential-affine form of the characteristic function of the transition probabilities. This paper provides a strict treatment and complete characterization of time-inhomogeneous the Hawkes processes. In a recent paper [21], a procedure for parameter optimization and for predicting the future profiles of retweet activity at different temporal resolutions was developed through a Time-Dependent Hawkes process. Hawkes [12] introduced the linear case, i.e., $\phi(\cdot)$ is linear, and the linear Hawkes process can be studied through the immigration–birth representation; see, e.g, Hawkes and Oakes [13]. The stability, law of large numbers, central limit theorem, large deviations, Bartlett spectrum, etc. have all been investigated and understood very well. It naturally generalizes the Poisson process, and it seizes both the self-exciting property and the clustering effect. Additionally, the Hawkes process is also a very versatile model suitable for statistical analysis. They explain why they are applicable to various fields: insurance, finance, social networks, neuroscience, criminology and many other fields. Almost all the literature on the application of the Hawkes process considers only the linear case [1, 3, 5, 16, 21, 33, 38]. Due to the lack of flexibility in immigration and birth representations and calculations, nonlinear Hawkes processes have been little studied, i.e., $\phi(\cdot)$ is nonlinear. However, some efforts have already been made in this direction. The nonlinear case was first introduced by Brémaud and Massoulié [2]. Recently, Zhu [34–38] examined some results for both linear and nonlinear models. The central limit theorem has been investigated in Zhu [36], and the large deviation principles have been obtained in Zhu [34] and Zhu [35]. Limit theorems and crude fractional diffusions as scaling limits for nearly unstable Hawkes processes were obtained in Jaisson and Rosenbaum [14, 15]. Zhu [38] has also researched applications to financial mathematics. Some variations and extensions of the Hawkes process have been studied by Dassios and Zhao [3], Zhu [39], Karabash and Zhu [16], Mehrdad and Zhu [22], and Ferro, Leiva, and Möller [6]. In the paper of Seol [25], he regards the arrival time τ_n, i.e., the inverse process of the Hawkes process, and investigates the limit theorems for τ_n. The moderate deviation principle for marked Hawkes processes was studied in Seol [26], and limit theorems for the compensator of Hawkes processes were studied in Seol [27]. In addition to the large time limits of Gao and Zhu [8] and Gao and Zhu [9–11], some progress toward asymptotic results was seen. There have been studies in the literature extending and modifying the classic Hawkes process. First, the baseline intensity may be chosen to be time-inhomogeneous over time (see Gao, Zhou, and Zhu [8]). Second, the immigrants may arrive following a Cox process with shot noise intensity; this model is known as the dynamic contagion model (see Dassios and Zhao [3]). Third, the immigrants may arrive following a renewal process or a Poisson process, which is a generalization of the classical Hawkes process. This is known as the renewal Hawkes process (see Wheatley, Filimonov, and Sorrette [33]). Recently, Seol [28] studied the limit of the inverse Markov-Hawks process by introducing an inverse Markov-Hawks process that combines the features of several existing models of self-excitation processes. An extended version of the inverse Markovian-Hawkes model was studied in Seol [29]. With recent advances in storage technology, data-driven models are attracting attention. Contrasting with the continuous setting, in practice, events are often recorded in a discrete-time manner. It is more important for data to be collected in fixed steps or to only show results where the data has been aggregated. For example, continuous-time Hawkes processes can model irregularly spaced events in time, whereas modeling events that are evenly spaced in time requires a discrete-time Hawkes process, which is not applicable to many domains. It is suitable. The

Hawkes process is generally considered to be a continuous time-setting process. However, data is often enrolled in a discrete-time scheme. Seol [24] proposed a 0-1 discrete Hawkes process that starts with empty history and studied the law of large numbers, the central limit theorem, and invariance principles. Recently, Wang [30, 31] studied the limit behaviors of a discrete-time Hawkes process with random marks and proved the large and moderate deviations for a discrete-time Hawkes process with marks. Wang and He [32] investigated the precise deviations for the discrete Hawkes processes.

The intensity of the conditional event arrival rate, or the Hawkes process, tends to spike in response to an event and toward a target level in the absence of an event. Hawkes process is widely used in various fields, but its distributional characteristics are not well understood. We develop these properties by exploiting the fact that the two-dimensional process is composed of a Hawkes process, and its intensity is Markov. The structure of the relevant infinitesimal generator leads to a Dynkin formula and closed expressions for the moments of the Hawkes intensity. We indicate a transform of the Hawkes process that satisfies a certain partial integral differential equation. The solution to that equation turns out to be an exponentially affine function, i.e., the initial value of the two-dimensional process, whose coefficients fill the ODE system. By analyzing the transform, we can obtain an ODE that characterizes the probability distribution of the Hawkes process. We derive closed formulae for the moments of the process. Transform, distribution, and moment formulae have computational tractability for a range of applications in portfolio credit risk. The time-inhomogeneous Hawkes process is used primarily in enterprise portfolios to model accumulated losses. For examples, see [8, 21, 23]. The jump times represent default times, and the jump magnitudes represent the random losses at default. This formulation captures the impact of a default on the surviving names. It also includes the negative correlation between base rates and recovery rates. The transform formulae facilitate the valuation, hedging, and calibration of credit derivatives in the portfolio. These are securities whose coverage is a specific function of portfolio losses and provide insurance against default losses in the portfolio [4, 5, 7].

In this paper, we mainly consider time-inhomogeneous Hawkes processes with exponential exciting functions, i.e., $h(t) = \alpha(t)e^{-\beta(t)t}$. Let

$$Z_t = \sum_{\tau_k < t} \alpha(t) e^{-\beta(t)(t-\tau_k)}, \tag{1.3}$$

where $\tau_k' s$ are the arrivals of the Hawkes process with intensity $\lambda_t = \nu(t) + Z_t$ at time t. It is easy to see that Z_t is Markovian with a generator

$$\mathcal{L}f(z) = -\beta(t) z \frac{\partial f}{\partial z} + (\nu(t) + z)[f(z + \alpha(t)) - f(z)]. \tag{1.4}$$

Since $Z(t)$ is positive and mean-reverting, it can be viewed as the short rate process, and the price of a zero- coupon bond is given as a function of the short rate process $Z(t)$ as follows:

$$u(z,t) := \mathbb{E}[\exp(-\int_0^t Z_s ds)|Z_0 = z]. \tag{1.5}$$

The main goal of this paper is to obtain the distributional results for time-inhomogeneous Hawkes processes derived from the distributional properties of underlying processes. In particular, we study the moments, and Laplace transform of time-inhomogeneous Hawkes processes and moreover, we derive the formulae both for pricing zero-coupon bonds in the case when the short-term interest rate is driven by inhomogeneous Hawkes processes and for pricing and measuring default risk by exploiting the properties of a time-inhomogeneous Hawkes process N driven by a default intensity Z. We also illustrate the numerical results for the zero- coupon bond associated with time-inhomogeneous Hawkes processes.

The structure of this paper is organized as follows: Some auxiliary results and main theorems are stated in this section. In particular, we derived several expressions for financial models based on the properties of time-inhomogeneous Hawkes processes, with some numerical results. The proofs for the main theorems and auxiliary results are contained in Section 3.

2. The statement of the main results

This section states the main results of this paper. The first is devoted to the distributional properties of time-inhomogeneous Hawkes processes to generate computational tractability for a financial application. In particular, moments and Laplace transform of time-inhomogeneous Hawkes processes are obtained by the distributional properties of underlying processes. Based on the distributional properties of time-inhomogeneous Hawkes processes, applications to finance and insurance are obtained with some numerical results. We start with the assumptions we will use throughout the paper.

Assumption 2.1. Let $\alpha(t), \beta(t)$, and $\nu(t)$ be the continuous functions with the following conditions:

[1.] $\alpha(t) \geq 0$ and $\beta(t) \geq 0$.

[2.] $\nu(t) \geq 0$.

We first state the distributional properties, such as moments and transform, of the underlying process Z of time-inhomogeneous Hawkes processes.

2.1. *Moments and transform of the process Z*

We are interested in studying the first and second moments of the process Z_t and in addition, the Laplace transform for the process Z_t is obtained. This process can be characterized as the short rate process in the interest rate models or defaults in the risk model in the literature on finance and insurance.

Theorem 2.1. [The first and second moments in Z] Suppose that Z is the stochastic process as defined in (1.3). Then the first moment of the process, Z_t is

$$\mathbb{E}[Z_t] = u^{-1}(t)\int_0^t u(\zeta)\nu(\zeta)\alpha(\zeta)d\zeta + z_0 u^{-1}(t) \tag{2.1}$$

and the second moment of the process, Z_t is

$$\begin{aligned}\mathbb{E}[Z_t^2] =& w^{-1}(t)\int_0^t (2\nu(\xi)\alpha(\xi) + \alpha^2(\xi))[\int_0^\xi w(\zeta)\alpha(\zeta)\nu(\zeta) \\ &+ z_0 u(\zeta)d\zeta]d\xi + w^{-1}(t)\int_0^t \nu(\xi)\alpha^2(\xi)w(\xi)d\xi + w^{-1}(t)z_0^2.\end{aligned} \tag{2.2}$$

Moreover, the variance is

$$\begin{aligned}Var[Z_t] =& w^{-1}(t)\int_0^t (2\nu(\xi)\alpha(\xi)+\alpha^2(\xi))[\int_0^{\xi} w(\zeta)\alpha(\zeta)\nu(\zeta)\\&+z_0u(\zeta)d\zeta]d\xi + w^{-1}(t)\int_0^t \nu(\xi)\alpha^2(\xi)w(\xi)d\xi + w^{-1}(t)z_0^2\\&-[u^{-1}(t)\int_0^t u(\zeta)\nu(\zeta)\alpha(\zeta)d\zeta + z_0u^{-1}(t)]^2\end{aligned} \tag{2.3}$$

where $u(t) := \exp[\int_0^t(\beta(\xi)-\alpha(\xi))d\xi]$ and $w(t) = u^2(t)$.

The following are the immediate lemma of these relationships :

Lemma 2.1. Suppose that Z is the stochastic process as defined in (1.3).

(i) $\mathbb{E}[Z_t|Z_s = z] = \frac{\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta+z}{u(s,t)}$.

(ii) $\mathbb{E}[Z_tZ_s] = u^{-1}(s,t)\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta\left[u^{-1}(s)\int_0^s u(\zeta)\nu(\zeta)\alpha(\zeta)d\zeta + z_0u^{-1}(s)\right]$

$$+u^{-1}(s,t)\Big[w^{-1}(s)\int_0^s(\nu(\xi)\alpha(\xi)+\alpha^2(\xi))[\int_0^{\xi} w(\zeta)\alpha(\zeta)\nu(\zeta)$$

$$+z_0u(\zeta)d\zeta]d\xi + w^{-1}(s)\int_0^s \nu(\xi)\alpha^2(\xi)w(\xi)d\xi + w^{-1}(s)z_0^2]\Big]$$

where $u(s,t) := \exp[\int_s^t(\beta(\xi)-\alpha(\xi))d\xi]$, $u(0,t) := u(t)$, and $w(t) = u^2(t)$.

If we consider homogeneous Hawkes processes as letting $\alpha(t) := \alpha, \beta(t) := \beta$ and $\nu(t) := \nu$, then we have the following immediate proposition :

Proposition 2.1. Suppose that Z is the stochastic process defined in (1.3). Then we have

(i) $\mathbb{E}[Z_t] = \frac{\nu\alpha}{\beta-\alpha} + (z_0 - \frac{\nu\alpha}{\beta-\alpha})e^{-(\beta-\alpha)t}$.

(ii) $\mathbb{E}[Z_t^2] = \frac{(2\nu\alpha+\alpha^2)z_0}{\beta-\alpha}e^{-(\beta-\alpha)t} + \frac{\nu\alpha^2(2\nu+2\beta-\alpha)}{4(\beta-\alpha)^2} - \frac{(2\nu\alpha+\alpha^2)(\nu\alpha+2z_0)}{2(\beta-\alpha)}\cdot te^{-2(\beta-\alpha)t}$
$+[z_0^2 - \frac{(2\nu\alpha+\alpha^2)(\nu\alpha-4z_0)-2\nu\alpha^2(\beta-\alpha)}{4(\beta-\alpha)^2}]e^{-2(\beta-\alpha)t}$.

(iii) $\mathbb{E}[Z_t|Z_s = z] = \frac{\alpha\nu}{\beta-\alpha} + \left(z - \frac{\alpha\nu}{\beta-\alpha}\right)e^{-(\beta-\alpha)(t-s)}$.

(iv) $\mathbb{E}[Z_tZ_s] = \frac{\alpha^2\nu^2}{(\beta-\alpha)^2}\left(1-e^{-(\beta-\alpha)(t-s)}\right) + \frac{\alpha\nu}{\beta-\alpha}\left(1-e^{-(\beta-\alpha)(t-s)}\right)\left(z_0 - \frac{\alpha\nu}{\beta-\alpha}\right)e^{-(\beta-\alpha)t}$
$+\frac{(2\nu\alpha+\alpha^2)z_0}{\beta-\alpha}e^{-(\beta-\alpha)(2t-s)} + \frac{\nu\alpha^2(2\nu+2\beta-\alpha)}{4(\beta-\alpha)^2}e^{-(\beta-\alpha)(t-s)}$
$+\left[z_0^2 - \frac{(2\nu\alpha+\alpha^2)(\nu\alpha+2z_0)t}{2(\beta-\alpha)} - \frac{(2\nu\alpha+\alpha^2)(\nu\alpha-4z_0)-2\nu\alpha^2(\beta-\alpha)}{4(\beta-\alpha)^2}\right]e^{-(\beta-\alpha)(3t-s)}$.

The following result is about the Laplace transform for the process Z.

Theorem 2.2 (Transform)**.** Suppose that Z is the stochastic process as defined in (1.3). For any θ, the Laplace transform of Z_t satisfies $E[\exp(-\theta Z_t)] = \exp(a(t)\lambda + b(t))$, where $a(t)$ and $b(t)$ satisfies the ordinary differential equations

$$\begin{cases} a'(t) + \beta(t)a(t) - e^{a(t)\alpha(t)} + 1 = 0 \\ b'(t) - \nu(t)e^{a(t)\alpha(t)} + \nu(t) = 0 \end{cases} \tag{2.4}$$

with boundary conditions

$$a(0) = -\theta \text{ and } b(0) = 0. \tag{2.5}$$

Based on the distributional properties of process Z, we obtain the followings distributional properties for time-inhomogeneous Hawkes processes :

2.2. Moments and transform of Hawkes processes

Theorem 2.3. [The first moment for N] Suppose that Z is the stochastic process as defined in (1.3) and N is time-inhomogeneous Hawkes process with exponentially exciting functions. Then the first moment of N is

$$\mathbb{E}[N(t)] = \int_0^t [\int_0^s u(\xi)\nu(\xi)\alpha(\xi)d\xi + z_0 + \nu(s)u(s)]u^{-1}(s)ds. \tag{2.6}$$

If we consider homogeneous Hawkes processes as $\alpha(t) := \alpha$, $\beta(t) := \beta$, and $\nu(t) := \nu$, then we have the following immediate proposition :

Proposition 2.2. Suppose that Z is the stochastic process as defined in (1.3) and N is the homogeneous Hawkes process with exponentially exciting functions.

$$\mathbb{E}[N(t)] = \frac{\nu\beta t}{\beta-\alpha} + \frac{z_0(\beta-\alpha)-\nu\alpha}{(\beta-\alpha)^2}(1-e^{-(\beta-\alpha)t}).$$

Theorem 2.4 (Transform)**.** Suppose that Z is the stochastic process as defined in (1.3) and N is inhomogeneous Hawkes processes with exponentially exciting functions. For any θ, the Laplace transform of N_t satisfies $E[\exp(-\theta N_t)] = \exp(c(t)\lambda + d(t))$, where $c(t)$ and $d(t)$ satisfy the ordinary differential equations

$$\begin{cases} c'(t) + \beta(t)c(t) - e^{c(t)\alpha(t)} + 1 = 0 \\ d'(t) - \nu(t)e^{c(t)\alpha(t)} + \nu(t) = 0 \end{cases} \tag{2.7}$$

with boundary conditions

$$c(0) = -\theta \text{ and } d(0) = -\nu_0\theta.$$

Finally, we study the applications to mathematical finance of time-inhomogeneous Hawkes processes based on some distributional properties and some numerical results.

Theorem 2.5. Let Z_t be the stochastic process as defined in (1.3). Let

$$u(z,t) := \mathbb{E}[\exp(-\int_0^t Z_s ds)|Z_0 = z] \tag{2.8}$$

with the following boundary condition:

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u - zu \\ u(z,0) = 1, \end{cases} \tag{2.9}$$

where $\mathcal{L}$ is defined in (1.4), then the process $u(z,t)$ satisfies $u(z,t) = \exp(A(t)z + B(t))$, where $A(t)$ and $B(t)$ satisfy the ordinary differential equations

$$\begin{cases} A'(t) + \beta(t)A(t) - e^{A(t)\alpha(t)} + 2 = 0 \\ B'(t) - \nu(t)e^{A(t)\alpha(t)} + \nu(t) = 0 \end{cases} \tag{2.10}$$

with boundary conditions

$$A(0) = 0 \text{ and } B(0) = 0. \tag{2.11}$$

2.3. The price of zero coupon bond

We now consider the problem of the valuation of a zero-coupon bond. That is, we assume that the dynamic of the interest rate is given by a time-inhomogeneous Hawkes process under risk- neutral probability, and we compute the price of a zero- coupon bond as defined by the short rate Z_t in the interest rate model. Based on Theorem 2.5, we have the following applications in finance :

Proposition 2.3. Let $r_t = Z_t$ be the stochastic model as defined in (1.3). Then the price of the zero-coupon bond satisfies

$$\mathbb{E}[\exp(-\int_0^t Z_s ds)|Z_0 = z] = \exp(A(t)z + B(t)) \tag{2.12}$$

where $A(t)$ and $B(t)$ satisfiy the ordinary differential equations

$$\begin{cases} A'(t) + \beta(t)A(t) - e^{A(t)\alpha(t)} + 2 = 0 \\ B'(t) - \nu(t)e^{A(t)\alpha(t)} + \nu(t) = 0 \end{cases} \tag{2.13}$$

with boundary conditions

$$A(0) = 0 \text{ and } B(0) = 0. \tag{2.14}$$

3. Numerical results

In this section, we illustrate the zero coupon bond $u(z,t)$ obtained in Theorem 2.5 by some numerical examples. Let us first assume that

$$\alpha(t) = \beta(t) = \nu(t) := t, \tag{3.1}$$

then the price of the zero coupon bond $u(z,t)$ satisfies $u(z,t) = \exp(A(t)z + B(t))$, where $A(t)$ and $B(t)$ satisfy the ordinary differential equations

$$\begin{cases} A'(t) + t \cdot A(t) - e^{A(t)\cdot t} + 2 = 0 \\ B'(t) - t \cdot e^{A(t)\cdot t} + t = 0 \end{cases} \tag{3.2}$$

with boundary conditions

$$A(0) = 0 \text{ and } B(0) = 0.$$

The summary statistics of the zero coupon bond $u(z,t)$ for the case $z = 0, 1, 2, 3, 4$ are given in Figures 1, 2 and Table 1.

Table 1. Illustration of the zero coupon bond $U(z,t)$.

$U(z,t)$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$z = 0$	1	0.835867	0.249213	0.0267053	0.0012382
$z = 1$	1	0.331194	0.0847802	0.0128266	0.000746301
$z = 2$	1	0.131228	0.0288415	0.00616064	0.000449819
$z = 3$	1	0.0519964	0.00981166	0.00295897	0.00027112
$z = 4$	1	0.0206025	0.00333784	0.0014212	0.000163413

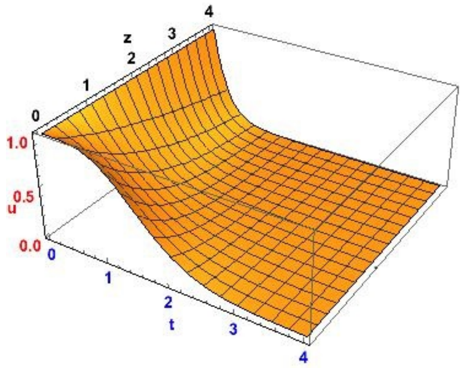

Figure 1. This is the 3D plot of the zero coupon bond as a function of the zero default intensity z and the present time t.

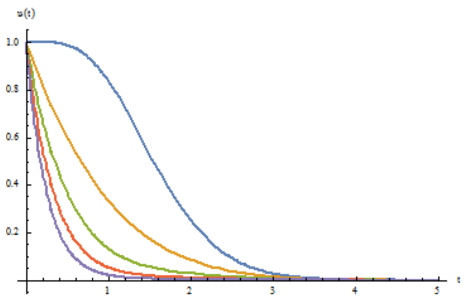

Figure 2. This is a plot of the zero coupon bond $U(z,t)$ against the present time t for fixed zero default intensity z=0, 1, 2, 3, 4. The blue, orange, green, red, purple lines denote the cases when z=0, 1, 2, 3, 4. We can see from the plot that when z=0, the shape of decay for the zero coupon bond is not exponential. Otherwise, the shape of decay for the zero-coupon bond is exponential.

Remark 3.1 (Comparison with the time-homogeneous Hawkes process)**.** Suppose that $h(t) = \alpha e^{-\beta t}$ and

$$Z_t = \sum_{\tau_k < t} \alpha e^{-\beta(t-\tau_k)}$$

where τ_k's are the arrival of the Hawkes process with intensity $\lambda_t = \nu + Z_t$ i.e., time-homogeneous case. $u(z,t)$ has a linear graph.

In terms of representing changes over time, exponential decay assumes that changes occur at an accelerating rate as time progresses, whereas linear decay assumes a constant rate of change over time. In financial contexts, where the rate of depreciation or appreciation often accelerates or decelerates, exponential decay provides a more realistic depiction of such changes. In modeling the change in value, exponential decay provides a more accurate representation as it can capture the accelerating or decelerating nature of value changes. While linear decay simplifies the modeling process by assuming a constant rate of change, it may fail to accurately reflect real-world fluctuations in value, which can be more accurately captured by exponential decay. Exponential decay enjoys strong theoretical and empirical backing. It is a widely used mathematical concept that accurately describes various natural phenomena, including the time-value decay of assets in financial markets. In contrast, linear decay may lack the theoretical robustness and empirical support that exponential decay benefits from.

4. The proofs of the main results

In this section, we give some proofs of the main results regarding on both interest rates and Hawkes processes.

4.1. Proof of results for the process Z

Proof of Theorem 2.1. To prove this, we start by noting that each jump of Z is of size $\alpha(t)$ and that, between the jumps, Z decays exponentially at rate $\beta(t)$. It follows that

$$f(Z_t) = f(Z_0) - \int_0^t \beta(s) Z_s f'(Z_s) ds + \int_0^t [f(Z_s + \alpha(s)) - f(Z_s)] dN_s. \tag{4.1}$$

Within the last integral, the integrand is a continuous function of the left continuous adaptive process and is therefore predictable. Thus, the meaning of intensity

$$\mathbb{E} f(Z_t) = f(Z_0) - \mathbb{E} \int_0^t \beta(s) Z_s f'(Z_s) ds + \mathbb{E} \int_0^t Z_s [f(Z_s + \alpha(s)) - f(Z_s)] ds. \tag{4.2}$$

Thus, we have

$$f(Z_t) = f(Z_0) + \int_0^t \mathbb{E} \mathcal{L} f(Z_s) ds. \tag{4.3}$$

Taking $f(z) = z$ and $f(z) = z^2$ give us two explicit forms

$$\mathbb{E}[Z_t] = z_0 + \mathbb{E}[\int_0^t \mathcal{L} Z_s ds] \tag{4.4}$$

and

$$\mathbb{E}[Z_t^2] = z_0^2 + \mathbb{E}[\int_0^t \mathcal{L} Z_s^2 ds]. \tag{4.5}$$

(i) Since $\mathcal{L} z = -\beta(t) z + (\nu(t) + z)\alpha(t)$, we have

$$\mathbb{E}[Z_t] = z_0 + \mathbb{E}[\int_0^t \mathcal{L} Z_s ds] = z_0 + \int_0^t -\beta(s)\mathbb{E}[Z_s] + (\nu(s) + \mathbb{E}[Z_s])\alpha(s) ds. \tag{4.6}$$

Taking the derivative with respect to t on both sides and letting $m(t) := \mathbb{E}[Z_t]$ provide

$$\frac{d}{dt} m(t) = -(\beta(t) - \alpha(t)) m(t) + \nu(t)\alpha(t) \ \text{ and } \ m(0) = z_0. \tag{4.7}$$

Let the integrating factor

$$u(t) := \exp[\int_0^t (\beta(\xi) - \alpha(\xi)) d\xi]. \tag{4.8}$$

Solving a differential equation with an initial value yields

$$m(t)=\frac{\int_0^t u(\zeta)v(\zeta)\alpha(\zeta)d\zeta+z_0}{u(t)}. \tag{4.9}$$

Thus, we have

$$\begin{aligned}\mathbb{E}[Z_t]=&\exp[-\int_0^t(\beta(\xi)-\alpha(\xi))d\xi]\int_0^t\exp[\int_0^\xi(\beta(\zeta)-\alpha(\zeta))d\zeta]v(\xi)\alpha(\xi)d\xi\\&+z_0\exp[-\int_0^t(\beta(\xi)-\alpha(\xi))d\xi].\end{aligned} \tag{4.10}$$

(ii) Since $\mathcal{L}z^2=-2\beta(t)z^2+(v(t)+z)(2z\alpha(t)+\alpha^2(t))$, we have

$$\begin{aligned}\mathbb{E}[Z_t^2]=&z_0^2+\mathbb{E}[\int_0^t\mathcal{L}Z_s^2ds]=z_0^2+\int_0^t-(2\beta(s)-2\alpha(s))\mathbb{E}[Z_s^2]\\&+(2v(s)\alpha(s)+\alpha^2(s))\mathbb{E}[Z_s]+v(s)\alpha^2(s)ds.\end{aligned} \tag{4.11}$$

Taking the derivative with respect to t on both sides and letting $n(t):=\mathbb{E}[Z_t^2]$ provide

$$\frac{d}{dt}n(t)=-2(\beta(t)-\alpha(t))n(t)+(2v(t)\alpha(t)+\alpha^2(t))m(t)+v(t)\alpha^2(t), \tag{4.12}$$

and so

$$\frac{d}{dt}n(t)+2(\beta(t)-\alpha(t))n(t)=(2v(t)\alpha(t)+\alpha^2(t))m(t)+v(t)\alpha^2(t) \tag{4.13}$$

with the initial value $n(0)=z_0^2$.

Let the integrating factor

$$w(t)=u(t)^2:=\exp[2\int_0^t(\beta(\xi)-\alpha(\xi))d\xi]. \tag{4.14}$$

Solving a differential equation with $w(t)$ yields

$$(w(t)n(t))'=(2v(t)\alpha(t)+\alpha^2(t))m(t)w(t)+v(t)\alpha^2(t)w(t) \tag{4.15}$$

and so

$$\begin{aligned}w(t)n(t)&=\int_0^t(2v(\xi)\alpha(\xi)+\alpha^2(\xi))m(\xi)w(\xi)d\xi+\int_0^t v(\xi)\alpha^2(\xi)w(\xi)d\xi+C\\&=\int_0^t(2v(\xi)\alpha(\xi)+\alpha^2(\xi))[\int_0^\xi w(\zeta)\alpha(\zeta)v(\zeta)+z_0u(\zeta)d\zeta]d\xi+\int_0^t v(\xi)\alpha^2(\xi)w(\xi)d\xi+C.\end{aligned} \tag{4.16}$$

Thus

$$\mathbb{E}[Z_t^2]=w^{-1}(t)\Big[\int_0^t(2v(\xi)\alpha(\xi)+\alpha^2(\xi))[\int_0^\xi w(t)\alpha(\zeta)v(\zeta)$$

$$+z_0u(\zeta)d\zeta]d\xi + \int_0^t \nu(\xi)\alpha^2(\xi)w(\xi)d\xi + C\Big]. \tag{4.17}$$

Therefore, by the boundary condition

$$\begin{aligned}\mathbb{E}[Z_t^2] =& w^{-1}(t)\Big[\int_0^t (2\nu(\xi)\alpha(\xi) + \alpha^2(\xi))[\int_0^\xi w(t)\alpha(\zeta)\nu(\zeta) \\ &+z_0u(\zeta)d\zeta]d\xi + \int_0^t \nu(\xi)\alpha^2(\xi)\nu(\xi)d\xi + w^{-1}(t)z_0^2\Big].\end{aligned} \tag{4.18}$$

□

Proof of Lemma 2.1. (i) Using Theorem 2.1, we obtain

$$\mathbb{E}[Z_t|Z_s = z] = z + \mathbb{E}[\int_s^t \mathcal{L}Z_u du] = z + \int_s^t \mathbb{E}[\mathcal{L}Z_u]du.$$

Taking the derivative with respect to both sides and letting $l(t) := \mathbb{E}[Z_t|Z_s = z]$ provide

$$\frac{d}{dt}l(t) = -(\beta(t) - \alpha(t))l(t) + \nu(t)\alpha(t), \tag{4.19}$$

and so

$$\frac{d}{dt}l(t) + (\beta(t) - \alpha(t))l(t) = \nu(t)\alpha(t) \tag{4.20}$$

with the initial value $l(s) = z$.

Let the integrating factor $u(s,t) := \exp[\int_s^t (\beta(\xi) - \alpha(\xi))d\xi]$. Solving a differential equation with $u(s,t)$ yields

$$(u(s,t)l(t))' = u(s,t)\nu(t)\alpha(t) \tag{4.21}$$

and so

$$u(s,t)l(t) = \int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta + C. \tag{4.22}$$

Thus

$$\mathbb{E}[Z_t|Z_s = z] = u^{-1}(s,t)[\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta + C]. \tag{4.23}$$

Therefore, by the boundary condition

$$\mathbb{E}[Z_t|Z_s = z] = u^{-1}(s,t)[\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta + z]. \tag{4.24}$$

(ii) Using the previous result in (i),

$$\mathbb{E}[Z_tZ_s] = \mathbb{E}[\mathbb{E}[Z_t|Z_s]Z_s]$$

$$=\mathbb{E}\left[u^{-1}(s,t)\left(\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta+Z_s\right)Z_s\right] \tag{4.25}$$

$$=u^{-1}(s,t)\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta\mathbb{E}[Z_s]+u^{-1}(s,t)\mathbb{E}[Z_s^2]. \tag{4.26}$$

Using Theorem 2.1, we have

$$\begin{aligned}\mathbb{E}[Z_tZ_s] =&u^{-1}(s,t)\int_s^t u(s,\zeta)\nu(\zeta)\alpha(\zeta)d\zeta\left[u^{-1}(s)\int_0^s u(\zeta)\nu(\zeta)\alpha(\zeta)d\zeta+z_0u^{-1}(s)\right]\\ &+u^{-1}(s,t)\left[w^{-1}(s)\int_0^s(\nu(\xi)\alpha(\xi)+\alpha^2(\xi))[\int_0^\xi w(\zeta)\alpha(\zeta)\nu(\zeta)\right.\\ &\left.+z_0u(\zeta)d\zeta]d\xi+w^{-1}(s)\int_0^s\nu(\xi)\alpha^2(\xi)w(\xi)d\xi+w^{-1}(s)z_0^2\right].\end{aligned} \tag{4.27}$$

□

Proof of Theorem 2.2. Let us recall that for a pair $(Z_t, N(t))$, the generator is

$$\mathcal{L}f(z)=-\beta(t)z\frac{\partial f}{\partial z}+(\nu(t)+z)[f(z+\alpha(t))-f(z)]. \tag{4.28}$$

Let $u(t,z):=u(\theta,t,z):=\mathbb{E}[\exp(-\theta Z_t)|Z_0=z]$. We consider

$$f(t,Z_t,N_t)=\mathbb{E}[\exp(-\theta Z_t)|Z_t,N(t)]. \tag{4.29}$$

Note that $f(t,Z_t,N_t)_{t\le T}$ is a martingale only if

$$\frac{\partial f}{\partial t}-\mathcal{L}f=0 \tag{4.30}$$

and $f(T,Z_T,N_T)=\exp(-\theta Z_T)$. Let $f(t,z,n)=u(t,z)\exp(-\theta n)$ and make the time change $t\to T-t$ to change the backward equation to the forward equation. We have

$$\begin{cases}\frac{\partial u}{\partial t}=-\beta(t)z\frac{\partial u}{\partial z}+(\nu(t)+z)[u(t,z+\alpha(t))-u(t,z)]\\ u(0,z)=e^{-\theta z}.\end{cases} \tag{4.31}$$

So the above equations are affine, and let us take $u(\theta,t,z)=\exp(a(t)z+b(t))$. Then we have

$$z[a'(t)+\beta(t)a(t)-\exp(a(t)\alpha(t))+1]+b'(t)-\nu(t)[\exp(a(t)\alpha(t))-1]=0 \tag{4.32}$$

which implies

$$\begin{cases}a'(t)+\beta(t)a(t)-e^{a(t)\alpha(t)}+1=0\\ b'(t)-\nu(t)e^{a(t)\alpha(t)}+\nu(t)=0\end{cases} \tag{4.33}$$

with boundary conditions

$$a(0)=-\theta \text{ and } b(0)=0. \tag{4.34}$$

□

4.2. Proofs of results for Hawkes processes

Proof of Theorem 2.3. Note first that

$$dZ_t = -\beta(t)Z_t dt + \alpha(t)dN(t), \tag{4.35}$$

which implies that

$$N(t) = \int_0^t \frac{dZ_s}{\alpha(s)} + \int_0^t \frac{\beta(s)}{\alpha(s)} Z_s ds. \tag{4.36}$$

Then, taking expectations to from sides and recalling that $m(t) := \mathbb{E}[Z_t]$, give us

$$\begin{aligned} \mathbb{E}[N(t)] &= \int_0^t \frac{dm(s)}{\alpha(s)} + \int_0^t \frac{\beta(s)}{\alpha(s)} m(s)ds \\ &= \int_0^t \frac{(\alpha(s)-\beta(s))m(s)+\nu(s)\alpha(s)}{\alpha(s)}ds + \int_0^t \frac{\beta(s)}{\alpha(s)} m(s)ds \\ &= \int_0^t (m(s)+\nu(s))ds. \end{aligned} \tag{4.37}$$

Thus

$$\mathbb{E}[N(t)] = \int_0^t u^{-1}(s)[\int_0^s u(\xi)\nu(\xi)\alpha(\xi)d\xi + z_0 + \nu(s)u(s)]ds. \tag{4.38}$$

□

Proof of Theorem 2.4. Let us recall that for a pair $(Z_t, N(t))$, the generator is

$$\mathcal{L}f(z) = -\beta(t)z\frac{\partial f}{\partial z} + (\nu(t)+z)[f(z+\alpha(t)) - f(z)]. \tag{4.39}$$

Let $u(t,z) := u(\theta,t,z) := \mathbb{E}[\exp(-\theta N_t)|N_0 = \nu_0 + z]$. We consider

$$f(t, Z_t, N_t) = \mathbb{E}[\exp(-\theta N_t)|Z_t, N(t)]. \tag{4.40}$$

Note that $f(t, Z_t, N_t)_{t\leq T}$ is a martingale only if

$$\frac{\partial f}{\partial t} - \mathcal{L}f = 0 \tag{4.41}$$

and $f(T, Z_T, N_T) = \exp(-\theta N_T)$. Let $f(t,z,n) = u(t,z)\exp(-\theta n)$ and make the time change $t \to T-t$ to change the backward equation to the forward equation. We have

$$\begin{cases} \frac{\partial u}{\partial t} = -\beta(t)z\frac{\partial u}{\partial z} + (\nu(t)+z)[u(t,z+\alpha(t)) - u(t,z)] \\ u(0,z) = e^{-\theta(z+\nu_o)}. \end{cases} \tag{4.42}$$

So the above equations are affine, and let us take $u(\theta,t,z) = \exp(c(t)z + d(t))$. Then we have

$$\lambda[c'(t) + \beta(t)c(t) - \exp(c(t)\alpha(t)) + 1] + d'(t) - \nu(t)[\exp(c(t)\alpha(t)) - 1] = 0 \tag{4.43}$$

which implies

$$\begin{cases} c'(t) + \beta(t)c(t) - e^{c(t)\alpha(t)} + 1 = 0 \\ d'(t) - \nu(t)e^{c(t)\alpha(t)} + \nu(t) = 0 \end{cases} \tag{4.44}$$

with boundary conditions

$$c(0) = -\theta \text{ and } d(0) = -\nu_0\theta. \tag{4.45}$$

□

4.3. Proof of results for applications to finance

Proof of Theorem 2.5. Let us recall that for a pair $(Z_t, N(t))$, note that for a Markovian Hawkes process N with intensity being λ_{t-} at time t, the Markov process Z has the infinitesimal generator as

$$\mathcal{L}f(z) = -\beta(t)z\frac{\partial f}{\partial z} + (\nu(t)+z)[f(z+\alpha(t)) - f(z)]. \tag{4.46}$$

Let

$$u(z,t) := \mathbb{E}[\exp(-\int_0^t Z_s ds)|Z_0 = z]. \tag{4.47}$$

We consider

$$f(t, Z_t, N_t) = \mathbb{E}[\exp(-\int_0^t Z_s ds)|Z_t, N(t)]. \tag{4.48}$$

Note that $f(t, Z_t, N_t)_{t\leq T}$ is a martingale only if

$$\frac{\partial f}{\partial t} - \mathcal{L}f + zf = 0. \tag{4.49}$$

Make the time change $t \to T - t$ to change the backward equation to the forward equation. We have

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u - zu = -\beta(t)z\frac{\partial u}{\partial z} + (\nu(t)+z)[u(t, z+\alpha(t)) - u(t,z)] - zu \\ u(0,z) = 1. \end{cases} \tag{4.50}$$

So the above equations are affine and let us take Ansatz $u(\theta, t, z) = \exp(A(t)z + B(t))$. Then, substituting the Ansatz into the Eq (4.50), we have

$$z\Big[A'(t) + \beta(t)A(t) - \exp(A(t)\alpha(t)) + 2\Big] + B'(t) - \nu(t)\Big[\exp(A(t)\alpha(t)) - 1\Big] = 0$$

which implies

$$\begin{cases} A'(t) + \beta(t)A(t) - e^{A(t)\alpha(t)} + 2 = 0 \\ B'(t) - \nu(t)e^{A(t)\alpha(t)} + \nu(t) = 0 \end{cases} \tag{4.51}$$

with boundary conditions

$$A(0) = 0 \text{ and } B(0) = 0. \tag{4.52}$$

□

Proof of Corollary 2.3. From Theorem 2.5, letting $r_t = Z_t$, we conclude that the price of a zero- coupon bond satisfies the given ordinary differential equation. Thus, the proof of Corollary 2.3 is complete. □

5. Conclusions

The Hawkes process has wide applications in neuroscience, seismology, DNA modeling, finance and many other fields. It has both self-exciting and clustering properties, making it very attractive for some financial applications. In particular, the self-exciting property and clustering properties of the Hawkes process make it a strong candidate in modeling for correlation fundamental modeling and credit derivatives evaluation in finance, while the Hawkes process is distinguished by its rich structural properties, making it a preferred candidate for computation in all kinds of finance application. This fact has been explored and illustrated in [4] for the time-homogeneous case, which covers most of the related applications. However, there are many situations that require time-dependent parameters, such as model calibration. In this paper, we obtained the distributional results for time-inhomogeneous Hawkes processes, derived from the distributional properties of underlying processes. In particular, we study the moments and Laplace transform of time-inhomogeneous Hawkes processes, and furthermore, we derive the formulae both for pricing zero-coupon bonds in the case when the short- term interest rate is driven by inhomogeneous Hawkes processes and for pricing and measuring default risk by exploiting the properties of a time-inhomogeneous Hawkes process N driven by a default intensity Z. We also illustrate the numerical results for the zero- coupon bond associated with time-inhomogeneous Hawkes processes. So the first step in this paper is to devote attention to the distributional properties of time-inhomogeneous Hawkes processes to generate computational tractability for a financial application. In particular, the moments and Laplace transform of time-inhomogeneous Hawkes processes are obtained by the distributional properties of the underlying processes. Based on the distributional properties of time-inhomogeneous Hawkes processes, applications to finance and insurance are obtained with some numerical results.

Author contributions

Youngsoo Seol: Conceptualization, Formal analysis, Investigation, Methodology, Software, Visualization, Writing-original draft, Writing-review editing; Suhyun Lee: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Visualization, Writing-original draft, Writing-review editing; Mikyoung Ha: Investigation, Methodology, Software, Visualization, Writing-original draft, Writing-review editing; Young-Ju Lee: Conceptualization, Data curation, Formal analysis, Methodology, Software, Validation, Visualization, Writing-original draft, Writing-review editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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