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Research article

Exponentiated extended extreme value distribution: Properties, estimation, and applications in applied fields

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Abstract: The proposed article introduces a novel three-parameter lifetime model called an exponentiated extended extreme-value (EEEV) distribution model. The EEEV distribution is characterized by increasing or bathtub-shaped hazard rates, which can be advantageous in the context of reliability. Various statistical properties of the distribution have been derived. The article discusses four estimation methods, namely, maximum likelihood, least squares, weighted least squares, and Cramér-von Mises, for EEEV distribution parameter estimation. A simulation study was carried out to examine the performance of the new model estimators based on the four estimation methods by using the average bias, mean squared errors, relative absolute biases, and root mean square error. The flexibility and significance of the EEEV distribution are demonstrated by analyzing three real-world datasets from the fields of medicine and engineering. The EEEV distribution exhibits high adaptability and outperforms several well-known statistical models in terms of performance.

Keywords: exponentiated extended extreme value distribution; mathematical statistics; hazard rate; estimation methods; simulations; lifetime data
Mathematics Subject Classification: 60E05, 62F10, 62H12

1. Introduction

Data analysis plays a critical role in numerous scientific disciplines, including reliability analysis, health sciences, economics, industry, and environmental studies, among others. To effectively conduct data analysis, it is essential to utilize appropriate models and statistical distributions. The development of new distributions has expanded researchers' options of suitable models, allowing them to accurately examine data characteristics and patterns. Additionally, the introduction of new distributions has increased the accuracy of analysis and predictions, enhancing the understanding of data and the ability to forecast future outcomes. Recently, several methods have emerged to facilitate the derivation of new distributions from existing ones, leading to the creation of more adaptable and improved models that better align with real-world data. In this context, we refer the reader to the literature cited in [1–9]. Xu et al. [10] proposed a model to evaluate the reliability of multicomponent systems in dynamic environments, accounting for component correlations and shared environmental effects. Wang et al. [11] conducted a study that analyzed clustered panel count data, which are commonly found in biomedical studies with multiple observation points and potential correlations within clusters. They introduced two semiparametric models to address these correlations and prevent biased estimations; these models were supported by simulation studies and a real-world application showcasing their efficacy. Xu et al. [12] introduced a bivariate Wiener model to capture the degradation patterns of two key performance characteristics of permanent magnet brakes. They also presented an objective Bayesian method for analyzing degradation data with small sample sizes.

Many newly introduced probability distributions include a hazard rate (HR) function that is important in the analysis of lifetime data, particularly in studies related to survival and reliability. For instance, in the field of reliability engineering, numerous real-life datasets exhibit a distinctive bathtub-shaped HR, which indicates varying behaviors across different phases. In the initial phase, known as the infant mortality phase, the HR starts at a high level, indicating a greater likelihood of early failures in the life cycle of the system. Subsequently, as the system enters the normal life phase, the HR stabilizes and remains relatively constant, indicating a consistent HR during this period. Finally, in the wear-out phase, the HR increases, indicating an escalating probability of failures as the system ages and its components deteriorate. Furthermore, the HR function offers valuable insights into the timing and frequency of failures in medical data. It has extensive applications, such as survival analysis, clinical trial analysis, disease progression modeling, and risk assessment [13, 14].

Cho et al. [15] introduced the exponentiated extreme-value (EEV) distribution, which is defined by the cumulative distribution function (CDF) as follows:

$$F(x) = \left(1 - e^{-e^{\frac{x-\theta}{\sigma}}}\right)^{\lambda}, \quad -\infty < x < \infty, \quad \sigma, \lambda > 0, \quad -\infty < \theta < \infty. \tag{1.1}$$

Cho et al. [15] examined the characteristics of the EEV distribution and provided approximate maximum likelihood estimators (MLEs) for the scale and location parameters by using multiply type-II censored samples. The estimators were evaluated based on the mean squared error (MSE) for different censored samples. One major drawback of the work mentioned above is a lack of HR function, rendering it inadequate for many applications, particularly those involving engineering and medical data. Several conventional distributions cannot model the failure rate patterns observed in real-world data with high accuracy, specifically the characteristic bathtub-shaped trend commonly found in engineering and medical data. While the EEV distribution offers some flexibility, it lacks the ability to fully capture these patterns. To address this limitation, we propose a new three-parameter lifetime model called the exponentiated extended extreme-value (EEEV) distribution. By incorporating an additional parameter, the EEEV distribution can effectively model both increasing and bathtub-shaped HRs, making it a more versatile tool for lifetime data analysis.

In this research paper, we propose a novel distribution by adding an additional variable to the CDF of the EEV distribution. We refer to this new distribution as the EEEV distribution, which possesses several advantageous properties. First, there is the flexibility in representing increasing or bathtub-shaped trends in data. Second, it facilitates the realization of the necessary flexibility when analyzing real-world data, particularly in fields such as engineering and medicine. Third, due to its inclusion of only three parameters, this model is easy to implement. Fourth, its parameters can be estimated by using various methods, and, here, simulation results were used to compare the performance of these estimators. Finally, the EEEV distribution consistently outperformed other competing distributions in terms of goodness of fit. Although the EEEV distribution provides flexibility in the modeling of different failure rate patterns, it is not suitable for all types of data, particularly those exhibiting a decreasing trend. Furthermore, the proposed model was only applied to complete datasets, but its applicability to censored datasets, such as type-II progressive censoring, generalized hybrid censoring, and adaptive type-II progressive censoring schemes, will be explored in future research.

The paper is organized as follows. Section 2 explores the distribution of the proposed EEEVs. Section 3 establishes some of its statistical properties. Section 4 presents four estimation methods for calculating the EEEV distribution parameters. Section 5 shows the simulation results that were obtained by using four different estimating methods. Section 6 discusses the applicability of the EEEV distribution to three real datasets, illustrating its significance and flexibility. Finally, Section 7 provides a summary of the conclusion.

2. The EEEV distribution

Within this section, we propose a new model that extends the EEV distribution by incorporating an additional variable into the base CDF for the EEV distribution. The additional variable enhances the flexibility of the new distribution in the modeling of various types of data in different fields, consistently providing a better fit than its competitors. The resulting model is referred to as the EEEV distribution model. The CDF and probability density function (PDF) of the EEEV distribution with the parameter vectors denoted by $\Theta = (\delta, \gamma, \eta)$ can be expressed as follows:

$$F(x;\Theta) = \left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta}, \quad x > 0, \quad \eta, \delta > 0, \quad -\infty < \gamma < \infty, \tag{2.1}$$

and

$$f(x;\Theta) = \eta\,\delta\,(1+\delta x)\,\mathrm{e}^{\delta x-\gamma}\,\mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta-1}, \tag{2.2}$$

where η, γ, and δ are the shape, location, and scale parameters, respectively.

The survival function and the HR function of the EEEV distribution are expressed as follows:

$$S(x;\Theta) = 1 - \left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta}, \tag{2.3}$$

and

$$h(x;\Theta) = \frac{\eta\,\delta\,(1+\delta x)\,\mathrm{e}^{\delta x-\gamma}\,\mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta-1}}{1 - \left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta}}. \tag{2.4}$$

Figures 1 and 2 depict the graphical performance of the PDF and HR function for the EEEV distribution for various parameter options. Figure 1 shows that the PDF of the EEEV distribution can be unimodal-shaped (see Figure 1(a)) or decreasing (see Figure 1(b)). On the other hand, Figure 2 indicates that the HR function can be increasing (see Figure 2(a)) or bathtub-shaped (see Figure 2(b)).

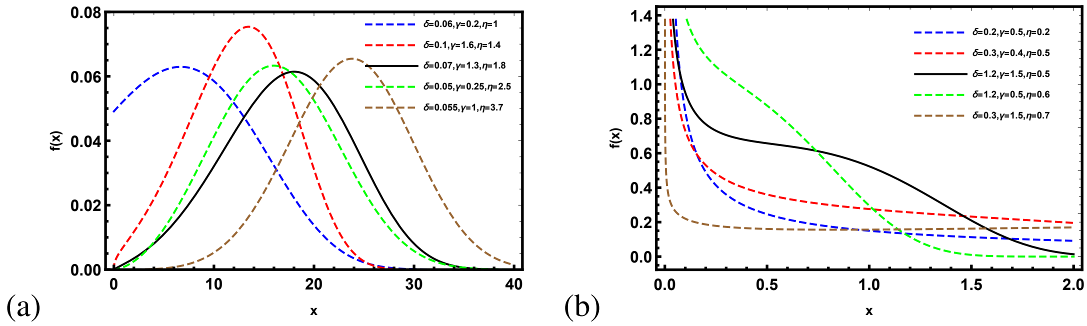

Figure 1. PDF results for EEEV distribution for different choices of η, γ, and δ.

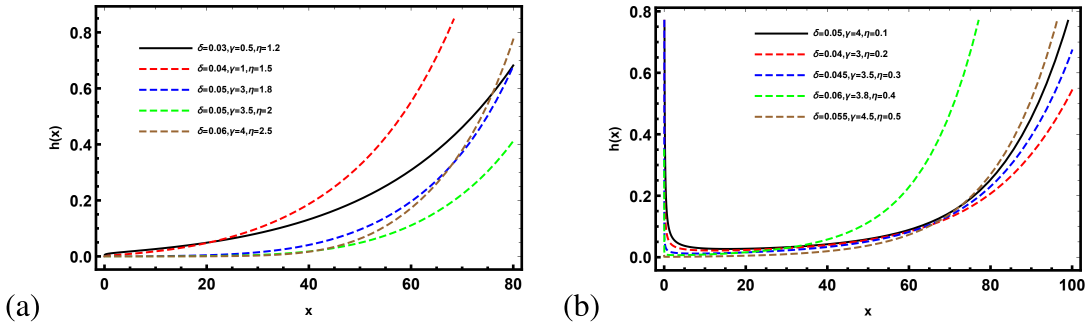

Figure 2. HR function results for EEEV distribution for different choices of η, γ, and δ.

3. Statistical properties

3.1. Quantile and mode

To find the qth quantile (x_q) of the EEEV distribution, one can solve the following equation:

$$F(x_q) = q, \tag{3.1}$$

Thus, solving Eq (3.1) yields

$$x_q \, e^{\delta \, x_q} = \frac{-e^{\gamma}}{\delta} \, \log(1 - q^{\frac{1}{\eta}}). \tag{3.2}$$

Let $u = x_q \, e^{\delta \, x_q}$; it is possible to express x_q in terms of u when the value of δ is positive, as follows:

$$x_q = \frac{1}{\delta} \, F(\delta \, u), \tag{3.3}$$

where

$$F(w) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \, k^{k-2} \, w^k}{(k-1)!}.$$

We have verified Eq (3.3) and the power-series expansion for $F(w) = ProductLog[w]$ by using Wolfram Mathematica software, which provides $F(w)$ as the main solution for z in $w = z\,\mathrm{e}^z$. We get

$$\begin{aligned} F(w) \;=\;& w - w^2 + \frac{3w^3}{2} - \frac{8w^4}{3} + \frac{125w^5}{24} - \frac{54w^6}{5} + \frac{16807w^7}{720} - \frac{16384w^8}{315} + \frac{531441w^9}{4480} - \frac{156250w^{10}}{567} \\ & + \frac{2357947691w^{11}}{3628800} - \frac{2985984w^{12}}{1925} + \frac{1792160394037w^{13}}{479001600} - \frac{7909306972w^{14}}{868725} \\ & + \frac{320361328125w^{15}}{14350336} - \frac{35184372088832w^{16}}{638512875} + \frac{286242305150981579 3w^{17}}{20922789888000} \\ & - \frac{5083731656658w^{18}}{14889875} + O(w^{19}). \end{aligned}$$

Consequently, it is feasible to represent x_q in relation to u by using Eq (3.3) given that $x_q = \sum_{k=1}^{\infty} a_k\, u^k$, where $a_k = \frac{(-1)^{k+1}\, k^{k-2}}{(k-1)!}\, \delta^{k-1}$ and the condition of convergence of this sum is $-\log(1 - q^{\frac{1}{\eta}}) < \mathrm{e}^{-(\gamma+1)}$ (see [16, 17]). Thus, the quantile for the EEEV distribution is as follows:

$$x_q = \sum_{k=1}^{\infty} a_k \Big(\frac{-\mathrm{e}^{\gamma}}{\delta}\, \log(1 - q^{\frac{1}{\eta}}) \Big)^k, \qquad 0 < q < 1. \tag{3.4}$$

Applying $q = 0.25,\ 0.5,\ 0.75$ in Eq (3.4), we obtain the first quartile, the median, and the third quartile of the EEEV distribution, respectively. Also, the skewness (SK) and kurtosis (KU) can be determined according to quartiles as follows:

$$Sk = \frac{Q(0.75) - 2\,Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)},$$

and

$$Ku = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)}.$$

The mode is determined by finding the solution to the following nonlinear equation:

$$\eta\, \delta^2\, \mathrm{e}^{\delta x - \gamma}\, \mathrm{e}^{-\delta x \mathrm{e}^{\delta x - \gamma}} \left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta-1} \left(\left((1 + \delta x)^2\, \mathrm{e}^{\delta x - \gamma}\right) \Big(\frac{\eta - 1}{\mathrm{e}^{\delta x \mathrm{e}^{\delta x-\gamma}} - 1} - 1 \Big) + (2 + \delta x) \right) = 0. \tag{3.5}$$

In general, Eq (3.5) lacks an analytical solution, which implies that the mode of EEEV distribution must be estimated by using numerical methods.

3.2. Ordinary moments

Using Eq (2.2), the sth moment of the EEEV distribution can be determined as follows:

$$\mu_s' = \eta\, \delta \int_0^{\infty} x^s\, (1 + \delta x)\, \mathrm{e}^{\delta x - \gamma}\, \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}} \left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta-1} dx. \tag{3.6}$$

Apply the binomial series expansion of $\left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta-1}$, given by

$$\left(1 - \mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta-1} = \sum_{i=0}^{\infty} \frac{(-1)^{-i}\, \Gamma(\eta)}{i!\, \Gamma(\eta - i)}\, \mathrm{e}^{-i \delta x \mathrm{e}^{\delta x-\gamma}}. \tag{3.7}$$

By substituting Eq (3.7) into Eq (3.6), we obtain

$$\mu_s' = \eta\,\delta \sum_{i=0}^{\infty} \frac{(-1)^{-i}\,\Gamma(\eta)}{i!\,\Gamma(\eta - i)} \int_0^{\infty} x^s\,(1 + \delta\,x)\,\mathrm{e}^{\delta\,x-\gamma}\,\mathrm{e}^{-(1+i)\,\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}}\,dx. \tag{3.8}$$

By applying the substitution $u = \delta\,x\,\mathrm{e}^{\delta\,x-\gamma}$ and employing the same method as previously described, it can be shown that $x = \sum_{j=1}^{\infty} a_j\,(\frac{\mathrm{e}^{\gamma}}{\delta}\,u)^j$, where $a_j = \frac{(-1)^{j+1}\,j^{j-2}}{(j-1)!}\,\delta^{j-1}$. This sum's convergence requirement is that $\frac{\mathrm{e}^{\gamma}}{\delta}\,u < \frac{1}{\mathrm{e}\,\delta}$ (see [17]). Therefore, the sth moment of the EEEV distribution can be written as

$$\mu_s' = \eta \sum_{i=0}^{\infty} \frac{(-1)^{-i}\,\Gamma(\eta)}{i!\,\Gamma(\eta - i)} \int_0^{\mathrm{e}^{-(\gamma+1)}} \Big(\sum_{j=0}^{\infty} a_j\,(\frac{\mathrm{e}^{\gamma}}{\delta}\,u)^j\Big)^s\,\mathrm{e}^{-(1+i)\,u}\,du, \tag{3.9}$$

but

$$\Big(\sum_{j=1}^{\infty} a_j\Big(\frac{\mathrm{e}^{\gamma}}{\delta}\,u\Big)^j\Big)^s = \sum_{j_1,\,j_2,\,\ldots,\,j_s=1}^{\infty} A_{j_1,\,j_2,\,\ldots,\,j_s}\Big(\frac{\mathrm{e}^{\gamma}}{\delta}\,u\Big)^{m_s},$$

where

$$A_{j_1,\,j_2,\ldots,\,j_s} = a_{j_1}\,a_{j_2}\ldots a_{j_s}, \qquad \textbf{and} \qquad m_s = a_{j_1} + a_{j_2} + \ldots + a_{j_s}.$$

Returning to Eq (3.9) gives

$$\mu_s' = \sum_{i=0}^{\infty} \sum_{j_1,\,j_2,\,\ldots,\,j_s=1}^{\infty} A_{j_1,\,j_2,\,\ldots,\,j_s}\,\frac{(-1)^{-i}\,\Gamma(\eta+1)}{i!\,\Gamma(\eta - i)}\,(\frac{\mathrm{e}^{\gamma}}{\delta})^{m_s} \int_0^{\mathrm{e}^{-(\gamma+1)}} u^{m_s}\,\mathrm{e}^{-(1+i)\,u}\,du,$$

After solving the integral presented above, the resulting solution is as follows:

$$\begin{aligned} \mu_s' \;&=\; \sum_{i=0}^{\infty} \sum_{j_1,\,j_2,\,\ldots,\,j_s=0}^{\infty} A_{j_1,\,j_2,\,\ldots,\,j_s}\,\frac{(-1)^{-i}\,\Gamma(\eta+1)}{i!\,\Gamma(\eta - i)\,(1+i)^{m_s+1}}\Big(\frac{\mathrm{e}^{\gamma}}{\delta}\Big)^{m_s} \\ &\quad\times\Big(\Gamma(m_s+1) - \Gamma\Big(m_s+1,(1+i)\,\mathrm{e}^{-(\gamma+1)}\Big)\Big). \end{aligned} \tag{3.10}$$

Based on the first four ordinary moments of the EEEV distribution, the measures of SK and KU can be derived as follows:

$$SK = \frac{\mu_3' - 3\,\mu_1'\,\mu_2' + 2\,(\mu_1')^3}{\Big(\mu_2' - (\mu_1')^2\Big)^{\frac{3}{2}}},$$

and

$$KU = \frac{\mu_4' - 4\,\mu_1'\,\mu_3' + 6\,(\mu_1')^2\,\mu_2' - 3\,(\mu_1')^4}{\Big(\mu_2' - (\mu_1')^2\Big)^2}.$$

Table 1 provides the values of the first four moments, variance, SK, and KU for the EEEV distribution, considering various combinations of η, γ, and δ.

Table 1. The first four ordinary moments, SK, and KU of the EEEV distribution for different values of η, γ, and δ.

η	γ	δ	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	Variance	SK	KU
0.05	0.25	2.25	15.7775	287.012	5771.23	125347.	38.0825	0.17533	2.69041
0.25	0.25	2.25	3.1555	11.4805	46.1698	200.556	1.5233	0.17533	2.69041
0.45	0.25	2.25	1.75306	3.54336	7.91664	19.1049	0.47015	0.17533	2.69041
0.65	0.25	2.25	1.21365	1.6983	2.62687	4.38876	0.22534	0.17533	2.69041
2.55	0.25	2.25	0.30936	0.11035	0.04351	0.01853	0.01464	0.17533	2.69041
0.45	0.05	2.25	1.57074	2.87727	5.86865	12.997	0.41004	0.23256	2.71288
0.45	0.25	2.25	1.75306	3.54336	7.91664	19.1049	0.47015	0.17533	2.69041
0.45	0.75	2.25	2.25948	5.73786	15.8451	46.7126	0.63263	0.04316	2.68527
0.45	1.25	2.25	2.83354	8.83346	29.5368	104.398	0.80452	-0.07261	2.73617
0.45	1.5	1.25	2.5935	7.98134	27.1849	99.5165	1.25511	-0.01774	2.46311
0.45	2.35	2.25	4.29882	19.6529	94.2261	469.725	1.17305	-0.27045	2.95185
0.45	0.25	0.15	0.32438	0.4367	0.79478	1.69636	0.33148	2.29542	8.25998
0.45	0.25	0.85	1.14133	1.90683	3.82028	8.60174	0.60419	0.56385	2.66745
0.45	0.25	1.25	1.38594	2.49182	5.1907	11.9674	0.57099	0.35801	2.5789
0.45	0.25	2.35	1.77914	3.62706	8.14864	19.7394	0.46173	0.16779	2.70302
0.45	0.25	5.55	2.24874	5.36591	13.4809	35.446	0.30909	0.14095	2.9169

According to Table 1, we can see that the KU of the EEEV distribution falls within the range of $(2.46311, 8.25998)$, whereas the SK can exhibit either negative or positive values, making it suitable for modeling skewed data.

3.3. Order statistics

Assume that $X_1, ..., X_n$ denote a random sample from EEEV distribution and let $X_{1:n}, ..., X_{n:n}$ be the corresponding order statistics (OS). The PDF of the kth-OS is given by

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)}\, f(x)\,[F(x)]^{k-1}\,[1 - F(x)]^{n-k}, \tag{3.11}$$

where $B(k, n-k+1)$ is the beta function. Applying the binomial expansion of $[1 - F(x; \underline{\Theta})]^{n-k}$, Eq (3.11) can be expressed as follows:

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} f(x)\,[F(x)]^{k+l-1}. \tag{3.12}$$

By substituting Eqs (2.1) and (2.2) into Eq (3.12), we obtain

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \eta\,\delta\,(1+\delta\,x)\,\mathrm{e}^{\delta\,x-\gamma}\,\mathrm{e}^{-\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}} \left(1 - \mathrm{e}^{-\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}}\right)^{(k+l)\eta-1}.$$

As a result,

$$f_{k:n}(x) = \sum_{l=0}^{n-k} \frac{(-1)^l\, n!}{k!\,(k-1)!\,(n-k-l)!\,(k+l)}\, f(x;\delta,\gamma,\eta'), \tag{3.13}$$

where $f(x;\delta,\gamma,\eta^{'})$ is the PDF of the EEEV distribution with the parameters $\delta, \gamma, \eta' = (k+l)\,\eta$. Utilizing Eq (3.10), the rth moment of the kth-OS for the EEEV distribution is given by

$$\begin{aligned} \mu_r'^{(k:n)} &= \sum_{l=0}^{n-k} \sum_{i=0}^{\infty} \sum_{j_1, j_2, \ldots, j_s=0}^{\infty} \frac{(-1)^{l-i}\; n!\,\Gamma(\eta+1)}{i!k!\,(k-1)!\,(n-k-l)!\,\Gamma(\eta-i)\,(1+i)^{m_s+1}\,(k+l)} \Big(\frac{\mathrm{e}^{\gamma}}{\delta}\Big)^{m_s} \\ &\quad \times \Big(\Gamma(m_s+1) - \Gamma\Big(m_s+1, (1+i)\,\mathrm{e}^{-(\gamma+1)}\Big)\Big). \end{aligned}$$

3.4. Incomplete moments

The definition of incomplete moments is as follows:

$$m_s(x) = \int_0^t f(x)\,dx. \tag{3.14}$$

$$m_s(x) = \eta\,\delta \int_0^t (1+\delta\,x)\,\mathrm{e}^{\delta\,x-\gamma}\,\mathrm{e}^{-\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}} \Big(1-\mathrm{e}^{-\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}}\Big)^{\eta-1} dx. \tag{3.15}$$

By utilizing the expansion $\Big(1-\mathrm{e}^{-\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}}\Big)^{\eta-1}$ and substituting $u = \delta\,x\,\mathrm{e}^{\delta\,x-\gamma}$, as explained in the previous subsection, we obtain

$$m_s(x) = \sum_{i=0}^{\infty} \sum_{j_1, j_2, \ldots, j_s=0}^{\infty} A_{j_1, j_2, \ldots, j_s} \frac{(-1)^{-i}\,\Gamma(\eta+1)}{i!\,\Gamma(\eta-i)} \Big(\frac{\mathrm{e}^{\gamma}}{\delta}\Big)^{m_s} \int_0^{s\,t\,\mathrm{e}^{(s\,t-\gamma)}} u^{m_s}\,\mathrm{e}^{-(1+i)\,u}\,du.$$

Upon solving the integral presented above, the resulting solution is as follows:

$$m_s(x) = \sum_{i=0}^{\infty} \sum_{j_1, j_2, \ldots, j_s=0}^{\infty} A_{j_1, j_2, \ldots, j_s} \frac{(-1)^{-i}\,\Gamma(\eta+1)}{i!\,\Gamma(\eta-i)\,(1+i)^{m_s+1}} \Big(\frac{\mathrm{e}^{\gamma}}{\delta}\Big)^{m_s} \gamma_1,$$

where $\gamma_1 = \Gamma(m_s+1) - \Gamma\Big(m_s+1, (1+i)\,s\,t\,\mathrm{e}^{(s\,t-\gamma)}\Big), \quad \Re(m_s) > -1.$

3.5. Entropy

The Rényi entropy of X for the EEEV distribution is defined as follows:

$$I_R(\rho) = \frac{1}{1-\rho} \log \int_0^{\infty} \Big(f(x;\Theta)\Big)^{\rho} dx, \qquad \rho > 0, \quad \rho \neq 1. \tag{3.16}$$

By substituting Eq (2.2) into Eq (3.16), we get

$$I_R(\rho) = \frac{1}{1-\rho} \log \int_0^{\infty} (\eta\,\delta)^{\rho}\,(1+\delta\,x)^{\rho}\,\mathrm{e}^{(\delta\,x-\gamma)\rho}\,\mathrm{e}^{-\rho\,\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}} \Big(1-\mathrm{e}^{-\delta\,x\,\mathrm{e}^{\delta\,x-\gamma}}\Big)^{\rho(\eta-1)}. \tag{3.17}$$

By employing series expansions for $(1+\delta x)^{\rho}$, $e^{-\rho\,\delta\,x\,e^{\delta x-\gamma}}$, and $\left(1-e^{-\delta\,x\,e^{\delta x-\gamma}}\right)^{\rho\,(\eta-1)}$, we obtain the Renyi entropy for the EEEV distribution:

$$I_R(\rho)=\frac{1}{1-\rho}\,\log\sum_{i,\,j,\,k,\,l=0}^{\infty}\frac{(-1)^{j+k+l}\binom{\rho}{i}\binom{\rho\,(\eta-1)}{k}\eta^{\rho}\,\rho^{j}\,k^{l}\,\delta^{\rho+d}\,e^{-d\,\gamma}}{j!\,l!}\,I(x;\,\delta,\,d), \tag{3.18}$$

where $I(x;\,\delta,\,d)=\int_0^{\infty}x^{d}\,e^{\delta\,d\,x}$, $\quad d=\rho+j+l.$

4. Methods of estimation

Within this section, we estimate the unknown parameters δ, γ, and η of the EEEV distribution by utilizing four different estimation methods. The aim is to illustrate the performance variations among different estimators of the EEEV distribution for different combinations of parameters and sample sizes. The estimation methods employed in this study include the use of MLEs, least-squares estimators (LSEs), weighted LSEs (WLSEs), and Cramér-von Mises estimators (CVMEs). The MLE is a consistent method that converges to true parameter values as the sample size is increased. It is efficient and widely applicable, but it can be sensitive to initial parameter choices and computationally challenging for complex models. The LSE offers simplicity and robustness but may lack efficiency compared to the MLE. The WLSE improves upon the LSE by incorporating weights based on data variance, but it requires knowledge of the variance structure. The CVME, based on the empirical distribution function, demonstrates robustness and reduced sensitivity to tail behavior, but it can be computationally complex. The choice of the most suitable method depends on factors such as sample size, parameter values, data characteristics, and computational resources [18, 19].

4.1. MLE

Assuming that $x_1,\,x_2,\ldots,x_n$ denote n independent random variables with the EEEV distribution. Then, the log-likelihood function of the EEEV distribution, denoted as $L(\Theta;\,x)$, can be expressed as follows:

$$\begin{aligned}L(\Theta;\,x) \;&=\; n\log\delta+n\log\eta-n\gamma+\delta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left(1+\delta x_i\right)-\delta\sum_{i=1}^{n}x_i e^{\delta x_i-\gamma}\\&\quad+(\eta-1)\sum_{i=1}^{n}\log\left(1-e^{-\delta x_i e^{\delta x_i-\gamma}}\right).\end{aligned} \tag{4.1}$$

The MLEs $\hat{\delta}$, $\hat{\gamma}$, and $\hat{\eta}$ can by calculated by maximizing Eq (4.1) by using numerical methods, such as NMaximize in the Wolfram Mathematica software. To obtain the normal equations of $L(\Theta;\,x)$, we can derive the first partial derivatives of $L(\Theta;\,x)$ with respect to δ, γ, and η, respectively. By setting these derivatives equal to zero, we have the following equations:

$$\begin{aligned}\frac{\partial L(\Theta;\,x)}{\partial\delta} \;&=\; \frac{n}{\delta}+\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{x_i}{1+\delta x_i}-\sum_{i=1}^{n}x_i\,(1+\delta x_i)\,e^{\delta x_i-\gamma}\\&\quad+(\eta-1)\sum_{i=1}^{n}\frac{x_i\,(1+\delta x_i)\,e^{(\delta x_i-\gamma)-\delta x_i e^{\delta x_i-\gamma}}}{1-e^{-\delta x_i e^{\delta x_i-\gamma}}}.\end{aligned} \tag{4.2}$$

$$\frac{\partial L(\Theta;\, x)}{\partial \gamma} = -n\, + \delta \sum_{i=1}^{n} x_i\, \mathrm{e}^{\delta x_i - \gamma} -\, (\eta - 1)\, \sum_{i=1}^{n} \frac{\delta x_i\, \mathrm{e}^{(\delta x_i - \gamma) - \delta x_i \mathrm{e}^{\delta x_i - \gamma}}}{1 - \mathrm{e}^{-\delta x_i \mathrm{e}^{\delta x_i - \gamma}}}. \tag{4.3}$$

$$\frac{\partial L(\Theta;\, x)}{\partial \eta} = \frac{n}{\eta} + \left(1 - \mathrm{e}^{-\delta x_i \mathrm{e}^{\delta x_i - \gamma}}\right). \tag{4.4}$$

Solving Eqs (4.2)–(4.4) by using numerical methods, such as FindRoot in Wolfram Mathematica software, we can obtain the estimators of the EEEV parameters by the MLE method.

4.2. LSE

Assume that $x_1,\, x_2, \ldots, x_n$ represent the OS of a random sample of size n drawn from the EEEV distribution. Thus, the LSE [20] of the EEEV parameters can be obtained by minimizing the following function:

$$\begin{aligned} LS(\Theta) &= \sum_{i=1}^{n} \left(F((x_{i:n}|\Theta)) - \frac{i}{n+1} \right)^2 \\ &= \sum_{i=1}^{n} \left(\left(1 - \mathrm{e}^{-\delta\, x_{i:n} \mathrm{e}^{\delta\, x_{i:n} - \gamma}}\right)^{\eta} - \frac{i}{n+1} \right)^2. \end{aligned} \tag{4.5}$$

The previous equation's solution can be obtained by using numerical methods, such as NMinimize in the Wolfram Mathematica software.

Similarly, the LSEs $\hat{\delta},\, \hat{\gamma}$, and $\hat{\eta}$ can also be determined by solving the following nonlinear equations:

$$\sum_{i=1}^{n} \left(\left(1 - \mathrm{e}^{-\delta\, x_{i:n} \mathrm{e}^{\delta\, x_{i:n} - \gamma}}\right)^{\eta} - \frac{i}{n+1} \right) I_s\,(x_{i:n}|\Theta) = 0, \quad s = 1,\, 2,\, 3, \tag{4.6}$$

where

$$I_1\,(x_{i:n}|\Theta) = \frac{\partial\, F((x_{i:n}|\Theta))}{\partial\, \delta} = \eta\, x_i\,(1 + \delta x_i)\,(1 - w_i)^{\eta - 1}\, \mathrm{e}^{\delta x_i - \gamma}, \tag{4.7}$$

$$I_2\,(x_{i:n}|\Theta) = \frac{\partial\, F((x_{i:n}|\Theta))}{\partial\, \gamma} = \delta\, \eta\, x_i\,(1 - w_i)^{\eta - 1}\, \mathrm{e}^{\delta x_i - \gamma - w_i}, \tag{4.8}$$

$$I_3\,(x_{i:n}|\Theta) = \frac{\partial\, F((x_{i:n}|\Theta))}{\partial\, \eta} = (1 - w_i)^{\eta}\, \log(1 - w_i), \tag{4.9}$$

and $w_i = \mathrm{e}^{-\delta\, x_i\, \mathrm{e}^{\delta\, x_i - \gamma}}$.

The solution of $I_s\,(x_{i:n}|\Theta)$ for $s = 1,\, 2,\, 3$ can be obtained numerically.

4.3. WLSE

Let $x_1,\, x_2, \ldots, x_n$ represent the OS of a random sample of size n drawn from the EEEV distribution. Thus, the WLSE [20] of the EEEV parameters can be obtained by minimizing the following function:

$$W(\Theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F((x_{i:n}|\Theta)) - \frac{i}{n+1} \right)^2$$

$$= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(\left(1 - \mathrm{e}^{-\delta\, x_{i:n} \mathrm{e}^{\delta\, x_{i:n} - \gamma}}\right)^{\eta} - \frac{i}{n+1} \right)^2 . \tag{4.10}$$

Moreover, the WLSEs $\hat{\delta}$, $\hat{\gamma}$, and $\hat{\eta}$ can by obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(\left(1 - \mathrm{e}^{-\delta\, x_{i:n} \mathrm{e}^{\delta\, x_{i:n} - \gamma}}\right)^{\eta} - \frac{i}{n+1} \right) I_s\left(x_{i:n}|\Theta\right) = 0, \quad s = 1,\, 2,\, 3,$$

where $I_s\left(x_{i:n}|\Theta\right)$, $s = 1, 2, 3$ have been defined in Eqs (4.7)–(4.9).

4.4. CVME

Let $x_1,\ x_2, \ldots, x_n$ represent the OS of a random sample of size n drawn from the EEEV distribution. Thus, the CVME [21] of the EEEV parameters can be obtained by minimizing the following function:

$$\begin{aligned} CV(\Theta) &= \frac{1}{12\,n} + \sum_{i=1}^{n} \left(F((x_{i:n}|\Theta)) - \frac{2\,i-1}{2\,n} \right)^2 \\ &= \frac{1}{12\,n} + \sum_{i=1}^{n} \left(\left(1 - \mathrm{e}^{-\delta\, x_{i:n} \mathrm{e}^{\delta\, x_{i:n} - \gamma}}\right)^{\eta} - \frac{2\,i-1}{2\,n} \right)^2 . \end{aligned} \tag{4.11}$$

Furthermore, the CVMEs $\hat{\delta}$, $\hat{\gamma}$, and $\hat{\eta}$ can by obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n} \left(\left(1 - \mathrm{e}^{-\delta\, x_{i:n} \mathrm{e}^{\delta\, x_{i:n} - \gamma}}\right)^{\eta} - \frac{2\,i-1}{2\,n} \right) I_s\left(x_{i:n}|\Theta\right) = 0, \quad s = 1,\, 2,\, 3,$$

where $I_s\left(x_{i:n}|\Theta\right)$, $s = 1, 2, 3$ have been defined in Eqs (4.7)–(4.9).

5. Simulation analysis

Within this section, we detail a simulation study that was conducted to illustrate the effectiveness of different estimators by using data generated from the EEEV distribution. Version 12.3 of Wolfram Mathematica was used for all simulated studies and graphical representations. The numerical procedures were executed by following the algorithm below:

Step 1: Generate random samples of varying sizes, $n = \{25, 60, 100, 200, 300\}$, from the inverse CDF of the EEEV distribution, or by utilizing Eq (3.4).

Step 2: Use four sets of parameters to generate the simulation results: set 1 ($\delta = 0.05, \gamma = 1.8, \eta = 0.8$), set 2 ($\delta = 0.1, \gamma = 1.5, \eta = 0.6$), set 3 ($\delta = 0.25, \gamma = 2.3, \eta = 0.9$), and set 4 ($\delta = 0.5, \gamma = 2.6, \eta = 1.3$).

Step 3: Repeat the process 2000 times for each sample.

Step 4: Calculate the average absolute biases (ABs), MSEs, mean relative errors (MREs), and root mean square error (RMSEs) for the four sets. The computational formulas for the AB, MSE, MRE, and RMSE are provided as follows:

$$\mathbf{AB} = \frac{\sum_{j=1}^{n} |\hat{\Theta}_j - \Theta|}{n},$$

$$\textbf{MSE} = \frac{\sum_{j=1}^{n} (\hat{\Theta}_j - \Theta)^2}{n},$$

$$\textbf{MRE} = \frac{1}{n} \sum_{j=1}^{n} \frac{|\hat{\Theta}_j - \Theta|}{n},$$

and

$$\textbf{RMSE} = \sqrt{\frac{\sum_{j=1}^{n} (\hat{\Theta}_j - \Theta)^2}{n}},$$

where $\hat{\Theta} = (\hat{\delta}, \hat{\gamma}, \hat{\eta})'$.

The AB, MSE, MRE, and RMSE results were calculated for all sets; the results are shown in Tables 2 and 3. While Figures 3–6 provide graphical representations of the numerical values. Based on the results of the simulation analysis, as displayed in Tables 2 and 3, we can draw several important conclusions:

- As the sample size n increased, a clear trend in bias reduction was observed across all of the estimating techniques being examined.
- It can be deduced that all estimation methods demonstrate the consistency property, meaning that, as the sample size n increases, the estimators tend to approach the true parameter values.
- All of the estimation techniques performed very well on the task of estimating the EEEV parameters.
- The plots depicted in Figures 3–6 illustrate that the AB, MSE, MRE, and RMSE values diminish to zero with increasing sample size, irrespective of the parameter combinations. These graphical representations suggest that the MLE is the most efficient approach for estimating the parameters of the EEEV distribution.

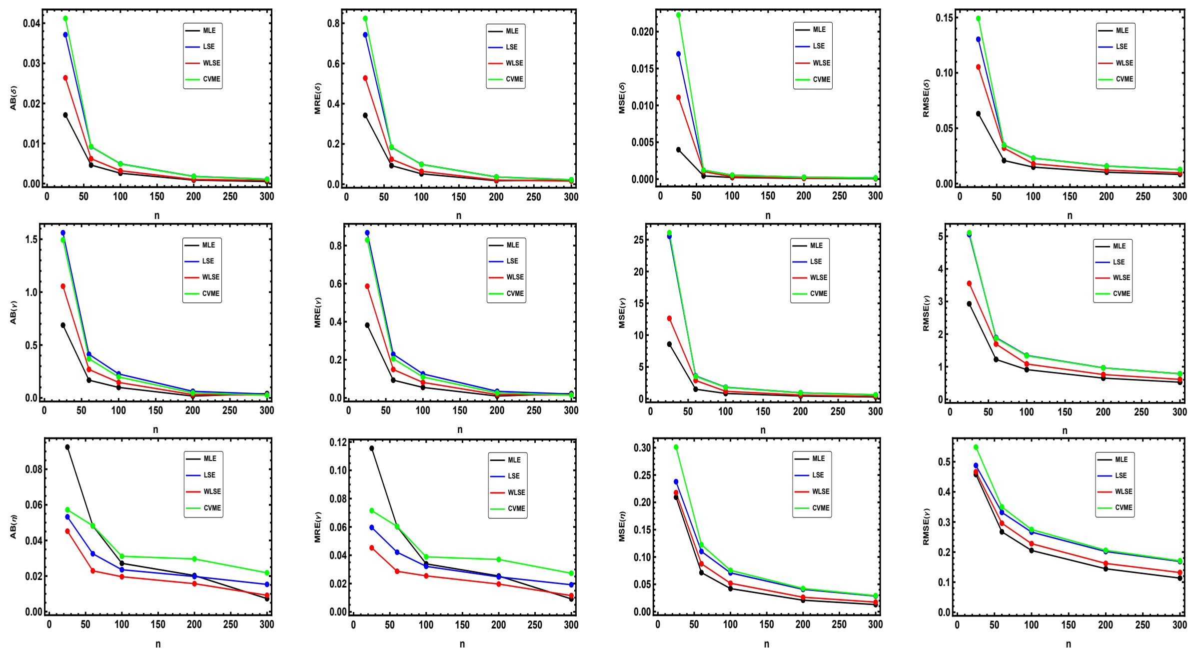

Figure 3. The AB, MSE, MRE, and RMSE results for the EEEV distribution for various values of n when $\delta = 0.05$, $\gamma = 1.8$, $\eta = 0.8$.

Table 2. Simulation results for the MLE, LSE, WLSE, and CVME methods for (δ = 0.05, γ = 1.8, η = 0.8) and (δ = 0.1, γ = 1.5, η = 0.6) when n = 25, 60, 100, 200, 300.

			δ = 0.05, γ = 1.8, η = 0.8					δ = 0.1, γ = 1.5, η = 0.6				
n	Method	Est. Par.	Average	AB	MSE	MRE	RMSE	Average	AB	MSE	MRE	RMSE
		$\hat{\delta}$	0.0671	0.0171	0.00399	0.34197	0.06313	0.13217	0.03217	0.0093	0.32172	0.09642
	MLE	$\hat{\gamma}$	2.48756	0.68756	8.58016	0.38198	2.92919	2.08876	0.58876	4.57726	0.39251	2.13945
		$\hat{\eta}$	0.89245	0.09245	0.2094	0.11556	0.4576	0.63913	0.03913	0.07125	0.06521	0.26693
		$\hat{\delta}$	0.08713	0.03713	0.01698	0.74257	0.13031	0.15793	0.05793	0.03375	0.57928	0.1837
	LSE	$\hat{\gamma}$	3.36142	1.56142	25.5415	0.86746	5.05386	2.66636	1.16636	12.5284	0.77757	3.53955
25		$\hat{\eta}$	0.7843	0.05324	0.23782	0.5962	0.48767	0.5629	0.0371	0.07536	0.06183	0.27451
		$\hat{\delta}$	0.07637	0.02637	0.01109	0.52741	0.10533	0.14395	0.04395	0.02089	0.43947	0.14453
	WLSE	$\hat{\gamma}$	2.85585	1.05585	12.6336	0.58658	3.55438	2.39486	0.89486	8.26169	0.59657	2.87432
		$\hat{\eta}$	0.80104	0.04521	0.21774	0.04526	0.46662	0.57615	0.02385	0.06807	0.03975	0.2609
		$\hat{\delta}$	0.0912	0.0412	0.02226	0.824	0.1492	0.15984	0.05984	0.04124	0.59844	0.20307
	CVME	$\hat{\gamma}$	3.29307	1.49307	26.0844	0.82948	5.10729	2.55771	1.05771	13.0611	0.70514	3.61401
		$\hat{\eta}$	0.85724	0.05724	0.30114	0.07155	0.54876	0.61304	0.03304	0.09029	0.04574	0.30048
		$\hat{\delta}$	0.05461	0.00461	0.00043	0.09226	0.02081	0.11054	0.01054	0.00175	0.10542	0.04188
	MLE	$\hat{\gamma}$	1.96756	0.16756	1.49478	0.09309	1.22261	1.69266	0.19266	1.258	0.12844	1.1216
		$\hat{\eta}$	0.8482	0.0482	0.0714	0.06025	0.26721	0.62114	0.02114	0.02783	0.03524	0.16681
		$\hat{\delta}$	0.05919	0.00919	0.00121	0.18374	0.03479	0.11489	0.01489	0.00385	0.14895	0.06206
	LSE	$\hat{\gamma}$	2.21243	0.41243	3.58126	0.22913	1.89242	1.80852	0.30852	2.3887	0.20568	1.54554
60		$\hat{\eta}$	0.8163	0.03251	0.11012	0.04213	0.33184	0.59962	0.01538	0.03661	0.03735	0.19134
		$\hat{\delta}$	0.0562	0.0062	0.00103	0.12394	0.03204	0.11018	0.01018	0.00241	0.10176	0.04907
	WLSE	$\hat{\gamma}$	2.06897	0.26897	2.87373	0.14943	1.69521	1.70949	0.20949	1.65289	0.13966	1.28565
		$\hat{\eta}$	0.82292	0.02292	0.08756	0.02865	0.29591	0.60547	0.01547	0.0303	0.02562	0.17407
		$\hat{\delta}$	0.05925	0.00925	0.00121	0.18494	0.03481	0.11464	0.01464	0.00377	0.14641	0.06137
	CVME	$\hat{\gamma}$	2.17052	0.37052	3.51659	0.20584	1.87526	1.75379	0.25379	2.28894	0.16919	1.51293
		$\hat{\eta}$	0.84828	0.04828	0.12254	0.06035	0.35005	0.62152	0.02152	0.03986	0.03586	0.19966
		$\hat{\delta}$	0.05258	0.00258	0.00022	0.05154	0.01485	0.10513	0.00513	0.00087	0.05128	0.02948
	MLE	$\hat{\gamma}$	1.89902	0.09902	0.82896	0.05501	0.91047	1.57888	0.07888	0.68684	0.05259	0.82876
		$\hat{\eta}$	0.82713	0.02713	0.04215	0.03391	0.20531	0.61606	0.01606	0.01601	0.02677	0.12652
		$\hat{\delta}$	0.05493	0.00493	0.00053	0.09859	0.02298	0.10758	0.00758	0.00192	0.07581	0.04376
	LSE	$\hat{\gamma}$	2.02658	0.22658	1.81476	0.12588	1.34713	1.63644	0.13644	1.34157	0.09096	1.15826
100		$\hat{\eta}$	0.81175	0.02354	0.07091	0.03215	0.2663	0.60706	0.00706	0.02461	0.01176	0.15686
		$\hat{\delta}$	0.05321	0.00321	0.00032	0.06426	0.01794	0.10463	0.00463	0.0012	0.0463	0.03463
	WLSE	$\hat{\gamma}$	1.94653	0.14653	1.17544	0.08141	1.08418	1.57663	0.07663	0.91315	0.05109	0.95559
		$\hat{\eta}$	0.81262	0.01962	0.05194	0.02543	0.22789	0.60988	0.00988	0.01888	0.01647	0.13739
		$\hat{\delta}$	0.0549	0.0049	0.00053	0.09805	0.02294	0.10745	0.00745	0.00189	0.07452	0.04353
	CVME	$\hat{\gamma}$	1.99827	0.19827	1.78684	0.11015	1.33673	1.60401	0.10401	1.31253	0.06934	1.14566
		$\hat{\eta}$	0.83111	0.03111	0.0756	0.03888	0.27496	0.62044	0.02044	0.02611	0.03406	0.16158
		$\hat{\delta}$	0.05086	0.00086	0.0001	0.01723	0.01022	0.10249	0.00249	0.00039	0.02488	0.01986
	MLE	$\hat{\gamma}$	1.81744	0.01744	0.42212	0.00969	0.64971	1.53915	0.03915	0.32636	0.0261	0.57128
		$\hat{\eta}$	0.82027	0.02027	0.02087	0.02534	0.14446	0.6073	0.0073	0.00742	0.01216	0.08613
		$\hat{\delta}$	0.0518	0.0018	0.00025	0.03603	0.01586	0.10355	0.00355	0.00096	0.0355	0.03102
	LSE	$\hat{\gamma}$	1.86225	0.06225	0.93335	0.03458	0.9661	1.55992	0.05992	0.71091	0.03995	0.84315
200		$\hat{\eta}$	0.81975	0.01975	0.0407	0.02469	0.20174	0.60473	0.00473	0.01385	0.00788	0.11771
		$\hat{\delta}$	0.05103	0.00103	0.00015	0.02051	0.01215	0.10231	0.00231	0.00059	0.02305	0.02422
	WLSE	$\hat{\gamma}$	1.83104	0.03104	0.57806	0.01725	0.7603	1.5381	0.0381	0.45972	0.0254	0.67803
		$\hat{\eta}$	0.81572	0.01572	0.02633	0.01965	0.16225	0.60483	0.00483	0.00969	0.00805	0.09844
		$\hat{\delta}$	0.05178	0.00178	0.00025	0.0357	0.01584	0.10346	0.00346	0.00096	0.03459	0.03094
	CVME	$\hat{\gamma}$	1.84793	0.04793	0.92833	0.02663	0.9635	1.54287	0.04287	0.70442	0.02858	0.8393
		$\hat{\eta}$	0.82961	0.02961	0.04235	0.03702	0.2058	0.61149	0.01149	0.01431	0.01915	0.11964
		$\hat{\delta}$	0.05101	0.00051	0.00007	0.02011	0.00828	0.10187	0.00187	0.00027	0.01868	0.01655
	MLE	$\hat{\gamma}$	1.84075	0.04075	0.27529	0.02264	0.52468	1.53059	0.03059	0.22801	0.02039	0.47751
		$\hat{\eta}$	0.8073	0.0073	0.01286	0.00913	0.11342	0.60418	0.00418	0.00488	0.00697	0.06985
		$\hat{\delta}$	0.05109	0.00109	0.00016	0.02176	0.01261	0.10238	0.00238	0.00059	0.0238	0.02436
	LSE	$\hat{\gamma}$	1.83615	0.03615	0.61449	0.02008	0.78389	1.53926	0.03926	0.45913	0.02617	0.67759
300		$\hat{\eta}$	0.81531	0.01531	0.02846	0.01914	0.16869	0.60311	0.00311	0.00898	0.00518	0.09478
		$\hat{\delta}$	0.05079	0.00079	0.00009	0.0158	0.00972	0.10162	0.00162	0.00037	0.01617	0.01928
	WLSE	$\hat{\gamma}$	1.82891	0.02891	0.37413	0.01606	0.61166	1.52674	0.02674	0.29961	0.01783	0.54737
		$\hat{\eta}$	0.80918	0.00918	0.01743	0.01148	0.13201	0.6027	0.0027	0.00608	0.00449	0.07794
		$\hat{\delta}$	0.05108	0.00108	0.00016	0.02153	0.01261	0.10232	0.00232	0.00059	0.02319	0.02432
	CVME	$\hat{\gamma}$	1.82662	0.02662	0.61249	0.01479	0.78262	1.52786	0.02786	0.45626	0.01857	0.67547
		$\hat{\eta}$	0.82182	0.02182	0.02925	0.02727	0.17103	0.60761	0.00761	0.00918	0.01268	0.09583

Table 3. Simulation results for the MLE, LSE, WLSE, and CVME methods for ($\delta = 0.25, \gamma = 2.3, \eta = 0.9$) and ($\delta = 0.5, \gamma = 2.6, \eta = 1.3$) when n = 25, 60, 100, 200, 300.

			$\delta = 0.25, \gamma = 2.3, \eta = 0.9$					$\delta = 0.5, \gamma = 2.6, \eta = 1.3$				
n	Method	Est. Par.	Average	AB	MSE	MRE	RMSE	Average	AB	MSE	MRE	RMSE
25	MLE	$\hat{\delta}$	0.35092	0.10092	0.12491	0.40367	0.35342	0.79137	0.29137	1.0852	0.58273	1.04173
		$\hat{\gamma}$	3.27032	0.97032	14.6829	0.42188	3.83183	4.22957	1.62957	38.2691	0.62676	6.1862
		$\hat{\eta}$	1.04184	0.14184	0.43378	0.1576	0.65862	1.66951	0.36951	1.98997	0.28424	1.41066
	LSE	$\hat{\delta}$	0.48518	0.23518	0.50414	0.94073	0.71003	0.97518	0.47518	2.12617	0.95037	1.45814
		$\hat{\gamma}$	4.69185	2.39185	51.8132	1.03993	7.19814	5.31577	2.71577	70.2814	1.04453	8.3834
		$\hat{\eta}$	0.9382	0.06982	0.52559	0.07545	0.72498	1.54415	0.24415	2.10624	0.18781	1.45129
	WLSE	$\hat{\delta}$	0.40129	0.15129	0.24664	0.60516	0.49663	0.82816	0.32816	1.20848	0.65632	1.09931
		$\hat{\gamma}$	3.86563	1.56563	26.9407	0.68071	5.19044	4.49194	1.89194	40.6753	0.72767	6.37772
		$\hat{\eta}$	0.94883	0.05983	0.45286	0.06826	0.67295	1.53852	0.23852	1.8327	0.18348	1.35377
	CVME	$\hat{\delta}$	0.50049	0.25049	0.57997	1.00195	0.76155	1.02994	0.52994	2.63037	1.05988	1.62184
		$\hat{\gamma}$	4.71112	2.41112	56.704	1.04831	7.53021	5.50417	2.90417	82.6703	1.11699	9.09232
		$\hat{\eta}$	1.03185	0.13185	0.67443	0.1465	0.82123	1.7139	0.4139	2.86944	0.31839	1.69394
60	MLE	$\hat{\delta}$	0.27446	0.02446	0.01234	0.09785	0.11108	0.56227	0.06227	0.08046	0.12453	0.28366
		$\hat{\gamma}$	2.52462	0.22462	2.2402	0.09766	1.49673	2.94318	0.34318	3.86849	0.13199	1.96685
		$\hat{\eta}$	0.98249	0.08249	0.16374	0.09166	0.40465	1.44809	0.14809	0.56166	0.11391	0.74944
	LSE	$\hat{\delta}$	0.30697	0.05697	0.04338	0.22786	0.20829	0.65157	0.15157	0.34572	0.30313	0.58798
		$\hat{\gamma}$	2.90419	0.60419	6.43102	0.26269	2.53595	3.49419	0.89419	13.2909	0.34392	3.64567
		$\hat{\eta}$	0.95662	0.05662	0.24218	0.06291	0.49212	1.42573	0.18765	0.83869	0.012785	0.9158
	WLSE	$\hat{\delta}$	0.28499	0.03499	0.02133	0.13994	0.14606	0.59009	0.09009	0.11912	0.18019	0.34513
		$\hat{\gamma}$	2.67374	0.37374	3.56645	0.16249	1.8885	3.13818	0.53818	5.58451	0.20699	2.36316
		$\hat{\eta}$	0.95121	0.05121	0.18393	0.0569	0.42887	1.40737	0.10737	0.63103	0.08259	0.79437
	CVME	$\hat{\delta}$	0.30752	0.05752	0.04366	0.2301	0.20895	0.65889	0.15889	0.37627	0.31778	0.61341
		$\hat{\gamma}$	2.86611	0.56611	6.36025	0.24613	2.52195	3.49912	0.89912	14.0149	0.34581	3.74364
		$\hat{\eta}$	0.99657	0.09657	0.27374	0.1073	0.5232	1.48448	0.18448	0.94552	0.14191	0.97238
100	MLE	$\hat{\delta}$	0.2615	0.0115	0.00574	0.04601	0.07574	0.52559	0.02559	0.02565	0.05119	0.16017
		$\hat{\gamma}$	2.39524	0.09524	1.12555	0.04141	1.06092	2.7324	0.1324	1.47605	0.05092	1.21493
		$\hat{\eta}$	0.94712	0.04712	0.07902	0.05236	0.2811	1.40069	0.10069	0.27375	0.07745	0.52321
	LSE	$\hat{\delta}$	0.27398	0.02398	0.01645	0.09593	0.12824	0.56115	0.06115	0.09196	0.1223	0.30324
		$\hat{\gamma}$	2.53265	0.23265	2.81444	0.10115	1.67763	2.94524	0.34524	4.48555	0.13278	2.11791
		$\hat{\eta}$	0.95453	0.05453	0.15666	0.06058	0.3958	1.44611	0.14611	0.60773	0.11239	0.77957
	WLSE	$\hat{\delta}$	0.26464	0.01464	0.00872	0.05854	0.09339	0.5355	0.0355	0.04283	0.07101	0.20695
		$\hat{\gamma}$	2.4418	0.1418	1.64835	0.06165	1.28388	2.80378	0.20378	2.34048	0.07838	1.52986
		$\hat{\eta}$	0.94064	0.04064	0.10859	0.04515	0.32953	1.39714	0.09714	0.3648	0.07473	0.60398
	CVME	$\hat{\delta}$	0.27404	0.02404	0.01645	0.09615	0.12826	0.56352	0.06352	0.09347	0.12704	0.30574
		$\hat{\gamma}$	2.50734	0.20734	2.78855	0.09015	1.66989	2.93868	0.33868	4.50827	0.13026	2.12327
		$\hat{\eta}$	0.97825	0.07825	0.16893	0.08694	0.411	1.48196	0.18196	0.65944	0.13997	0.81206
200	MLE	$\hat{\delta}$	0.25594	0.00594	0.00247	0.02376	0.04967	0.5104	0.0104	0.01162	0.0208	0.10778
		$\hat{\gamma}$	2.3518	0.0518	0.50164	0.02252	0.70827	2.64302	0.04302	0.69743	0.01655	0.83512
		$\hat{\eta}$	0.92026	0.02026	0.03086	0.02251	0.17566	1.35763	0.05763	0.12527	0.04433	0.35393
	LSE	$\hat{\delta}$	0.26056	0.01056	0.00694	0.04223	0.0833	0.52011	0.02011	0.03298	0.04023	0.18159
		$\hat{\gamma}$	2.39257	0.09257	1.32685	0.04025	1.15189	2.68862	0.08862	1.84477	0.03408	1.35822
		$\hat{\eta}$	0.93858	0.03858	0.0836	0.04287	0.28914	1.41239	0.11239	0.31598	0.08645	0.56212
	WLSE	$\hat{\delta}$	0.25696	0.00696	0.00389	0.02783	0.06239	0.51098	0.01098	0.01756	0.02195	0.1325
		$\hat{\gamma}$	2.36454	0.06454	0.76947	0.02806	0.8772	2.64555	0.04555	1.02578	0.01752	1.01281
		$\hat{\eta}$	0.92212	0.02212	0.04676	0.02457	0.21625	1.36954	0.06954	0.17521	0.05349	0.41858
	CVME	$\hat{\delta}$	0.26058	0.01058	0.00694	0.04231	0.0833	0.52113	0.02113	0.0332	0.04227	0.18221
		$\hat{\gamma}$	2.37999	0.07999	1.32131	0.03478	1.14948	2.68472	0.08472	1.84808	0.03258	1.35944
		$\hat{\eta}$	0.9501	0.0501	0.08717	0.05567	0.29524	1.42931	0.12931	0.32994	0.09947	0.5744
300	MLE	$\hat{\delta}$	0.25333	0.00333	0.00169	0.01332	0.04111	0.50356	0.00356	0.00713	0.00712	0.08444
		$\hat{\gamma}$	2.32383	0.02383	0.35088	0.01036	0.59235	2.6023	0.0023	0.4417	0.00088	0.66461
		$\hat{\eta}$	0.91716	0.01716	0.0228	0.01907	0.15099	1.34918	0.04918	0.07844	0.03783	0.28008
	LSE	$\hat{\delta}$	0.25826	0.00826	0.00437	0.03304	0.06608	0.51192	0.01192	0.02053	0.02384	0.14327
		$\hat{\gamma}$	2.37853	0.07853	0.8412	0.03414	0.91717	2.64535	0.04535	1.20856	0.01744	1.09934
		$\hat{\eta}$	0.92049	0.02049	0.0497	0.02276	0.22295	1.38797	0.08797	0.22245	0.06767	0.47165
	WLSE	$\hat{\delta}$	0.25451	0.00451	0.00243	0.01806	0.0493	0.50501	0.00501	0.01096	0.01003	0.1047
		$\hat{\gamma}$	2.34027	0.04027	0.49181	0.01751	0.70129	2.61116	0.01116	0.66555	0.00429	0.81581
		$\hat{\eta}$	0.91548	0.01548	0.03059	0.0172	0.17489	1.35743	0.05743	0.11838	0.04418	0.34406
	CVME	$\hat{\delta}$	0.25829	0.00829	0.00437	0.03315	0.06609	0.51257	0.01257	0.02061	0.02514	0.14355
		$\hat{\gamma}$	2.37035	0.07035	0.83881	0.03059	0.91587	2.6426	0.0426	1.20953	0.01638	1.09979
		$\hat{\eta}$	0.92788	0.02788	0.05107	0.03098	0.22598	1.39889	0.09889	0.22906	0.07607	0.4786

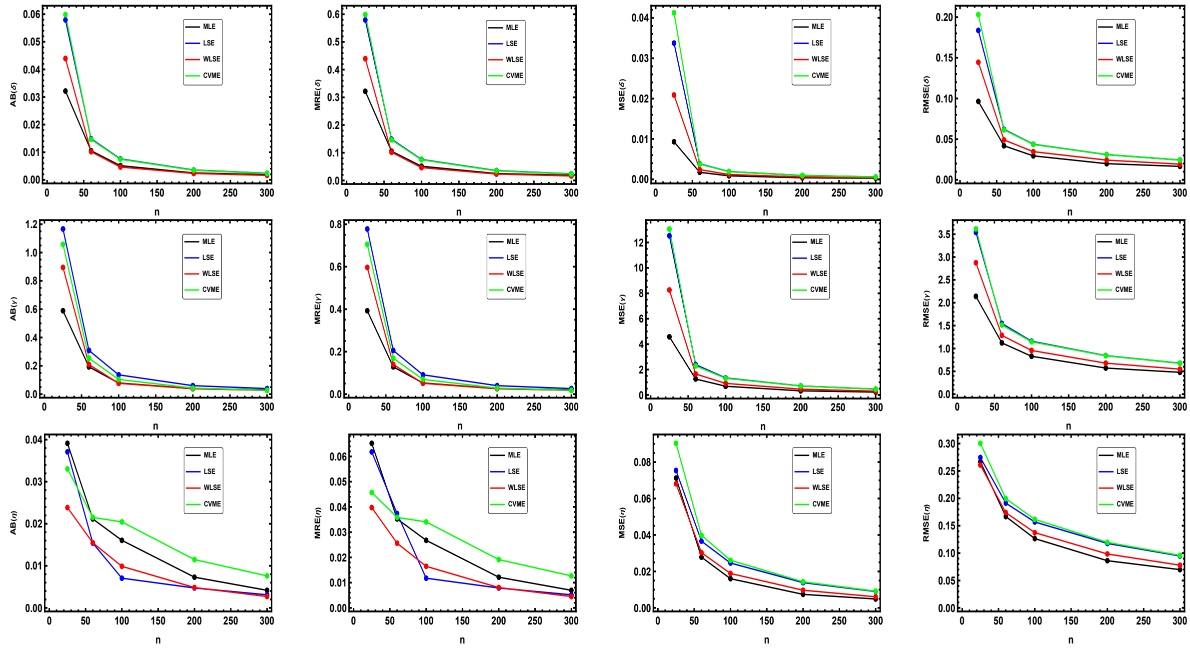

Figure 4. The AB, MSE, MRE, and RMSE results for the EEEV distribution for various values of n when $\delta = 0.1$, $\gamma = 1.5$, $\eta = 0.6$.

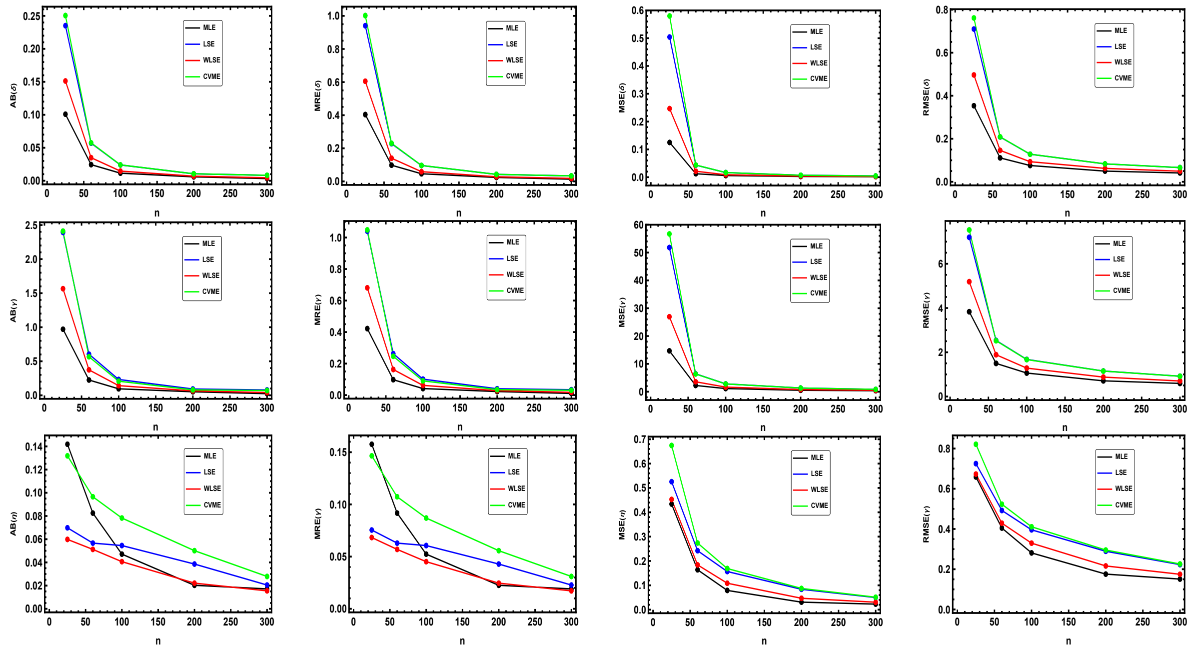

Figure 5. The AB, MSE, MRE, and RMSE results for the EEEV distribution for various values of n when $\delta = 0.25$, $\gamma = 2.3$, $\eta = 0.9$.

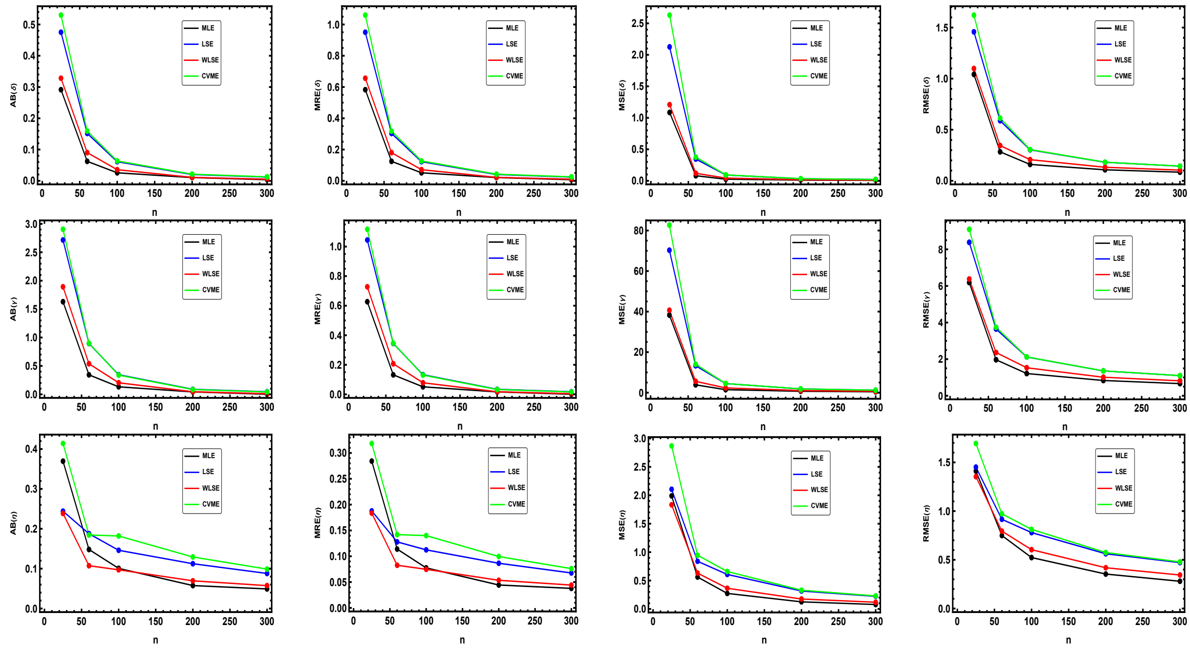

Figure 6. The AB, MSE, MRE, and RMSE results for the EEEV distribution for various values of n when $\delta = 0.5$, $\gamma = 2.6$, $\eta = 1.3$.

6. Applications

In this section, we demonstrate the versatility and significance of the EEEV distribution in the modeling of real-life data by employing three datasets from the fields of medicine and engineering. The first dataset, originally presented by Boag [22], details the ages (in months) of 18 patients who passed away due to causes other than cancer. The second dataset, introduced by Aarset [23], consists of the failure times of 50 electronic devices that were subjected to life tests starting from time zero. The third dataset, provided by Murthy et al. [24], revolves around the service times (in thousand hours) of 63 aircraft windshields of a specific model. Figures 7–9 illustrate the total time test (TTT) plots [23] for these three real datasets. Based on Figures 7(a) and 8(a), we can conclude that the empirical HR functions for the first and second datasets exhibited bathtub curves. On the other hand, Figure 9(a) suggests that the third dataset has an increasing HR.

In this analysis, we compared our model with six other competitive lifetime models, i.e., the exponentiated Weibull (EW) [25], power generalized Weibull (PGW) [26], EEV [15], exponentiated Nadarajah-Haghighi (ENH) [27], alpha logarithmic transformed Weibull (ALTW) [28], and logistic Nadarajah-Haghighi (LNH) [29] distributions. To assess the suitability of these competing distributions, we utilized various goodness-of-fit analytical measures, including the log-likelihood (LL), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and Cramér-von Mises (W^*), Anderson-Darling (A^*), and Kolmogorov-Smirnov (KS) statistics, along with their corresponding P-value. All calculations were performed by using the Wolfram Mathematica software, as well as the graph construction.

Tables 4, 7, and 10 present the estimated EEEV parameters obtained from four different estimation methods, along with the corresponding goodness-of-fit measures for the Boag, Aarset, and aircraft windshield datasets, respectively. By examining the P-values in Tables 4, 7, and 10, it is suggested that the MLE is suitable for estimating the EEEV parameters for the Boag data, while the WLSE is recommended for the Aarset data. As for the aircraft windshield data, it is advised to employ the CVME method to estimate the EEEV parameters. Tables 5, 8, and 11 provide the MLE method estimates for the EEEV parameters, as well as the estimates for all of the compared distributions based on the Boag, Aarset, and aircraft windshield data. Tables 6, 9, and 12 present the comparison statistics for the Boag, Aarset, and aircraft windshield data. These statistics consist of LL, AIC, BIC, CAIC, HQIC, A^*, W^*, and KS values with corresponding P-value. The values presented in Tables 6, 9, and 12 indicate that the EEEV distribution outperformed other competing models. This is evidenced by the fact that the EEEV distribution had the lowest values across all measures, as well as the highest P-value.

The estimated PDF, survival function, and HR function, along with the TTT plot for the EEEV distribution and all other considered models, are depicted in Figures 5–9 for the three datasets. The findings presented in Tables 5, 8, and 11 indicate that the EEEV distribution is the most suitable model for fitting the three datasets among all of the investigated distribution models. These findings are further supported by the graphical representations in Figures 7–9.

Table 4. The estimates of the EEEV parameters and goodness-of-fit measures for the Boag data.

Method	δ	γ	η	AIC	A^*	W^*	KS	P-value
MLEs	0.01743	2.62844	0.47484	186.715	0.1288	0.01754	0.08915	0.99881
LSEs	0.02493	4.21745	0.33478	187.254	0.15944	0.01941	0.10188	0.99214
WLSEs	0.0209	3.51947	0.36866	187.088	0.15425	0.02114	0.10682	0.98638
CRVMEs	0.0246	3.95979	0.37316	187.204	0.15054	0.01589	0.08229	0.99971

Table 5. Estimated parameters for the EEEV distribution and other fitted distributions for the Boag data.

Distribution	CDF	MLE of the parameters
EEEV	$\left(1-\mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta}$, $x>0$; $\delta,\gamma,\eta>0$	$\hat{\delta}=0.01743$, $\hat{\gamma}=2.62844$, $\hat{\eta}=0.47484$
EW	$\left(1-\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^{\alpha}}\right)^{\theta}$, $x>0$; $\sigma,\alpha,\theta>0$	$\hat{\sigma}=143.317$, $\hat{\alpha}=5.40643$, $\hat{\theta}=0.14$
PGW	$1-\mathrm{e}^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}$, $x>0$; $\lambda,\beta,\alpha>0$	$\hat{\lambda}=3.30541\times10^{-4}$, $\hat{\beta}=0.93287$, $\hat{\alpha}=38.4569$
EEV	$\left(1-\mathrm{e}^{-\mathrm{e}^{\frac{x-\theta}{\sigma}}}\right)^{\lambda}$, $x\in\Re$; $\sigma,\lambda>0$; $\theta\in\Re$	$\hat{\sigma}=243.088$, $\hat{\theta}=-376.646$, $\hat{\lambda}=316.176$
ENH	$\left(1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}\right)^{\beta}$, $x>0$; $\lambda,\alpha,\beta>0$	$\hat{\lambda}=3.39624\times10^{-4}$, $\hat{\alpha}=25.5174$, $\hat{\beta}=0.79492$
ALTW	$1-\frac{\log\left(\alpha-(\alpha-1)\left(1-\mathrm{e}^{-\lambda x^{\beta}}\right)\right)}{\log(\alpha)}$, $x>0$; $\alpha,\lambda,\beta>0$	$\hat{\alpha}=7.97175\times10^{6}$, $\hat{\lambda}=0.40239$, $\hat{\beta}=0.73571$
LNH	$\frac{\left((\lambda x+1)^{\alpha}-1\right)^{\gamma}}{1+\left((\lambda x+1)^{\alpha}-1\right)^{\gamma}}$, $x>0$; $\alpha,\lambda,\gamma>0$	$\hat{\alpha}=4452.57$, $\hat{\lambda}=3.33326\times10^{-6}$, $\hat{\gamma}=0.99148$

Table 6. Discrimination measures for the EEEV distribution and other competing distributions for the Boag data.

Distribution	LL	AIC	CAIC	BIC	HQIC	A^*	W^*	KS	P-value
EEEV	-90.3573	186.715	188.429	189.386	187.083	0.1288	0.01754	0.08915	0.99881
EW	-90.8083	187.617	189.331	190.288	187.985	0.27977	0.0483	0.14525	0.84207
PGW	-91.9461	189.892	191.606	192.563	190.26	0.49861	0.08063	0.15535	0.77779
EEV	-94.3796	194.759	196.473	197.43	195.127	0.30166	0.04266	0.10852	0.98384
ENH	-91.7872	189.574	191.289	192.246	189.943	0.56923	0.11327	0.18993	0.53473
ALTW	-90.838	187.676	189.39	190.347	188.044	0.26158	0.04533	0.14565	0.83967
LNH	-93.9015	193.803	195.517	196.474	194.171	0.80132	0.14853	0.21699	0.36491

Table 7. The estimates of the EEEV parameters and goodness-of-fit measures for the Aarset data.

Method	δ	γ	η	AIC	A^*	W^*	KS	P-value
MLEs	0.08099	8.719	0.21721	452.193	1.54936	0.21139	0.14323	0.25654
LSEs	0.0734	8.71268	0.18199	457.309	1.12336	0.1177	0.14026	0.27891
WLSEs	0.07329	8.13016	0.21355	452.757	1.21875	0.15883	0.12743	0.39126
CRVMEs	0.07194	8.4662	0.1899	456.338	1.08172	0.11632	0.13388	0.33159

Table 8. Estimated parameters for the EEEV distribution and other fitted distributions for the Aarset data.

Distribution	CDF	MLE of the parameters
EEEV	$\left(1-e^{-\delta x e^{\delta x-\gamma}}\right)^{\eta},\ x>0;\ \delta,\gamma,\eta>0$	$\hat{\delta}=0.08099$, $\hat{\gamma}=8.719$, $\hat{\eta}=0.21721$
EW	$\left(1-e^{-\left(\frac{x}{\sigma}\right)^{\alpha}}\right)^{\theta},\ x>0;\ \sigma,\alpha,\theta>0$	$\hat{\sigma}=91.7152$, $\hat{\alpha}=5.16712$, $\hat{\theta}=0.13253$
PGW	$1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}},\ x>0;\ \lambda,\beta,\alpha>0$	$\hat{\lambda}=0.00179$, $\hat{\beta}=0.89214$, $\hat{\alpha}=12.4692$
EEV	$\left(1-e^{-e^{\frac{x-\theta}{\sigma}}}\right)^{\lambda},\ x\in\Re;\ \sigma,\lambda>0;\ \theta\in\Re$	$\hat{\sigma}=2.50259$, $\hat{\theta}=89.8441$, $\hat{\lambda}=0.05625$
ENH	$\left(1-e^{1-(1+\lambda x)^{\alpha}}\right)^{\beta},\ x>0;\ \lambda,\alpha,\beta>0$	$\hat{\lambda}=3.2702\times10^{-4}$, $\hat{\alpha}=36.963$, $\hat{\beta}=0.67336$
ALTW	$1-\frac{\log\left(\alpha-(\alpha-1)\left(1-e^{-\lambda x^{\beta}}\right)\right)}{\log(\alpha)},\ x>0;\ \alpha,\lambda,\beta>0$	$\hat{\alpha}=6.72977\times10^{9}$, $\hat{\lambda}=0.72573$, $\hat{\beta}=0.75982$
LNH	$\frac{\left((\lambda x+1)^{\alpha}-1\right)^{\gamma}}{1+\left((\lambda x+1)^{\alpha}-1\right)^{\gamma}},\ x>0;\ \alpha,\lambda,\gamma>0$	$\hat{\alpha}=270.79$, $\hat{\lambda}=1.04928\times10^{-4}$, $\hat{\gamma}=0.74349$

Table 9. Discrimination measures for the EEEV distribution and other competing distributions for the Aarset data.

Distribution	LL	AIC	CAIC	BIC	HQIC	A^*	W^*	KS	P-value
EEEV	-223.096	452.193	452.714	457.929	454.377	1.54936	0.21139	0.14323	0.25654
EW	-228.506	463.012	463.534	468.748	465.196	3.32963	0.54406	0.206	0.02871
PGW	-235.576	477.152	477.674	482.888	479.336	3.48817	0.47986	0.1896	0.05493
EEV	-239.225	484.449	484.971	490.185	486.634	1.91921	0.28121	0.16717	0.12225
ENH	-233.402	472.804	473.326	478.54	474.989	3.25763	0.57281	0.20848	0.02591
ALTW	-225.448	456.896	457.418	462.633	459.081	3.41246	0.48076	0.18678	0.06108
LNH	-239.529	485.058	485.579	490.794	487.242	3.81132	0.72771	0.22928	0.01042

Table 10. The estimates of the EEEV parameters and goodness-of-fit measures for the aircraft windshield data.

Method	δ	γ	η	AIC	A^*	W^*	KS	P-value
MLEs	0.2939	0.38145	1.02884	202.355	0.27061	0.03964	0.06804	0.93246
LSEs	0.41537	1.15305	0.88251	203.766	0.33906	0.03147	0.05887	0.98114
WLSEs	0.3448	0.74329	0.94181	202.557	0.27615	0.037	0.06403	0.95844
CRVMEs	0.41966	1.13735	0.91225	204.301	0.38116	0.03042	0.05786	0.98427

Table 11. Estimated parameters for the EEEV distribution and other fitted distributions for the aircraft windshield data.

Distribution	CDF	MLE of the parameters
EEEV	$\left(1-\mathrm{e}^{-\delta x \mathrm{e}^{\delta x-\gamma}}\right)^{\eta},\ \ x>0;\ \ \delta,\gamma,\eta>0$	$\hat{\delta}=0.2939,\ \hat{\gamma}=0.38145,\ \hat{\eta}=1.02884$
EW	$\left(1-\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^{\alpha}}\right)^{\theta},\ \ x>0;\ \ \sigma,\alpha,\theta>0$	$\hat{\sigma}=3.42894,\ \hat{\alpha}=3.17339,\ \hat{\theta}=0.37166$
PGW	$1-\mathrm{e}^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}},\ \ x>0;\ \ \lambda,\beta,\alpha>0$	$\hat{\lambda}=0.03607,\ \hat{\beta}=1.29399,\ \hat{\alpha}=6.44458$
EEV	$\left(1-\mathrm{e}^{-\mathrm{e}^{\frac{x-\theta}{\sigma}}}\right)^{\lambda},\ \ x\in\Re;\ \ \sigma,\lambda>0;\ \ \theta\in\Re$	$\hat{\sigma}=8.99851,\ \hat{\theta}=-17.441,\ \hat{\lambda}=3874.54$
ENH	$\left(1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}\right)^{\beta},\ \ x>0;\ \ \lambda,\alpha,\beta>0$	$\hat{\lambda}=0.00401,\ \hat{\alpha}=83.026,\ \hat{\beta}=1.27068$
ALTW	$1-\frac{\log\left(\alpha-(\alpha-1)\left(1-\mathrm{e}^{-\lambda x^{\beta}}\right)\right)}{\log(\alpha)},\ \ x>0;\ \ \alpha,\lambda,\beta>0$	$\hat{\alpha}=76.795,\ \hat{\lambda}=1.02757,\ \hat{\beta}=1.18699$
LNH	$\frac{\left((\lambda x+1)^{\alpha}-1\right)^{\gamma}}{1+\left((\lambda x+1)^{\alpha}-1\right)^{\gamma}},\ \ x>0;\ \ \alpha,\lambda,\gamma>0$	$\hat{\alpha}=22783.3,\ \hat{\lambda}=1.68692\times10^{-5},\ \hat{\gamma}=1.59565$

Table 12. Discrimination measures for the EEEV distribution and other competing distributions for the aircraft windshield data.

Distribution	LL	AIC	CAIC	BIC	HQIC	A^*	W^*	KS	P-value
EEEV	-98.1775	202.355	202.762	208.784	204.884	0.27061	0.03964	0.06804	0.93246
EW	-98.3272	202.654	203.061	209.084	205.183	0.31054	0.04736	0.07604	0.85955
PGW	-98.4754	202.951	203.358	209.38	205.479	0.32891	0.04653	0.07955	0.82017
EEV	-101.243	208.485	208.892	214.914	211.014	0.29761	0.04203	0.07506	0.86983
ENH	-98.6164	203.233	203.64	209.662	205.762	0.41203	0.07274	0.09721	0.59099
ALTW	-98.4323	202.865	203.271	209.294	205.393	0.27959	0.04224	0.07571	0.86305
LNH	-102.64	211.28	211.687	217.71	213.809	0.92644	0.13163	0.12073	0.31745

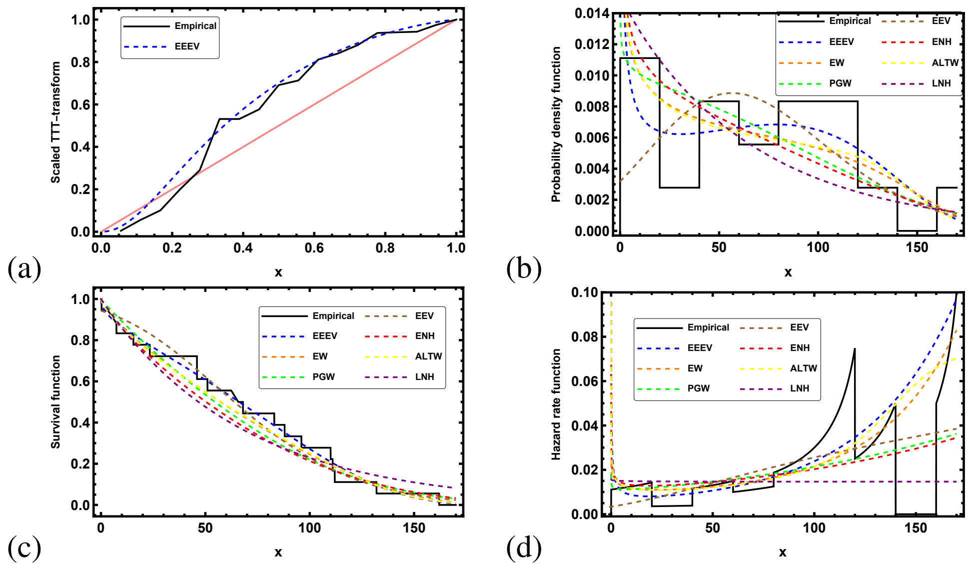

Figure 7. (a) TTT plot, estimated (b) PDFs, (c) survival functions, and (d) HR functions for the Boag data.

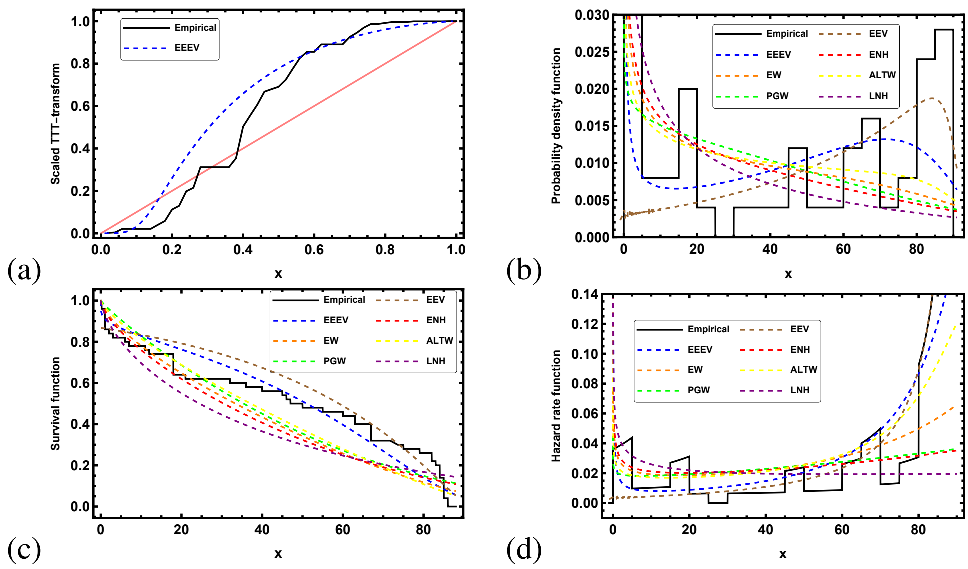

Figure 8. (a) TTT plot, estimated (b) PDFs, (c) survival functions, and (d) HR functions for the Aarset data.

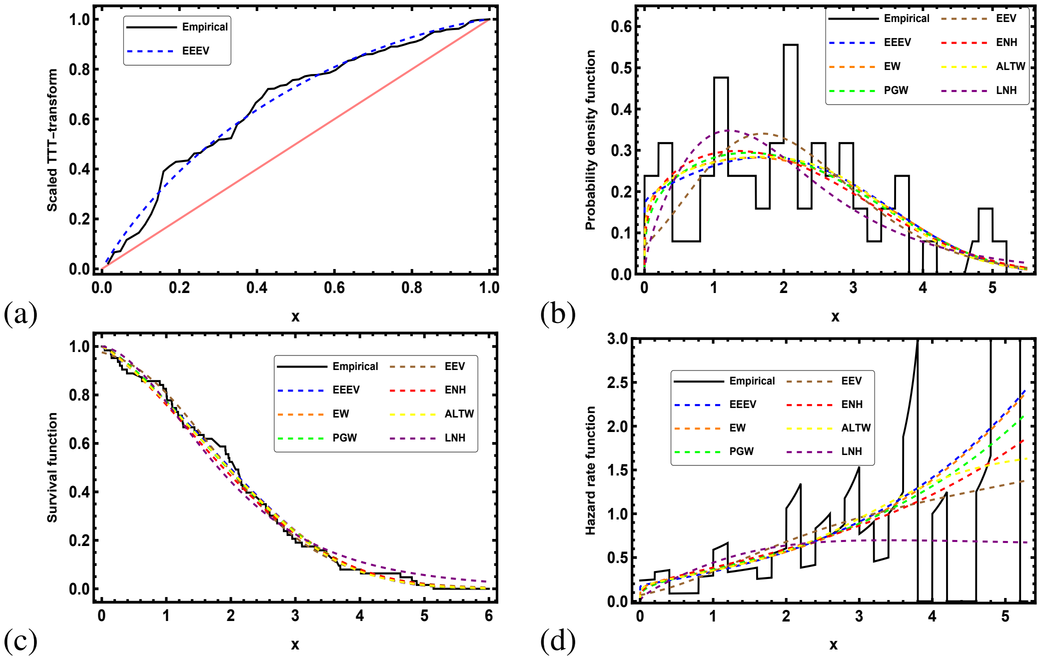

Figure 9. (a) TTT plot, estimated (b) PDFs, (c) survival functions, and (d) HR functions for the aircraft windshield data.

7. Conclusions

In this paper, we have proposed and examined the EEEV distribution as an extension of the EEV distribution. Its associated HR function can be bathtub-shaped or increasing. Some of its statistical properties have been derived. The estimation of the parameters for the EEEV distribution was performed by using four different estimation methods. The behaviors of these estimators were assessed via simulation. The practical applicability of the EEEV distribution has been illustrated by analyzing three real-life datasets from the fields of medicine and engineering. The analytical measures indicated that our EEEV distribution provided a good fit compared to other competing distributions.

Author contributions

M. G. M. Ghazal: Conceptualization, Data curation, Methodology, Investigation, Software, Writing-review & editing; Yusra A. Tashkandy: Conceptualization, Investigation, Methodology, Project administration, Funding acquisition, Writing-original draft; Oluwafemi Samson Balogun: Conceptualization, Data curation, Writing-original draft, Formal analysis, Investigation; M. E. Bakr: Methodology, Investigation, Funding acquisition, Writing an original draft. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

There is no conflict of interest declared by the authors.

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