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Research article

An E-extra iteration method for solving reduced biquaternion matrix equation $AX + XB = C$

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Abstract: This paper focuses on the solution of the reduced biquaternion equation $AX + XB = C$ using the E-extra iteration method. By utilizing the complex decomposition of a reduced biquaternion matrix, we transform the equation into a complex matrix equation. Subsequently, we analyze the convergence of this method and provide guidelines for selecting optimal parameters. Finally, numerical examples are presented to demonstrate the efficacy of our algorithm.

Keywords: reduced biquaternion; E-extra iteration method; complex decomposition; convergence analysis; numerical example
Mathematics Subject Classification: 15A33, 65F05

1. Introduction

Currently, the matrix equation $AX + XB = C$ represents a critically important class of equations with significant practical applications. This assertion is underscored by a plethora of studies that highlight its utility in various domains, including but not limited to, the development of matrix black box algorithms, applications in mechanics and control systems, thermodynamics, and vibration theory, among others [1–6]. The investigation of iteration methods for solving equations within the real and complex domains, as well as quaternions, has garnered considerable attention among scholars, highlighting its status as a significant research area [7–15]. For example, Zhou [16] applies the conjugate gradient method and linear projection operator to solve two matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, Bai [17] introduced the Hermitian and skew-Hermitian splitting (HSS) iteration method, while Zhou [18] et al. advanced the modified Hermitian and skew-Hermitian splitting (MHSS) iteration technique tailored for resolving complex linear systems. Furthermore, Zhou [19] et al. and Benzi [20] contributed to the field by establishing a generalization of the Hermitian and skew-Hermitian splitting iteration methods (GHSS) and introducing a generalization of the positive-definite and skew-Hermitian splitting (GPSS) iteration methods, respectively. Li [21] discussed the outer and inner iterative schemes to solve the complex symmetric matrix equation. Zhang [22] et al. established two new non-symmetric positive definite and semi-definite splitting (NPSS) iterations for solving the sub-positive definite matrix equation $AX = B$ over the quaternion field. Additionally, Ma [23] proposed an E-extra iteration method for solving the continuous Sylvester equations, further enriching the repertoire of iterative methods in the field. Nonetheless, there has been relatively little research on the reduced biquaternion iteration method. To address this gap, and drawing from reference [23] for inspiration, this paper investigates the use of the E-extra iteration method for solving the reduced biquaternion matrix equation

$$AX + XB = C,$$

where A, B, and C are known reduced biquaternion matrices, and X represents the unknown reduced biquaternion matrix to be solved. And E-extra iteration method is based on the Euler-extrapolation technique. It serves as an iterative approach for solving the continuous Sylvester equation $AX+XB = F$ in the complex domain, where $A = W + iT$, $B = U + iV$, and W, T, U, and V are symmetric positive semidefinite matrices.

Obviously, reduced biquaternions represent a generalization of complex numbers. Let $a = a_0 + a_1 i + a_2 j + a_3 k$, where $i^2 = -j^2 = k^2 = -1$, $ij = ji = k$, $jk = kj = i$, $ki = ik = -j$. Reduced biquaternions, being a type of exchange algebra, find extensive application in numerous practical problems, as evidenced by previous studies and so on [24–27].

Throughout this paper, let R be the real number field, C be the complex number field, and Q_{RB} respect the sets of all reduced biquaternions. $C^{m\times n}$ respect the set of all $m\times n$ complex matrices, $Q_{RB}^{m\times n}$ respect the set of all $m\times n$ reduced biquaternions matrices. A^T, $\rho(A)$, $\|A\|_F$, $\mathrm{Re}(A)$, and $\mathrm{Im}(A)$ represent the transpose, the spectral radius, the matrix Frobenius norm, and the real and imaginary parts of A, respectively. Let $A = [a_1, a_2, \cdots, a_n] \in C^{m\times n}$, where $a_s \in C^m$ is the sth column of the matrix A, $s = 1, 2, \cdots, n$, and the vec operator of A is defined to be $vec(A) = [a_1^T, a_2^T, \cdots, a_n^T]^T \in C^{mn\times 1}$. For $A = (a_{st}) \in C^{m\times n}$, $B = (b_{st}) \in C^{p\times q}$, and the symbol $A \otimes B = (a_{st}B) \in C^{mp\times nq}$ stands for the Kronecker product of A and B. Subsequently, this paper proceeds by presenting the pertinent definitions and lemmas.

Definition 1. *[28] For any $b \in Q_{RB}$, b can be uniquely expressed as $b = c_0 + c_1 j$, where $c_0, c_1 \in C$, $c_0 = b_0 + b_1 i$, $c_1 = b_2 + b_3 i$. Consequently, the Frobenius norm of b can be expressed as $\|b\|_F = \sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2}$.*

Definition 2. *[28] For any $A \in Q_{RB}^{m\times n}$, A can be uniquely expressed as $A = A_1 + A_2 j$, where $A_1, A_2 \in C^{m\times n}$, and the Frobenius norm of A can be expressed as*

$$\begin{aligned}\|A\|_F &= \left(\sum_{s=1}^{m}\sum_{t=1}^{n}\|a_{st}\|_F^2\right)^{\frac{1}{2}} \\ &= \left(\|\mathrm{Re}(A_1)\|_F^2 + \|\mathrm{Im}(A_1)\|_F^2 + \|\mathrm{Re}(A_2)\|_F^2 + \|\mathrm{Im}(A_2)\|_F^2\right)^{\frac{1}{2}}.\end{aligned}$$

For example, let $A = \begin{bmatrix} 1 & i \\ j & k \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & i \end{bmatrix} j$, *then* $\|A\|_F^2 = 1 + 1 + 1 + 1 = 4$.

Lemms 1. *[23] Assume that* $H, G \in R^{2n^2\times 2n^2}$ *are symmetric and positive definite matrices, and let* $\theta \in [0, \frac{\pi}{2}]$, *then* $H\cos\theta + G\sin\theta$ *is symmetric positive definite.*

This paper specifically addresses the following problem.

Problem 1. *Let* $A, B, C \in Q_{RB}^{n\times n}$, *and* B *is a pure reduced biquaternion matrix. Find out* $X \in Q_{RB}^{n\times n}$, *such that* $AX + XB = C$. *If the set of solutions is empty, find out the least-norm solution* $X \in Q_{RB}^{n\times n}$, *such that* $\min\limits_{X\in Q_{RB}^{n\times n}} \|AX + XB - C\|_F$.

2. The solutions of problem

Let $X \in Q_{RB}^{n\times n}$, it can be uniquely expressed as

$$X = X_1 + X_2 j,$$

where $X_i \in C^{n\times n} (i = 1, 2)$. Let $A, B, C \in Q_{RB}^{n\times n}$, these can also be uniquely expressed as

$$A = A_1 + A_2 j, B = B_1 + B_2 j, C = C_1 + C_2 j,$$

where $A_i, B_i, C_i \in C^{n\times n} (i = 1, 2)$. So the reduced biquaternion matrix equation

$$AX + XB = C \tag{2.1}$$

can be expressed as

$$(A_1 + A_2 j)(X_1 + X_2 j) + (X_1 + X_2 j)(B_1 + B_2 j) = C_1 + C_2 j, \tag{2.2}$$

according to the definition of reduced biquaternion, for any complex number $z \in C$, we have $z\cdot j = j\cdot z$. So the Eq (2.2) can be expressed as

$$(A_1X_1 + A_2X_2 + X_1B_1 + X_2B_2) + (A_1X_2 + A_2X_1 + X_2B_1 + X_1B_2)j = C_1 + C_2 j,$$

and due to the unique complex decomposition of a reduced biquaternion matrix, we have

$$\begin{cases} A_1X_1 + A_2X_2 + X_1B_1 + X_2B_2 = C_1, \\ A_2X_1 + A_1X_2 + X_2B_1 + X_1B_2 = C_2. \end{cases}$$

Assume that B is a pure reduced biquaternion matrix, then $B_1 = O$(zero matrix), and we can also write it as

$$\begin{bmatrix} A_2 & A_1 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} B_2 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{2.3}$$

Denote

$$\tilde{A} = \begin{bmatrix} A_2 & A_1 \\ A_1 & A_2 \end{bmatrix} \in C^{2n\times 2n}, \tilde{X} = \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \in C^{2n\times n}, \tilde{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \in C^{2n\times n}. \tag{2.4}$$

Assume that

$$\tilde{A} = A_R + iA_I, B_2 = B_R + iB_I,$$

where $A_R, A_I \in R^{2n\times 2n}$ are real, symmetric, and positive definite matrices, and $B_R, B_I \in R^{n\times n}$ are real, symmetric, and positive semidefinite matrices, then the Eq (2.3) can be expressed as

$$(A_R + iA_I)\tilde{X} + \tilde{X}(B_R + iB_I) = \tilde{C}. \tag{2.5}$$

Therefore, by introducing the Euler formula: $e^{-i\theta} = \cos\theta - i\sin\theta(\theta \in [0, \frac{\pi}{2}])$ and multiplying both sides of Eq (2.5) by $e^{-i\theta}$, we obtain

$$(\cos\theta - i\sin\theta)(A_R + iA_I)\tilde{X} + \tilde{X}(B_R + iB_I)(\cos\theta - i\sin\theta) = e^{-i\theta}\tilde{C}.$$

Expanding and rearranging the above equation, we have

$$\begin{aligned}&(A_R\cos\theta + A_I\sin\theta)\tilde{X} + \tilde{X}(B_R\cos\theta + B_I\sin\theta)\\ &= i(A_R\sin\theta - A_I\cos\theta)\tilde{X} + i\tilde{X}(B_R\sin\theta - B_I\cos\theta) + e^{-i\theta}\tilde{C}.\end{aligned}$$

In summary, for the problem, we propose the E-extra iteration method to solve the reduced biquaternion matrix equation $AX + XB = C$.

Algorithm 1. *Reduced biquaternion E-extra iterative method:*

Step 1. *Given matrices* $A, B, C \in Q_{RB}^{n\times n}$, *then these matrices can be uniquely expressed as* $A = A_1 + A_2 j, B = B_1 + B_2 j, C = C_1 + C_2 j$.

Step 2. *Write the complex matrix* $\tilde{A}, \tilde{C}$ *according to* (2.4), *then write the real matrix* A_R, A_I, B_R, B_I.

Step 3. *Given the initial matrix* $\tilde{X}^{(0)} \in C^{2n\times n}$, *the allowable error is* $0 \le \varepsilon \ll 1$. *Let* $k := 1$.

Step 4. *Calculate*

$$\begin{aligned}&(A_R\cos\theta + A_I\sin\theta)\tilde{X}^{(k)} + \tilde{X}^{(k)}(B_R\cos\theta + B_I\sin\theta)\\ &= i(A_R\sin\theta - A_I\cos\theta)\tilde{X}^{(k-1)} + i\tilde{X}^{(k-1)}(B_R\sin\theta - B_I\cos\theta) + e^{-i\theta}\tilde{C}.\end{aligned} \tag{2.6}$$

Step 5. *If*

$$\frac{\left\|\tilde{C} - \tilde{A}\tilde{X}^{(k)} - \tilde{X}^{(k)}B_2\right\|_F}{\left\|\tilde{C}\right\|_F} \le \varepsilon,$$

then stop the iteration and output $\tilde{X}^{(k)}$ *as the approximate solution; otherwise, proceed to Step 6.*

Step 6. *Let* $k := k + 1$, *and proceed to Step 4.*

Denote

$$H = I_n \otimes A_R + B_R^T \otimes I_{2n}, G = I_n \otimes A_I + B_I^T \otimes I_{2n},$$

where I_n is an $n \times n$ identity matrix, I_{2n} is a $2n \times 2n$ identity matrix, and

$$x = \mathrm{vec}(\tilde{X}), c = \mathrm{vec}(\tilde{C}).$$

Using the Kronecker product, Eq (2.6) can be expressed as follows:

$$(H\cos\theta + G\sin\theta)x^{(k)} = i(H\sin\theta - G\cos\theta)x^{(k-1)} + e^{-i\theta}c.$$

It can be verified that matrices H and G are symmetric and positive-definite. As stated in Lemma 1, it follows that the matrix $H\cos\theta + G\sin\theta$ is a symmetric positive definite matrix. We can express the iterative formula Eq (2.6) of the E-extra iteration method for reduced biquaternion in the following fixed-point form:

$$x^{(k)} = A(\theta)x^{(k-1)} + C(\theta)^{-1}c,$$

where

$$\begin{aligned}A(\theta) &= i(H\cos\theta + G\sin\theta)^{-1}(H\sin\theta - G\cos\theta),\\ C(\theta)^{-1} &= (H\cos\theta + G\sin\theta)^{-1}e^{-i\theta}.\end{aligned} \tag{2.7}$$

3. Convergence analysis

Then, we discuss the convergence of the E-extra iteration method and present the following relevant lemmas:

Lemms 2. *[23] Assume that $H, G \in R^{2n^2\times 2n^2}$ are symmetric and positive definite matrices, and let $\theta \in [0, \frac{\pi}{2}]$, then the eigenvalues of the iterative matrix $A(\theta)$ is $\lambda = i\frac{\sin\theta-\mu\cos\theta}{\cos\theta+\mu\sin\theta}$, and the corresponding eigenvectors are given by*

$$\rho(A(\theta)) = \max\left\{\left|\frac{\sin\theta - \mu_{\min}\cos\theta}{\cos\theta + \mu_{\min}\sin\theta}\right|, \left|\frac{\sin\theta - \mu_{\max}\cos\theta}{\cos\theta + \mu_{\max}\sin\theta}\right|\right\},$$

where μ is the generalized eigenvalue of the matrix pair (H, G), and $\mu_{\min}$ and $\mu_{\max}$ correspond to the minimum and maximum generalized eigenvalues of the matrix pair (H, G), respectively. If $\rho(A(\theta)) < 1, \forall\theta \in [0, \frac{\pi}{2}]$, the E-extra iteration method converges.

Lemms 3. *[23] Assume that $H, G \in R^{2n^2\times 2n^2}$ are symmetric and positive definite matrices. If the parameters θ satisfying*

$$\begin{cases} \left\{\theta \in \left(0, \frac{\pi}{2}\right) | \frac{1-\mu_{\min}}{1+\mu_{\min}} < \cot\theta < \frac{\mu_{\max}+1}{\mu_{\max}-1}\right\}, \mu_{\min} \leqslant 1, \mu_{\max} > 1, \\ \left\{\theta \in \left(0, \frac{\pi}{2}\right] | \frac{1-\mu_{\min}}{1+\mu_{\min}} < \cot\theta < \frac{\mu_{\max}+1}{\mu_{\max}-1}\right\}, \mu_{\min} > 1, \mu_{\max} > 1, \\ \left\{\theta \in \left(0, \frac{\pi}{2}\right) | \frac{1-\mu_{\min}}{1+\mu_{\min}} < \cot\theta\right\}, \mu_{\max} = 1, \\ \left\{\theta \in \left[0, \frac{\pi}{2}\right] | \frac{1-\mu_{\min}}{1+\mu_{\min}} < \cot\theta\right\}, \mu_{\max} < 1, \end{cases}$$

then the E-extra iteration method converges. Moreover, the optimal iteration parameter θ_* is

$$\begin{aligned} \theta_* = &\left\{\operatorname{arc}\tan\left(\frac{\mu_{\min}\mu_{\max}-1+\sqrt{\left(1+\mu_{\min}^2\right)\left(1+\mu_{\max}^2\right)}}{\mu_{\min}+\mu_{\max}}\right) \in \left[0, \frac{\pi}{2}\right)\right\} \\ &\cup \left\{\operatorname{arc}\cot\left(\frac{1-\mu_{\min}\mu_{\max}+\sqrt{\left(1+\mu_{\min}^2\right)\left(1+\mu_{\max}^2\right)}}{\mu_{\min}+\mu_{\max}}\right) \in \left(0, \frac{\pi}{2}\right]\right\}, \end{aligned}$$

and the corresponding optimal convergence factor is

$$\rho\left(A\left(\theta_*\right)\right) = \left|\frac{\sin\theta_* - \mu_{\min}\cos\theta_*}{\cos\theta_* + \mu_{\min}\sin\theta_*}\right| \left(= \left|\frac{\mu_{\max}\cos\theta_* - \sin\theta_*}{\cos\theta_* + \mu_{\max}\sin\theta_*}\right|\right).$$

In conclusion, regarding the solution to the problem, we have the following conclusions:

Theorem 1. *Given matrices $A, B, C \in Q_{RB}^{n\times n}$, the sufficient condition for the existence of a solution to the reduced biquaternion matrix equation $AX + XB = C$ is $\rho(A(\theta)) < 1, \forall\theta \in [0, \frac{\pi}{2}]$. Consequently,*

$$\frac{\left\|\tilde{C} - \tilde{A}\tilde{X}^{(k)} - \tilde{X}^{(k)}B_2\right\|_F}{\left\|\tilde{C}\right\|_F} \leq \varepsilon.$$

When the equation has a solution, the solution is given by

$$X = X_1 + X_2 j. \tag{3.1}$$

In the case of no solution, the least-norm solution of Eq (2.1) *remains* (3.1). *Where*

$$\tilde{A}=\begin{bmatrix} A_2 & A_1 \\ A_1 & A_2 \end{bmatrix}\in C^{2n\times 2n},\tilde{X}=\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}\in C^{2n\times n},\tilde{C}=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\in C^{2n\times n},$$

$$A(\theta)=i(H\cos\theta+G\sin\theta)^{-1}(H\sin\theta-G\cos\theta),$$

$X_1 \in C^{n\times n}$ *refers to the lower part of the block matrix* $\tilde{X}^{(k)}$, $X_2 \in C^{n\times n}$ *refers to the upper part of the block matrix* $\tilde{X}^{(k)}$.

Proof. By Eqs (2.1) and (2.3), we have

$$\|AX+XB-C\|_F \Leftrightarrow \left\|\begin{bmatrix} A_2 & A_1 \\ A_1 & A_2 \end{bmatrix}\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}+\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}B_2-\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\right\|_F.$$

This means that the reduced biquaternion least-norm problem for Eq (2.1) has a solution if and only if the complex least-norm problem for Eq (2.3) has a solution.

By Lemmas 1–3, when $\rho(A(\theta)) < 1, \forall\theta \in [0, \frac{\pi}{2}]$, the E-extra iteration method converges, implying that the system of Eq (2.3) has a solution. Therefore, $\frac{\|\tilde{C}-\tilde{A}\tilde{X}^{(k)}-\tilde{X}^{(k)}B_2\|_F}{\|\tilde{C}\|_F} \leq \varepsilon$ in this case. It follows that the solution of Eq (2.1) is given by (3.1). □

4. Numerical examples

The following example was executed by writing an M-file and running it on Matlab R2022b. And all the computations are run on a personal computer with a 64-bit Win10 operating system, a 12th Gen Intel(R) Core(TM) i3-12100F CPU at 3.30 GHz, and 16.00GB of 3200MHz of RAM memory. In actual computations, we use the identity matrix as the initial guess and the stopping criterion

$$\text{res}=\frac{\left\|\tilde{C}-\tilde{A}\tilde{X}^{(k)}-\tilde{X}^{(k)}B_2\right\|_F}{\left\|\tilde{C}\right\|_F}<10^{-8},$$

where $\tilde{X}^{(k)}$ is the current approximate. If the number of iteration steps exceeds 1000, the iteration is terminated. The obtained results validate the effectiveness of this algorithm.

Example 1. *Let*

$$A=\begin{bmatrix} -2+i+6j+2k & 2-j & 0 & 0 \\ 2-j & -2+i+6j+2k & 2-j & 0 \\ 0 & 2-j & -2+i+6j+2k & 2-j \\ 0 & 0 & 2-j & -2+i+6j+2k \end{bmatrix},$$

$$B=\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & j+k & 0 & 0 \\ 0 & 0 & j+k & 0 \\ 0 & 0 & 0 & j \end{bmatrix}, C=\begin{bmatrix} 1-j & 2-2j & 3-3j & 4-4j \\ 2-2j & 1-j & 4-4j & 3-3j \\ 3-3j & 4-4j & 1-j & 2-2j \\ 4-4j & 3-3j & 2-2j & 1-j \end{bmatrix},$$

consider the reduced biquaternion equation $AX+XB=C$ by the E-extra iteration method, where $X \in Q_{RB}^{4\times 4}$.

First, these matrices A, B, and C can be uniquely expressed as

$$A_1 = \begin{bmatrix} -2+i & 2 & 0 & 0 \\ 2 & -2+i & 2 & 0 \\ 0 & 2 & -2+i & 2 \\ 0 & 0 & 2 & -2+i \end{bmatrix}, A_2 = \begin{bmatrix} 6+2i & -1 & 0 & 0 \\ -1 & 6+2i & -1 & 0 \\ 0 & -1 & 6+2i & -1 \\ 0 & 0 & -1 & 6+2i \end{bmatrix},$$

$$B = j \cdot B_2 = j \cdot \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}, C_2 = -\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

According to (2.4)*, we can derive the expressions for matrices $\tilde{A}$ and $\tilde{C}$. Moreover, we have*

$$A_R = \begin{bmatrix} A_{21} & A_{11} \\ A_{11} & A_{21} \end{bmatrix}, A_I = \begin{bmatrix} A_{22} & A_{12} \\ A_{12} & A_{22} \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 2 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}, A_{21} = \begin{bmatrix} 6 & -1 & 0 & 0 \\ -1 & 6 & -1 & 0 \\ 0 & -1 & 6 & -1 \\ 0 & 0 & -1 & 6 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_{22} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$B_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that A_R and A_I are real symmetric positive definite matrices, B_R and B_I are real symmetric positive semidefinite matrices. In this case, the optimal iteration parameter is $\theta_* \approx$ 0.5529*. The number of iteration steps (denoted by 'k'), the elapsed CPU times in seconds (denoted by 't_{cpu}'), the iterative residual (denoted by 'res'). When the iteration residual satisfies* $\mathrm{res} < 10^{-8}$*, it is considered the k-th approximate solution to Eq* (2.1)*. The main results are as follows: $k = 24$, $t_{cpu} \approx 0.0001$s,* $\mathrm{res} \approx 5.3157\mathrm{e}-08$,

$$X_1^{(24)} \approx \begin{bmatrix} 0.5119-0.2142i & -0.5119+0.2142i & 0.6689-0.1314i & -0.6689+0.1314i \\ 0.6784-0.3013i & -0.6784+0.3013i & 0.7170-0.1713i & -0.7170+0.1713i \\ 0.3909-0.2373i & -0.3909+0.2373i & 0.5393-0.1434i & -0.5393+0.1434i \\ 0.3192-0.1500i & -0.3192+0.1500i & 0.2821-0.0791i & -0.2821+0.0791i \end{bmatrix},$$

$$X_2^{(24)} \approx \begin{bmatrix} 0.2698-0.2091i & -0.2698+0.2091i & 0.3192-0.1500i & -0.3192+0.1500i \\ 0.5254-0.3777i & -0.5254+0.3777i & 0.3909-0.2373i & -0.3909+0.2373i \\ 0.7166-0.4479i & -0.7166+0.4479i & 0.6784-0.3013i & -0.6784+0.3013i \\ 0.6840-0.3390i & -0.6840+0.3390i & 0.5119-0.2142i & -0.5119+0.2142i \end{bmatrix}.$$

Therefore, $X = X_1^{(24)} + X_2^{(24)} j$ *is an approximate solution to Eq* (2.1).

Example 2. *Given* $n \times n$ *reduced biquaternion matrices*

$$A = \begin{bmatrix} -2+i+6j+2k & 2-j & & \\ 2-j & -2+i+6j+2k & \ddots & \\ & \ddots & \ddots & 2-j \\ & & 2-j & -2+i+6j+2k \end{bmatrix},$$

$$B = \begin{bmatrix} k & & & & \\ & j+k & & & \\ & & \ddots & & \\ & & & j+k & \\ & & & & j \end{bmatrix},$$

$$C = \begin{bmatrix} 1-j & 2-2j & \cdots & n-nj \\ 2-2j & 1-j & \cdots & n-1-(n-1)j \\ \vdots & \vdots & \ddots & \vdots \\ n-nj & n-1-(n-1)j & \cdots & 1-j \end{bmatrix},$$

consider the reduced biquaternion equation $AX + XB = C$ *by the E-extra iteration method, where* $X \in Q_{RB}^{n\times n}$.

First, these matrices A, B, and C can be uniquely expressed as

$$A_1 = \begin{bmatrix} -2+i & 2 & & \\ 2 & -2+i & \ddots & \\ & \ddots & \ddots & 2 \\ & & 2 & -2+i \end{bmatrix}, A_2 = \begin{bmatrix} 6+2i & -1 & & \\ -1 & 6+2i & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 6+2i \end{bmatrix},$$

$$B = j \cdot B_2 = j \cdot \begin{bmatrix} i & & & & \\ & 1+i & & & \\ & & \ddots & & \\ & & & 1+i & \\ & & & & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 1 & \cdots & n-1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n-1 & \cdots & 1 \end{bmatrix}, C_2 = -\begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 1 & \cdots & n-1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n-1 & \cdots & 1 \end{bmatrix}.$$

According to (2.4)*, we can derive the expressions for matrices $\tilde{A}$ and $\tilde{C}$. Moreover, we have*

$$A_R = \begin{bmatrix} A_{21} & A_{11} \\ A_{11} & A_{21} \end{bmatrix}, A_I = \begin{bmatrix} A_{22} & A_{12} \\ A_{12} & A_{22} \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -2 & 2 & & \\ 2 & -2 & \ddots & \\ & \ddots & \ddots & 2 \\ & & 2 & -2 \end{bmatrix}, A_{21} = \begin{bmatrix} 6 & -1 & & \\ -1 & 6 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 6 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, A_{22} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{bmatrix},$$

$$B_R = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, B_I = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix}.$$

It is easy to verify that A_R and A_I are real symmetric positive definite matrices, B_R and B_I are real symmetric positive semidefinite matrices. The number of iteration steps (denoted by 'cs'), the elapsed CPU times in seconds (denoted by 't_{cpu}'), the iterative residual (denoted by 'res'). When the iteration residual satisfies $\text{res} < 10^{-8}$*, it is regarded as the approximate solution of Eq* (2.1)*. For matrices of different orders, the required time is shown in the figure below, and the calculation results are presented in the following table.*

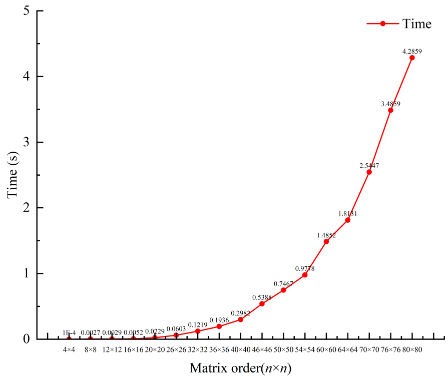

Figure 1. CPU times under different matrix order.

Table 1. θ_*, cs, t_{cpu} and res for Example 2.

$n \times n$	θ_*	cs	t_{cpu}	res
16×16	0.5837	23	0.0052s	5.9527e-08
32×32	0.5860	22	0.1219s	6.1540e-08
50×50	0.5865	21	0.7467s	7.3943e-08
64×64	0.5866	21	1.8131s	5.8504e-08
80×80	0.5867	20	4.2859s	8.5967e-08

5. Conclusions

In this work, we utilize the complex decomposition of matrices and the multiplication rules of reduced biquaternion matrices to transform the reduced biquaternion matrix equation $AX + XB = C$ into a matrix equation over the complex field. We then explore the solution of such matrix equations based on the E-extra iterative method, derive the convergence of the E-extra iterative method, and provide guidelines for choosing the optimal parameters. Finally, we also provide numerical examples, which illustrate that our algorithms are effective and workable.

Author contributions

Jiaxin Lan: Writing-original draft, Writing-review, Editing, Funding acquisition; Jingpin Huang: Funding acquisition, Writing-review, Editing, Supervision; Yun Wang: Funding acquisition, Writing-review, Editing, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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