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Research article

Theorems of existence and uniqueness for pointwise-slant immersions in Kenmotsu space forms

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Abstract: The present paper aims to demonstrate the theorems of existence and uniqueness for pointwise slant immersions in Kenmotsu space forms. Some substantial results are given in this direction. Also, we offer non-trivial examples of pointwise slant submanifolds of an almost contact-metric manifold.

Keywords: Kenmotsu manifold; sectional curvature; Kenmotsu space form; pointwise-slant immersion
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1. Introduction

B.-Y Chen originally explored the idea of slant submanifolds in [9, 10], expanding upon the ideas of real and holomorphic submanifolds [12]. Additionally, A. Lotta [22] defined and examined the slant submanifolds in almost contact metric manifolds, and he established certain properties of such submanifolds. After that, this topic was researched in a number of structures on Riemannian manifolds (see [3, 6, 7, 13, 19]).

From this viewpoint, the theory of existence and uniqueness for slant immersions emerged in complex space forms [14, 15], in Sasakian space forms [5], in cosymplectic space forms [20], and in Kenmotsu space forms [23].

Otherwise, F. Etayo [17] introduced the concept of pointwise slant submanifolds, also known as quasi-slant submanifolds, as a generalization of slant submanifolds. Afterwards, pointwise slant submanifolds of almost Hermitian manifolds were studied by Chen and Garay [11]. Subsequently, many geometries investigated this concept (see [18, 24–27]).

In an analogous manner, [1, 2] presented the existence and uniqueness theorems for pointwise slant immersions in complex space forms and in Sasakian space forms, respectively.

In the framework of previous papers, in this paper, we study these theorems in Kenmotsu space forms.

The structure of the paper is as follows: In Section 2, we review some fundamental definitions and formulas that will come in handy later. Next, we review some results for the pointwise-slant submanifold of an almost contact-metric manifold, and we offer a general example of such a submanifold in Section 3. Establishing the existence and uniqueness theorems for pointwise-slant immersions into Kenmotsu space forms is the focus of Section 4.

2. Preliminaries

A smooth manifold $\tilde{N}$ of dimension $(2r + 1)$ is said to be an *almost contact metric manifold* if it equipped with the almost contact metric structure (ψ, ξ, η, g), which included a $(1, 1)$ tensor field ψ, a vector field ξ, a $1-$form η, and a Riemannian metric g on $\tilde{N}$, which satisfies the following conditions: [4]

$$\psi^2 U = -U + \eta(U)\xi, \;\; \psi\xi = 0, \;\; \eta(\psi U) = 0, \;\; \eta(\xi) = 1, \tag{2.1}$$

and

$$g(\psi U, \psi V) = g(U, V) - \eta(U)\eta(V), \;\; g(U, \xi) = \eta(U), \tag{2.2}$$

for all $U, V \in \Omega(T\tilde{N}^{2r+1})$, where $\Omega(T\tilde{N}^{2r+1})$ denotes the Lie algebra of smooth vector fields on $\tilde{N}^{2r+1}$.

Definition 2.1. *[21] An almost contact metric manifold $\tilde{N}^{2r+1}$ is said to be a Kenmotsu manifold if*

$$(\tilde{\nabla}_U \psi)V = g(\psi U, V)\xi - \eta(V)\psi U, \tag{2.3}$$

and

$$\tilde{\nabla}_U \xi = U - \eta(U)\xi, \tag{2.4}$$

for any $U, V \in \Omega(T\tilde{N}^{2r+1})$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{N}^{2r+1}$ with respect to the Riemannian metric g.

The following formula is known to provide the covariant derivative of the tensor field ψ:

$$(\tilde{\nabla}_U \psi)V = \tilde{\nabla}_U \psi V - \psi \tilde{\nabla}_U V, \tag{2.5}$$

for all $U, V \in \Omega(T\tilde{N}^{2r+1})$.

The Kenmotsu manifold $\tilde{N}^{2r+1}$ with the constant ψ-sectional curvature k is referred to as a *Kenmotsu space form* and is represented by $\tilde{N}^{2r+1}(k)$ when the curvature tensor $\tilde{R}$ is given by:

$$\begin{aligned} \tilde{R}(U, V)W = \frac{k-3}{4}\Big\{g(V, W)U - g(U, W)V\Big\} + \frac{k+1}{4}\Big\{\eta(U)\eta(W)V \\ - \eta(V)\eta(W)U + \eta(V)g(U, W)\xi - \eta(U)g(V, W)\xi \\ + g(\psi V, W)\psi U - g(\psi U, W)\psi V + 2g(U, \psi V)\psi W\Big\}, \end{aligned} \tag{2.6}$$

for any $U, V, W \in \Omega(T\tilde{\mathcal{N}}^{2r+1})$ [21].

With the same Riemannian metric g induced on an almost contact metric manifold $\tilde{\mathcal{N}}^{2r+1}$, let $\mathcal{N}$ be a submanifold of dimension $(s+1)$ in $\tilde{\mathcal{N}}^{2r+1}$. The tangent bundle of $\mathcal{N}$ is represented by $T\mathcal{N}$, and the set of all vector fields normal to $\mathcal{N}$ is denoted by $T^{\perp}\mathcal{N}$. The Riemannian connection $\tilde{\nabla}$ of $\tilde{\mathcal{N}}^{2r+1}$ induces the connections ∇ and $\nabla^{\perp}$ on $T\mathcal{N}$ and $T^{\perp}\mathcal{N}$ of $\mathcal{N}$, respectively, governed by the Gauss and Weingarten formulas as follows:

$$\tilde{\nabla}_U V = \nabla_U V + \alpha(U, V), \tag{2.7}$$

$$\tilde{\nabla}_U \zeta = -A_\zeta U + \nabla^{\perp}_U \zeta, \tag{2.8}$$

for any $U, V \in \Omega(T\mathcal{N})$ and $\zeta \in \Omega(T^{\perp}\mathcal{N})$, where α and A_ζ are the second fundamental form of $\mathcal{N}$ and the shape operator corresponding to ζ, respectively. Their relationship is based on

$$g(\alpha(U, V), \zeta) = g(A_\zeta U, V). \tag{2.9}$$

For the second fundamental form α, the covariant derivative $\overline{\nabla}\alpha$ is given by

$$(\overline{\nabla}_U \alpha)(V, W) = \nabla^{\perp}_U \alpha(V, W) - \alpha(\nabla_U V, W) - \alpha(V, \nabla_U W). \tag{2.10}$$

The curvature tensors of the connections ∇ and $\nabla^{\perp}$ on $\mathcal{N}$ are denoted by R and $R^{\perp}$. Then, the Gauss, Ricci, and Codazzi equations are provided, respectively, by Chen in [8] as follows:

$$\begin{aligned}\tilde{R}(U, V; W, X) = R(U, V; W, X) &+ g(\alpha(U, W), \alpha(V, X)) \\ &- g(\alpha(U, X), \alpha(V, W)),\end{aligned} \tag{2.11}$$

$$\tilde{R}(U, V; \zeta, \omega) = R^{\perp}(U, V; \zeta, \omega) - g([A_\zeta, A_\omega]U, V), \tag{2.12}$$

and

$$(\tilde{R}(U, V)W)^{\perp} = (\overline{\nabla}_U \alpha)(V, W) - (\overline{\nabla}_V \alpha)(U, W), \tag{2.13}$$

for any $U, V, W, X \in \Omega(T\mathcal{N})$ and $\zeta, \omega \in \Omega(T^{\perp}\mathcal{N})$, where $(\tilde{R}(U, V)W)^{\perp}$ denotes the normal component of $\tilde{R}(U, V)W$.

Let us now decompose ψU for any tangent vector $U \in \Omega(T\mathcal{N})$ into tangent and normal parts as follows:

$$\psi U = TU + NU. \tag{2.14}$$

In a similar vein, we decompose $\psi\zeta$ for any normal vector $\zeta \in \Omega(T^{\perp}\mathcal{N})$ into tangent and normal parts as follows:

$$\psi\zeta = t\zeta + n\zeta. \tag{2.15}$$

The covariant derivatives of tensor fields in (2.14) are characterized by:

$$(\nabla_U T)V = \nabla_U TV - T(\nabla_U V), \tag{2.16}$$

$$(\nabla_U N)V = \nabla^{\perp}_U NV - N(\nabla_U V). \tag{2.17}$$

Immediately, from (2.3), (2.4), (2.7) and (2.8), we obtain

$$(\nabla_U T)V = A_{NV}U + t\alpha(U, V) + g(TU, V)\xi - \eta(V)TU, \tag{2.18}$$

$$(\nabla_U N)V = n\alpha(U, V) - \alpha(U, TV) - \eta(V)NU, \tag{2.19}$$

for any $U, V \in \Omega(T\mathcal{N})$.

3. Pointwise slant submanifolds of an almost contact metric manifold

We revisit certain findings regarding pointwise slant submanifolds of an almost contact metric manifold $\tilde{\mathcal{N}}$ in this section.

For a submanifold $\mathcal{N}$ of an almost contact metric manifold $\tilde{\mathcal{N}}$, the angle $\theta(U)$ between ψU and $T_p\mathcal{N}$ for a non-zero vector $U \in T_p\mathcal{N}$ and for each point $p \in \mathcal{N}$ is known as the *Wirtinger angle*, and $\mathcal{N}$ is called a *pointwise θ-slant submanifold* of $\tilde{\mathcal{N}}$ if $\theta(U)$ is independent of the selection of $U \in T_p\mathcal{N}$. In this instance, $\theta(U)$ is called the *slant function* of the pointwise θ-slant submanifold $\mathcal{N}$. If θ is globally constant, a pointwise θ-slant submanifold $\mathcal{N}$ of $\tilde{\mathcal{N}}$ is referred to as *slant*. It is also referred to as an *invariant* (resp., *anti-invariant*) if $\theta = 0$ (resp., $\theta = \frac{\pi}{2}$), and it is called a *proper pointwise slant* whenever $\theta \neq 0, \frac{\pi}{2}$ and θ are not constant on $\mathcal{N}$ ([9, 10]).

Now, we provide non-trivial examples of pointwise slant submanifolds of an almost contact metric manifold.

Example 3.1. Let $\mathbb{R}^7$ be the Euclidean 7-space with the usual cartesian coordinates (x_i, y_j, z), $1 \leq i, j \leq 3$. We define the structure (ψ, ξ, η, g) on $\mathbb{R}^7$ as follows:

$$\psi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i},\ \ \psi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j},\ \ \psi\Big(\frac{\partial}{\partial z}\Big) = 0,$$

with $\xi = \dfrac{\partial}{\partial z}$, $\eta = dz$ and the usual Euclidean metric tensor $g = \sum_{i,j=1}^{3}(dx_i^2 + dy_j^2) + dz^2$ on $\mathbb{R}^7$. For any vector field $U = \lambda_i \dfrac{\partial}{\partial x_i} + \mu_j \dfrac{\partial}{\partial y_j} + \nu \dfrac{\partial}{\partial z}$ in $T\,\mathbb{R}^7$, we have

$$\eta(U) = dz(\lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z}) = \nu = g(U, \xi) = g(\lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z}, \frac{\partial}{\partial z}),$$

$$\psi U = -\lambda_i \frac{\partial}{\partial y_i} + \mu_j \frac{\partial}{\partial x_j},\ \ \eta(\psi U) = 0,$$

$$\psi U^2 = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -U + \nu \frac{\partial}{\partial z} = -U + \eta(U)\xi,$$

$$g(U,U) = g(\lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z}, \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z}) = \lambda_i^2 + \mu_j^2 + \nu^2,$$

and

$$g(\psi U, \psi U) = g(-\lambda_i \frac{\partial}{\partial y_i} + \mu_j \frac{\partial}{\partial x_j}, -\lambda_i \frac{\partial}{\partial y_i} + \mu_j \frac{\partial}{\partial x_j}) = \lambda_i^2 + \mu_j^2.$$

Thus,

$$g(\psi U, \psi U) = g(U,U) - \eta^2(U).$$

Hence, the defined structure (ψ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$.

Consider a submanifold $\mathcal{N}$ of $\mathbb{R}^7$ given by the following immersion:

$$f(p, q, z) = (p\cos q,\ q\cos p,\ \frac{p^2+q^2}{2},\ p\sin q,\ q\sin p,\ \frac{p^2-q^2}{2},\ z),$$

for any p, q non vanishing real valued functions. Thus, the tangent space of $\mathcal{N}$ is generated by the following vectors:

$$\begin{aligned}
V_1 &= \cos q \frac{\partial}{\partial x_1} - q\sin p \frac{\partial}{\partial x_2} + p \frac{\partial}{\partial x_3} + \sin q \frac{\partial}{\partial y_1} + q\cos p \frac{\partial}{\partial y_2} + p \frac{\partial}{\partial y_3},\\
V_2 &= -p\sin q \frac{\partial}{\partial x_1} + \cos p \frac{\partial}{\partial x_2} + q \frac{\partial}{\partial x_3} + p\cos q \frac{\partial}{\partial y_1} + \sin p \frac{\partial}{\partial y_2} - q \frac{\partial}{\partial y_3},\\
V_3 &= \frac{\partial}{\partial z}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\psi V_1 &= -\cos q \frac{\partial}{\partial y_1} + q\sin p \frac{\partial}{\partial y_2} - p \frac{\partial}{\partial y_3} + \sin q \frac{\partial}{\partial x_1} + q\cos p \frac{\partial}{\partial x_2} + p \frac{\partial}{\partial x_3},\\
\psi V_2 &= p\sin q \frac{\partial}{\partial y_1} - \cos p \frac{\partial}{\partial y_2} - q \frac{\partial}{\partial y_3} + p\cos q \frac{\partial}{\partial x_1} + \sin p \frac{\partial}{\partial x_2} - q \frac{\partial}{\partial x_3},\\
\psi V_3 &= 0.
\end{aligned}$$

By simple calculation, we infer that $\mathcal{N}$ is a 3-dimensional proper pointwise slant submanifold of $\mathbb{R}^7$ such that the vector field ξ is tangent to $\mathcal{N}$ with slant function $\theta = \cos^{-1}\left(\frac{p-q(1+2p)}{\sqrt{2p^2+q^2+1}\sqrt{2q^2+p^2+1}}\right)$, as $p, q\, (p \neq q)$ are non-vanishing real valued functions on $\mathcal{N}$.

Example 3.2. Let $\mathbb{R}^{11}$ be the Euclidean 11-space with the usual cartesian coordinates (x_i, y_j, z), $1 \leq i, j \leq 5$ and the same almost contact metric structure (ψ, ξ, η, g) as mentioned in the previous example. Given a submanifold $\mathcal{N}$ of $\mathbb{R}^{11}$, which is defined by the following immersion:

$$f(p, q, z) = (p+q,\ -p,\ e^q,\ \sin p,\ \cos q,\ p-q,\ q,\ e^p,\ \cos p,\ \sin q,\ z),$$

where p, q are non-vanishing real-valued functions. The tangent space of $\mathcal{N}$ is spanned by the following vectors:

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \cos p \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_1} + e^p \frac{\partial}{\partial y_3} - \sin p \frac{\partial}{\partial y_4}, \\
V_2 &= \frac{\partial}{\partial x_1} + e^q \frac{\partial}{\partial x_3} - \sin q \frac{\partial}{\partial x_5} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \cos q \frac{\partial}{\partial y_5}, \\
V_3 &= \frac{\partial}{\partial z}.
\end{aligned}$$

Then, the slant function is provided by $\theta = \cos^{-1}\left(\frac{3+e^{p+q}}{\sqrt{4+e^{2p}}\sqrt{4+e^{2q}}}\right)$. Hence, $\mathcal{N}$ is a 3-dimensional proper pointwise slant submanifold of $\mathbb{R}^{11}$.

For a pointwise θ-slant submanifold $\mathcal{N}$ of an almost contact metric manifold $\tilde{\mathcal{N}}$, we remember the following significant results from [27].

Theorem 3.1. *Let $\mathcal{N}$ be a submanifold of an almost contact metric manifold $\tilde{\mathcal{N}}$, such that the vector field ξ is tangent to $\mathcal{N}$. Then, $\mathcal{N}$ is a pointwise θ-slant submanifold of $\tilde{\mathcal{N}}$ if and only if*

$$T^2 U = -\cos^2\theta(U - \eta(U)\xi), \tag{3.1}$$

for any $U \in \Omega(T\mathcal{N})$, where θ is the slant function on $\mathcal{N}$.

Corollary 3.1. *Let $\mathcal{N}$ be a pointwise θ-slant submanifold of $\tilde{\mathcal{N}}$ with ξ tangent to $\mathcal{N}$. Then, we have*

$$g(TU, TV) = \cos^2\theta(g(U, V) - \eta(U)\eta(V)), \tag{3.2}$$

$$g(NU, NV) = \sin^2\theta(g(U, V) - \eta(U)\eta(V)), \tag{3.3}$$

for any $U, V \in \Omega(T\mathcal{N})$.

In addition, there is another relation achieved in the pointwise θ-slant submanifold $\mathcal{N}$ of $\tilde{\mathcal{N}}$ given by:

$$(a)\quad tNU = -\sin^2\theta(U - \eta(U)\xi) \qquad (b)\quad nNU = -NTU, \tag{3.4}$$

for any $U \in \Omega(T\mathcal{N})$.

From now on, suppose that $\mathcal{N}$ is a pointwise θ-slant submanifold in Kenmotsu space of form $\tilde{\mathcal{N}}^{2r+1}(k)$ with $\xi \in \Omega(T\mathcal{N})$. Therefore, we can obtain the orthogonal direct decomposition $T\mathcal{N} = \mathcal{D} \oplus \langle\xi\rangle$ if we denote the orthogonal distribution to ξ in $T\mathcal{N}$ by $\mathcal{D}$.

For each $U \in \Omega(T\mathcal{N})$, for simplicity, we set

$$U^* = (\csc\theta)NU, \tag{3.5}$$

where $\theta \neq 0$ be a slant function. Let σ be a symmetric bilinear $T\mathcal{N}$-valued form on $\mathcal{N}$ defined as follows:

$$\sigma(U, V) = t\alpha(U, V), \tag{3.6}$$

for any $U, V \in \Omega(T\mathcal{N})$. In particular, using (2.4) and (2.7), the above expression reduces to

$$\sigma(U, \xi) = 0. \tag{3.7}$$

Then, from (2.14) and (3.5) in (3.6), we find

$$\psi\sigma(U, V) = T\sigma(U, V) + (\sin\theta)\sigma^*(U, V). \tag{3.8}$$

In view of (2.15) and (3.6), we receive that

$$\psi\alpha(U, V) = \sigma(U, V) + \delta^*(U, V),$$

where δ is a symmetric bilinear $\mathcal{D}$-valued form on $\mathcal{N}$, and it is defined as $\delta^*(U, V) = n\alpha(U, V)$. Operating the almost contact structure ψ on the above equation with (2.1), (2.15) and (3.8), we observe that

$$-\alpha(U, V) = T\sigma(U, V) + (\sin\theta)\sigma^*(U, V) + t\delta^*(U, V) + n\delta^*(U, V).$$

Comparing the tangential and the normal parts in the above expression, we obtain

$$T\sigma(U, V) = -t\delta^*(U, V),$$

and

$$\alpha(U, V) = -(\sin\theta)\sigma^*(U, V) - n\delta^*(U, V).$$

Making use of (3.1), (3.4) and (3.5), we find

$$\delta(U, V) = (\csc\theta)T\sigma(U, V),$$

and

$$\alpha(U, V) = -(\csc\theta)\sigma^*(U, V), \tag{3.9}$$

In view of (2.14) and (3.5), the above relation becomes

$$\alpha(U, V) = (\csc^2\theta)(T\sigma(U, V) - \psi\sigma(U, V)). \tag{3.10}$$

On the other hand, taking the inner product of (2.18) with $W \in \Omega(T\mathcal{N})$ and using (2.2), (2.9) and (3.6), we obtain

$$\begin{aligned} g((\nabla_U T)V, W) = {} & g(\sigma(U, W), NV) + g(\sigma(U, V), W) \\ & + \eta(W)g(TU, V) - \eta(V)g(TU, W). \end{aligned}$$

Then, from (2.14) and (2.15), we get

$$\begin{aligned} g((\nabla_U T)V, W) = {} & g(\sigma(U, V), W) - g(\sigma(U, W), V) \\ & + \eta(W)g(TU, V) - \eta(V)g(TU, W), \end{aligned}$$

for any $U, V, W \in \Omega(T\mathcal{N})$. Now, we want to derive the equation of Gauss and Codazzi for a $(s+1)$ dimensional pointwise θ-slant submanifold $\mathcal{N}$ in Kenmotsu space form $\tilde{\mathcal{N}}^{2r+1}(k)$. First, by (2.6), (2.11) and (2.14), we find

$$\begin{aligned}
&R(U,V;W,X) - g(\alpha(U,X),\alpha(V,W)) + g(\alpha(U,W),\alpha(V,X))\\
&= \frac{k-3}{4}\Big\{g(U,X)g(V,W) - g(U,W)g(V,X)\Big\} + \frac{k+1}{4}\Big\{\eta(U)\eta(W)g(V,X)\\
&- \eta(V)\eta(W)g(U,X) + \eta(V)\eta(X)g(U,W) - \eta(U)\eta(X)g(V,W)\\
&+ g(TU,X)g(TV,W) - g(TU,W)g(TV,X) + 2g(U,TV)g(TW,X)\Big\}.
\end{aligned}$$

But, from (3.3) and (3.9), we conclude that

$$\begin{aligned}
R(U,V;W,X) = {} & \csc^2\theta\Big\{g(\sigma(U,X),\sigma(V,W)) - g(\sigma(U,W),\sigma(V,X))\Big\}\\
&+ \frac{k-3}{4}\Big\{g(U,X)g(V,W) - g(U,W)g(V,X)\Big\}\\
&+ \frac{k+1}{4}\Big\{\eta(U)\eta(W)g(V,X) - \eta(V)\eta(W)g(U,X)\\
&+ \eta(V)\eta(X)g(U,W) - \eta(U)\eta(X)g(V,W)\\
&+ g(TU,X)g(TV,W) - g(TU,W)g(TV,X)\\
&+ 2g(U,TV)g(TW,X)\Big\},
\end{aligned}$$

for any $U, V, W, X \in \Omega(T\mathcal{N})$, which is the Gauss equation of $\mathcal{N}$ in $\tilde{\mathcal{N}}^{2r+1}(k)$.

Second, for the Codazzi equation, we take the normal parts of (2.6), and we get

$$\begin{aligned}
\Big(\tilde{R}(U,V)W\Big)^{\perp} = {} & \frac{k+1}{4}\Big\{g(TV,W)NU - g(TU,W)NV\\
&+ 2g(U,TV)NW\Big\}.
\end{aligned} \tag{3.11}$$

Moreover, from (3.5) and (3.9), we have

$$\nabla^{\perp}_{U}\Big(\alpha(V,W)\Big) = -(\csc^2\theta)\nabla^{\perp}_{U}N\sigma(Y,Z) + 2(\csc^2\theta\cot\theta)U(\theta)F\sigma(V,W),$$

which implies that

$$\begin{aligned}
\nabla^{\perp}_{U}\Big(\alpha(V,W)\Big) = {} & (\csc^2\theta)\Big[-n\alpha(U,\sigma(V,W)) + \alpha(X,T\sigma(Y,Z))\\
&- N((\nabla_U\sigma)(V,W)) + 2(\cot\theta)U(\theta)N\sigma(V,W)\Big],
\end{aligned} \tag{3.12}$$

by using (2.17) and (2.19). Also, using (3.5) and (3.9), we derive

$$\alpha(\nabla_U V, W) = -(\csc^2\theta)N\sigma(\nabla_U V, W), \tag{3.13}$$

and

$$\alpha(V, \nabla_U W) = -(\csc^2\theta)N\sigma(V, \nabla_U W). \tag{3.14}$$

Applying the Eqs (3.12)–(3.14) to (2.10), we find

$$(\overline{\nabla}_U\alpha)(V,W) = (\csc^2\theta)\Big[-n\alpha(U,\sigma(V,W)) + \alpha(U,T\sigma(V,W)) - N((\nabla_U\sigma)(V,W)) + 2(\cot\theta)U(\theta)N\sigma(V,W)\Big].$$

But, by (3.4), (3.5) and (3.9), the previous expression takes the form

$$(\overline{\nabla}_U\alpha)(V,W) = -(\csc^2\theta)\Big[(\csc^2\theta)NT\sigma(U,\sigma(V,W)) + (\csc^2\theta)N\sigma(U,T\sigma(V,W)) + N((\nabla_U\sigma)(V,W)) - 2(\cot\theta)U(\theta)N\sigma(V,W)\Big]. \tag{3.15}$$

Replacing U by V in the above relation, we derive

$$(\overline{\nabla}_V\alpha)(U,W) = -(\csc^2\theta)\Big[(\csc^2\theta)NT\sigma(V,\sigma(U,W)) + (\csc^2\theta)N\sigma(V,T\sigma(U,W)) + N((\nabla_V\sigma)(U,W)) - 2(\cot\theta)V(\theta)N\sigma(U,W)\Big]. \tag{3.16}$$

Hence, substituting the Eqs (3.11), (3.15) and (3.16) in the Codazzi equation, we arrive at

$$\begin{aligned}&(\nabla_U\sigma)(V,W) - 2(\cot\theta)U(\theta)\sigma(V,W)\\ &\quad + (\csc^2\theta)\{T\sigma(U,\sigma(V,W)) + \sigma(U,T\sigma(V,W))\}\\ &\quad + \frac{k+1}{4}(\sin^2\theta)\{g(U,TV)(W-\eta(W)\xi) + g(U,TW)(V-\eta(V)\xi))\}\\ &= (\nabla_V\sigma)(U,W) - 2(\cot\theta)V(\theta)\sigma(U,W)\\ &\quad + (\csc^2\theta)\{T\sigma(Y,\sigma(U,W)) + \sigma(Y,T\sigma(U,W))\}\\ &\quad + \frac{k+1}{4}(\sin^2\theta)\{ g(V,TU)(W-\eta(W)\xi) + g(V,TW)(U-\eta(U)\xi)\},\end{aligned}$$

for any $U,V,W \in \Omega(T\mathcal{N})$.

The previous equations lead to the following existence and uniqueness theorems for pointwise θ-slant immersion into a Kenmotsu space form.

4. Existence and uniqueness theorems

Theorem 4.1. (Existence) *Let $\mathcal{N}$ be a $(s+1)$-dimensional Riemannian manifold that is simply connected and has a metric tensor g attached to it. Assuming k to be constant, let us consider an endomorphism T of the tangent bundle $T\mathcal{N}^{s+1}$, a unit global vector field ξ, a dual 1-form η of ξ, a symmetric bilinear $T\mathcal{N}^{s+1}$-valued form σ on $\mathcal{N}^{s+1}$, and a differential real valued function θ defined on $\mathcal{N}^{s+1}$, where $0 < \theta \leq \frac{\pi}{2}$, such that the following relationships hold:*

$$T^2U = -(\cos^2\theta)(U - \eta(U)\xi), \tag{4.1}$$

$$g(TU,V) = -g(U,TY), \tag{4.2}$$

$$T(\xi) = 0, \quad g(\sigma(U,V),\xi) = 0, \quad \nabla_U \xi = U - \eta(X)\xi, \tag{4.3}$$

$$\sigma(U,\xi) = 0, \tag{4.4}$$

$$\begin{aligned} g((\nabla_U T)Y, Z) &= g(\sigma(U,V),W) - g(\sigma(U,W),V) \\ &\quad + \eta(W)g(TU,V) + \eta(V)g(U,TW), \end{aligned} \tag{4.5}$$

$$\begin{aligned} R(U,V;W,X) &= \csc^2\theta\Big\{g(\sigma(U,X),\sigma(V,W)) - g(\sigma(U,W),\sigma(V,X))\Big\} \\ &\quad + \frac{k-3}{4}\Big\{g(U,X)g(V,W) - g(U,W)g(V,X)\Big\} \\ &\quad + \frac{k+1}{4}\Big\{\eta(U)\eta(W)g(V,X) - \eta(V)\eta(W)g(U,X) \\ &\quad + \eta(V)\eta(X)g(U,W) - \eta(U)\eta(X)g(V,W) \\ &\quad + g(TU,X)g(TV,W) - g(TU,W)g(TV,X) \\ &\quad + 2g(U,TV)g(TW,X)\Big\}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &(\nabla_U\sigma)(V,W) - 2(\cot\theta)U(\theta)\sigma(V,W) \\ &\qquad + (\csc^2\theta)\{T\sigma(U,\sigma(V,W)) + \sigma(U,T\sigma(V,W))\} \\ &\qquad + \frac{k+1}{4}(\sin^2\theta)\{g(U,TV)(W-\eta(W)\xi) + g(U,TW)(V-\eta(V)\xi))\} \\ &= (\nabla_V\sigma)(U,W) - 2(\cot\theta)V(\theta)\sigma(U,W) \\ &\qquad + (\csc^2\theta)\{T\sigma(Y,\sigma(U,W)) + \sigma(Y,T\sigma(U,W))\} \\ &\qquad + \frac{k+1}{4}(\sin^2\theta)\{\ g(V,TU)(W-\eta(W)\xi) + g(V,TW)(U-\eta(U)\xi)\}, \end{aligned} \tag{4.7}$$

with every $U,V,W,X \in \Omega(T\mathcal{N}^{s+1})$. Then there exists a pointwise θ-slant isometric immersion of $\mathcal{N}^{s+1}$ into a Kenmotsu space form $\tilde{\mathcal{N}}^{2r+1}(k)$, and the second fundamental form α of $\mathcal{N}^{s+1}$ is given by the relation

$$\alpha(U,V) = (\csc^2\theta)(T\sigma(U,V) - \psi\sigma(U,V)). \tag{4.8}$$

Proof. We assume that $\mathcal{N}^{s+1}$, k, T, ξ, η, σ and θ verify the relations mentioned above. Let us assume a Whitney sum of $T\mathcal{N}^{s+1} \oplus \mathcal{D}$. For each $U \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$ we identify $(U,0)$ by U, $(0,W)$ by W^*, and $\acute{\xi} = (\xi,0)$ with ξ.

Represent the product metric on $T\mathcal{N}^{s+1} \oplus \mathcal{D}$ by $\acute{g}$. Thus, if we put $\acute{\eta}$ as the dual 1-form of $\acute{\xi}$, then $\acute{\eta}(U,W) = \eta(U)$, for any $U \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$.

The endomorphism $\acute{\psi}$ on $T\mathcal{N}^{s+1} \oplus \mathcal{D}$ is defined by

$$\acute{\psi}(U,0) = (TU, (\sin\theta)(U - \eta(U)\xi)), \quad \acute{\psi}(0,W) = (-(\sin\theta)W, -TW), \tag{4.9}$$

for any $U \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$. Therefore, it is immediately to clear that $\acute{\psi}^2(U,0) = -(U,0) + \acute{\eta}(U,0)\acute{\xi}$ and $\acute{\psi}^2(0,W) = -(0,W)$, which gives $\acute{\psi}^2(U,W) = -(U,W) + \acute{\eta}(U,W)\acute{\xi}$ for any $U \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$. So, (4.1), (4.2) and (4.9) imply that $(\acute{\phi}, \acute{\eta}, \acute{\xi}, \acute{g})$ is an almost contact metric structure on $T\mathcal{N}^{s+1} \oplus \mathcal{D}$.

Now, we can define a $(\mathcal{D})^*$-valued symmetric bilinear form α on $T\mathcal{N}^{s+1}$, an endomorphism A on $T\mathcal{N}^{s+1}$, and a metric connection $\nabla^\perp$ of the vector bundle $(\mathcal{D})^*$ over $\mathcal{N}^{s+1}$ by the following relations:

$$\alpha(U,V) = -(\csc\theta)\sigma^*(U,V), \tag{4.10}$$

$$A_{W^*}U = (\csc\theta)\{(\nabla_U T)W - \sigma(U,W) - g(TU,W)\xi\}, \tag{4.11}$$

$$\begin{aligned}\nabla^\perp_U W^* &= (\nabla_U W - \eta(\nabla_U W)\xi)^* - (\cot\theta)U(\theta)W^* \\ &\quad + (\csc^2\theta)\{(T\sigma(U,W))^* + \sigma^*(U,TW)\},\end{aligned} \tag{4.12}$$

for $U, V \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$.

Let $\acute{\nabla}$ be the canonical connection on $T\mathcal{N}^{s+1} \oplus \mathcal{D}$ as inferred from Eqs (4.9)–(4.12). After that, by using (4.1), (4.3), (4.4) and (4.9), we derive

$$\begin{aligned}&(\acute{\nabla}_{(U,0)}\acute{\psi})(V,0) = \acute{g}(\acute{\psi}(U,0),(V,0))\acute{\xi} - \acute{\eta}(V,0)\acute{\psi}(U,0),\\ &(\acute{\nabla}_{(U,0)}\acute{\psi})(0,W) = 0,\end{aligned}$$

for any $U, V \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$.

Let $R^\perp$ be the curvature tensor correlated with the connection $\nabla^\perp$ on $(\mathcal{D})^*$ given by

$$R^\perp(U,V)Z^* = \nabla^\perp_U\nabla^\perp_V W^* - \nabla^\perp_V\nabla^\perp_U W^* - \nabla^\perp_{[U,V]}W^*,$$

for every $U, V \in \Omega(T\mathcal{N}^{s+1})$ and $W \in \Omega(\mathcal{D})$.

Hence, by using (2.16), (4.2), (4.3), (4.7) and (4.12) with direct arithmetic, we obtain

$$\begin{aligned}R^\perp(U,V)W^* &= (\csc^2\theta)[V(\theta) - U(\theta)]W^* + \Big\{R(U,V)W - \eta(R(U,V)W)\xi\Big\}^* \\ &\quad + \frac{k+1}{4}\Big\{T[g(V,TW)U - g(U,TW)V - 2g(U,TV)W] \\ &\quad + [g(V,T^2W)(U - \eta(U)\xi) - g(U,T^2W)(V - \eta(V)\xi) \\ &\quad - 2g(U,TV)TW]\Big\}^* \\ &\quad + \csc^2\theta\Big\{(\nabla_U T)\sigma(V,W) - (\nabla_V T)\sigma(U,W) - \eta(\nabla_U(T\sigma(V,W)))\xi \\ &\quad + \eta(\nabla_V(T\sigma(U,W)))\xi - \sigma(U,(\nabla_V T)W) + \sigma(V,\nabla_U T)W) \\ &\quad - \eta(\nabla_U(\sigma(V,TW)))\xi + \eta(\nabla_V(\sigma(U,TW)))\xi\Big\}^* \\ &\quad + \Big\{\eta(U)\eta(\nabla_V W)\xi - \eta(V)\eta(\nabla_U W)\xi - \eta(\nabla_V W)U + \eta(\nabla_U W)V\Big\}^*.\end{aligned} \tag{4.13}$$

Furthermore, (4.3), (4.5), (4.10) and (4.11) imply

$$g([A_{W^*}, A_{X^*}]U, V) = \csc^2\theta\Big\{g((\nabla_U T)X, (\nabla_V T)W) - g((\nabla_U T)W, (\nabla_V T)X)$$

$$
\begin{aligned}
&+ g((\nabla_U T)W, \sigma(V, X)) + g((\nabla_V T)X, \sigma(U, W)) \\
&- g((\nabla_U T)X, \sigma(V, W)) - g((\nabla_V T)W, \sigma(U, X)) \qquad (4.14)\\
&+ g(\sigma(U, X), \sigma(V, W)) - g(\sigma(U, W), \sigma(V, X)) \\
&+ g(TU, Z)g(TV, W) - g(TU, W)g(TV, X)) \\
&- \eta((\nabla_V T)W)g(TU, X) - \eta((\nabla_U T)X)g(TV, W) \\
&+ \eta((\nabla_V T)X)g(TU, W) + \eta((\nabla_U T)W)g(TV, X)\Big\}.
\end{aligned}
$$

By (4.2), we get

$$g(\sigma(V, W), TX) + g(T\sigma(V, W), X) = 0.$$

For any $U \in \Omega(T\mathcal{N}^{s+1})$, if we take the covariant derivative of the above expression with respect to U and use (4.2), we obtain

$$g(\sigma(V, W), (\nabla_U T)X) + g((\nabla_U T)\sigma(V, W), X) = 0.$$

Additionally, from (4.5), we observe that

$$
\begin{aligned}
g((\nabla_U T)W, (\nabla_V T)X) &= g((\nabla_U T)W, \sigma(V, X)) - g(\sigma(V, (\nabla_U T)W), X) \\
&\quad + \eta((\nabla_U T)W)g(TV, X) + \eta(X)g(V, T((\nabla_U T)W)).
\end{aligned}
$$

Thus, using a straightforward computation and the above relations in (4.13) and (4.14), we obtain that

$$
\begin{aligned}
&g(R^{\perp}(U, V)W^*, X^*) - g([A_{W^*}, A_{X^*}]U, V) \\
&\quad = \frac{k+1}{4}\Big[(\sin^2\theta)\{g(U, X)g(V, W) - g(U, W)g(V, X)\} - 2g(U, TV)g(TW, X)\Big] \\
&\quad + (\csc^2\theta)\big[V(\theta) - U(\theta)\big]\, g(W, X),
\end{aligned}
$$

for any $U, V, W, X \in \Omega(T\mathcal{N}^{s+1})$. We note that $(\mathcal{N}^{s+1}, A, \nabla^{\perp})$ satisfies the equation of Ricci for a pointwise θ-slant submanifold $\mathcal{N}^{s+1}$ of dimension $(s+1)$ in the Kenmotsu space form $\tilde{\mathcal{N}}^{2r+1}(k)$ based on the equation above with (2.6), (4.1) and (4.2). $(\mathcal{N}^{s+1}, \alpha)$ satisfies the equations of Gauss and Codazzi, respectively, for a pointwise θ-slant submanifold $\mathcal{N}^{s+1}$ of $\tilde{\mathcal{N}}^{2r+1}(k)$, according to (4.6) and (4.7). Therefore, we have a vector bundle $T\mathcal{N}^{s+1} \oplus \mathcal{D}$ over $\mathcal{N}^{s+1}$ equipped with the product metric $\acute{g}$, the second fundamental form α, the shape operator A, and the connections $\nabla^{\perp}$ and $\acute{\nabla}$ satisfy the structure equations of a pointwise θ-slant submanifold $\mathcal{N}^{s+1}$ of $\tilde{\mathcal{N}}^{2r+1}(k)$. Thus, we find that there is a pintwise θ-slant isometric immersion from $\mathcal{N}^{s+1}$ into $\tilde{\mathcal{N}}^{2r+1}(k)$, whose second fundamental form α is given by the relation (4.8) by applying Theorem 1 of [16].

□

The necessary conditions to look into the pointwise θ-slant immersion uniqueness property are provided by the following result:

Theorem 4.2. (Uniqueness Theorem) *Consider the two pointwise θ-slant isometric immersions x^1, x^2 : $\mathcal{N}^{s+1} \to \mathcal{N}^{2r+1}(k)$ from the connected Riemannian manifold $\mathcal{N}^{s+1}$ to a Kenmotsu space form $\tilde{\mathcal{N}}^{2r+1}(k)$ with the slant function θ $(0 < \theta \leq \frac{\pi}{2})$. Let α_1 and α_2 represent the second fundamental forms of x^1 and*

x^2, respectively. Let us assume that there is a vector field $\acute{\xi}$ on $\mathcal{N}^{s+1}$ such that $x^i_{*p}(\acute{\xi}_p) = \xi_{x^i(p)}$, for any point $p \in \mathcal{N}$ and $i = 1, 2$. Allow us to

$$g(\alpha_1(U, V), \psi x^1_* W) = g(\alpha_2(U, V), \psi x^2_* W), \tag{4.15}$$

with every $U, V, W \in \Omega(T\mathcal{N}^{s+1})$. Additionally, we take it for granted that at least one of the subsequent prerequisites is met:

(i) $\theta = \frac{\pi}{2}$.
(ii) There exists a point p in $\mathcal{N}$ such that $T_1 = T_2$.
(iii) $k \neq -1$.

Then $T_1 = T_2$, and there exists an isometry γ of $\tilde{\mathcal{N}}^{2r+1}(k)$ such that $x^1 = \gamma(x^2)$.

Proof. By taking $\acute{\xi}$ in the orthonormal frame tangent to $\mathcal{N}$, the proof of this theorem is similar to the uniqueness theorem in complex space forms (see [1, 14]). □

5. Conclusions

In this paper, we established the theorems of existence and uniqueness for pointwise slant immersions in Kenmotsu space forms. Firstly, we reviewed the definition of pointwise slant submanifold of an almost contact-metric manifold and we provided non-trivial examples of such submanifold. Then, we proved the Gauss and Codazzi equations of the pointwise slant submanifold in Kenmotsu space form which leads to prove the existence and uniqueness theorems.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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