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Research article

Associative memories based on delayed fractional-order neural networks and application to explaining-lesson skills assessment of normal students: from the perspective of multiple $O(t^{-\alpha})$ stability

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Abstract: This paper discusses associative memories based on time-varying delayed fractional-order neural networks (DFNNs) with a type of piecewise nonlinear activation function from the perspective of multiple $O(t^{-\alpha})$ stability. Some sufficient conditions are gained to assure the existence of 5^n equilibria for n-neuron DFNNs with the proposed piecewise nonlinear activation functions. Additionally, the criteria ensure the existence of at least 3^n equilibria that are locally multiple $O(t^{-\alpha})$ stable. Furthermore, we apply these results to a more generic situation, revealing that DFNNs can attain $(2k + 1)^n$ equilibria, and among them, $(k + 1)^n$ equilibria are locally $O(t^{-\alpha})$ stable. Here, the parameter k is highly dependent on the sinusoidal function frequency in the expanded activation functions. Such DFNNs are well-suited to synthesize high-capacity associative memories; the design process is given via singular value decomposition. Ultimately, four illustrative examples, including applying neurodynamic associative memory to the explaining-lesson skills assessment of normal students, are supplied to validate the efficacy of the results.

Keywords: associative memory; fractional-order neural networks; multiple $O(t^{-\alpha})$ stability; piecewise nonlinear activation functions; time-varying delays
Mathematics Subject Classification: 34A08

1. Introduction

Associative memory is a content-addressing process similar to the brain that is aimed at storing a set of patterns in a manner resembling stable equilibrium points. This enables reliable retrieval of stored patterns with an initial probe containing adequate pattern-related information. In neural networks, associative memory can not only learn and store the correlations between different input patterns but also be used for prediction and the generation of sequence data. In addition, associative memory can be applied to categorize input patterns or recognize new patterns that are similar to the learned patterns, which is meaningful for tasks such as image recognition, speech recognition, and text categorization. In tackling associative memory, the multistable property is one of the key issues. By increasing their storage capacity, neural networks can store and retrieve large amounts of information more efficiently. At present, many studies have been carried out on neurodynamic associative memory [1–3]. Overall, the essence of neural network-based associative memory is to transform any input vector set into an output vector set related to patterns through nonlinear mapping. As a result, an inevitable requirement is that the corresponding network model possess multiple locally stable equilibria [4, 5].

Recently, the analysis and design of fractional-order neural networks (FNNs) have been extensively implemented in many domains, including physics, engineering [6, 7], control systems [8, 9], and biological sciences. Fractional calculus, serving as its theoretical foundation, enables neural networks to better adapt to and address practical issues in these diverse domains. The emergence of FNNs signifies an impressive innovation in the field of neural networks. Compared to integer-order neural networks, FNNs offer advantages in enhancing degrees of freedom and providing better descriptions for processes with memory or hereditary features. Moreover, researchers can better comprehend the power-law phenomenon in fractional-order systems. This phenomenon may be misinterpreted in integer-order systems. But within the framework of fractional-order systems, it can more correctly account for the power-law decay of state variables, providing a more refined modeling of the actual observational result. Therefore, the exploration of the dynamic behavior of FNNs is amusing and challenging. Over the past decade, lots of researchers have achieved important and intriguing results in FNNs (see, e.g., [10–15]). Chen and Chen [10] studied the global $O(t^{-\alpha})$ stability for time-varying delayed fractional-order neural networks (DFNNs).

In the application area of associative memory, multistability proves significantly superior to monostability in terms of providing a greater number of patterns. Additionally, a larger number of stable equilibria usually indicates higher storage capacity. As a result, the analysis of multistable systems has captured the interest of many scholars [16–19]. From the perspective of system integration, attaining asymptotic stability is a prerequisite for achieving decent performance in FNNs. $O(t^{-\alpha})$ stability refers to a specific asymptotic stability property of fractional-order systems, where α is the order of the fractional term. This stability characterizes the unusual evolution of fractional dynamical systems; here, the system trajectory can converge to a steady state at a rate of $t^{-\alpha}$. Consequently, it is crucial to analyze and comprehend the multiple $O(t^{-\alpha})$ stability of FNNs which can help to evaluate the performance of FNNs and guide the development of network design and control methods.

It is well known that the quantity of equilibria is intimately linked to the type of activation functions in the multistability analysis. Accordingly, designing an activation function with excellent performance is important and indispensable. In previous analyses of multistability, researchers observed that neural networks with certain non-monotonic activation functions [16] possess more equilibrium points. Actually, there have been several fruitful works in the area of multistability analysis of neural networks with diverse activation functions, which are mainly based on non-decreasing or piecewise linear assumptions. Besides, several recently published literatures have concentrated on smooth activation functions like Gaussian function [20, 21], sigmoidal function, Mexican hat function [22] and Morita-like function [23]. Nevertheless, it is vital to highlight that analyzing the dynamic behavior of smooth activation functions is more intricate than that of piecewise linear activation functions, owing to the heightened nonlinearity inherent in smooth activation functions. In [24], Liu et al. introduced a category of piecewise nonlinear activation function defined as follows:

$$f(r)=\begin{cases}-1, & -\infty<r<-\frac{\pi}{2},\\ (-1)^{k+1}\sin((2k-1)r), & -\frac{\pi}{2}\le r\le\frac{\pi}{2},\\ 1, & \frac{\pi}{2}<r<+\infty,\end{cases} \tag{1.1}$$

where $k>0$ is an integer. When $k=1$, the function (1.1) is a type of sigmoidal function. When $k=2$, the activation function (1.1) turns to

$$f(r)=\begin{cases}-1, & -\infty<r<-\frac{\pi}{2},\\ -\sin(3r), & -\frac{\pi}{2}\le r\le\frac{\pi}{2},\\ 1, & \frac{\pi}{2}<r<+\infty.\end{cases} \tag{1.2}$$

The conditions are obtained to guarantee that the investigative neural networks with piecewise nonlinear activation functions (1.2) have a total of 5^n equilibria. To the extent that the activation functions devised in [24] are almost always in integer-order systems, however, related situations in fractional-order systems have rarely been explored. Thus, it is of great interest to investigate the multistable properties of FNNs with activation functions (1.1) or (1.2).

It is worth pointing out that time delays, in particular time-varying delays, are prevalent in neural networks owing to the restricted propagation speed of signals as well as the finite switching speed of neuron amplifiers. Time delays complicate the dynamic behavior of neural networks and even cause instability or oscillations in originally stable networks. As such, it is essential and vital to look into the multistability of DFNNs. Currently, many scholars are researching the multistability issues of DFNNs, which has yielded some noteworthy achievements. In [25], Wan and Liu explored the multiple stability for time-varying DFNNs. The criteria are established to guarantee there are $\prod_{i=1}^{n}(2M_i+1)$ equilibria, and among them, $\prod_{i=1}^{n}(M_i+1)$ equilibria are locally $O(t^{-\alpha})$ stable. In [26], the authors introduced Gaussian activation functions to analyze the multiple stability of Cohen-Grossberg neural networks with delays. It is worth clarifying that there is very little literature exploring DFNNs and the activation function described in (1.2). This motivation aroused our interest in the multistability analysis of time-varying DFNNs with activation functions (1.1) or (1.2).

As indicated in the above analysis, this paper is dedicated to inquiring into associative memories from the perspective of multiple $O(t^{-\alpha})$ stability of DFNNs with piecewise nonlinear activation function (1.2). In general, the strengths of this paper can be generalized as follows: (1) Some sufficient criteria are deduced to ensure the existence of 5^n equilibria by means of Brouwer's fixed point theorem. (2) Several invariant sets are got, and the multiple $O(t^{-\alpha})$ stability for DFNNs with activation function (1.2) is disclosed. (3) This paper offers a handy and useful approach to enhance the quantity of stable equilibria for FNNs by elevating the value of k within the proposed sinusoidal function, which can be used for high-capacity associative memories. (4) The results of this paper are complementary to existing analyses of related associative memories.

Notations. Consider $C([t_0-\sigma,t_0],R^n)$ as the Banach space of continuous functions mapping $[t_0-\sigma,t_0]$ into $D\subset R^n$, where the norm is given by $\|x\|=\sqrt{\sum_{i=1}^{n}x_i^2}$. For $\phi\in C([t_0-\sigma,t_0],R^n)$, let $\|\phi\|_M=\sup_{t_0-\sigma\le s\le t_0}\|\phi(s)\|$.

2. Preliminaries

First of all, we show some definitions of fractional-order calculus.

Definition 2.1. [27] *For a function $F(t)$, its fractional integral $I_{t_0}^{\alpha}(\cdot)$ is defined as*

$$I_{t_0}^{\alpha}F(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F(s)\,\mathrm{d}s,$$

where $\alpha \in (0,1)$, $\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1}\exp(-u)\,\mathrm{d}u$ is the Gamma function.

Definition 2.2. [27] *For a differentiable function $F(t)$, its Caputo derivative of α order (when $0 < \alpha < 1$) is described as*

$${}^{C}D_{t_0}^{\alpha}F(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\left(\frac{\mathrm{d}F(s)}{\mathrm{d}s}\right)\mathrm{d}s.$$

Next, this paper will take into account a general class of time-varying DFNNs with an activation function (1.2) as follows:

$${}^{C}D_{t_0}^{\alpha}x_i(t) = -\beta_i x_i(t) + \sum_{j=1}^{n}\rho_{ij}f_j(x_j(t)) + \sum_{j=1}^{n}\varphi_{ij}f_j(x_j(t-\tau_{ij}(t))) + u_i, \quad i = 1,2,...,n, \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in R^n$ denotes the state vector, $A = \mathrm{diag}(\beta_1, \beta_2, ..., \beta_n)$ stands for the neuron self-inhibition matrix with $\beta_i > 0$, $B = (\rho_{ij})_{n\times n}$ and $C = (\varphi_{ij})_{n\times n}$ stand for connection weight matrices, $f(\cdot) = (f_1(\cdot), f_2(\cdot), ..., f_n(\cdot))^T$ is the activation function, and u_i is input. $\tau_{ij}(\cdot)$ is a time-varying delay that meets $0 \le \tau_{ij}(t) \le \sigma = \max_{1\le i,j\le n}\left\{\sup_{t\ge t_0}\tau_{ij}(t)\right\}$, where $\sigma > 0$ is a constant. The initial value of the neural network (2.1) is endowed with

$$x(t_0+s) = \phi(s), s \in [t_0-\sigma, t_0], \tag{2.2}$$

where $\phi(s) = (\phi_1(s), \phi_2(s), ..., \phi_n(s))^T \in C([t_0-\sigma, t_0], R^n)$.

In what follows, we introduce some definitions and lemmas that will be employed to investigate the multistability of DFNNs.

Definition 2.3. [28] *If a constant vector $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$ satisfies*

$$-\beta_i x_i^* + \sum_{j=1}^{n}\rho_{ij}f_j(x_j^*) + \sum_{j=1}^{n}\varphi_{ij}f_j(x_j^*) + u_i = 0, \quad i = 1,2,...,n,$$

then x^* can be called an equilibrium point of (2.1).

Definition 2.4. [14] *Suppose that $x^* \in \mathcal{D}$ is an equilibrium point of (2.1) as well as that each $\mathcal{D} \in \mathcal{R}^n$ is positively invariant. Then (2.1) is regarded as locally $O(t^{-\alpha})$ stable if*

$$\|x(t) - x^*\| \le \frac{\Lambda\varsigma^{\alpha}\|\phi - x^*\|_M}{(t-t_0+\varsigma)^{\alpha}},$$

where $\Lambda \ge 1, \varsigma \ge \sigma$ are constants, and $t \ge t_0$.

Lemma 2.1. [17] *Suppose that $\mathcal{W}(t)$ is differentiable on $[t_0, +\infty)$ and $\alpha \in (0, 1)$. Assume $\mathcal{W}(t) < 0$ ($\mathcal{W}(t) \le 0$) for $t_0 \le t < \bar{t}$ and $\mathcal{W}(\bar{t}) = 0$ hold, then*

$$ {}^{C}D^{\alpha}_{t_0}\mathcal{W}(t)|_{t=\bar{t}} > 0 \ \ ({}^{C}D^{\alpha}_{t_0}\mathcal{W}(t)|_{t=\bar{t}} \ge 0). $$

Lemma 2.2. [10] *For function $\mathcal{P}(t) \in C^1([0, +\infty), \mathcal{R})$ as well as $0 < \alpha < 1$,*

$$ {}^{C}D^{\alpha}_{t_0}|\mathcal{P}(t)| \le sign(\mathcal{P}(t))^{C}D^{\alpha}_{t_0}\mathcal{P}(t), t \ge t_0, $$

holds almost everywhere.

Lemma 2.3. [25] *Select $\varsigma > 0$ is a constant, and $0 < \alpha < 1$. Suppose $G(t) \ge 0$ is a continuous function on $[t_0, +\infty)$, then*

$$ {}^{C}D^{\alpha}_{t_0}H(t) \le (t - t_0 + \varsigma)^{\alpha}\ {}^{C}D^{\alpha}_{t_0}G(t) + \frac{1 - \alpha + 2\alpha^2 - \alpha^3}{\varsigma^{\alpha}\Gamma(2 - \alpha)}\hat{H}(t), $$

for $t \ge t_0$, where $H(t) = (t - t_0 + \varsigma)^{\alpha}G(t), \hat{H}(t) = (t - t_0 + \varsigma)^{\alpha}\hat{G}(t), \hat{G}(t) = sup_{t_0 \le s \le t}G(s)$.

3. Multiple $O(t^{-\alpha})$ stability analysis

We are going to delve into the multistability of time-varying DFNNs (2.1) in this section. The existence and stability of multiple equilibria of DFNNs (2.1) are also proved as follows:

3.1. Existence of multiple equilibria

For any given interval $I \subset R$, let $I^0 = \emptyset$ and $I^1 = I$, then denote

$$ \begin{aligned} (-\infty, -\frac{\pi}{2}) &= (-\infty, -\frac{\pi}{2})^1 \times [-\frac{\pi}{2}, -\frac{\pi}{6}]^0 \times (-\frac{\pi}{6}, \frac{\pi}{6})^0 \times [\frac{\pi}{6}, \frac{\pi}{2}] \times (\frac{\pi}{2}, +\infty)^0, \\ [-\frac{\pi}{2}, -\frac{\pi}{6}] &= (-\infty, -\frac{\pi}{2})^0 \times [-\frac{\pi}{2}, -\frac{\pi}{6}]^1 \times (-\frac{\pi}{6}, \frac{\pi}{6})^0 \times [\frac{\pi}{6}, \frac{\pi}{2}]^0 \times (\frac{\pi}{2}, +\infty)^0, \\ (-\frac{\pi}{6}, \frac{\pi}{6}) &= (-\infty, -\frac{\pi}{2})^0 \times [-\frac{\pi}{2}, -\frac{\pi}{6}]^0 \times (-\frac{\pi}{6}, \frac{\pi}{6})^1 \times [\frac{\pi}{6}, \frac{\pi}{2}]^0 \times (\frac{\pi}{2}, +\infty)^0, \\ [\frac{\pi}{6}, \frac{\pi}{2}] &= (-\infty, -\frac{\pi}{2})^0 \times [-\frac{\pi}{2}, -\frac{\pi}{6}]^0 \times (-\frac{\pi}{6}, \frac{\pi}{6})^0 \times [\frac{\pi}{6}, \frac{\pi}{2}]^1 \times (\frac{\pi}{2}, +\infty)^0, \\ (\frac{\pi}{2}, +\infty) &= (-\infty, -\frac{\pi}{2})^0 \times [-\frac{\pi}{2}, -\frac{\pi}{6}]^0 \times (-\frac{\pi}{6}, \frac{\pi}{6})^0 \times [\frac{\pi}{6}, \frac{\pi}{2}]^1 \times (\frac{\pi}{2}, +\infty)^1, \end{aligned} $$

and let

$$ \begin{aligned} \Theta =& \Big\{ \prod_{i=1}^{n}(-\infty, -\frac{\pi}{2})^{\delta_1^{(i)}} \times [-\frac{\pi}{2}, -\frac{\pi}{6}]^{\delta_2^{(i)}} \times (-\frac{\pi}{6}, \frac{\pi}{6})^{\delta_3^{(i)}} \times [\frac{\pi}{6}, \frac{\pi}{2}]^{\delta_4^{(i)}} \times (\frac{\pi}{2}, +\infty)^{\delta_5^{(i)}}, (\delta_1^{(i)}, \delta_2^{(i)}, \delta_3^{(i)}, \delta_4^{(i)}, \delta_5^{(i)}) \\ &=(1, 0, 0, 0, 0) \text{ or } (0, 1, 0, 0, 0) \text{ or } (0, 0, 1, 0, 0) \text{ or } (0, 0, 0, 1, 0) \text{ or } (0, 0, 0, 0, 1)\Big\}. \end{aligned} $$

Hence, we can know that there are 5^n regions in Θ. Suppose that ι is sufficiently small and meets $0 < \iota \ll \min\left\{\frac{2}{\pi}, \beta_i/(\sum_{j=1}^{n}\left|\rho_{ij}\right| + \sum_{j=1}^{n}\left|\varphi_{ij}\right| + |u_i|)\right\}$. Then, denote a set

$$ \begin{aligned} \Theta_\iota =& \Big\{ \prod_{i=1}^{n}[-\frac{1}{\iota}, -\frac{\pi}{2} - \iota]^{\delta_1^{(i)}} \times [-\frac{\pi}{2} + \iota, -\frac{\pi}{6} - \iota]^{\delta_2^{(i)}} \times [-\frac{\pi}{6} + \iota, \frac{\pi}{6} - \iota]^{\delta_3^{(i)}} \times [\frac{\pi}{6} + \iota, \frac{\pi}{2} - \iota]^{\delta_4^{(i)}} \times [\frac{\pi}{2} + \iota, \frac{1}{\iota}]^{\delta_5^{(i)}}, \\ &(\delta_1^{(i)}, \delta_2^{(i)}, \delta_3^{(i)}, \delta_4^{(i)}, \delta_5^{(i)}) = (1, 0, 0, 0, 0) \text{ or } (0, 1, 0, 0, 0) \text{ or } (0, 0, 1, 0, 0) \text{ or } (0, 0, 0, 1, 0) \text{ or } (0, 0, 0, 0, 1)\Big\}. \end{aligned} $$

Consequently, any subset $\Theta(s) \in \Theta_\iota$ is a bounded and closed set, where $s \in \{1, 2, ..., 5^n\}$.

Remark 3.1. In view of Figure 1, by means of the geometric properties of the activation function (1.2), it can be seen that we divide the interval $(-\infty, +\infty)$ into five parts: $(-\infty, -\frac{\pi}{2}) \cup [-\frac{\pi}{2}, -\frac{\pi}{6}] \cup (-\frac{\pi}{6}, \frac{\pi}{6}) \cup [\frac{\pi}{6}, \frac{\pi}{2}] \cup (\frac{\pi}{2}, +\infty)$. As a result, the region $\prod_{i=1}^{n}(-\infty, +\infty)$ can be divided into 5^n subsets.

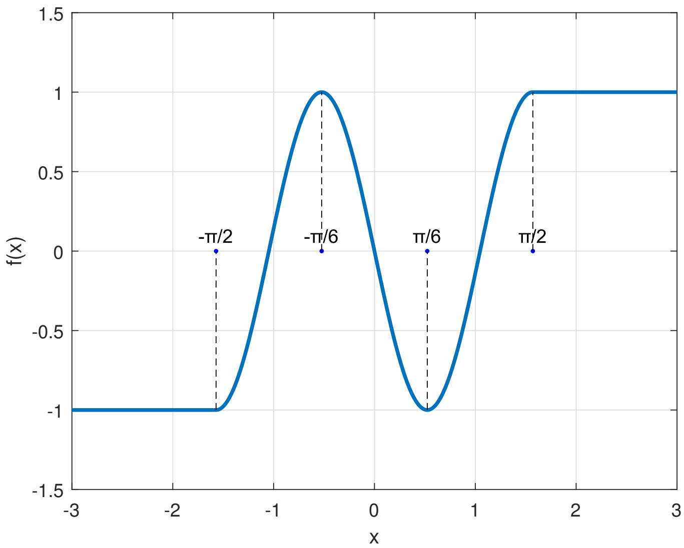

Figure 1. The activation function defined in (1.2).

Theorem 3.1. *Assume that the following conditions meet*

$$\frac{\pi}{2}\beta_i - \rho_{ii} - \varphi_{ii} + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i < 0, \tag{3.1}$$

$$-\frac{\pi}{2}\beta_i + \rho_{ii} + \varphi_{ii} - \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i > 0, \tag{3.2}$$

for $i = 1, 2, ..., n$. Then DFNNs (2.1) with activation functions (1.2) can possess at least 5^n equilibria in Θ_ι.

Proof. From (3.1) and the fact that $f_i(-\frac{\pi}{2}) = f_i(\frac{\pi}{6}) = -1$, we know

$$-\frac{\pi}{6}\beta_i - \rho_{ii} - \varphi_{ii} + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i < 0. \tag{3.3}$$

Taking arbitrarily a region $\tilde{\Phi}_\iota \in \Theta_\iota$, which is denoted by:

$$\tilde{\Phi}_\iota = \prod_{i\in N_1}[-\frac{1}{\iota}, -\frac{\pi}{2} - \iota] \times \prod_{i\in N_2}[-\frac{\pi}{2} + \iota, -\frac{\pi}{6} - \iota] \times \prod_{i\in N_3}[-\frac{\pi}{6} + \iota, \frac{\pi}{6} - \iota] \times \prod_{i\in N_4}[\frac{\pi}{6} + \iota, \frac{\pi}{2} - \iota] \times \prod_{i\in N_5}[\frac{\pi}{2} + \iota, \frac{1}{\iota}] \subset \Theta_\iota,$$

where $N_i \in \{1, 2, ..., n\}$ and $N_i \cap N_j = \emptyset (i \neq j, i, j = 1, 2, 3, 4, 5)$, $N_1 \cup N_2 \cup N_3 \cup N_4 \cup N_5 = \{1, 2, ..., n\}$.

Then, we are about to prove that there is an equilibrium point in $\tilde{\Phi}_\iota$ for (2.1) with the activation function (1.2).

Choose a point $(\xi_1, \xi_2, ..., \xi_n)^T \in \tilde{\Phi}_\iota$ and fix $\xi_1, ..., \xi_{i-1}, \xi_{i+1}, ..., \xi_n$ except ξ_i. Define a function

$$Z_i(x) = -\beta_i x + (\rho_{ii} + \varphi_{ii})f_i(x) + \sum_{j\neq i, j=1}^{n} (\rho_{ij} + \varphi_{ij})f_j(\xi_j) + u_i. \tag{3.4}$$

In the argument that follows, we categorize the discussion into five situations.

Situation 1. $i \in N_1$. Since $f_i(-\frac{\pi}{2} - \iota) = -1$, $|f_j(\xi_j)| \leq 1$, based on (3.1) and the definition of ι, we get

$$\begin{aligned} Z_i(-\frac{1}{\iota}) &= \frac{\beta_i}{\iota} - (\rho_{ii} + \varphi_{ii}) + \sum_{j\neq i, j=1}^{n} (\rho_{ij} + \varphi_{ij})f_j(\xi_j) + u_i \\ &\geq \frac{\beta_i}{\iota} - \sum_{j=1}^{n} |\rho_{ij}| - \sum_{j=1}^{n} |\varphi_{ij}| - |u_i| > 0, \\ Z_i(-\frac{\pi}{2} - \iota) &\leq \frac{\pi}{2}\beta_i + \beta_i\iota - \rho_{ii} - \varphi_{ii} + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i \leq 0. \end{aligned}$$

Hence, thanks to the continuity of $Z_i(x)$, there exists a $\bar{\xi}_i \in [-\frac{1}{\iota}, -\frac{\pi}{2} - \iota]$ such that $Z_i(\bar{\xi}_i) = 0$.

Situation 2. $i \in N_2$. Owing to $f_i(-\frac{\pi}{2}) = -1$, $f_i(-\frac{\pi}{6}) = 1$, under the facts $\beta_i > 0$ and (3.2),

$$\frac{\pi}{6}\beta_i + \rho_{ii} + \varphi_{ii} - \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i > 0, \tag{3.5}$$

so, according to (3.1) and (3.5), we can have

$$\begin{aligned} Z_i(-\frac{\pi}{2} + \iota) &\leq \frac{\pi}{2}\beta_i + (\rho_{ii} + \varphi_{ii})f_i(-\frac{\pi}{2} + \iota) - \beta_i\iota + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i \leq 0, \\ Z_i(-\frac{\pi}{6} - \iota) &\geq \frac{\pi}{6}\beta_i + (\rho_{ii} + \varphi_{ii})f_i(-\frac{\pi}{6} - \iota) + \beta_i\iota - \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i \geq 0, \end{aligned}$$

hence, there exists a $\bar{\xi}_i \in [-\frac{\pi}{2} + \iota, -\frac{\pi}{6} - \iota]$ such that $Z_i(\bar{\xi}_i) = 0$.

Situation 3. $i \in N_3$. $f_i(-\frac{\pi}{6}) = 1$, and $f_i(\frac{\pi}{6}) = -1$, from (3.1), we have

$$-\frac{\pi}{6}\beta_i - \rho_{ii} - \varphi_{ii} + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i < 0, \tag{3.6}$$

due to (3.5) and (3.6),

$$\begin{aligned} Z_i(-\frac{\pi}{6} + \iota) &\geq \frac{\pi}{6}\beta_i + (\rho_{ii} + \varphi_{ii})f_i(-\frac{\pi}{6} + \iota) - \beta_i\iota + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i \geq 0, \\ Z_i(\frac{\pi}{6} - \iota) &\leq -\frac{\pi}{6}\beta_i + (\rho_{ii} + \varphi_{ii})f_i(\frac{\pi}{6} - \iota) + \beta_i\iota + \sum_{j\neq i, j=1}^{n} |\rho_{ij} + \varphi_{ij}| + u_i \leq 0, \end{aligned}$$

so there exists a $\bar{\xi}_i \in [-\frac{\pi}{6} + \iota, \frac{\pi}{6} - \iota]$ such that $Z_i(\bar{\xi}_i) = 0$.

Situation 4. $i \in N_4$. $f_i(\frac{\pi}{6}) = -1$, and $f_i(\frac{\pi}{2}) = 1$, from (3.2) and (3.6), we can obtain

$$Z_i(\frac{\pi}{6}+\iota) \le -\frac{\pi}{6}\beta_i + (\rho_{ii}+\varphi_{ii})f_i(\frac{\pi}{6}+\iota) - \beta_i\iota + \sum_{j\neq i,j=1}^{n}|\rho_{ij}+\varphi_{ij}| + u_i \le 0,$$
$$Z_i(\frac{\pi}{2}-\iota) \ge -\frac{\pi}{2}\beta_i + (\rho_{ii}+\varphi_{ii})f_i(\frac{\pi}{2}-\iota) - \beta_i\iota - \sum_{j\neq i,j=1}^{n}|\rho_{ij}+\varphi_{ij}| + u_i \ge 0,$$

similarly, there exists a $\bar{\xi}_i \in [\frac{\pi}{6}+\iota, \frac{\pi}{2}-\iota]$ such that $Z_i(\bar{\xi}_i) = 0$.

Situation 5. $i \in N_5$. $f_i(\frac{\pi}{2}+\iota) = 1$, and $|f_j(\xi_j)| \le 1$, from (3.2), we can know

$$\begin{aligned} Z_i(\frac{\pi}{2}+\iota) &\ge -\frac{\pi}{2}\beta_i + (\rho_{ii}+\varphi_{ii}) - \beta_i\iota - \sum_{j\neq i,j=1}^{n}|\rho_{ij}+\varphi_{ij}| + u_i \ge 0, \\ Z_i(\frac{1}{\iota}) &= -\frac{\beta_i}{\iota} + (\rho_{ii}+\varphi_{ii}) + \sum_{j\neq i,j=1}^{n}(\rho_{ij}+\varphi_{ij})f_j(\xi_j) + u_i \\ &< -\frac{\beta_i}{\iota} + \sum_{j=1}^{n}|\rho_{ij}| + \sum_{j=1}^{n}|\varphi_{ij}| + |u_i| < 0, \end{aligned}$$

therefore, there exists a $\bar{\xi}_i \in [\frac{\pi}{6}+\iota, \frac{\pi}{2}-\iota]$ such that $Z_i(\bar{\xi}_i) = 0$.

That is, it is possible to draw the conclusion that there exists at least one zero about each interval in $\tilde{\Phi}_\iota$ for $Z_i(x)$. Define a continuous mapping $\Xi : \Theta^{(1)} \to \Theta^{(1)}$, $\Xi(x_1, x_2, ..., x_n) = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)^T$. Taking advantage of Brouwer's fixed point theorem, there exists a fixed point $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$ of Ξ, which simultaneously acts as the equilibrium point of DFNNs (2.1) in $\tilde{\Phi}_\iota \in \Theta_\iota$. Hence, for DFNNs (2.1) with activation function (1.2), there are a minimum of 5^n equilibria in Θ_ι. □

3.2. Stability of multiple equilibria

We will explore the stability of multiple equilibria of DFNNs (2.1) with activation function (1.2) in this subsection. For this purpose, some positive invariant sets of DFNNs (2.1) need to be determined in order to analyze the stability of equilibria.

Denote

$$\bar{\Theta}_\iota = \{\prod_{i=1}^{n}[-\frac{1}{\iota}, -\frac{\pi}{2}-\iota]^{\delta_1^{(i)}} \times [-\frac{\pi}{6}+\iota, \frac{\pi}{6}-\iota]^{\delta_2^{(i)}} \times [\frac{\pi}{2}+\iota, \frac{1}{\iota}]^{\delta_3^{(i)}}, (\delta_1^{(i)}, \delta_2^{(i)}, \delta_3^{(i)}) = (1,0,0) \text{ or } (0,1,0) \text{ or } (0,0,1)\},$$

apparently, the set $\bar{\Theta}_\iota$ has 3^n regions.

Pick any region of $\bar{\Theta}_\iota$,

$$\bar{\Phi}_L^\iota = \prod_{i\in L_1}[-\frac{1}{\iota}, -\frac{\pi}{2}-\iota] \times \prod_{i\in L_2}[-\frac{\pi}{6}+\iota, \frac{\pi}{6}-\iota] \times \prod_{i\in L_3}[\frac{\pi}{2}+\iota, \frac{1}{\iota}] \subset \bar{\Theta}_\iota,$$

where $L_1 \cup L_2 \cup L_3 = \{1, 2, ..., n\}, L_i \cap L_j = \emptyset$ $(i \neq j, i, j = 1, 2, 3)$.

Theorem 3.2. *Suppose that DFNNs (2.1) with activation functions (1.2) satisfy:*

$$\frac{\pi}{2}\beta_i - \rho_{ii} + \sum_{j\neq i,j=1}^{n} |\rho_{ij}| + \sum_{j=1}^{n} |\varphi_{ij}| + u_i < 0, \tag{3.7}$$

$$-\frac{\pi}{2}\beta_i + \rho_{ii} - \sum_{j\neq i,j=1}^{n} |\rho_{ij}| - \sum_{j=1}^{n} |\varphi_{ij}| + u_i > 0, \tag{3.8}$$

for $i = 1, 2, ..., n$. Then each region $\bar{\Phi}^{\iota}_{L} \in \bar{\Theta}_{\iota}$ is a positively invariant set of DFNNs (2.1).

Proof. From (3.7) and (3.8), the following conditions can be obtained

$$-\frac{\pi}{6}\beta_i - \rho_{ii} + \sum_{j\neq i,j=1}^{n} |\rho_{ij}| + \sum_{j=1}^{n} |\varphi_{ij}| + u_i < 0, \tag{3.9}$$

$$\frac{\pi}{6}\beta_i + \rho_{ii} - \sum_{j\neq i,j=1}^{n} |\rho_{ij}| - \sum_{j=1}^{n} |\varphi_{ij}| + u_i > 0. \tag{3.10}$$

Assume that $x(t)$ is the solution of DFNNs (2.1) with the initial value (2.2). In the following, we will certify that each subspace $\bar{\Phi}^{\iota}_{L} \in \bar{\Theta}_{\iota}$ is positively invariant. Thus, for a given subspace $\bar{\Phi}^{\iota}_{L}$, if the initial condition $\phi(t_0) \in \bar{\Phi}^{\iota}_{L}$, then $x(t) \in \bar{\Phi}^{\iota}_{L}$ for all $t \geq t_0$. If not, three cases will be considered.

Case 1. $i \in L_1$. There exists a $\bar{t}_1 > 0$ such that $x_i(\bar{t}_1) < -\frac{1}{\iota}$, or $x_i(\bar{t}_1) > -\frac{\pi}{2} - \iota$. Assume that $x_i(\bar{t}_1) > -\frac{\pi}{2} - \iota$ without loss of generality. Then

$$\begin{cases} x_i(t) = -\dfrac{\pi}{2} - \iota, & t = \bar{t}_1, \\ x_i(t) < -\dfrac{\pi}{2} - \iota, & t_0 \leq t < \bar{t}_1. \end{cases}$$

Denote $\mathcal{W}_1(t) = x_i(t) - (-\frac{\pi}{2} - \iota)$, according to Lemma 2.1,

$${}^{C}D^{\alpha}_{t_0}\mathcal{W}_1(t)|_{t=\bar{t}_1} = {}^{C} D^{\alpha}_{t_0} x_i(\bar{t}_1) > 0, \tag{3.11}$$

on the other hand, owing to (3.7), $f_i(-\frac{\pi}{2} - \iota) = -1$, and the sufficiently small positive ι,

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0} x_i(\bar{t}_1) &= -\beta_i x_i(\bar{t}_1) + \sum_{j=1}^{n} \rho_{ij} f_j(x_j(\bar{t}_1)) + \sum_{j=1}^{n} \varphi_{ij} f_j(x_j(\bar{t}_1 - \tau_{ij}(\bar{t}_1))) + u_i \\ &\leq \frac{\pi}{2}\beta_i + \beta_i\iota - \rho_{ii} + \sum_{j\neq i,j=1}^{n} |\rho_{ij}| + \sum_{j=1}^{n} |\varphi_{ij}| + u_i \leq 0, \end{aligned}$$

which shows that it is a paradox with (3.11).

So we can get that $x_i(t) \leq -\frac{\pi}{2} - \iota$. Similarly, we can prove $x_i(t) \geq -\frac{1}{\iota}$.

Case 2. $i \in L_2$. There exists a $\bar{t}_2 > 0$ such that $x_i(\bar{t}_2) < -\frac{\pi}{6} + \iota$, or $x_i(\bar{t}_2) > \frac{\pi}{6} - \iota$. Using the same approach as Case 1, assume that $x_i(\bar{t}_3) < -\frac{\pi}{6} + \iota$. Then

$$\begin{cases} x_i(t) = -\dfrac{\pi}{6} + \iota, & t = \bar{t}_2, \\ x_i(t) > -\dfrac{\pi}{6} + \iota, & t_0 \leq t < \bar{t}_2. \end{cases}$$

Denote $\mathcal{W}_2(t) = -\frac{\pi}{6} + \iota - x_i(\bar{t}_2)$, based on Lemma 2.1,

$$^{C}D_{t_0}^{\alpha}\mathcal{W}_2(t)|_{t=\bar{t}_2} = -^{C}D_{t_0}^{\alpha}x_i(\bar{t}_2) > 0, \tag{3.12}$$

on the other hand, in view of (3.10) and $|f_j(x_j)| \le 1$,

$$\begin{aligned} ^{C}D_{t_0}^{\alpha}x_i(\bar{t}_2) &= -\beta_i x_i(\bar{t}_2) + \sum_{j=1}^{n}\rho_{ij}f_j(x_j(\bar{t}_2)) + \sum_{j=1}^{n}\varphi_{ij}f_j(x_j(\bar{t}_2 - \tau_{ij}(\bar{t}_2))) + u_i \\ &\ge \frac{\pi}{6}\beta_i - \beta_i\iota + \rho_{ii}f_i(-\frac{\pi}{6} + \iota) - \sum_{j\ne i, j=1}^{n}|\rho_{ij}| - \sum_{j=1}^{n}|\varphi_{ij}| + u_i \ge 0, \end{aligned}$$

this contradicts (3.12). So we can get that $x_i(t) \ge -\frac{\pi}{6} + \iota$. Similarly, we can prove $x_i(t) \le \frac{\pi}{6} - \iota$.

Case 3. $i \in L_3$. There exists a $\bar{t}_3 > 0$ such that $x_i(\bar{t}_3) < \frac{\pi}{2} + \iota$, or $x_i(\bar{t}_3) > \frac{1}{\iota}$. Assume that $x_i(\bar{t}_3) < \frac{\pi}{2} + \iota$. Then

$$\begin{cases} x_i(t) = \dfrac{\pi}{2} + \iota, & t = \bar{t}_3, \\ x_i(t) > \dfrac{\pi}{2} + \iota, & t_0 \le t < \bar{t}_3. \end{cases}$$

Let $\mathcal{W}_3(t) = \frac{\pi}{2} + \iota - x_i(\bar{t}_3)$. According to Lemma 2.1,

$$^{C}D_{t_0}^{\alpha}\mathcal{W}_3(t)|_{t=\bar{t}_3} = -^{C}D_{t_0}^{\alpha}x_i(\bar{t}_3) > 0, \tag{3.13}$$

which implies $x_i(\bar{t}_3) < 0$. On the other hand, on account of (3.8),

$$\begin{aligned} ^{C}D_{t_0}^{\alpha}x_i(\bar{t}_3) &= -\beta_i x_i(\bar{t}_3) + \sum_{j=1}^{n}\rho_{ij}f_j(x_j(\bar{t}_3)) + \sum_{j=1}^{n}\varphi_{ij}f_j(x_j(\bar{t}_3 - \tau_{ij}(\bar{t}_3))) + u_i \\ &\ge -\frac{\pi}{2}\beta_i - \beta_i\iota + \rho_{ii} - \sum_{j\ne i, j=1}^{n}|\rho_{ij}| - \sum_{j=1}^{n}|\varphi_{ij}| + u_i \ge 0, \end{aligned}$$

this is contradicted by $x_i(\bar{t}_3) < 0$. Hence, we can get that $x_i(t) \ge \frac{\pi}{2} + \iota$. Similarly, we can prove $x_i(t) \le \frac{1}{\iota}$.

Summing up, we conclude that the corresponding solution $x_i(t)$ would always stay in $\bar{\Phi}_{\delta}^{\iota} \in \bar{\Theta}_{\iota}$, which suggests that each subset $\bar{\Phi}_{\delta}^{\iota} \in \bar{\Theta}_{\iota}$ is a positive invariant set of DFNNs (2.1). □

Below, the stability of DFNNs (2.1) with activation functions (1.2) will be discussed.

Theorem 3.3. *Under the conditions (3.7)–(3.8), further suppose that there are n positive constants* $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ *and* $\varsigma > \sigma$ *satisfying the following condition:*

$$\beta_i - \chi_i - \frac{3}{\varepsilon_i}\sum_{j=1, j\ne i}^{n}|\rho_{ij}|\varepsilon_j - \frac{3}{\varepsilon_i}\sum_{j=1}^{n}|\varphi_{ij}|\varepsilon_j(\frac{\varsigma}{\varsigma - \sigma})^{\alpha} - \frac{1 - \alpha + 2\alpha^2 - \alpha^3}{\varsigma^{\alpha}\Gamma(2 - \alpha)} > 0,$$

for $i = 1, 2, ..., n$, *where* $\chi_i = max_{1\le i\le n}\{0, -3\rho_{ii}\}$. *Then, there are* 3^n *locally* $O(t^{-\alpha})$ *stable equilibria in* $\bar{\Theta}_{\iota}$ *for DFNNs (2.1).*

Proof. Based on Theorem 3.2, we get that there are 3^n subsets in Θ_ι and each $\tilde{\Phi}_\iota \in \Theta_\iota$ is positively invariant. Hence, all we need to do is verify that x^* is a locally $O(t^{-\alpha})$ stable equilibrium point when $\tilde{\Phi}_\iota \in \Theta_\iota$.

Let

$$\mathcal{Y}(t) = x(t) - x^*, \quad z(t) = \max_{1\le i\le n}\left\{\frac{|\mathcal{Y}_i(t)|}{\varepsilon_i}\right\}.$$

Since $z(t) = \max_{1\le i\le n}\left\{\frac{|y_i(t)|}{\varepsilon_i}\right\}$, thus, an index $k \in \{1, 2, ..., n\}$ exists such that $z(t) = \frac{|\mathcal{Y}_k(t)|}{\varepsilon_k}$.

Substituting $x(t) = \mathcal{Y}(t) + x^*$ into (2.1), it shows that

$${}^C D^\alpha_{t_0}\mathcal{Y}_i(t) = -\beta_i\mathcal{Y}_i(t) + \sum_{j=1}^n \rho_{ij}F_j(\mathcal{Y}_j(t)) + \sum_{j=1}^n \varphi_{ij}F_j(\mathcal{Y}_j(t-\tau_{ij}(t))),$$

where $F_j(\mathcal{Y}_j(t)) = f_j(\mathcal{Y}_j(t) + x_j^*) - f_j(x_j^*)$, $F_j(\mathcal{Y}_j(t-\tau_{ij}(t))) = f_j(\mathcal{Y}_j(t-\tau_{ij}(t)) + x_j^*) - f_j(x_j^*)$.

Then, according to Lemma 2.2, we get that

$$\begin{aligned}
{}^C D^\alpha_{t_0}|\mathcal{Y}_i(t)| \le& \operatorname{sign}(\mathcal{Y}_i(t)) \cdot {}^C D^\alpha_{t_0}\mathcal{Y}_i(t)\\
\le& \operatorname{sign}(\mathcal{Y}_i(t)) \cdot \Big(-\beta_i\mathcal{Y}_i(t) + \sum_{j=1}^n \rho_{ij}F_j(\mathcal{Y}_j(t)) + \sum_{j=1}^n \varphi_{ij}F_j(\mathcal{Y}_j(t-\tau_{ij}(t))\Big)\\
\le& -\beta_i|\mathcal{Y}_i(t)| + \rho_{ii}\frac{f_i(x_i(t)) - f_i(x_i^*)}{x_i(t) - x_i^*}|\mathcal{Y}_i(t)| + \sum_{j=1,j\ne i}^n \rho_{ij}\frac{f_j(x_j(t)) - f_j(x_j^*)}{x_j(t) - x_j^*}|\mathcal{Y}_j(t)|\\
&+ \sum_{j=1}^n \varphi_{ij}\frac{f_j(x_j(t-\tau_{ij}(t))) - f_j(x_j^*)}{x_j(t-\tau_{ij}(t)) - x_j^*}|\mathcal{Y}_j(t-\tau_{ij}(t))|.
\end{aligned} \tag{3.14}$$

Next, recalling (1.2), under the Lagrange mean value theorem, there exists $\xi^*_{1j} \in (x_j^*, x_j(t)), \xi^*_{2j} \in (x_j^*, x_j(t-\tau_{ij}(t)))$ such that $f_j'(\xi^*_{1j}) = \frac{f_j(x_j(t))-f_j(x_j^*)}{x_j(t)-x_j^*} \in (-3, 0), f_j'(\xi^*_{2j}) = \frac{f_j(x_j(t-\tau_{ij}(t)))-f_j(x_j^*)}{x_j(t-\tau_{ij}(t))-x_j^*} \in (-3, 0)$, $i, j = 1, 2, ..., n$.

Consider the term $\rho_{ii}\frac{f_i(x_i(t))-f_i(x_i^*)}{x_i(t)-x_i^*}|\mathcal{Y}_i(t)|$, if $\rho_{ii} \ge 0$, $\rho_{ii}\frac{f_i(x_i(t))-f_i(x_i^*)}{x_i(t)-x_i^*}|\mathcal{Y}_i(t)| \le 0$; if $\rho_{ii} < 0$, $\rho_{ii}\frac{f_i(x_i(t))-f_i(x_i^*)}{x_i(t)-x_i^*}|\mathcal{Y}_i(t)| \le -3\rho_{ii}|\mathcal{Y}_i(t)|$. In conclusion, $\rho_{ii}\frac{f_i(x_i(t))-f_i(x_i^*)}{x_i(t)-x_i^*}|\mathcal{Y}_i(t)| \le \chi_i|\mathcal{Y}_i(t)|$.

Combining with (3.14), we can have

$${}^C D^\alpha_{t_0}|\mathcal{Y}_i(t)| \le (-\beta_i + \chi_i)|\mathcal{Y}_i(t)| + 3\sum_{j=1,j\ne i}^n |\rho_{ij}||\mathcal{Y}_j(t)| + 3\sum_{j=1}^n |\varphi_{ij}||\mathcal{Y}_j(t-\tau_{ij}(t))|. \tag{3.15}$$

Invoking (3.15) and the definition of $z(t)$,

$$\begin{aligned}
{}^C D^\alpha_{t_0}z(t) = \frac{1}{\varepsilon_\kappa} \cdot {}^C D^\alpha_{t_0}|\mathcal{Y}_\kappa(t)| \le& \frac{1}{\varepsilon_\kappa}\Big[(-\beta_\kappa + \chi_\kappa)|\mathcal{Y}_\kappa(t)| + 3\sum_{j=1,j\ne\kappa}^n |\rho_{\kappa j}||\mathcal{Y}_j(t)| + 3\sum_{j=1}^n |\varphi_{\kappa j}||\mathcal{Y}_j(t-\tau_{\kappa j}(t))|\Big]\\
\le& (-\beta_\kappa + \chi_\kappa)z(t) + \frac{3}{\varepsilon_\kappa}\Big[\sum_{j=1,j\ne\kappa}^n |\rho_{\kappa j}|\varepsilon_j z(t) + \sum_{j=1}^n |\varphi_{\kappa j}|\varepsilon_j z(t-\tau_{\kappa j}(t))\Big].
\end{aligned} \tag{3.16}$$

In view of Lemma 2.3, let

$$\begin{aligned}\varpi(t) &= (t - t_0 + \varsigma)^{\alpha} z(t),\\ \hat{\varpi}(t) &= (t - t_0 + \varsigma)^{\alpha} \hat{z}(t),\\ \hat{z}(t) &= \sup_{t_0-\sigma\le s\le t} z(s).\end{aligned}$$

Then,

$$ {}^{C}D_{t_0}^{\alpha}\varpi(t) \le (t - t_0 + \varsigma)^{\alpha}\, {}^{C}D_{t_0}^{\alpha} z(t) + \frac{1-\alpha+2\alpha^2-\alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)}\hat{\varpi}(t), \tag{3.17}$$

substituting (3.16) into (3.17), it yields

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha}\varpi(t) \le&(-\beta_\kappa + \chi_\kappa)\varpi(t) + \frac{3}{\varepsilon_\kappa}\Big[\sum_{j=1,j\neq\kappa}^{n}|\rho_{\kappa j}|\varepsilon_j\varpi(t) + \sum_{j=1}^{n}|\varphi_{\kappa j}|\varepsilon_j\varpi(t-\tau_{\kappa j}(t))\\ &\times\frac{(t-t_0+\varsigma)^{\alpha}}{(t-\tau_{\kappa j}(t)-t_0+\varsigma)^{\alpha}}\Big]+\frac{1-\alpha+2\alpha^2-\alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)}\hat{\varpi}(t).\end{aligned}\tag{3.18}$$

Note that

$$\varpi(t-\tau_{\kappa j}(t)) \le (t-\tau_{\kappa j}(t)-t_0+\varsigma)^{\alpha}\hat{z}(t) \le \hat{\varpi}(t),$$

and

$$\frac{t-t_0+\varsigma}{t-\tau_{\kappa j}(t)-t_0+\varsigma} \le \frac{\varsigma}{\varsigma-\tau_{\kappa j}(t)} \le \frac{\varsigma}{\varsigma-\sigma},$$

hence, (3.18) turns into

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha}\varpi(t) \le& -\Big(\beta_\kappa - \chi_\kappa - \frac{3}{\varepsilon_\kappa}\sum_{j=1,j\neq\kappa}^{n}|\rho_{\kappa j}|\varepsilon_j\Big)\varpi(t)\\ &+\frac{3}{\varepsilon_\kappa}\sum_{j=1}^{n}|\varphi_{\kappa j}|\varepsilon_j(\frac{\varsigma}{\varsigma-\sigma})^{\alpha}\hat{\varpi}(t) + \frac{1-\alpha+2\alpha^2-\alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)}\hat{\varpi}(t).\end{aligned}$$

When $\hat{\varpi}(t) = \varpi(t)$ holds for $t \ge t_0$, it implies that

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha}\varpi(t) \le& -\Big[\beta_\kappa - \chi_\kappa - \frac{3}{\varepsilon_\kappa}\sum_{j=1,j\neq\kappa}^{n}|\rho_{\kappa j}|\varepsilon_j - \frac{3}{\varepsilon_\kappa}\sum_{j=1}^{n}|\varphi_{\kappa j}|\varepsilon_j(\frac{\varsigma}{\varsigma-\sigma})^{\alpha} - \frac{1-\alpha+2\alpha^2-\alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)}\Big]\varpi(t)\\ \le& -\Upsilon\varpi(t),\end{aligned}\tag{3.19}$$

where $\Upsilon = \min_{1\le i\le n}\Big(\beta_i - \chi_i - \frac{3}{\varepsilon_i}\sum_{j=1,j\neq i}^{n}|\rho_{ij}|\varepsilon_j - \frac{3}{\varepsilon_i}\sum_{j=1}^{n}|\varphi_{ij}|\varepsilon_j(\frac{\varsigma}{\varsigma-\sigma})^{\alpha} - \frac{1-\alpha+2\alpha^2-\alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)}\Big) > 0$.

Next, we are about to demonstrate that $\hat{\varpi}(t) \le \hat{\varpi}(t_0)$, for $t \ge t_0$. Otherwise, there must be some $\tilde{t} > t_0$, then $\hat{\varpi}(\tilde{t}) = \varpi(\tilde{t}) > \hat{\varpi}(t_0) \ge 0$.

Now, denote $\hbar(t) = \varpi(t) - \hat{\varpi}(t)$, then

$$\begin{cases}\hbar(t) = 0, & t = \tilde{t},\\ \hbar(t) \le 0, & t < \tilde{t}.\end{cases}$$

Based on Lemma 2.1, we obtain that

$$ {}^{C}D_{t_0}^{\alpha}\hbar(t)|_{t=\tilde{t}} = {}^{C}D_{t_0}^{\alpha}\varpi(t)|_{t=\tilde{t}} - {}^{C}D_{t_0}^{\alpha}\hat{\varpi}(t)|_{t=\tilde{t}} \ge 0,$$

from (3.19), we get that ${}^{C}D_{t_0}^{\alpha}\varpi(t) < 0$, thereupon, we get that

$$ {}^{C}D_{t_0}^{\alpha}\hat{\varpi}(t)|_{t=\tilde{t}} \leq {}^{C}D_{t_0}^{\alpha}\varpi(t)|_{t=\tilde{t}} < 0. \tag{3.20}$$

On the other side, it's worth noting that the property of $\hat{\varpi}(t)$ is that when $t_0 \leq t \leq \tilde{t}$, $\frac{d\hat{\varpi}(t)}{dt} \geq 0$, and $\frac{d\hat{\varpi}(t)}{dt} \not\equiv 0$, hence

$$ {}^{C}D_{t_0}^{\alpha}\hat{\varpi}(t)|_{t=\tilde{t}} = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\,(\frac{d\hat{\varpi}(s)}{ds})ds > 0,$$

which conflicts with (3.20). Therefore, $\hat{\varpi}(t) \leq \hat{\varpi}(t_0)$ holds for $t \geq t_0$.

Recalling the norms $\|x\|$ and $z(t)$, it follows

$$\|x(t) - x^*\| = \|\mathcal{Y}(t)\| \leq \sqrt{n}\varepsilon_\kappa z(t).$$

Then

$$\begin{aligned}
\|x(t) - x^*\| &\leq \sqrt{n}\|\varepsilon\|\frac{\varpi(t)}{(t-t_0+\varsigma)^\alpha} \leq \sqrt{n}\|\varepsilon\|\frac{\hat{\varpi}(t_0)}{(t-t_0+\varsigma)^\alpha} \\
&\leq \sqrt{n}\|\varepsilon\|\frac{\varsigma^\alpha\|\phi - x^*\|_M}{\check{\varepsilon}(t-t_0+\varsigma)^\alpha} \\
&= \frac{\Lambda\varsigma^\alpha\|\phi - x^*\|_M}{(t-t_0+\varsigma)^\alpha},
\end{aligned}$$

where $\Lambda = \frac{\sqrt{n}\|\varepsilon\|}{\check{\varepsilon}} \geq 1$, $\check{\varepsilon} = \min_{1\leq i\leq n}\{\varepsilon_i\}$.

Therefore, x^* is a locally $O(t^{-\alpha})$ stable equilibrium point; this accomplishes the proof. □

Remark 3.2. Compared to the study of piecewise nonlinear activation functions (1.2) in integer-order neural networks in [24], this paper extends activation functions (1.2) to FNNs, thereby enhancing the generalizability of the results.

Remark 3.3. In contrast with the multiple $O(t^{-\alpha})$ stability of FNNs studied in [25, 26], the piecewise nonlinear activation functions considered in this paper can achieve more stable equilibrium points. This implies the possibility of achieving greater storage capacity, thus leading to better performance in the application of associative memory.

The above mainly discusses the existence and stability of DFNNs (2.1) with $k = 2$ in the proposed activation function (1.2). In order to attain a greater number of stable equilibria, we promote the activation function in a more generalized scenario. The subsequent theorem elucidates the conditions under $k \geq 2$ in the activation function (1.1).

Theorem 3.4. *Under the conditions (3.7)–(3.8), further suppose that there are n positive constants $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ and $\varsigma > \sigma$ satisfying the following condition:*

$$\beta_i - \chi_i - \frac{2k-1}{\varepsilon_i}\sum_{j=1, j\neq i}^{n}|\rho_{ij}|\varepsilon_j - \frac{2k-1}{\varepsilon_i}\sum_{j=1}^{n}|\varphi_{ij}|\varepsilon_j(\frac{\varsigma}{\varsigma-\sigma})^\alpha - \frac{1-\alpha+2\alpha^2-\alpha^3}{\varsigma^\alpha\Gamma(2-\alpha)} > 0,$$

for $i = 1, 2, ..., n$. Then, there are $(2k+1)^n$ equilibria, out of which $(k+1)^n$ equilibria exist local $O(t^{-\alpha})$ stability for DFNNs (2.1) with activation function (1.1). The positively invariant set of DFNNs (2.1) *with activation function (1.1) is*

$$\Theta^* =\Big\{ \prod_{i=1}^{n}(-\infty, -\frac{\pi}{2})^{\delta_1^{(i)}} \times [-\frac{\pi}{2} + \frac{\pi}{k}, -\frac{\pi}{2} + \frac{2\pi}{k}]^{\delta_2^{(i)}} \times \cdots \times [\frac{\pi}{2} - \frac{2\pi}{k}, \frac{\pi}{2} - \frac{\pi}{k}]^{\delta_k^{(i)}} \times (\frac{\pi}{2}, +\infty)^{\delta_{k+1}^{(i)}},$$
$$(\delta_1^{(i)}, \delta_2^{(i)}, ..., \delta_{k+1}^{(i)}) = (1, 0, ..., 0) \text{ or } (0, 1, ..., 0) \text{ or... or } (0, 0, ..., 1)\Big\}.$$

Proof. Since the proof process is similar to Theorems 3.1–3.3, we omit it here. □

Remark 3.4. From Theorem 3.4, we can see that there is a connection between the amount of stable equilibria and the frequency k of the sinusoidal functions. The result shows that adding the frequency of the sinusoidal functions leads to an expansion of the quantity of stable equilibria.

4. Synthesis of neural networks

In what follows, a design procedure for DFNNs (2.1) is introduced.

4.1. Synthesis problem

Let $\mathcal{B} = \{-1, 1\}$ and $\mathcal{B}^n = \Big\{\eta \in R^n, \eta = (\eta_1, \eta_2, ..., \eta_n)^T, \eta_i \in \mathcal{B}, i = 1, 2, ..., n.\Big\}$. For a given positive integer r, design neurodynamic associative memory utilizing DFNNs (2.1), with r vectors denoted as $\eta_1, \eta_2, ..., \eta_r$ acting as memory patterns, such that: (a) The vectors $\eta_1, \eta_2, ..., \eta_r$ represent memory vectors. (b) The system complies with the conditions outlined in Theorem 3.3. (c) Minimize the presence of spurious memory patterns.

4.2. Design procedure

With r vectors denoted as $\eta_1, \eta_2, ..., \eta_r \in \mathcal{B}^n$, we proceed as follows:

(1) Choose matrix A as the identity matrix E.

(2) Take a real constant $\gamma > 1$ and choose r vectors $\epsilon_1, \epsilon_2, ..., \epsilon_r$ such that $\epsilon_i = \gamma\eta_i$, and $\epsilon_i = (B + C)\eta_i + I, i = 1, 2, ..., r.$

(3) Let $Q = [\eta_1 - \eta_r, \eta_2 - \eta_r, ..., \eta_{r-1} - \eta_r]$, and $\mathcal{M} = [\epsilon_1 - \epsilon_r, \epsilon_2 - \epsilon_r, ..., \epsilon_{r-1} - \epsilon_r]$. Performing singular value decomposition on Q yields

$$Q = t[\mathcal{U}_1, \mathcal{U}_2]\begin{bmatrix}\mathcal{D} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathcal{V}_1^T\\ \mathcal{V}_2^T\end{bmatrix},$$

where matrix $\mathcal{D} = \operatorname{diag}(d_1, d_2, ..., d_m)$ and $d_i > 0$ are singular values, as well as $m = \operatorname{rank}(Q), i = 1, 2, ..., m.$ $\mathcal{U}_1^T\mathcal{U}_1 = \mathcal{V}_1\mathcal{V}_1^T = \mathcal{D}\mathcal{D}^{-1} = E.\mathcal{U}_1 \in R^{n\times m}, \mathcal{U}_2 \in R^{n\times(r-m-1)}, \mathcal{V}_1 \in R^{m\times m}$.

(4) Compute the sum matrix $T = (h_{ij})$ of B, C, $T = \mathcal{M}\mathcal{V}_1\mathcal{D}\mathcal{D}^{-1}\mathcal{U}_1^T + W\mathcal{U}_2^T$, where W is an arbitrary $n \times (r - m - 1)$ matrix.

(5) Choose B and C such that $\rho_{ij} + \varphi_{ij} = h_{ij}$.

(6) The vector of neuron inputs is calculated from $I = \epsilon_r - T\eta_r$.

Remark 4.1. In [29], Theorem 3.2 solved the validity of the above-mentioned design process, and the authors employed this design procedure applied to the associative memory of integer-order neural networks in [3]. In contrast, this paper applies that design process to the associative memory of DFNNs.

5. Illustrative examples

Here, four examples are provided to demonstrate the validity of the theoretical results.

Example 5.1. Consider the following 2-dimensional DFNNs with $\alpha = 0.98$:

$$\begin{cases} {}^C D_{t_0}^{\alpha} x_1(t) = -x_1(t) + 2.9 f_1(x_1(t)) + 0.1 f_2(x_2(t)) + 0.1 f_1(x_1(t-\tau_{11}(t))) + 0.1 f_2(x_2(t-\tau_{12}(t))) + 0.9, \\ {}^C D_{t_0}^{\alpha} x_2(t) = -x_2(t) + 0.1 f_1(x_1(t)) + 2.7 f_2(x_2(t)) + 0.1 f_1(x_1(t-\tau_{21}(t))) + 0.1 f_2(x_2(t-\tau_{22}(t))) + 1, \end{cases} \tag{5.1}$$

where $\tau_{11}(t) = \tau_{21}(t) = \frac{t}{1+t}$, $\tau_{12}(t) = \tau_{22}(t) = \frac{e^t}{1+e^t}$, and

$$f(x) = \begin{cases} -1, & -\infty < x < -\frac{\pi}{2}, \\ -\sin(3x), & -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < +\infty. \end{cases} \tag{5.2}$$

For the following calculation formulas:

$$\frac{\pi}{2}\beta_1 - \rho_{11} + |\rho_{12}| + |\varphi_{11}| + |\varphi_{12}| + u_1 = \frac{\pi}{2} - 1.6 \approx -0.0292 < 0,$$

$$-\frac{\pi}{2}\beta_2 + \rho_{22} - |\rho_{21}| - (|\varphi_{21}| + |\varphi_{22}|) + u_2 = 3.4 - \frac{\pi}{2} \approx 1.8292 > 0,$$

and pick $\varepsilon_1 = \varepsilon_2 = 1$, and $\varsigma = 50 > \sigma = 1$, such that

$$\beta_i - \chi_i - \frac{3}{\varepsilon_i}\sum_{j=1, j\neq i}^{n} |\rho_{ij}|\varepsilon_j - \frac{3}{\varepsilon_i}\sum_{j=1}^{n} |\varphi_{ij}|\varepsilon_j \left(\frac{\varsigma}{\varsigma - \sigma}\right)^{\alpha} - \frac{1 - \alpha + 2\alpha^2 - \alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)} = 0.0661 > 0.$$

As a result, we can see that the criteria introduced in Theorems 3.1–3.3 are met for DFNNs (5.1). Accordingly, there are $5^2 = 25$ equilibria for (5.1), among which $3^2 = 9$ equilibria are locally $O(t^{-\alpha})$ stable. Utilizing MATLAB, we can see the evolution behaviors of DFNNs (5.1) in Figures 2 and 3.

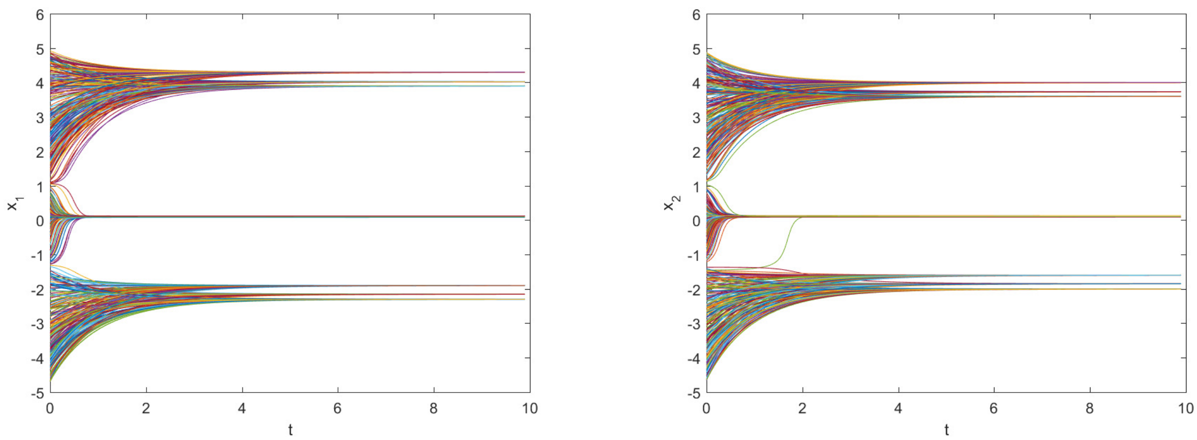

Figure 2. Transient behaviors of DFNNs (5.1).

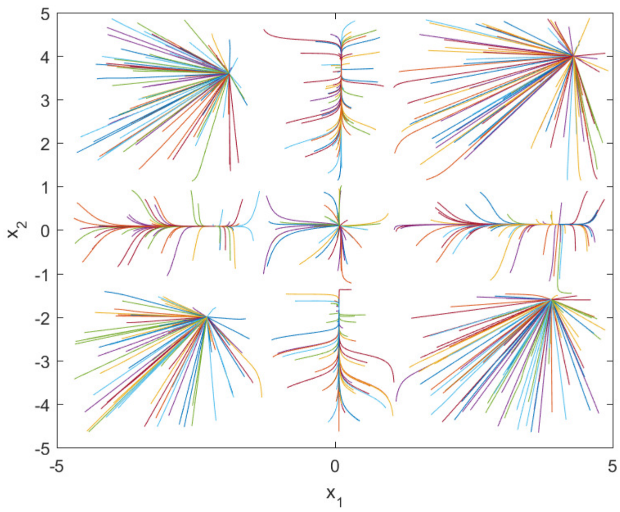

Figure 3. Phase trajectory of DFNNs (5.1).

Example 5.2. Consider the following 2-dimensional DFNNs with $\alpha = 0.98$:

$$\begin{cases} {}^{C}D^{\alpha}_{t_0}x_1(t) = -x_1(t) + 2.9f_1(x_1(t)) + 0.1f_2(x_2(t)) + 0.1f_1(x_1(t-\tau_{11}(t))) + 0.1f_2(x_2(t-\tau_{12}(t))) + 0.9, \\ {}^{C}D^{\alpha}_{t_0}x_2(t) = -x_2(t) + 0.1f_1(x_1(t)) + 2.7f_2(x_2(t)) + 0.1f_1(x_1(t-\tau_{21}(t))) + 0.1f_2(x_2(t-\tau_{22}(t))) + 1, \end{cases} \tag{5.3}$$

where $\tau_{11}(t) = \tau_{21}(t) = \frac{t}{1+t}$, $\tau_{12}(t) = \tau_{22}(t) = \frac{e^t}{1+e^t}$, and

$$f(x) = \begin{cases} -1, & -\infty < x < -\frac{\pi}{2}, \\ \sin(5x), & -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < +\infty. \end{cases} \tag{5.4}$$

For the following calculation formulas:

$$\frac{\pi}{2}\beta_1 - \rho_{11} + |\rho_{12}| + |\varphi_{11}| + |\varphi_{12}| + u_1 = \frac{\pi}{2} - 1.6 \approx -0.0292 < 0,$$

$$-\frac{\pi}{2}\beta_2 + \rho_{22} - |\rho_{21}| - (|\varphi_{21}| + |\varphi_{22}|) + u_2 = 3.4 - \frac{\pi}{2} \approx 1.8292 > 0,$$

and pick $\varepsilon_1 = \varepsilon_2 = 1$, and $\varsigma = 50 > \sigma = 1$, such that

$$\beta_i - \chi_i - \frac{3}{\varepsilon_i}\sum_{j=1, j\neq i}^{n} |\rho_{ij}|\varepsilon_j - \frac{3}{\varepsilon_i}\sum_{j=1}^{n} |\varphi_{ij}|\varepsilon_j(\frac{\varsigma}{\varsigma - \sigma})^{\alpha} - \frac{1 - \alpha + 2\alpha^2 - \alpha^3}{\varsigma^{\alpha}\Gamma(2-\alpha)} = 0.0661 > 0.$$

As a result, we can see that the criteria introduced in Theorems 3.1–3.3 are met for DFNNs (5.3). Accordingly, there are $7^2 = 49$ equilibria for (5.3), among which $4^2 = 16$ equilibria are locally $O(t^{-\alpha})$ stable. Using MATLAB, we can see the evolution behaviors of DFNNs (5.3) in Figures 4 and 5.

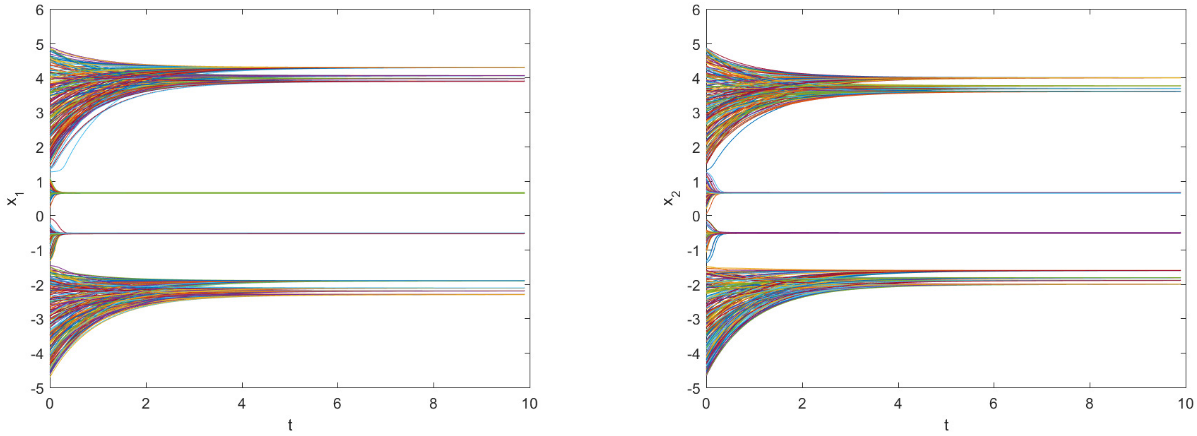

Figure 4. Transient behaviors of DFNNs (5.3).

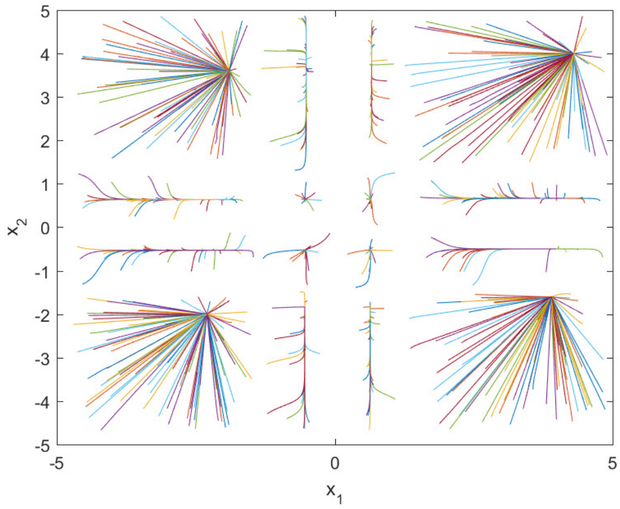

Figure 5. Phase trajectory of DFNNs (5.3).

Example 5.3. Next, we verify neurodynamic associative memory. Assuming the target pattern that need to be memorized is a 6×3–pixel image, as shown in Figure 6, in which a black pixel represents '-1', and a white pixel represents '1', that is, the target pattern is $(-1, -1, -1,\ 1,\ 1,\ 1;\ 1,\ 1, -1, -1, -1, -1; -1, -1, -1,\ 1,\ 1,\ 1)^T$. Here, we will synthesize associative memory by adopting model (5.1) with the activation function (5.2) in Example 1, and then we can get 9 locally $O(t^{-\alpha})$ stable equilibria, which are designated as $X_1, X_2, ..., X_9$. By applying MATLAB, these equilibria are $X_1 = (-1.9, 3.6)$, $X_2 = (-2.151, 0.086)$, $X_3 = (-2.3, -2.0)$, $X_4 = (0.119, 3.7341)$, $X_5 = (0.0848, 0.1025)$, $X_6 = (0.0705, -1.842)$, $X_7 = (4.1, 4.0)$, $X_8 = (3.8236, 0.1306)$, and $X_9 = (3.7, -1.6)$. Besides, define $X_i = (X_{i1}, X_{i2})^T, i = 1, 2, ..., n$.

By calculation,

$$f(X_{11}) = -1, f(X_{12}) = 1, f(X_{21}) = -1, f(X_{22}) = -0.2551, f(X_{31}) = -1, f(X_{32}) = -1,$$

$$f(X_{41}) = -0.3294, f(X_{42}) = 1, f(X_{51}) = -0.2517, f(X_{52}) = -0.3027, f(X_{61}) = -0.2099, f(X_{62}) = -1,$$

$$f(X_{71}) = -1, f(X_{72}) = 1, f(X_{81}) = 1, f(X_{82}) = -0.3819, f(X_{91}) = 1, f(X_{92}) = -1.$$

To get a better display, define nine vectors: $e_i = (e_{i1}, e_{i2})^T, i = 1, 2, ..., 9$. The details are as follows:

$$e_1 = (0, -2)^T, e_2 = (0, 1.2551)^T, e_3 = (2, 2)^T,$$

$$e_4 = (1.3294, 0)^T, e_5 = (-0.7483, -0.6973)^T, e_6 = (-0.7901, 0)^T,$$
$$e_7 = (0, -2)^T, e_8 = (-2, 1.3819)^T, e_9 = (0, 2)^T.$$

Define $\mathcal{J}_i = (\mathcal{J}_{i1}, \mathcal{J}_{i2})^T = (f(X_{i1}) + e_{i1}, f(X_{i2}) + e_{i2})^T$. Let $\mathcal{J} = (\mathcal{J}_1; \mathcal{J}_2; \mathcal{J}_3; \mathcal{J}_4; \mathcal{J}_5; \mathcal{J}_6; \mathcal{J}_7; \mathcal{J}_8; \mathcal{J}_9) = (-1, -1, -1, 1, 1, 1; 1, 1, -1, -1, -1, -1; -1, -1, -1, 1, 1, 1)^T$. By the aid of the proposed design procedure in Section 4, the matrices B and C need to satisfy $B + C = \begin{bmatrix} 3 & 0.2 \\ 0.2 & 2.8 \end{bmatrix}$. Then we can pick the same connection weights ρ_{ij}, φ_{ij} as (5.1) in Example 1. The corresponding evolutionary pattern is shown in Figure 7.

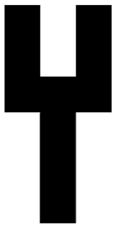

Figure 6. Target pattern of 6×3–pixel image.

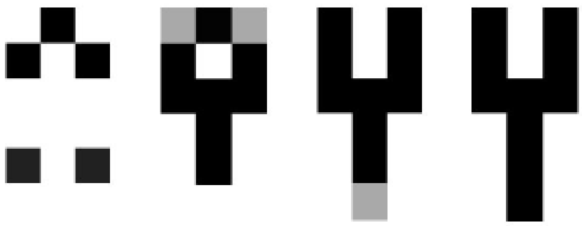

Figure 7. Evolutionary pattern based on DFNNs (5.1) with the activation function (5.2).

Example 5.4. To further demonstrate the universality and applicability of theoretical results, we apply neurodynamic associative memory to the explaining-lesson skills assessment of normal students. In China, in order to further carry out the new normal construction, actively adapt and serve the needs of educational reform and development for teacher training, whose essence is also to promote the reform of teacher education training modes and test the results of teaching and training of basic teaching skills for normal students, explaining-lesson skills of normal students have received high attention from the governments and universities. Generally, the explaining-lesson process in China includes the following aspects: (1) textbook analysis, (2) student analysis, (3) determining teaching objectives and key and difficult points, (4) determining teaching and learning methods, (5) selecting teaching aids and learning aids, and (6) designing the teaching process. Based on the explaining-lesson process, there are nine indexes for evaluating explaining-lesson skills: (I) textbook analysis, (II) teaching objectives, (III) teaching priorities, (IV) teaching difficulties, (V) teaching methods, (VI) teaching process, (VII) blackboard design, (VIII) post-class reflection, and (IX) explaining-lesson modes. For a specific normal student, which index has the most significant impact on his or her explaining-lesson skill is crucial for personalized discrimination and improvement of his or her teaching skill. From a signal processing perspective, this is essentially a nonlinear classification problem.

We know that high-dimensional mapping can effectively solve nonlinear classification problems. Firstly, based on the format of the data frames in the explaining-lesson videos, we extract the data fields that need to be converted into matrices through the algorithm in [30]. Then we determine the dimension n of the matrix by way of the characteristics and requirements of the data fields, where the choice of dimension n depends on the specific application scenario and requirement. The next question is how to map data fields to this $n \times n$ matrix according to a certain rule. Here, this rule will be the neurodynamic associative memory that is going to be explored. Based on DFNNs (5.1) with the activation function (5.2) in Example 1 (i.e., as an associative memory network), the network model is used to construct the structural elements of high-dimensional mapping, where 9 local $O(t^{-\alpha})$ stable equilibria represent the corresponding membership degree (i.e., corresponding to the previously stated nine indexes (I)-(IX)). Once adaptive structural elements are obtained, the samples are preprocessed before classification so as to reduce the impact of irrelevant interference information on classification accuracy. Neurodynamic associative memory based on DFNNs (5.1) with the activation function (5.2) is a very useful intelligent classifier.

Figure 8 describes three scene segments of explaining-lesson for normal students, Figure 9 displays normalization a preprocessing of samples before classification; and Figure 10 shows a certain index that has the most significant impact on the explaining-lesson skill of a specific normal student.

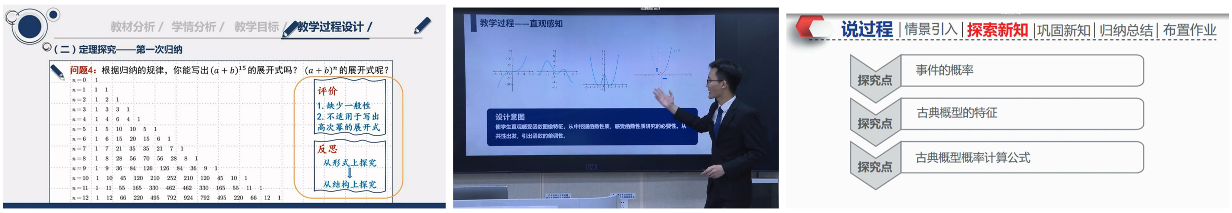

Figure 8. Three scene segments of explaining-lesson for normal students.

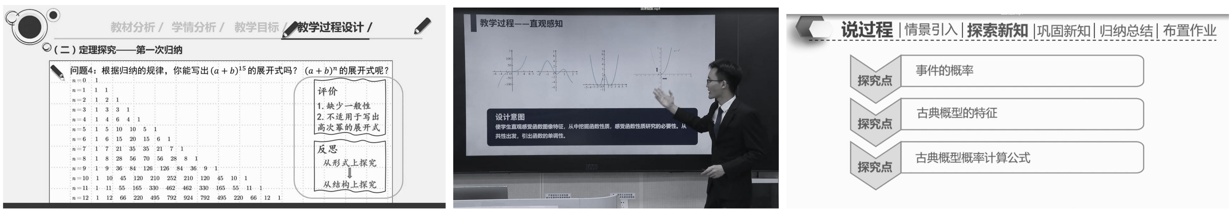

Figure 9. The normalization preprocessing of samples.

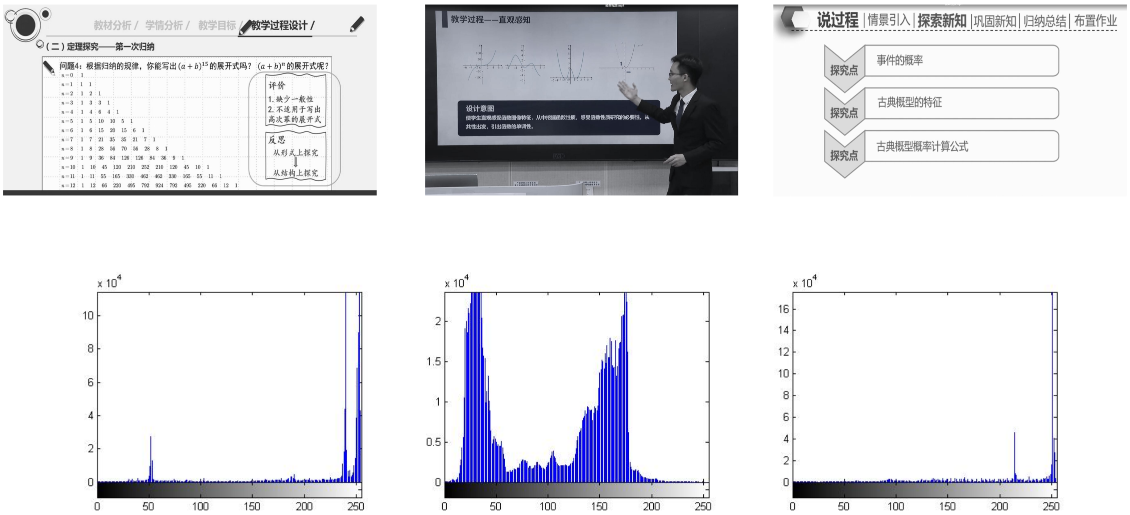

Figure 10. The most influential index on the explaining-lesson skills of specific normal students.

6. Conclusions

This paper investigates the multiple $O(t^{-\alpha})$ stability of DFNNs with a type of piecewise nonlinear activation function. Some sufficient conditions are acquired to assure that there exist 5^n equilibria for DFNNs (2.1), and 3^n equilibria of them are locally multiple $O(t^{-\alpha})$ stable. Furthermore, we apply these results to a more general case, revealing that the time-varying DFNNs can attain $(2k + 1)^n$ equilibria, among which $(k + 1)^n$ equilibria exhibit $O(t^{-\alpha})$ stability. Here, the parameter k is highly dependent on the frequency of the sinusoidal functions in the expanded activation functions. Accordingly, this work offers effective assistance in obtaining a larger storage capacity for the application of DFNNs in associative memory.

Author contributions

Jiangwei Ke: Conceptualization, formal analysis, methodology, writing-original draft, writing-review and editing; Jine Zhang: Software, validation and supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. Z. G. Zeng, J. Wang, Design and analysis of high-capacity associative memories based on a class of discrete-time recurrent neural networks, *IEEE Trans. Syst. Man Cybern. B*, **38** (2008), 1525–1536. https://doi.org/10.1109/tsmcb.2008.927717
2. Z. G. Zeng, J. Wang, Associative memories based on continuous-time cellular neural networks designed using space-invariant cloning templates, *Neural Netw.*, **22** (2009), 651–657. https://doi.org/10.1016/j.neunet.2009.06.031
3. G. Bao, Z. G. Zeng, Analysis and design of associative memories based on recurrent neural network with discontinuous activation functions, *Neurocomputing*, **77** (2012), 701–707. https://doi.org/10.1016/j.neucom.2011.08.026
4. J. Park, H. Y. Kim , Y. Park, S. W. Lee, A synthesis procedure for associative memories based on space-varying cellular neural networks, *Neural Netw.*, **14** (2001), 107–113. https://doi.org/10.1016/S0893-6080(00)00086-1
5. Z. G. Zeng, J. Wang, Analysis and design of associative memories based on recurrent neural networks with linear saturation activation functions and time-varying delays, *Neural Comput.*, **19** (2007), 2149–2182. https://doi.org/10.1016/j.neucom.2011.08.026
6. R. Rakkiyappan, G. Velmurugan, J. D. Cao, Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays, *Nonlinear Dyn.*, **78** (2014), 2823–2836. https://doi.org/10.1016/j.neucom.2017.03.042
7. A. L. Wu, Z. G. Zeng, Global Mittag-Leffler stabilization of fractional-order memristive neural networks, *IEEE Trans. Neural Netw. Learn Syst.*, **28** (2015), 206–217. https://doi.org/10.1109/tnnls.2015.2506738
8. W. Zhang, R. Wu, J. Cao, A. Alsaedi, T. Hayat, Synchronization for stochastic fractional-order memristive BAM neural networks with multiple delays, *Fractal Fract.*, **7** (2023), 678. https://doi.org/10.3390/fractalfract7090678
9. H. B. Bao, J. D. Cao, Projective synchronization of fractional-order memristor-based neural networks, *Neural Netw.*, **63** (2015), 1–9. https://doi.org/10.1016/j.neunet.2014.10.007
10. B. S. Chen, J. J. Chen, Global asymptotical ω-periodicity of a fractional-order non-autonomous neural networks, *Neural Netw.*, **68** (2015), 78–88. https://doi.org/10.1016/j.neunet.2015.04.006

11. C. Y. Chen, S. Zhu, Y. C. Wei, C. Y. Yang, Finite-time stability of delayed memristor-based fractional-order neural networks, *IEEE Trans Cybern.*, **50** (2018), 1607–1616. https://doi.org/10.1109/tcyb.2018.2876901

12. J. E. Zhang, Linear-type discontinuous control of fixed-deviation stabilization and synchronization for fractional-order neurodynamic systems with communication delays, *IEEE Access*, **6** (2018), 52570–52581. https://doi.org/10.1109/access.2018.2870979

13. J. J. Chen, B. S. Chen, Z. G. Zeng, $O(t^{-\alpha})$ synchronization and Mittag-Leffler synchronization for the fractional-order memristive neural networks with delays and discontinuous neuron activations, *Neural Netw.*, **100** (2018), 10–24. https://doi.org/10.1016/j.neunet.2018.01.004

14. B. S. Chen, J. J. Chen, Global $O(t^{-\alpha})$ stability and global asymptotical periodicity for a non-autonomous fractional-order neural networks with time-varying delays, *Neural Netw.*, **73** (2016), 47–57. https://doi.org/10.1016/j.neunet.2015.09.007

15. P. Liu, M. X. Kong, Z. G. Zeng, Projective synchronization analysis of fractional-order neural networks with mixed time delays, *IEEE Trans. Cybern.*, **52** (2020), 6798–6808. https://doi.org/10.1109/tcyb.2020.3027755

16. X. B. Nie, J. D. Cao, Multistability of competitive neural networks with time-varying and distributed delays, *Nonlinear Anal. Real World Appl.*, **10** (2009), 928–942. https://doi.org/10.1016/j.nonrwa.2007.11.014

17. P. Liu, Z. G. Zeng, J. Wang, Multiple Mittag-Leffler stability of fractional-order recurrent neural networks, *IEEE Trans. Syst. Man Cybern. Syst.*, **47** (2017), 2279–2288. https://doi.org/10.1109/tsmc.2017.2651059

18. Z. G. Zeng, W. X. Zheng, Multistability of two kinds of recurrent neural networks with activation functions symmetrical about the origin on the phase plane, *IEEE Trans. Neural Netw. Learn. Syst.*, **24** (2013), 1749–1762. https://doi.org/10.1109/tnnls.2013.2262638

19. P. Liu, Z. G. Zeng, J. Wang, Multistability of recurrent neural networks with nonmonotonic activation functions and mixed time delays, *IEEE Trans. Syst. Man Cybern. Syst.*, **46** (2015), 512–523. https://doi.org/10.1109/tsmc.2015.2461191

20. P. P. Liu, X. B. Nie, J. L. Liang, J. D. Cao, Multiple Mittag-Leffler stability of fractional-order competitive neural networks with Gaussian activation functions, *Neural Netw.*, **108** (2018), 452–465 . https://doi.org/10.1016/j.neunet.2018.09.005

21. P. Liu, Z. G. Zeng, J. Wang, Complete stability of delayed recurrent neural networks with Gaussian activation functions, *Neural Netw.,* **85** (2017), 21–32. https://doi.org/10.1016/j.neunet.2016.09.006

22. P. Liu, Z. G. Zeng, J Wang, Multistability of delayed recurrent neural networks with Mexican hat activation functions, *Neural Comput.*, **29** (2017), 423–457. https://doi.org/10.1162/necoa00922

23. Y. Shen, S. Zhu, X. Liu, S. Wen, Multistability and associative memory of neural networks with morita-like activation functions, *Neural Netw.*, **142** (2021), 162–170. https://doi.org/10.1016/j.neunet.2021.04.035

24. Y. Liu, Z. Wang, Q. Ma, H. Shen, Multistability analysis of delayed recurrent neural networks with a class of piecewise nonlinear activation functions, *Neural Netw.*, **152** (2022), 80–89. https://doi.org/10.1016/j.neunet.2022.04.015
25. L. G. Wan, Z. X. Liu, Multiple $O(t^{-\alpha})$ stability for fractional-order neural networks with time-varying delays, *J Franklin Inst.*, **357** (2020), 12742–12766. https://doi.org/10.1016/j.jfranklin.2020.09.019
26. L. G. Wan, Z. X. Liu, Multiple $O(t^{-q})$ stability and instability of time-varying delayed fractional-order Cohen-Grossberg neural networks with Gaussian activation functions, *Neurocomputing*, **454** (2021), 212–227. https://doi.org/10.1016/j.neucom.2021.05.018
27. C. P. Li, F. R. Zhang, A survey on the stability of fractional differential equations, *Eur. Phys. J. Special Topics*, **193** (2011), 27–47. https://doi.org/10.1140/epjst/e2011-01379-1
28. A. L. Wu, L. Liu, T. W. Huang, Z. G. Zeng, Mittag-Leffler stability of fractional-order neural networks in the presence of generalized piecewise constant arguments, *Neural Netw.*, **85** (2017), 118–127. https://doi.org/10.1016/j.neunet.2016.10.002
29. D. Liu, A. N. Michel, Sparsely interconnected neural networks for associative memories with applications to cellular neural networks, *IEEE Trans. Circuits Syst.*, **41** (1994), 295–307. https://doi.org/10.1007/bfb0032156
30. S. S. Bucak, B. Gunsel, Video content representation by incremental non-negative matrix factorization, *Proc. Int. Conf. Image Proc.*, **2** (2007), 113–116. https://doi.org/10.1109/ICIP.2007.4379105