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Research article

The analysis of fractional neutral stochastic differential equations in $\mathfrak{L}^{\tilde{p}}$ space

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Abstract: After extensive examination, scholars have determined that many dynamic systems exhibit intricate connections not only with their current and past states but also with the delay function itself. As a result, their focus shifts towards fractional neutral stochastic differential equations, which find applications in diverse fields such as biology, physics, signal processing, economics, and others. The fundamental principles of existence and uniqueness of solutions to differential equations, which guarantee the presence of a solution and its uniqueness for a specified equation, are pivotal in both the mathematical and physical realms. A crucial approach for analyzing complex systems of differential equations is the utilization of the averaging principle, which simplifies problems by approximating existing ones. Applying contraction mapping principles, we present results concerning the concepts of existence and uniqueness for the solutions of fractional neutral stochastic differential equations. Additionally, we present Ulam-type stability and the averaging principle results within the framework of $\mathfrak{L}^{\tilde{p}}$ space. This exploration involved the utilization of Jensen's, Gröenwall-Bellman's, Hölder's, Burkholder-Davis-Gundy's inequalities, and the interval translation technique. Our findings are established within the context of the conformable fractional derivative, and we provide several examples to aid in comprehending the theoretical outcomes.

Keywords: fractional neutral stochastic differential equations; conformable fractional derivative; averaging principle; existence and uniqueness
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

Fractional calculus (Fr-Cal), a branch of calculus, extends the concepts of differentiation and integration to include non-integer orders. While traditional calculus restricts differentiation and integration to integer orders, Fr-Cal broadens these operations to encompass non-integer orders, facilitating more sophisticated analysis and modeling of phenomena that exhibit fractional behavior or dimensions. Fr-Cal serves as a robust tool for examining and modeling complex systems that display dynamics of non-integer order, such as fractional differential equations (FrDEs), fractional order control systems, and fractional order signal processing. It provides a more comprehensive framework for understanding and articulating phenomena that cannot be adequately captured through conventional integer-order calculus.

Fr-Cal offers several benefits and advantages in various fields [1, 2]:

- Many natural phenomena exhibit behaviors that cannot be fully described by integer-order differential equations (DEs). Fr-Cal provides a more accurate and flexible framework for modeling such complex systems, including those with memory effects, anomalous diffusion, and fractal geometries.
- Fr-Cal naturally accounts for memory effects in systems where the rate of change depends on past history rather than just the current state. This is particularly useful in fields like viscoelasticity, biology, and finance.
- Fractal geometry and dynamics play a crucial role in many systems, from chaotic systems to biological structures. Fr-Cal enables the description of fractal behaviors more accurately than traditional integer-order calculus.
- Fr-Cal has applications in signal processing, where it is used to design filters and analyze signals exhibiting non-integer-order characteristics. It helps in better understanding and processing signals in various applications such as telecommunications, image processing, and medical signal analysis.
- Fractional-order control systems offer advantages such as improved robustness, better stability, and enhanced performance compared to their integer-order counterparts. They find applications in areas such as robotics, automotive systems, and industrial automation.
- Fr-Cal is used to model anomalous transport phenomena where particles exhibit behaviors such as superdiffusion or subdiffusion. This has applications in fields such as environmental science, geophysics, and chemical engineering.
- Fr-Cal techniques are increasingly being applied to optimization problems and dynamical systems analysis, providing new insights and approaches for tackling complex optimization tasks and understanding system behavior.

Overall, Fr-Cal provides a more comprehensive and versatile mathematical framework that extends the capabilities of traditional integer-order calculus, enabling more accurate modeling, analysis, and control of complex systems across various disciplines.

Fractional derivatives (Fr-D) are mathematical operators that generalize traditional integer-order derivatives to non-integer orders. They are essential in various fields due to their ability to capture complex dynamics and behaviors that cannot be adequately described by integer-order derivatives alone. There are several types of Fr-D, each with its own mathematical definition and properties.

Some common types include [3, 4]:

- Riemann-Liouville derivative: This is one of the earliest definitions of Fr-D, introduced by Riemann and Liouville. It generalizes the concept of integer-order derivatives to non-integer orders through the use of an integral representation.
- Caputo fractional derivative (Cap-FrD): This derivative was proposed by Caputo as an alternative definition that avoids singularities at the initial time. It is defined by taking the integer part of the derivative of a function and then applying a fractional integration.
- Grünwald-Letnikov derivative: This derivative is based on a discretization of the Fr-D using a weighted sum of function values at discrete time points. It provides a numerical approximation of the Fr-D and is often used in computational methods.
- Conformable fractional derivative (Con-FrD): This derivative represents a relatively recent advancement in the field of Fr-Cal, providing an alternative approach to fractional differentiation. It was introduced to tackle certain limitations and challenges associated with traditional Fr-D. The Con-FrD inherits many properties from classical differentiation, including linearity, the product rule, and the chain rule. However, it also possesses unique properties that distinguish it from other Fr-D, such as the Riemann-Liouville or Caputo derivatives.

The Con-FrD offers several advantages [5, 6]:

- The Con-FrD is based on a simple and intuitive definition that extends the concept of Fr-D to a wider class of functions defined on intervals with constrained endpoint behavior. This makes it easier to understand and apply in practice compared to other types of Fr-D.
- One of the key advantages of the Con-FrD is its ability to preserve endpoint conditions, such as initial or boundary conditions, which are essential for many practical applications. This property allows for a more accurate representation of physical systems and ensures that solutions to FrDEs satisfy the necessary constraints.
- Con-FrD provides a generalization of Fr-Cal that encompasses both classical integer-order derivatives and traditional Fr-D. This allows for a unified treatment of DEs involving mixed integer and fractional orders, facilitating the analysis and solution of complex dynamical systems.
- The Con-FrD offers numerical stability and efficiency in computational implementations, making it suitable for numerical simulations and practical applications in engineering, physics, and other fields. Its simplicity and computational efficiency make it an attractive choice for solving DEs involving fractional orders.
- The Con-FrD has been successfully applied in control theory and signal processing, where it offers advantages such as improved stability, robustness, and performance compared to traditional integer-order and fractional-order control systems. Its ability to capture memory effects and long-range dependence makes it particularly useful in modeling and controlling complex dynamical systems.

In summary, the Con-FrD provides a flexible and potent tool for modeling, analyzing, and managing intricate systems across diverse scientific and engineering fields. Its benefits include simplicity, preservation of endpoint conditions, and computational efficiency.

In terms of a mapping $\varpi(\rho) : [0, \infty[\to \mathbb{R}$, the Con-FrD is [7]

$$\mathscr{T}_{\rho}^{\psi}\varpi(\rho) = \lim_{\epsilon \to 0} \frac{\varpi^{\lceil\psi\rceil-1}(\rho + \epsilon\rho^{\lceil\psi\rceil-\psi}) - \varpi^{\lceil\psi\rceil-1}(\rho)}{\epsilon},$$

where $\mathscr{T}_{\rho}^{\psi}$ is the Con-FrD with respect to time and $u-1<\psi\leq u,\ \rho>0,\ u\in\mathscr{N}$ and $\lceil\psi\rceil$, the lowest integer that is equal to or larger than ψ. In a certain instance, if $0<\psi\leq 1$, afterward, we acquire

$$\mathscr{T}_{\rho}^{\psi}\varpi(\rho)=\lim_{\epsilon\to 0}\frac{\varpi(\rho+\epsilon\rho^{1-\psi})-\varpi(\rho)}{\epsilon},\ \rho>0.$$

If $\varpi(\rho)$ is ψ-differentiable in some $(0,\Lambda)$, $\Lambda>0$ and $\lim_{\rho\to 0^{+}}\varpi^{\psi}(\rho)$ exists, then define $\varpi^{(\psi)}(0)=\lim_{\rho\to 0^{+}}\varpi^{(\psi)}(\rho)$.

An integral of a function $\varpi(\rho)$ that is conformable fractional $\varpi(\rho)$ starting from $\rho_0\geq 0$ is defined as [7]

$$\mathfrak{I}_{\psi}^{\rho_0}(\varpi)(\rho)=\int_{\rho_0}^{\rho}\frac{\varpi(\vartheta)}{(\vartheta-\rho_0)^{1-\psi}}\mathrm{d}\vartheta,\ \psi\in(0,\ 1].$$

The topics covered in [8–11] include its physical interpretation, the Leibniz rule, Gronwall's inequality, integration by parts, the chain rule, exponential functions, and Taylor series. By utilizing the Con-FrD in a gray system model, Ma et al. demonstrated its appropriateness and efficacy [12]. Moreover, significant research has been conducted on Ulam's stability, the Lotka-Volterra model, Sturm's theorems, and the variational iteration technique in [13–18]. The authors of [19, 20] have recently explored stochastic differential equations (SDEs) in the sense of conformable Itô, Lyapunov stability, existence results for solutions, exponential stability, and Ulam-type stability.

DEs with a random or stochastic component are known as SDEs. Within the fields of social and natural sciences, such as economics, physics, biology, chemistry, ecology, finance, and engineering, SDEs are extensively employed to model a wide range of phenomena.

Stochastic differential equations (SDEs) find applications in various fields, particularly in modeling systems where random fluctuations play a significant role. Some notable applications include [21, 22]:

- Finance: Perhaps one of the most well-known applications is in financial modeling, especially in options pricing, portfolio optimization, and risk management. SDEs, particularly geometric Brownian motion, are foundational in the Black-Scholes model for option pricing.
- Physics: In areas like statistical physics and quantum mechanics, SDEs are used to describe systems subject to random forces or fluctuations. For example, Langevin equations describe the motion of particles subjected to random forces, while the Fokker-Planck equation describes the evolution of probability distributions in systems with stochastic dynamics.
- Chemistry and biology: SDEs are employed in modeling chemical reactions, population dynamics, and biological processes where randomness and uncertainty play a role. For instance, in enzyme kinetics, SDEs can capture the stochastic nature of molecular collisions.
- Engineering: In control theory, SDEs are used to model systems with uncertain inputs or parameters, leading to robust control strategies. They're also utilized in signal processing for tasks like filtering noisy signals or estimating parameters in dynamic systems.
- Economics: SDEs are applied in modeling economic systems with uncertainty, such as in macroeconomic models, stochastic growth models, and models of asset pricing.
- Climate science: SDEs are used in climate modeling to represent stochastic processes such as weather fluctuations. They can capture the uncertainty in climate models due to chaotic dynamics and external forcings.

- Neuroscience: SDEs are employed in modeling neural systems, where they can capture the stochastic behavior of neurons and synaptic transmission. For instance, they're used in models of neuronal firing patterns and in understanding neural coding.
- Machine learning and data science: SDEs are increasingly being used in machine learning, particularly in probabilistic modeling and deep learning frameworks. They provide a formalism for modeling uncertainty in data and can be used for tasks such as time series forecasting, Bayesian inference, and generative modeling.

These are just a few examples, but the versatility of SDEs makes them applicable in numerous fields where randomness and uncertainty are inherent features of the systems being studied.

A DE featuring a Fr-D and a stochastic process is referred to as a fractional stochastic differential equation (FrSDE). These equations serve as models for systems exhibiting long-range dependencies, turbulence, economics, and biology. However, the concept of FrSDEs still presents numerous questions and challenges that require resolution. Issues such as the existence and uniqueness (EU) of solutions, stability and convergence, numerical techniques, and applications to physics, biology, finance, and other domains remain intriguing subjects of study. Several writers have recently been actively working on FrSDEs; for instance, Li and Xu [23] developed exponential stabilization for delay FrSDEs. The authors in [24, 25] also introduced novel criteria for evaluating stability within the $\mathfrak{L}^2$ space. Li and Peng [26] established controllability results for FrSDEs using Sadovskii's fixed-point theorem (FPT). Similarly, Cui and Yan [27] applied the same FPT in Hilbert spaces. In [28], the authors demonstrated the asymptotic and stability results in the $\mathfrak{L}^2$ space of FrSDEs in the sense of Cap-FrD. In [29], the authors presented results related to the EU and stability within the framework of Hyers-Ulam for FrSDEs of Cap-FrD, utilizing the Banach FPT. In [30], the authors explored stability in terms of exponential and EU results for fuzzy FrSDEs under the Lipschitzian condition. Similar results were established in [31, 32]. The researchers in [33, 34] established criteria for the EU of solutions for FrSDEs under various assumptions. Karczewska and Lizama [35], as well as Schnaubelt and Veraar [36], explored distinct aspects of stochastic Volterra equations and FrSDEs. Karczewska and Lizama elaborated on various findings concerning perturbations in stochastic Volterra equations while also discussing the EU results for FrSDEs. Schnaubelt and Veraar focused on demonstrating the path-wise continuity properties of solutions for the same model. Xiao and Wang [37] utilized the stopping time technique to investigate the stability of FrSDEs characterized by the Caputo type.

Fractional neutral stochastic differential equations (FrNSDEs) and neutral stochastic differential equations both involve delayed terms. However, neutral stochastic differential equations use ordinary derivatives, while FrNSDEs employ Fr-D to capture memory effects and long-range dependencies more accurately. FrNSDEs constitute a robust mathematical framework for modeling intricate systems characterized by random fluctuations, memory effects, and time delays. They integrate principles from Fr-Cal, stochastic calculus, and delay DEs. FrNSDEs manifest as DEs wherein Fr-D, stochastic processes, and delays coexist. These equations represent a type of DE where the function's derivative at any given time depends on both the function's values at that time and at prior instances.

Applications of FrNSDEs can be found in various fields, including [38–40]:

- FrNSDEs are used to model biological systems, where memory effects play a significant role, such as neuronal dynamics, gene regulation, and population dynamics with environmental fluctuations.
- In finance, FrNSDEs are employed to model stock price dynamics, option pricing, and risk

management, taking into account the long-range dependence and memory effects observed in financial time series data.
- FrNSDEs are applied in various areas of physics, including anomalous diffusion, where particles exhibit sub-diffusive or super-diffusive behavior due to non-local interactions or memory effects.
- FrNSDEs are utilized in engineering applications such as control systems, where memory effects and stochastic fluctuations need to be considered for accurate modeling and control design.
- FrNSDEs are used in signal processing applications, such as noise reduction and signal denoising, where signals with memory effects and stochastic fluctuations need to be analyzed and processed.
- FrNSDEs can be applied to model environmental systems, such as climate dynamics and ecological systems, where memory effects and stochastic fluctuations are important factors influencing system behavior.
- In economics, FrNSDEs are used to model macroeconomic variables, such as GDP growth and unemployment rates, taking into account memory effects and stochastic shocks.

Overall, FrNSDEs provide a flexible framework for modeling complex systems in various fields, allowing researchers to capture the interplay between memory effects, stochastic fluctuations, and system dynamics. Moreover, FrNSDEs, which involve derivatives with delays alongside the function itself, have attracted considerable research attention. A substantial body of work [41–45] has been dedicated to investigating various aspects of FrNSDEs, including their existence, controllability, optimal control, stability, and asymptotic estimations of solutions, including random periodic solutions.

The averaging principle has drawn interest from scholars in a variety of domains in recent decades. Because it may further balance between simple and complex systems and successfully simplify deterministic and stochastic systems, the averaging principle, as an approximate principle, can be of great interest to scholars. The fundamental idea behind the averaging principle is to provide an approximation theorem by reducing the complexity of the SDEs, partially substituting the original system, and providing the related optimal order convergence theorem. The averaging principle, an essential approximation theorem, can also, to some extent, be very effective in the average processes of complex equations in engineering mechanics, control, and mathematics. Stated differently, studying the characteristics of complicated equations becomes a straightforward problem thanks to the averaging principle. The non-periodic averaging principle of nonlinear systems, put forward by Bogolyubov and Krylov, is where the averaging principle got its start. Since then, the averaging idea has become widely known and acknowledged. Furthermore, it is important to note that Khasminskii [46] advanced and thoroughly examined the application of the averaging concept to SDEs. In recent times, the concept of averaging in FrSDEs has captured the attention of some scholars. For example, Luo et al. [47] investigated a class of FrSDE with time delays. Under some novel assumptions, the authors obtained an averaging principle for the solution of the considered system. The authors [48] investigated the averaging principle for FrSDE of Caputo-type driven by Brownian motion. The authors also presented approximation theorems in the sense of mean square [49, 50]. Abouagwa and Li [51] established the approximation theorem as an averaging principle for the solutions to FrSDE of Itô-Doob type with non-Lipschitz coefficients. Liu et al. [52] established the averaging principle for FrSDEs in the sense of Caputo-Hadamard. Yang et al. [53] discussed the averaging principle for FrSDEs driven by Lévy noise in the framework of the Hilfer derivative. Liu et al. [54] also investigated periodic averaging principles for various types of impulsive FrSDEs.

Motivated by the earlier findings, we first demonstrated the EU of solutions for a specific category of

FrNSDEs using the Banach FPT and Ulam-Hyers (UH) stability. In the subsequent section, we delve into investigating the averaging principle of a certain class of FrNSDEs. This investigation utilizes various mathematical tools, including Grönwall-Bellman inequality (G-B-I), Jensen inequality (J-I), Burkholder-Davis-Gundy inequality (BHDG-I), Hölder inequality (H-I), and the interval translation approach. To validate the mathematical framework, we also present a few numerical examples.

Moreover, previous research mostly focuses on investigating the EU and averaging principles of FrSDEs within the $\mathfrak{L}^2$ space and concerning Cap-FrD. When compared to the findings documented in [23–43, 47–54], this paper significantly contributes in four main aspects:

i. Differing from [23–37, 47–54], our study deals with a more generalized system as we present the outcomes of FrNSDEs, which exhibit broader generality compared to FrSDEs.

ii. Within this paper, we effectively establish results within the $\mathfrak{L}^{\tilde{p}}$ space. Previous studies predominantly center on the case where $\tilde{p}$ equals 2, as evident in publications like [23, 24, 26–43, 47–54].

iii. Unlike [23, 26–43], our research work focuses on establishing outcomes concerning the EU using Con-FrD.

iv. Unlike [47–54], our research work focuses on establishing outcomes concerning the averaging principles using Con-FrD.

We examined the following FrNSDEs:

$$\begin{cases} \quad \mathscr{T}_{\rho}^{\psi}[\mathfrak{I}(\rho) - \mathcal{Y}(\rho, \mathfrak{I}(\rho), \mathfrak{I}(\rho-\phi))] \\ = \mathcal{G}_1(\rho, \mathfrak{I}(\rho), \mathfrak{I}(\rho-\phi)) + \mathcal{G}_2(\rho, \mathfrak{I}(\rho), \mathfrak{I}(\rho-\phi))\frac{\mathrm{d}\mathscr{B}_{\rho}}{\mathrm{d}\rho}, \ \rho \in [0, \Upsilon] \\ \quad \mathfrak{I}(\rho) = \aleph(\rho), \ \rho \in [-\phi, 0]. \end{cases} \tag{1.1}$$

Here, ψ denotes the Con-FrD with $\psi \in (\frac{1}{2}, 1]$. The function $\mathcal{G}_1 : [\rho_0, \Upsilon] \times \mathbb{R}^{\kappa} \times \mathbb{R}^{\kappa} \to \mathbb{R}^{\kappa}$ and $\mathcal{G}_2 : [\rho_0, \Upsilon] \times \mathbb{R}^{\kappa} \times \mathbb{R}^{\kappa} \to \mathbb{R}^{\kappa\times\mathfrak{r}}$ are both measurable and continuous functions. We consider a complete probability space $(\Omega, \mathscr{F}, \mathfrak{P})$ defining the $\mathfrak{r}$-dimensional Brownian motion. Additionally, the function $\aleph : [-\phi, 0] \to \mathbb{R}^{\kappa}$ is continuous. We use $\|\cdot\|$ with $\Xi\|\aleph(\zeta)\|^2 < \infty$ as the norm for $\mathbb{R}^{\kappa}$.

Equation (1.1) represents a class of FrNSDEs that have been studied by various researchers; some of them are [55–58]. In this research study, we consider the range of Con-FrD to be within $(\frac{1}{2}, 1]$. When ψ reaches 1, Con-FrD transitions into a traditional integer-order derivative, so our established results are also valid for $\psi = 1$. Several studies have established a range of $(\frac{1}{2}, 1]$, such as [56, 59–63].

Selecting the range of the fractional order in FrSDEs depends on various factors, including the specific characteristics of the system being modeled and the desired properties of the resulting equations. Here are some considerations for selecting the range of the fractional order:

- Physical interpretation: Consider the physical interpretation of the fractional order. In some cases, the fractional order may correspond to the degree of memory or long-range dependence in the system. For example, a fractional order close to 1 may indicate strong memory effects, while a fractional order closer to $\frac{1}{2}$ may represent weaker memory effects.
- Modeling requirements: Assess the modeling requirements of the system. Depending on the behavior being modeled, you may need to choose a fractional order that best captures the

dynamics of interest. For instance, if the system exhibits significant memory effects or long-range dependence, a fractional order closer to 1 may be appropriate.

- Numerical stability: Consider the numerical stability of the solution methods used to solve the FrSDEs. Some numerical methods may have limitations on the range of the fractional order for which they provide accurate and stable solutions. Ensure that the chosen range is compatible with the numerical methods being employed.
- Experimental data: If available, utilize experimental data or empirical observations to guide the selection of the fractional order range. Analyze the behavior of the system and adjust the range of the fractional order to best match the observed dynamics.
- Sensitivity analysis: Perform sensitivity analysis to assess how variations in the fractional order affect the behavior of the system. This can help determine the robustness of the chosen range and identify any critical thresholds or transitions in behavior.

FrNSDEs encompass various classes based on the nature of their differential operators, delay terms, and stochastic components. Each class of FrNSDEs has its own properties, solution methods, and applications in different fields such as physics, biology, finance, and engineering. Understanding these classes aids in effectively modeling and analyzing systems with memory effects, delayed responses, and stochastic fluctuations. For more details about the types of FrNSDEs, refer to [59, 64].

The study adheres to this structure: In the next section, we employ certain fundamental concepts and demonstrate a lemma that forms the basis for establishing findings concerning FrNSDEs. Within the initial subsection of Section 3, we demonstrate the EU and Ulam-type stability of FrNSDE solutions, and in the subsequent subsection, we establish the averaging principle, accompanied by examples to elucidate our findings in Section 4. Finally, the conclusion is presented in Section 5.

2. Preliminaries

In this section, we discuss several foundational assumptions, definitions, and lemmas essential for this paper. Initially, we introduce certain assumptions that serve as the cornerstone of our findings. We assume that the coefficient $\mathcal{Y}$ satisfies $\|\mathcal{Y}(0,\aleph(0),\aleph(-\phi))\| < \infty$, and that the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ in Eq (1.1) are uniformly continuous when $\forall \varrho_1, \varrho_2, \alpha_1, \alpha_2, \varrho, \alpha \in \mathbb{R}^\kappa$, $\rho \in [-\phi, \Upsilon]$, there are $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3 > 0$ that meet the following requirements:

(σ_1):

$$\|\mathcal{Y}(\rho,\varrho_1,\varrho_2) - \mathcal{Y}(\rho,\alpha_1,\alpha_2)\| \le \mathscr{V}_1(\|\varrho_1 - \alpha_1\| + \|\varrho_2 - \alpha_2\|).$$

(σ_2):

$$\|\mathcal{G}_1(\rho,\varrho_1,\varrho_2) - \mathcal{G}_1(\rho,\alpha_1,\alpha_2)\| \vee \|\mathcal{G}_2(\rho,\varrho_1,\varrho_2) - \mathcal{G}_2(\rho,\alpha_1,\alpha_2)\| \le \mathscr{V}_2(\|\varrho_1 - \alpha_1\| + \|\varrho_2 - \alpha_2\|),$$

where $\mathcal{G}_1 \vee \mathcal{G}_2 = \max(\mathcal{G}_1, \mathcal{G}_2)$.

(σ_3):

$$\|\mathcal{G}_1(\rho,\varrho,\alpha)\| \vee \|\mathcal{G}_2(\rho,\varrho,\alpha)\| \le \mathscr{V}_3(1 + \|\varrho\| + \|\alpha\|).$$

(σ_4): The functions $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ are defined, and for $\Upsilon_1 \in [0,\Upsilon]$, $\rho \in [0,\Upsilon]$, and $\tilde{p} \ge 2$, we can find positively bounded functions $\mathscr{Y}_1(\Upsilon_1)$ and $\mathscr{Y}_2(\Upsilon_1)$ that satisfy

$$\frac{1}{\Upsilon_1}\int_0^{\Upsilon_1} \|\mathcal{G}_1(\rho,\varrho,\alpha) - \widetilde{\mathcal{G}}_1(\varrho,\alpha)\|^{\tilde{p}} d\rho \le \mathscr{Y}_1(\Upsilon_1)(1 + \|\varrho\|^{\tilde{p}} + \|\alpha\|^{\tilde{p}}),$$

$$\frac{1}{\Upsilon_1}\int_0^{\Upsilon_1}\|\mathcal{G}_2(\rho,\varrho,\alpha)-\widetilde{\mathcal{G}}_2(\varrho,\alpha)\|^{\tilde{p}}d\rho \le \mathscr{Y}_2(\Upsilon_1)(1+\|\varrho\|^{\tilde{p}}+\|\alpha\|^{\tilde{p}}),$$

where $\lim_{\Upsilon_1\to\infty}\mathscr{Y}_1(\Upsilon_1)=0$ and $\lim_{\Upsilon_1\to\infty}\mathscr{Y}_2(\Upsilon_1)=0$.

Definition 2.1. *Let* $\{\mathbb{U}(\rho)=\mathbb{U}(\rho+\omega), -\phi\le\omega\le 0\}$ *be the history of the state, and for all* $\varepsilon>0$, *we have the following inequality for* $\frac{1}{2}<\psi\le 1$ *and* $\rho\in[0,\Upsilon]$*:*

$$\left|\mathscr{T}_\rho^{\psi}[\mathbb{U}(\rho)-\mathcal{Y}(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))]-\mathcal{G}_1(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))-\mathcal{G}_2(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))\frac{\mathrm{d}\mathscr{B}_\rho}{\mathrm{d}\rho}\right|\le\varepsilon. \quad (2.1)$$

Remark 2.1. *A process* $\mathbb{U}(\rho)$ *is a solution of Eq* (2.1) *iff for all* $\varepsilon>0$, *there exists a function* $\mathbb{M}(\rho)$ *such that when* $|\mathbb{M}|<\sqrt{\varepsilon}$ *and*

$$\mathscr{T}_\rho^{\psi}[\mathbb{U}(\rho)-\mathcal{Y}(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))]=\mathcal{G}_1(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))+\mathcal{G}_2(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))\frac{\mathrm{d}\mathscr{B}_\rho}{\mathrm{d}\rho}+\mathbb{M}(\rho)$$

are satisfied.

Lemma 2.1. *Let* $\mathcal{K}(\rho)$, $\mathbb{I}(\rho)$ *be real continuous functions on* $[\rho_0,\rho_1]$, $\mathcal{R}(\rho)\ge 0$ *is an integrable function over interval* $[\rho_0,\rho_1]$ *and* $\mathcal{R}(\rho)\ge 0$ *is nondecreasing. If*

$$\mathcal{K}(\rho)\le\mathbb{I}(\rho)+\int_{\rho_0}^{\rho_1}\mathcal{K}(\vartheta)\mathcal{R}(\vartheta)\mathrm{d}\vartheta,\ \rho\in[\rho_0,\rho_1],$$

then

$$\mathcal{K}(\rho)\le\mathbb{I}(\rho)\exp\Big(\int_{\rho_0}^{\rho_1}\mathcal{R}(\vartheta)\mathrm{d}\vartheta\Big),\ \rho\in[\rho_0,\rho_1].$$

Definition 2.2. *The solution* $\mathfrak{J}(\rho)$ *of Eq* (1.1) *is called UH stable, if, for all* $\varepsilon>0$, *there exists a constant* $\mathscr{W}>0$ *such that for each process* $\mathbb{U}(\rho)$, *where* $\mathbb{U}(\rho)$ *is a solution of Eq* (2.1), *then*

$$\Xi\Big(\sup_{-\phi\le\rho\le\Upsilon}\left|\mathbb{U}(\rho)-\mathfrak{J}(\rho)\right|^2\Big)\le\mathscr{W}\varepsilon. \quad (2.2)$$

Lemma 2.2. *For each* $\Upsilon_1\in[0,\Upsilon]$, *we can establish the following growth conditions for* $\widetilde{\mathcal{G}}_2$ *based on assumptions* (σ_3) *and* (σ_4)*:*

$$\|\widetilde{\mathcal{G}}_2(\varrho,\alpha)\|^{\tilde{p}}\le\mathscr{V}_4\left(1+\|\varrho\|^{\tilde{p}}+\|\alpha\|^{\tilde{p}}\right),$$

where $\mathscr{V}_4=\left(2^{\tilde{p}-1}\mathscr{Y}_2(\Upsilon_1)+6^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}\right)$.

Proof. Considering J-I and assumptions $(\sigma_3),(\sigma_4)$, we derive the following result:

$$\begin{aligned}\|\widetilde{\mathcal{G}}_2(\varrho,\alpha)\|^{\tilde{p}} &\le 2^{\tilde{p}-1}\|\mathcal{G}_2(\rho,\varrho,\alpha)-\widetilde{\mathcal{G}}_2(\varrho,\alpha)\|^{\tilde{p}}+2^{\tilde{p}-1}\|\mathcal{G}_2(\rho,\varrho,\alpha)\|^{\tilde{p}}\\ &\le 2^{\tilde{p}-1}\mathscr{Y}_2(\Upsilon_1)\left(1+\|\varrho\|^{\tilde{p}}+\|\alpha\|^{\tilde{p}}\right)+2^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}(1+\|\varrho\|+\|\alpha\|)^{\tilde{p}}\\ &\le\left(2^{\tilde{p}-1}\mathscr{Y}_2(\Upsilon_1)+6^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}\right)\left(1+\|\varrho\|^{\tilde{p}}+\|\alpha\|^{\tilde{p}}\right).\end{aligned}$$

Lemma 2.3. *If there are real numbers* $\eta_1,\eta_2,\ldots,\eta_\iota$ *where* $\iota\in\mathscr{N}$ *and each* $\eta_\jmath\ge 0$ *for* $\jmath=1,2,\ldots,\iota$, *then*

$$\left(\sum_{\jmath=1}^{\iota}\eta_\jmath\right)^{\tilde{p}}\le\iota^{\tilde{p}-1}\sum_{\jmath=1}^{\iota}\eta_i^{\tilde{p}},\quad\forall\tilde{p}>1.$$

3. The main results

The results of the EU, UH stability, and the averaging principle are demonstrated in this part.

3.1. Existence and uniqueness

The contraction mapping theorem, with its key concept of contraction mappings, plays a fundamental role in establishing the EU of solutions to various mathematical problems. In this subsection, we investigate the EU and UH stability of the solution to Eq (1.1).

Definition 3.1. *If $\mathfrak{I}(\rho)$ is $\mathscr{F}(\rho)$-adapted and $\Xi\left[\int_{-\phi}^{\Upsilon}\|\mathfrak{I}(\rho)\|\mathrm{d}\rho\right] < \infty$, $\aleph(0) = \aleph_0$, and fulfill the below criteria, in this way an $\mathbb{R}^{\kappa}$-value stochastic process $\{\mathfrak{I}(\rho)\}_{-\phi\leq\rho\leq\Upsilon}$ is called a unique solution to Eq (1.1).*

$$\begin{cases}\mathfrak{I}(\rho) = & \aleph_0 - \mathcal{Y}(0,\aleph(0),\aleph(-\phi)) + \mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi)) + \int_0^{\rho}\vartheta^{\psi-1}\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\vartheta \\ & + \int_0^{\rho}\vartheta^{\psi-1}\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\mathscr{B}_{\vartheta},\ \rho\in[0,\Upsilon], \\ \mathfrak{I}(\rho) = & \aleph(\rho),\ \rho\in[-\phi,0].\end{cases} \tag{3.1}$$

Theorem 3.1. *If (σ_1) and (σ_2) hold true, then Eq (1.1) possesses a unique solution under the subsequent criteria:*

$$\lambda = 6^{\tilde{p}-1}2\mathscr{V}_1^{\tilde{p}} + 6^{\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}\rho^{\psi\tilde{p}}\left(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\right)^{\tilde{p}-1} + 2^{\tilde{p}}3^{\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}\rho^{\frac{(2\psi-1)\tilde{p}}{2}}\left(\frac{1}{2\psi-1}\right)^{\frac{\tilde{p}}{2}}\left(\frac{(\tilde{p}-1)^{1-\tilde{p}}\tilde{p}^{\tilde{p}+1}}{2}\right)^{\frac{\tilde{p}}{2}}, \tag{3.2}$$

where $\lambda < 1$ and λ is non-negative.

Proof. We form an operator $\beta : \xi \to \xi$ with the utilization of $\mathfrak{I}(\rho) = \aleph(\rho)$, $\rho\in[-\phi,0]$ and the ensuing equality remains valid.

$$\begin{aligned}\beta(\mathfrak{I}(\rho)) =& \aleph_0 - \mathcal{Y}(0,\aleph(0),\aleph(-\phi)) + \mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi)) \\ &+ \int_0^{\rho}\vartheta^{\psi-1}\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\vartheta + \int_0^{\rho}\vartheta^{\psi-1}\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\mathscr{B}_{\vartheta}.\end{aligned} \tag{3.3}$$

Step 1: We will first demonstrate that β maps ξ to ξ. Let $\mathfrak{I}(\rho)\in\xi$, where $\mathfrak{I}(\rho)$ is arbitrary. From the $\beta(\mathfrak{I}(\rho))$ with J-I for $\rho\in[0,\Upsilon]$, we have

$$\begin{aligned}\Xi[\|\beta(\mathfrak{I}(\rho))\|^{\tilde{p}}] \leq& 4^{\tilde{p}-1}\Xi[\|\aleph_0\|^{\tilde{p}}] + 4^{\tilde{p}-1}\Xi[\|\mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi)) - \mathcal{Y}(0,\aleph_0,\aleph(-\phi))\|^{\tilde{p}}] \\ &+ 4^{\tilde{p}-1}\Xi\left[\left\|\int_0^{\rho}\vartheta^{\psi-1}\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\vartheta\right\|^{\tilde{p}}\right] \\ &+ 4^{\tilde{p}-1}\Xi\left[\left\|\int_0^{\rho}\vartheta^{\psi-1}\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\mathscr{B}_{\vartheta}\right\|^{\tilde{p}}\right] \\ =& \ell_1 + \ell_2 + \ell_3 + \ell_4.\end{aligned} \tag{3.4}$$

By employing (σ_1) and the J-I, we acquire the subsequent outcome:

$$\begin{aligned}\ell_2 =& 4^{\tilde{p}-1}\Xi[\|\mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi)) - \mathcal{Y}(0,\aleph_0,\aleph(-\phi))\|^{\tilde{p}}] \\ \leq& 4^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\Xi[\|\mathfrak{I}(\rho)-\aleph_0\| + \|\mathfrak{I}(\rho-\phi)-\aleph(-\phi)\|]^{\tilde{p}}\end{aligned}$$

$$\leq 8^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\Big(\Xi[\|\mathfrak{I}(\rho)-\aleph_0\|^{\tilde{p}}]+\Xi[\|\mathfrak{I}(\rho-\phi)-\aleph(-\phi)\|^{\tilde{p}}]\Big)$$
$$\leq 4\cdot 8^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\Xi\|\mathfrak{I}\|^{\tilde{p}}. \tag{3.5}$$

By applying H-I, J-I, and (σ_3), we derive the following results:

$$\ell_3 = 4^{\tilde{p}-1}\Xi\Big[\Big\|\int_0^\rho \vartheta^{\psi-1}\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\vartheta\Big\|^{\tilde{p}}\Big]$$
$$\leq 4^{\tilde{p}-1}\Big(\int_0^\rho \vartheta^{\frac{\tilde{p}(\psi-1)}{\tilde{p}-1}}\mathrm{d}\vartheta\Big)^{\tilde{p}-1}\Xi\Big[\int_0^\rho \|\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\|^{\tilde{p}}\mathrm{d}\vartheta\Big]$$
$$\leq 4^{\tilde{p}-1}\rho^{\psi\tilde{p}-1}\Big(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\Big)^{\tilde{p}-1}\Xi\Big[\int_0^\rho \mathscr{V}_3^{\tilde{p}}(1+\|\mathfrak{I}(\vartheta)\|+\|\mathfrak{I}(\vartheta-\phi)\|)^{\tilde{p}}\mathrm{d}\vartheta\Big]$$
$$\leq 8^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}\rho^{\psi\tilde{p}}\Big(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\Big)^{\tilde{p}-1}(1+2^{\tilde{p}}\Xi[\|\mathfrak{I}\|^{\tilde{p}}]). \tag{3.6}$$

By utilizing the BHDG-I, J-I, and (σ_3), we get

$$\ell_4 \leq 4^{\tilde{p}-1}\Xi\Big[\sup_{\rho\in[-\phi,\Upsilon]}\Big\|\int_0^\rho \vartheta^{\psi-1}\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\mathrm{d}\mathscr{B}_\vartheta\Big\|^{\tilde{p}}\Big]$$
$$\leq \Big(\frac{(\tilde{p})^{\tilde{p}+1}}{2(\tilde{p}-1)^{\tilde{p}-1}}\Big)^{\frac{\tilde{p}}{2}}\Xi\Big[\int_0^\rho \vartheta^{2\psi-2}\Big\|\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\Big\|^2\mathrm{d}\vartheta\Big]^{\frac{\tilde{p}}{2}}4^{\tilde{p}-1}$$
$$\leq \frac{2^{\frac{\tilde{p}}{2}}4^{\tilde{p}-1}}{(2\psi-1)^{\frac{\tilde{p}}{2}}}\rho^{\frac{(2\psi-1)\tilde{p}}{2}}\mathscr{V}_3^{\tilde{p}}(2(\tilde{p}-1)^{1-\tilde{p}}\tilde{p}^{\tilde{p}+1})^{\frac{\tilde{p}}{2}}(1+2\Xi[\|\mathfrak{I}\|^2])^{\frac{\tilde{p}}{2}}$$
$$\leq \frac{8^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}\rho^{\frac{(2\psi-1)\tilde{p}}{2}}}{(2\psi-1)^{\frac{\tilde{p}}{2}}}(2(\tilde{p}-1)^{1-\tilde{p}}(\tilde{p})^{\tilde{p}+1})^{\frac{\tilde{p}}{2}}(1+2^{\frac{\tilde{p}}{2}}\Xi[\|\mathfrak{I}\|^{p}]). \tag{3.7}$$

Equations (3.5)–(3.7), when applied to (3.4), result in the following conclusions:

$$\Xi[\|\beta(\mathfrak{I}(\rho))\|^{\tilde{p}}] \leq 4^{\tilde{p}-1}\Xi[\big\|\aleph_0\big\|^{\tilde{p}}]+4\cdot 8^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\Xi[\|\mathfrak{I}\|^{\tilde{p}}]+(1+2^{\tilde{p}}\Xi[\|\mathfrak{I}\|^{\tilde{p}}])\mathscr{V}_3^{\tilde{p}}\rho^{\psi\tilde{p}}\Big(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\Big)^{\tilde{p}-1}8^{\tilde{p}-1}$$
$$+\frac{8^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}\rho^{\frac{(2\psi-1)\tilde{p}}{2}}}{(2\psi-1)^{\frac{\tilde{p}}{2}}}(2(\tilde{p}-1)^{1-\tilde{p}}\tilde{p}^{\tilde{p}+1})^{\frac{\tilde{p}}{2}}(1+2^{\frac{\tilde{p}}{2}}\Xi[\|\mathfrak{I}\|^{\tilde{p}}]). \tag{3.8}$$

When combined with the information from the preceding discussion, it becomes apparent that a constant $\mathscr{C}$ satisfies the following condition:

$$\Xi[\|\beta(\mathfrak{I}(\rho))\|^{\tilde{p}}] \leq \mathscr{C}(1+\Xi[\|\mathfrak{I}\|^{\tilde{p}}]).$$

In another way, β maps ξ to ξ.

Step 2: Next, we will demonstrate that the mapping β is contractive. For this demonstration, let $\mathfrak{I}(\rho)$ and $\mathscr{Q}(\rho)$ be arbitrary. We derive the following for all $\rho\in[0,\Upsilon]$ using Eq (3.3) and the J-I:

$$\Xi[\|\beta(\mathfrak{I}(\rho))-\beta(\mathscr{Q}(\rho))\|^{\tilde{p}}]$$
$$\leq 3^{\tilde{p}-1}\Xi[\|\mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))-\mathcal{Y}(\rho,\mathscr{Q}(\rho),\mathscr{Q}(\rho-\phi))\|^{\tilde{p}}]$$

$$+\,3^{\tilde{p}-1}\Xi\Big[\Big\|\int_0^{\rho}\vartheta^{\psi-1}(\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))-\mathcal{G}_1(\vartheta,\mathcal{Q}(\vartheta),\mathcal{Q}(\vartheta-\phi)))\mathrm{d}\vartheta\Big\|^{\tilde{p}}\Big]$$
$$+\,3^{\tilde{p}-1}\Xi\Big[\Big\|\int_0^{\rho}\vartheta^{\psi-1}(\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathcal{Q}(\vartheta),\mathcal{Q}(\vartheta-\phi)))\mathrm{d}\mathcal{B}_{\vartheta}\Big\|^{\tilde{p}}\Big]$$
$$=\mathscr{A}_5+\mathscr{A}_6+\mathscr{A}_7. \tag{3.9}$$

By utilizing J-I and (σ_1), the following outcome is attained:

$$\begin{aligned}\mathscr{A}_5&=3^{\tilde{p}-1}\Xi[\|\mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))-\mathcal{Y}(\rho,\mathcal{Q}(\rho),\mathcal{Q}(\rho-\phi))\|^{\tilde{p}}]\\&\le 3^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|+\|\mathfrak{I}(\rho-\phi)-\mathcal{Q}(\rho-\phi)\|]^{\tilde{p}}\\&\le 6^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\Big(\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}]+\Xi[\|\mathfrak{I}(\rho-\phi)-\mathcal{Q}(\rho-\phi)\|^{\tilde{p}}]\Big)\\&\le 2\cdot 6^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}\sup_{\rho\in[-\phi,\Upsilon]}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}].\end{aligned}\tag{3.10}$$

Applying H-I and (σ_2), we acquire

$$\begin{aligned}\mathscr{A}_6\le&3^{\tilde{p}-1}\Big(\int_0^{\rho}\vartheta^{\frac{\tilde{p}(\psi-1)}{\tilde{p}-1}}d\vartheta\Big)^{\tilde{p}-1}\Xi\Big[\int_0^{\rho}\|\mathcal{G}_1(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))-\mathcal{G}_1(\vartheta,\mathcal{Q}(\vartheta),\mathcal{Q}(\vartheta-\phi))\|^{\tilde{p}}\mathrm{d}\vartheta\Big]\\\le&3^{\tilde{p}-1}\rho^{\psi\tilde{p}-1}\Big(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\Big)^{\tilde{p}-1}\int_0^{\rho}\mathscr{V}_2^{\tilde{p}}\Xi[\|\mathfrak{I}(\vartheta)-\mathcal{Q}(\vartheta)\|+\|\mathfrak{I}(\vartheta-\phi)-\mathcal{Q}(\vartheta-\phi)\|^{\tilde{p}}\mathrm{d}\vartheta]\\\le&6^{\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}\rho^{\psi\tilde{p}}\Big(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\Big)^{\tilde{p}-1}\sup_{\rho\in[-\phi,\Upsilon]}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}].\end{aligned}\tag{3.11}$$

Nonetheless, employing (σ_2) alongside the BHDK-I yields

$$\begin{aligned}\mathscr{A}_7=&3^{\tilde{p}-1}\Xi\Big[\sup_{\rho\in[-\phi,\Upsilon]}\Big\|\int_0^{\rho}\vartheta^{\psi-1}\Big(\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathcal{Q}(\vartheta),\mathcal{Q}(\vartheta-\phi))\Big)\mathrm{d}\mathcal{B}_{\vartheta}\Big\|^{\tilde{p}}\Big]\\\le&3^{\tilde{p}-1}\Big(\frac{(\tilde{p})^{\tilde{p}+1}}{2(\tilde{p}-1)^{\tilde{p}-1}}\Big)^{\frac{\tilde{p}}{2}}\Xi\Big[\int_0^{\rho}\vartheta^{2\psi-2}\Big\|\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathcal{Q}(\vartheta),\mathcal{Q}(\vartheta-\phi))\|^2\mathrm{d}\vartheta\Big]^{\frac{\tilde{p}}{2}}\\\le&\frac{2^{\tilde{p}}3^{\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}\rho^{\frac{(2\psi-1)\tilde{p}}{2}}}{(2\psi-1)^{\frac{\tilde{p}}{2}}}\Big(\frac{\tilde{p}^{\tilde{p}+1}}{2(\tilde{p}-1)^{\tilde{p}-1}}\Big)^{\frac{\tilde{p}}{2}}\sup_{\rho\in[-\phi,\Upsilon]}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}].\end{aligned}\tag{3.12}$$

By applying Eqs (3.10)–(3.12) to (3.9), we derive the following results:

$$\begin{aligned}&\Xi[\|\beta(\mathfrak{I}(\rho))-\beta(\mathcal{Q}(\rho))\|^{\tilde{p}}]\\\le&\Big(2\cdot 6^{\tilde{p}-1}\mathscr{V}_1^{\tilde{p}}+6^{\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}\rho^{\psi\tilde{p}}\Big(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\Big)^{\tilde{p}-1}+\frac{2^{\tilde{p}}3^{\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}\rho^{\frac{(2\psi-1)\tilde{p}}{2}}}{(2\psi-1)^{\frac{\tilde{p}}{2}}}\Big(\frac{\tilde{p}^{\tilde{p}+1}}{2(\tilde{p}-1)^{\tilde{p}-1}}\Big)^{\frac{\tilde{p}}{2}}\Big)\sup_{\rho\in[-\phi,\Upsilon]}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}]\\\le&\lambda\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}},\end{aligned}\tag{3.13}$$

where $\sup_{\rho\in[-\phi,\Upsilon]}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}]$ represents the least upper bound of the set generated by expected values of $\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}$, that can be obtained in the range $[-\phi,\Upsilon]$, and $\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}$ is the upper bound for that set, so we get $\sup_{\rho\in[-\phi,\Upsilon]}\Xi[\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}]\le\|\mathfrak{I}(\rho)-\mathcal{Q}(\rho)\|^{\tilde{p}}$. Therefore, based on Eq (3.2), which ascertains that $\lambda<1$, the operator β represents a contractive mapping. Consequently, there exists a unique fixed point $\mathfrak{I}(\rho)\in\xi$ of this mapping, with the initial function $\mathfrak{I}(\rho)=\aleph(\rho)$, where $\rho\in[-\phi,0]$, as per the Banach FPT.

Lemma 3.1. *Let* $\mathbb{U}(\rho)$ *be a solution of Eq* (2.1). *Then, for* $\rho \in [0, \Upsilon]$, *we have the following:*

$$\Xi\Big(\Big|\mathbb{U}(\rho) - \mathcal{Y}(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi)) + \mathcal{Y}(0, \mathbb{U}(0), \mathbb{U}(-\phi)) - \mathbb{M}(\rho)\Big|^2\Big) \leq \varepsilon \frac{\Upsilon^{2\psi}}{2\psi - 1}.$$

Proof. For $\rho \in [0, \Upsilon]$ and $\psi \in (0, 1]$, we acquire as follows:

$$\begin{cases} & \mathscr{T}_\rho^{\psi}[\mathbb{U}(\rho) - \mathcal{Y}(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi))] \\ = & \mathcal{G}_1(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi)) + \mathcal{G}_2(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi))\frac{\mathrm{d}\mathscr{B}_\rho}{\mathrm{d}\rho} + \mathbb{Q}(\rho), \end{cases}$$

with initial value $\mathbb{U}(0) = \mathbb{U}_0 = \mathfrak{I}_0$. After that, the solution can be stated as follows:

$$\begin{cases} \mathbb{U}(\rho) = & \mathfrak{I}_0 - \mathcal{Y}(0, \aleph(0), \aleph(-\phi)) + \mathcal{Y}(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi)) \\ & + \int_0^\rho \vartheta^{\psi-1}\mathcal{G}_1(\vartheta, \mathbb{U}(\vartheta), \mathbb{U}(\vartheta-\phi))\mathrm{d}\vartheta + \int_0^\rho \vartheta^{\psi-1}\mathcal{G}_2(\vartheta, \mathbb{U}(\vartheta), \mathbb{U}(\vartheta-\phi))\mathrm{d}\mathscr{B}_\vartheta \\ & + \int_0^\rho \vartheta^{\psi-1}\mathbb{Q}(\rho)\mathrm{d}\vartheta. \end{cases}$$

By H-I, we get

$$\begin{aligned} & \Xi\Big(\Big|\mathbb{U}(\rho) - \mathcal{Y}(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi)) + \mathcal{Y}(0, \mathbb{U}(0), \mathbb{U}(-\phi)) - \mathbb{M}(\rho)\Big|^2\Big) \\ =& \Xi\Big(\Big|\int_0^\rho \vartheta^{\psi-1}\mathbb{Q}(\vartheta)\mathrm{d}\vartheta\Big|^2\Big) \leq \Big|\int_0^\rho \vartheta^{\psi-1}\mathbb{Q}(\vartheta)\mathrm{d}\vartheta\Big|^2 \\ \leq& \int_0^\rho \big|\mathbb{Q}(\vartheta)\big|^2\mathrm{d}\vartheta \int_0^\rho \vartheta^{2(\psi-1)}\mathrm{d}\vartheta \leq \varepsilon\rho\frac{\rho^{2\psi-1}}{2\psi-1} \\ \leq& \frac{\varepsilon\Upsilon^{2\psi}}{2\psi-1}. \end{aligned}$$

The proof is now complete.

Now, we will prove that the solution of Eq (1.1) is UH stable.

Theorem 3.2. *Suppose that* (σ_1)–(σ_3) *hold and* $\mathscr{V}_1 \in (0, \frac{1}{2})$, $\psi \in (\frac{1}{2}, 1]$. *Then, the solution of Eq* (1.1) *is UH stable on* $\rho \in [0, \Upsilon]$.

Proof. Let $\mathbb{U}(\rho)$ be a solution of Eq (2.1), and $\mathfrak{I}(\rho)$ be a solution of Eq (1.1). Note that $\mathbb{U}_0 = \mathfrak{I}_0$, from Lemma 3.1, $\Xi\Big(\int_a^b \mathbb{G}(\rho)\mathrm{d}\mathscr{B}_\rho\Big) = 0$, $\Xi\Big(\Big|\int_a^b \mathbb{G}(\rho)\mathrm{d}\mathscr{B}_\rho\Big|^2\Big) = \Xi\Big(\int_a^b |\mathbb{G}(\rho)|^2\mathrm{d}\rho\Big)$, (σ_1) and (σ_2), for $0 \leq \rho \leq \Upsilon$, we get

$$\begin{aligned} & \Xi\Big(\Big|\mathbb{U}(\rho) - \mathfrak{I}(\rho)\Big|^2\Big) \\ =& \Xi\Big(\Big|\mathbb{U}(\rho) - \mathbb{U}(\rho) + \mathbb{U}(\rho) - \mathfrak{I}(\rho)\Big|^2\Big) \\ \leq& \Xi\Bigg(\Bigg|\mathbb{U}(\rho) - \mathcal{Y}(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi)) + \mathcal{Y}(0, \mathbb{U}(0), \mathbb{U}(-\phi)) - \mathbb{M}(\rho) \\ & + \mathcal{Y}(\rho, \mathbb{U}(\rho), \mathbb{U}(\rho-\phi)) - \mathcal{Y}(0, \mathbb{U}(0), \mathbb{U}(-\phi)) + \mathcal{Y}(\rho, \mathfrak{I}(\rho), \mathfrak{I}(\rho-\phi)) + \mathcal{Y}(0, \mathfrak{I}(0), \mathfrak{I}(-\phi)) \\ & + \int_0^\rho \Big(\mathcal{G}_1(\vartheta, \mathbb{U}(\vartheta), \mathbb{U}(\vartheta-\phi)) - \mathcal{G}_1(\vartheta, \mathfrak{I}(\vartheta), \mathfrak{I}(\vartheta-\phi))\Big)\vartheta^{\psi-1}\mathrm{d}\vartheta \end{aligned}$$

$$+\int_0^\rho\Big(\mathcal{G}_2(\vartheta,\mathbb{U}(\vartheta),\mathbb{U}(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathfrak{I}(\vartheta),\mathfrak{I}(\vartheta-\phi))\Big)\vartheta^{\psi-1}\mathrm{d}\mathscr{B}_\vartheta\Big|^2\Big)$$

$$\leq 4\Xi\Big(\Big|\mathbb{U}(\rho)-\mathcal{Y}(\rho,\mathbb{U}(\rho),\mathbb{U}(\rho-\phi))+\mathcal{Y}(0,\mathbb{U}(0),\mathbb{U}(-\phi))-\mathbb{M}(\rho)\Big|^2+4\mathscr{V}_1^2\Xi\Big(\Big|\mathbb{U}(\rho)-\mathfrak{I}(\rho)\Big|^2\Big)$$

$$+4\rho\mathscr{V}_2^2\int_0^\rho\Xi\Big(\Big|\mathbb{U}(\vartheta)-\mathfrak{I}(\vartheta)\Big|^2\Big)\vartheta^{2(\psi-1)}\mathrm{d}\vartheta+4\mathscr{V}_2^2\int_0^\rho\Xi\Big(\Big|\mathbb{U}(\vartheta)-\mathfrak{I}(\vartheta)\Big|^2\Big)\vartheta^{2(\psi-1)}\mathrm{d}\vartheta$$

$$\leq\frac{4\varepsilon\Upsilon^{2\psi}}{2\psi-1}+4\mathscr{V}_1^2\Xi\Big(\Big|\mathbb{U}(\rho)-\mathfrak{I}(\rho)\Big|^2\Big)+4(1+\Upsilon)\mathscr{V}_2^2\int_0^\rho\Xi\Big(\Big|\mathbb{U}(\vartheta)-\mathfrak{I}(\vartheta)\Big|^2\Big)\vartheta^{2(\psi-1)}\mathrm{d}\vartheta. \tag{3.14}$$

So, from Eq (3.14), we acquire

$$\Xi\Big(\sup_{-\phi\leq\rho\leq\Upsilon}\Big|\mathbb{U}(\rho)-\mathfrak{I}(\rho)\Big|^2\Big)\leq\frac{4\varepsilon\Upsilon^{2\psi}}{2\psi-1}+4\mathscr{V}_1^2\Xi\Big(\sup_{-\phi\leq\rho\leq\Upsilon}\Big|\mathbb{U}(\rho)-\mathfrak{I}(\rho)\Big|^2\Big)$$
$$+4(1+\Upsilon)\mathscr{V}_2^2\int_0^\rho\Xi\Big(\sup_{-\phi\leq\rho\leq\Upsilon}\Big|\mathbb{U}(\vartheta)-\mathfrak{I}(\vartheta)\Big|^2\Big)\vartheta^{2(\psi-1)}\mathrm{d}\vartheta.$$

Hence, we have the following:

$$\Xi\Big(\sup_{-\phi\leq\rho\leq\Upsilon}\Big|\mathbb{U}(\rho)-\mathfrak{I}(\rho)\Big|^2\Big)$$
$$\leq\frac{4\varepsilon\Upsilon^{2\psi}}{(2\psi-1)(1-4\mathscr{V}_1^2)}+\frac{4}{(1-4\mathscr{V}_1^2)}(1+\Upsilon)\mathscr{V}_2^2\int_0^\rho\Xi\Big(\sup_{-\phi\leq\rho\leq\Upsilon}\Big|\mathbb{U}(\vartheta)-\mathfrak{I}(\vartheta)\Big|^2\Big)\vartheta^{2(\psi-1)}\mathrm{d}\vartheta.$$

Further, we have the following by using Lemma 2.1:

$$\Xi\Big(\sup_{-\phi\leq\rho\leq\Upsilon}\Big|\mathbb{U}(\rho)-\mathfrak{I}(\rho)\Big|^2\Big)\leq\frac{4\varepsilon\Upsilon^{2\psi}}{(2\psi-1)(1-4\mathscr{V}_1^2)}\exp\left(\frac{4}{(1-4\mathscr{V}_1^2)}(1+\Upsilon)\mathscr{V}_2^2\int_0^\rho\vartheta^{2(\psi-1)}\mathrm{d}\vartheta\right)$$
$$\leq\frac{4\varepsilon\Upsilon^{2\psi}}{(2\psi-1)(1-4\mathscr{V}_1^2)}\exp\left(\frac{4}{(1-4\mathscr{V}_1^2)}(1+\Upsilon)\mathscr{V}_2^2\Upsilon^{2\psi-1}\frac{1}{2\psi-1}\right)$$
$$=\mathscr{W}\varepsilon,$$

where $\mathscr{W}=\frac{4\Upsilon^{2\psi}}{(2\psi-1)(1-4\mathscr{V}_1^2)}\exp\left(\frac{4}{(1-4\mathscr{V}_1^2)}(1+\Upsilon)\mathscr{V}_2^2\Upsilon^{2\psi-1}\frac{1}{2\psi-1}\right)$.

This completes the proof.

3.2. Averaging principle result

In this section, we will first analyze the averaging principle of FrNSDEs with respect to $\mathfrak{L}^{\tilde{p}}$. Initially, we will investigate the standard expression of Eq (3.1).

$$\mathfrak{I}_\varepsilon(\rho)=\aleph_0-\mathcal{Y}(0,\aleph(0),\aleph(-\phi))+\mathcal{Y}(\rho,\mathfrak{I}_\varepsilon(\rho),\mathfrak{I}_\varepsilon(\rho-\phi))$$
$$+\varepsilon\int_0^\rho\vartheta^{\psi-1}\mathcal{G}_1(\vartheta,\mathfrak{I}_\varepsilon(\vartheta),\mathfrak{I}_\varepsilon(\vartheta-\phi))\mathrm{d}\vartheta+\sqrt{\varepsilon}\int_0^\rho\vartheta^{\psi-1}\mathcal{G}_1(\vartheta,\mathfrak{I}_\varepsilon(\vartheta),\mathfrak{I}_\varepsilon(\vartheta-\phi))\mathrm{d}\mathscr{B}_\vartheta. \tag{3.15}$$

For any positive small parameter ε within the range $(0, \varepsilon_0]$, where ε_0 is a predetermined constant, and under the assumptions (σ_1), (σ_2) and (σ_3) holding for $\mathcal{Y}$, $\mathcal{G}_1$ and $\mathcal{G}_2$, the averaged form of Eq (3.15) is presented below.

$$\begin{aligned}\mathfrak{J}^*_\varepsilon(\rho) =&\aleph_0 - \mathcal{Y}(0, \aleph(0), \aleph(-\phi)) + \mathcal{Y}(\rho, \mathfrak{J}^*_\varepsilon(\rho), \mathfrak{J}^*_\varepsilon(\rho-\phi))\\ &+ \varepsilon \int_0^\rho \vartheta^{\psi-1}\widetilde{\mathcal{G}}_1(\vartheta, \mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi))\mathrm{d}\vartheta\\ &+ \sqrt{\varepsilon}\int_0^\rho \vartheta^{\psi-1}\widetilde{\mathcal{G}}_2(\vartheta, \mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi))\mathrm{d}\mathscr{B}_\vartheta,\end{aligned} \tag{3.16}$$

where $\widetilde{\mathcal{G}}_1 : \mathbb{R}^\kappa \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$, $\widetilde{\mathcal{G}}_2 : \mathbb{R}^\kappa \times \mathbb{R}^\kappa \to \mathbb{R}^{\kappa\times\mathfrak{r}}$.

Theorem 3.3. *Assuming that conditions (σ_1) to (σ_4) hold, we can identify a corresponding ε_1 within the interval $(0, \varepsilon_0]$, along with constants $\varphi > 0$ and $\Psi \in (0, 1)$ that hold for all $\varepsilon \in (0, \varepsilon_1]$ when $\tilde{p} \in [2, (1-\psi)^{-1})$ and for $\mathbb{Z} > 0$, which is a sufficiently small value. The derivation of this formula proceeds as follows:*

$$\Xi\Big[\sup_{\rho\in[-\phi,\varphi\varepsilon^{-\Psi}]}\left\|\mathfrak{J}_\varepsilon(\rho) - \mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\Big] \le \mathbb{Z}. \tag{3.17}$$

Proof. We attain the subsequent result for any $\rho \in [0, \Upsilon]$ through Eqs (3.15) and (3.16).

$$\begin{aligned}\mathfrak{J}_\varepsilon(\rho) - \mathfrak{J}^*_\varepsilon(\rho) =&\mathcal{Y}(\rho, \mathfrak{J}_\varepsilon(\rho), \mathfrak{J}_\varepsilon(\rho-\phi)) - \mathcal{Y}(\rho, \mathfrak{J}^*_\varepsilon(\rho), \mathfrak{J}^*_\varepsilon(\rho-\phi))\\ &+ \varepsilon\int_0^\rho \vartheta^{\psi-1}(\mathcal{G}_1(\vartheta, \mathfrak{J}_\varepsilon(\vartheta), \mathfrak{J}_\varepsilon(\vartheta-\phi)) - \widetilde{\mathcal{G}}_1(\mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi)))\mathrm{d}\vartheta\\ &+ \sqrt{\varepsilon}\int_0^\rho \vartheta^{\psi-1}(\mathcal{G}_2(\vartheta, \mathfrak{J}_\varepsilon(\vartheta), \mathfrak{J}_\varepsilon(\vartheta-\phi)) - \widetilde{\mathcal{G}}_2(\mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi)))\mathrm{d}\mathscr{B}_\vartheta.\end{aligned} \tag{3.18}$$

When $\mathbb{A} \in (0, 1), \mathbb{B}_1, \mathbb{B}_2 \in \mathbb{R}^\kappa, \tilde{p} \ge 2$, we have

$$\|\mathbb{B}_1 + \mathbb{B}_2\|^{\tilde{p}} \le \frac{\|\mathbb{B}_1\|^{\tilde{p}}}{\mathbb{A}^{\tilde{p}-1}} + \frac{\|\mathbb{B}_2\|^{\tilde{p}}}{(1-\mathbb{A})^{\tilde{p}-1}}. \tag{3.19}$$

Let $\mathbb{A} = \mathscr{V}_1$. Upon applying Eq (3.18) in Eq (3.19), followed by the utilization of (σ_1) and J-I, we obtain the following outcome:

$$\begin{aligned}&\left\|\mathfrak{J}_\varepsilon(\rho) - \mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\\ \le&\mathscr{V}_1^{1-\tilde{p}}\left\|\mathcal{Y}(\rho, \mathfrak{J}_\varepsilon(\rho), \mathfrak{J}_\varepsilon(\rho-\phi)) - \mathcal{Y}(\rho, \mathfrak{J}^*_\varepsilon(\rho), \mathfrak{J}^*_\varepsilon(\rho-\phi))\right\|^{\tilde{p}}\\ &+ \frac{2^{\tilde{p}-1}}{(1-\mathscr{V}_1)^{\tilde{p}-1}}\left\|\varepsilon\int_0^\rho \vartheta^{\psi-1}\Big(\mathcal{G}_1(\vartheta, \mathfrak{J}_\varepsilon(\vartheta), \mathfrak{J}_\varepsilon(\vartheta-\phi)) - \widetilde{\mathcal{G}}_1(\mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\vartheta\right\|^{\tilde{p}}\\ &+ \frac{2^{\tilde{p}-1}}{(1-\mathscr{V}_1)^{\tilde{p}-1}}\left\|\sqrt{\varepsilon}\int_0^\rho \vartheta^{\psi-1}\Big(\mathcal{G}_2(\vartheta, \mathfrak{J}_\varepsilon(\vartheta), \mathfrak{J}_\varepsilon(\vartheta-\phi)) - \widetilde{\mathcal{G}}_2(\mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\mathscr{B}_\vartheta\right\|^{\tilde{p}}\\ \le&2^{\tilde{p}-1}\mathscr{V}_1(\left\|\mathfrak{J}_\varepsilon(\rho) - \mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}} + \left\|\mathfrak{J}_\varepsilon(\rho-\phi) - \mathfrak{J}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}})\\ &+ \frac{2^{\tilde{p}-1}\varepsilon^p}{(1-\mathscr{V}_1)^{\tilde{p}-1}}\left\|\int_0^\rho \vartheta^{\psi-1}\Big(\mathcal{G}_1(\vartheta, \mathfrak{J}_\varepsilon(\vartheta), \mathfrak{J}_\varepsilon(\vartheta-\phi)) - \widetilde{\mathcal{G}}_1(\mathfrak{J}^*_\varepsilon(\vartheta), \mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\vartheta\right\|^{\tilde{p}}\end{aligned}$$

$$+\frac{2^{\tilde{p}-1}\varepsilon^{\frac{p}{2}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}}\left\|\int_0^{\rho}\vartheta^{\psi-1}\Big(\mathcal{G}_2(\vartheta,\mathfrak{J}_\varepsilon(\vartheta),\mathfrak{J}_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_2(\mathfrak{J}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\mathscr{B}_\vartheta\right\|^{\tilde{p}}. \tag{3.20}$$

Utilizing Eq (3.20) in Eq (3.17),

$$\begin{aligned}
&\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{J}_\varepsilon(\rho)-\mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\Big]\\
\le&\frac{2^{p-1}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{J}_\varepsilon(\rho-\phi)-\mathscr{V}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\Big]\\
&+\frac{2^{\tilde{p}-1}\varepsilon^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\\
&\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\int_0^{\rho}\vartheta^{\psi-1}\Big(\mathcal{G}_1(\vartheta,\mathfrak{J}_\varepsilon(\vartheta),\mathfrak{J}_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_1(\mathfrak{J}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\vartheta\right\|^{\tilde{p}}\Big]\\
&+\frac{2^{\tilde{p}-1}\varepsilon^{\frac{\tilde{p}}{2}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\\
&\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\int_0^{\rho}\vartheta^{\psi-1}\Big(\mathcal{G}_2(\vartheta,\mathfrak{J}_\varepsilon(\vartheta),\mathfrak{J}_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_2(\mathscr{V}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\mathscr{B}_\vartheta\right\|^{\tilde{p}}\Big]\\
=&\mathfrak{V}_1+\mathfrak{V}_2+\mathfrak{V}_3.
\end{aligned} \tag{3.21}$$

From $\mathfrak{V}_1$,

$$\mathfrak{V}_1\le\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\Big(\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{J}_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\Big]+\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{J}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\Big]\Big). \tag{3.22}$$

From $\mathfrak{V}_2$,

$$\begin{aligned}
\mathfrak{V}_2\le&\frac{2^{2\tilde{p}-2}\varepsilon^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\\
&\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\int_0^{\rho}\vartheta^{\psi-1}\Big(\mathcal{G}_1(\vartheta,\mathfrak{J}_\varepsilon(\vartheta),\mathfrak{J}_\varepsilon(\vartheta-\phi))-\mathcal{G}_1(\vartheta,\mathfrak{J}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\vartheta\right\|^{\tilde{p}}\Big]\\
&+\frac{2^{2\tilde{p}-2}\varepsilon^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\\
&\Xi\Big[\sup_{0\le\rho\le\varkappa}\left\|\int_0^{\rho}\vartheta^{\psi-1}\Big(\mathcal{G}_1(\vartheta,\mathfrak{J}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_1(\mathfrak{J}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi))\Big)\mathrm{d}\vartheta\right\|^{\tilde{p}}\Big]\\
=&\mathfrak{V}_{21}+\mathfrak{V}_{22}.
\end{aligned} \tag{3.23}$$

Using H-I, J-I, and (σ_2) on $\mathfrak{V}_{21}$, the following result is obtained:

$$\begin{aligned}
\mathfrak{V}_{21}\le&\frac{2^{2\tilde{p}-2}\varepsilon^{p}}{(1-\mathscr{V}_1)^{\tilde{p}-1}\left(1-2^{\tilde{p}-1}\mathscr{V}_1\right)}\left(\int_0^{\varkappa}(\vartheta)^{\frac{(\psi-1)\tilde{p}}{\tilde{p}-1}}\mathrm{d}\vartheta\right)^{\tilde{p}-1}\\
&\Xi\left[\sup_{0\le\rho\le\varkappa}\int_0^{\rho}\left\|\mathcal{G}_1\left(\vartheta,\mathfrak{J}_\varepsilon(\vartheta),\mathfrak{J}_\varepsilon(\vartheta-\phi)\right)-\mathcal{G}_1\left(\vartheta,\mathfrak{J}^*_\varepsilon(\vartheta),\mathfrak{J}^*_\varepsilon(\vartheta-\phi)\right)\right\|^{\tilde{p}}\mathrm{d}\vartheta\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2^{3\tilde{p}-3}\varepsilon^{p}\varkappa^{\psi\tilde{p}-1}\mathscr{V}_2^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\left(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\right)^{\tilde{p}-1}\\
&\quad \Xi\Big[\sup_{0\leq\rho\leq\varkappa}\int_0^{\rho}\left\|\mathfrak{I}_\varepsilon(\vartheta)-\mathfrak{I}^*_\varepsilon(\vartheta)\right\|^{\tilde{p}}\mathrm{d}\vartheta\Big]+\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\int_0^{\rho}\left\|\mathfrak{I}_\varepsilon(\vartheta-\phi)-\mathfrak{I}^*_\varepsilon(\vartheta-\phi)\right\|^{\tilde{p}}\mathrm{d}\vartheta\Big]\\
&=\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1}\Bigg(\int_0^{\varkappa}\Xi\Big[\sup_{0\leq\wp\leq\vartheta}\left\|\mathfrak{I}_\varepsilon(\wp)-\mathfrak{I}^*_\varepsilon(\wp)\right\|^{\tilde{p}}\Big]\mathrm{d}\vartheta\\
&\quad+\int_0^{\varkappa}\Xi\Big[\sup_{0\leq\wp\leq\vartheta}\|\mathfrak{I}_\varepsilon(\wp-\phi)-\mathfrak{I}^*_\varepsilon(\wp-\phi)\|^{\tilde{p}}\Big]\mathrm{d}\vartheta\Bigg),
\end{aligned}
\tag{3.24}
$$

where $\mathfrak{M}_{21}=\frac{2^{3\tilde{p}-3}\mathscr{V}_2^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}\left(1-2^{\tilde{p}-1}\mathscr{V}_1\right)}\left(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\right)^{\tilde{p}-1}$.

Using H-I, J-I, and (σ_4) on $\mathfrak{V}_{22}$, we get the following result:

$$
\begin{aligned}
\mathfrak{V}_{22}&\leq\frac{2^{2\tilde{p}-2}\varepsilon^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\Big(\int_0^{\varkappa}(\varkappa-\vartheta)^{\frac{(\psi-1)\tilde{p}}{\tilde{p}-1}}\mathrm{d}\vartheta\Big)^{\tilde{p}-1}\\
&\quad\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\int_0^{\rho}\left\|\mathcal{G}_1(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_1(\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\right\|^{\tilde{p}}\mathrm{d}\vartheta\Big]\\
&\leq\frac{2^{2\tilde{p}-2}\varepsilon^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\left(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\right)^{\tilde{p}-1}\varkappa^{\psi\tilde{p}}\mathscr{Y}_1(\varkappa)(1+\Xi\left\|\mathfrak{I}^*_\varepsilon(\vartheta)\right\|^{\tilde{p}}+\Xi\left\|\mathfrak{I}^*_\varepsilon(\vartheta-\phi)\right\|^{\tilde{p}})\\
&=\mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}},
\end{aligned}
\tag{3.25}
$$

where $\mathfrak{M}_{22}=\frac{2^{2\tilde{p}-2}\mathscr{Y}_1(\varkappa)\left(1+\Xi\left\|\mathfrak{I}^*_\varepsilon(\vartheta)\right\|^{\tilde{p}}+\Xi\left\|\mathfrak{I}^*_\varepsilon(\vartheta-\phi)\right\|^{\tilde{p}}\right)}{(1-\mathscr{V}_1)^{\tilde{p}-1}\left(1-2^{\tilde{p}-1}\mathscr{V}_1\right)}\left(\frac{\tilde{p}-1}{\psi\tilde{p}-1}\right)^{\tilde{p}-1}$.

By employing J-I, $\mathfrak{V}_3$ yields the following:

$$
\begin{aligned}
\mathfrak{V}_3&\leq\frac{2^{2\tilde{p}-2}\varepsilon^{\frac{\tilde{p}}{2}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\\
&\quad\left(\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\Big\|\int_0^{\rho}\vartheta^{\psi-1}\Big[\mathcal{G}_2(\vartheta,\mathfrak{I}_\varepsilon(\vartheta),\mathfrak{I}_\varepsilon(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\Big]\mathrm{d}\mathscr{B}_\vartheta\Big\|^{\tilde{p}}\Big]\right)\\
&\quad+\frac{2^{2\tilde{p}-2}\varepsilon^{\frac{\tilde{p}}{2}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\\
&\quad\left(\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\Big\|\int_0^{\rho}\vartheta^{\psi-1}\Big[\mathcal{G}_2(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_2(\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\Big]\mathrm{d}\mathscr{B}_\vartheta\Big\|^{\tilde{p}}\Big]\right)\\
&=\mathfrak{V}_{31}+\mathfrak{V}_{32}.
\end{aligned}
\tag{3.26}
$$

Applying (σ_2), H-I, and BHDG-I to $\mathfrak{V}_{31}$ yields the following results:

$$
\begin{aligned}
\mathfrak{V}_{31}&\leq 2^{2\tilde{p}-2}\varepsilon^{\frac{\tilde{p}}{2}}\left(2(\tilde{p}-1)^{1-\tilde{p}}(\tilde{p})^{\tilde{p}+1}\right)^{\frac{\tilde{p}}{2}}\frac{1}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\tilde{p}_1)}\\
&\quad\Xi\left[\int_0^{\varkappa}\vartheta^{2\psi-2}\left\|\mathcal{G}_2(\vartheta,\mathfrak{I}_\varepsilon(\vartheta),\mathfrak{I}_\varepsilon(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\right\|^2\mathrm{d}\vartheta\right]^{\frac{\tilde{p}}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{2^{2\tilde{p}-2}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\left((\tilde{p})^{\tilde{p}+1}2(\tilde{p}-1)^{1-\tilde{p}}\right)^{\frac{\tilde{p}}{2}} \\
&\quad \Xi\left[\int_0^{\varkappa}\vartheta^{(\psi-1)\tilde{p}}\left\|\mathcal{G}_2(\vartheta,\mathfrak{I}_\varepsilon(\vartheta),\mathfrak{I}_\varepsilon(\vartheta-\phi))-\mathcal{G}_2(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\right\|^{\tilde{p}}\mathrm{d}\vartheta\right] \\
&\leq 2^{3\tilde{p}-3}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\mathscr{V}_2^{\tilde{p}}((\tilde{p})^{\tilde{p}+1}2(1-\tilde{p})^{\tilde{p}-1})^{\frac{\tilde{p}}{2}}\frac{1}{(1-\mathscr{V}_1)^{p-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\int_0^{\varkappa}\vartheta^{(\psi-1)\tilde{p}} \\
&\quad \Xi\Big[\sup_{0\leq\wp\leq S}\left[\left\|\mathfrak{I}_\varepsilon(\wp)-\mathfrak{I}^*_\varepsilon(\wp)\right\|^{\tilde{p}}+\left\|\mathfrak{I}_\varepsilon(\wp-\phi)-\mathfrak{I}^*_\varepsilon(\wp-\phi)\right\|^{\tilde{p}}\right]\mathrm{d}\vartheta\Big] \\
&\quad \mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\Bigg(\int_0^{\varkappa}\vartheta^{(\psi-1)\tilde{p}}\Xi\Big[\sup_{0\leq\wp\leq\vartheta}\left\|\mathfrak{I}_\varepsilon(\wp)-\mathfrak{I}^*_\varepsilon(\wp)\right\|^{\tilde{p}}\mathrm{d}\vartheta\Big] \\
&\quad +\int_0^{\varkappa}\vartheta^{(\psi-1)\tilde{p}}\Xi\Big[\sup_{0\leq\wp\leq\vartheta}\left\|\mathfrak{I}_\varepsilon(\wp-\phi)-\mathfrak{I}^*_\varepsilon(\wp-\phi)\right\|^{\tilde{p}}\mathrm{d}\vartheta\Big]\Bigg), \qquad (3.27)
\end{aligned}
$$

where $\mathfrak{M}_{31}=\frac{2^{3\tilde{p}-3}\mathscr{V}_2^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\left(\frac{(\tilde{p})^{\tilde{p}+1}}{2(\tilde{p}-1)^{\tilde{p}-1}}\right)^{\frac{\tilde{p}}{2}}$.

Once more, employing (σ_2), H-I, and BHDG-I on $\mathfrak{V}_{32}$ yields the following results:

$$
\begin{aligned}
\mathfrak{V}_{32} &\leq 2^{2\tilde{p}-2}(2(\tilde{p}-1)^{1-\tilde{p}-}(\tilde{p})^{\tilde{p}+1})^{\frac{\tilde{p}}{2}}\frac{1}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}\varepsilon^{\frac{\tilde{p}}{2}} \\
&\quad \Xi\left[\int_0^{\varkappa}\left\|\mathcal{G}_1(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))-\widetilde{\mathcal{G}}_2(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\right\|^2\vartheta^{2\psi-2}\mathrm{d}\vartheta\right]^{\frac{\tilde{p}}{2}} \\
&\leq 2^{2\tilde{p}-2}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\frac{1}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)}(2(\tilde{p}-1)^{\tilde{p}-1}(\tilde{p})^{\tilde{p}+1})^{\frac{\tilde{p}}{2}} \\
&\quad \Xi\Big[\int_0^{\varkappa}\vartheta^{(\psi-1)\tilde{p}}\left(\left\|\mathcal{G}_2(\vartheta,\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\right\|^{\tilde{p}}+\left\|\widetilde{\mathcal{G}}_2(\mathfrak{I}^*_\varepsilon(\vartheta),\mathfrak{I}^*_\varepsilon(\vartheta-\phi))\right\|^{\tilde{p}}\right)\mathrm{d}\vartheta\Big] \\
&\leq \frac{2^{3\tilde{p}-3}3^{\tilde{p}-1}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\mathscr{V}_3^{\tilde{p}}\left(\mathscr{V}_3^{\tilde{p}}+\mathscr{V}_4\right)^{\tilde{p}}}{(1-\mathscr{V}_1)^{\tilde{p}-1}(1-2^{\tilde{p}-1}\mathscr{V}_1)[(\psi-1)\tilde{p}+1]} \\
&\quad (2(\tilde{p}-1)^{1-\tilde{p}}(\tilde{p})^{\tilde{p}+1})^{\frac{\tilde{p}}{2}}(1+\Xi[\left\|\mathfrak{I}^*_\varepsilon(\vartheta)\right\|^{\tilde{p}}]+\Xi[\left\|\mathfrak{I}^*_\varepsilon(\vartheta-\phi)\right\|^{\tilde{p}}]) \\
&= \mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}, \qquad (3.28)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{M}_{32} = &\frac{1}{(1-\mathscr{V}_1)^{\tilde{p}-1}}2^{3\tilde{p}-3}3^{\tilde{p}-1}\mathscr{V}_3^{\tilde{p}}\left(\mathscr{V}_3^{\tilde{p}}+\mathscr{V}_4\right)^{\tilde{p}}\frac{1}{(1-2^{\tilde{p}-1}\mathscr{V}_1)((\psi-1)\tilde{p}+1)} \\
&(2(\tilde{p}-1)^{1-\tilde{p}}(\tilde{p})^{\tilde{p}+1})^{\frac{\tilde{p}}{2}}(1+\Xi[\left\|\mathfrak{I}^*_\varepsilon(\vartheta)\right\|^{\tilde{p}}]+\Xi[\left\|\mathfrak{I}^*_\varepsilon(\vartheta-\phi)\right\|^{\tilde{p}}]).
\end{aligned}
$$

By employing Eq (3.22) through (3.28) in Eq (3.21), we obtain the following outcomes:

$$
\begin{aligned}
&\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\left\|\mathfrak{I}_\varepsilon(\rho)-\mathfrak{I}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\Big] \\
&\leq \frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\|\mathfrak{I}_\varepsilon(\rho-\phi)\|^{\tilde{p}}\Big]+\Xi\Big[\sup_{0\leq\rho\leq\varkappa}\left\|\mathfrak{I}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\Big]\right)
\end{aligned}
$$

$$+\mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}}+\mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}$$
$$+\int_0^{\varkappa}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1}+\mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\vartheta^{(\psi-1)\tilde{p}}\right]\Xi\left[\sup_{0\le\wp\le\vartheta}\left\|\mathfrak{I}_{\varepsilon}(\wp)-\mathfrak{I}^*_{\varepsilon}(\wp)\right\|^{\tilde{p}}d\vartheta\right]$$
$$+\int_0^{\varkappa}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1}+\mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\vartheta^{(\psi-1)\tilde{p}}\right]\Xi\left[\sup_{0\le\wp\le\vartheta}\left\|\mathfrak{I}_{\varepsilon}(\wp-\phi)-\mathfrak{I}^*_{\varepsilon}(\wp-\phi)\right\|^{\tilde{p}}d\vartheta\right]. \tag{3.29}$$

Taking $\delta(\varkappa)=\Xi\left[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{I}_{\varepsilon}(\rho)-\mathfrak{I}^*_{\varepsilon}(\rho)\right\|^{\tilde{p}}\right]$ and $\Xi\left[\sup_{-\phi\le\rho\le 0}\|\mathfrak{I}_{\varepsilon}(\rho)-\mathfrak{I}^*_{\varepsilon}(\rho)\|^{\tilde{p}}\right]=0$.

Given the aforementioned assumptions, it is feasible.

$$\Xi\left[\sup_{0\le\wp\le\vartheta}\left\|\mathfrak{I}_{\varepsilon}(\wp-\phi)-\mathfrak{I}^*_{\varepsilon}(\wp-\phi)\right\|^{\tilde{p}}\right]=\delta(\vartheta-\phi).$$

Consequently,

$$\delta(\varkappa)\le\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\le\rho\le\varkappa}\|\mathfrak{I}_{\varepsilon}(\rho-\phi)\|^{\tilde{p}}\right]+\Xi\left[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{I}^*_{\varepsilon}(\rho-\phi)\right\|^{\tilde{p}}\right]\right)$$
$$+\mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}}+\mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}$$
$$+\int_0^{\varkappa}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1}+\mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\vartheta^{(\psi-1)\tilde{p}}\right](\delta(\vartheta)+\delta(\vartheta-\phi))d\vartheta. \tag{3.30}$$

Let $\gamma(\varkappa)=\sup_{\mho\in[-\phi,\varkappa]}\delta(\mho)$. Consequently, for all $\forall\varkappa\in[0,\Upsilon]$, we observe that $\delta(\vartheta)\le\gamma(\vartheta)$ and $\delta(\vartheta-\phi)\le\gamma(\vartheta)$.

Therefore, we derive the following results from Eq (3.29):

$$\delta(\varkappa)\le\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\le\rho\le\varkappa}\|\mathfrak{I}_{\varepsilon}(\rho-\phi)\|^{\tilde{p}}\right]+\Xi\left[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{I}^*_{\varepsilon}(\rho-\phi)\right\|^{\tilde{p}}\right]\right)$$
$$+\mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}}+\mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}$$
$$+2\int_0^{\varkappa}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1}+\mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{p}{2}-1}\vartheta^{(\psi-1)\tilde{p}}\right]\gamma(\vartheta)d\vartheta.$$

For all $\mho\in[0,\varkappa]$, we have

$$\delta(\mho)\le\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\le\rho\le\mho}\|\mathfrak{I}_{\varepsilon}(\rho-\phi)\|^{\tilde{p}}\right]+\Xi\left[\sup_{0\le\rho\le\mho}\left\|\mathfrak{I}^*_{\varepsilon}(\rho-\phi)\right\|^{\tilde{p}}\right]\right)$$
$$+\mathfrak{M}_{22}\varepsilon^{\tilde{p}}\mho^{\psi\tilde{p}}+\mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\mho^{\psi\tilde{p}-\frac{\tilde{p}}{2}}$$
$$+2\int_0^{\mho}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\mho^{\psi\tilde{p}-1}+\mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\mho^{\frac{\tilde{p}}{2}-1}(\mho-\vartheta)^{(\psi-1)\tilde{p}}\right]\gamma(\vartheta)d\vartheta$$
$$\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\le\rho\le\varkappa}\|\mathfrak{I}_{\varepsilon}(\rho-\phi)\|^{\tilde{p}}\right]+\Xi\left[\sup_{0\le\rho\le\varkappa}\left\|\mathfrak{I}^*_{\varepsilon}(\rho-\phi)\right\|^{\tilde{p}}\right]\right)$$
$$+\mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}}+\mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}$$
$$+2\int_0^{\varkappa}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1}+\mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\vartheta^{(\psi-1)\tilde{p}}\right]\gamma(\vartheta)d\vartheta.$$

As a result,

$$\begin{aligned}\gamma(\varkappa) =& \sup_{\mho\in[-\phi,\varkappa]} \delta(\mho)\\ \leq& \max\left\{\sup_{\mho\in[-\phi,0]} \delta(\mho), \sup_{\mho\in[0,\varkappa]} \delta(\mho)\right\}\\ \leq& \frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\leq\rho\leq\varkappa} \|\mathfrak{J}_\varepsilon(\rho-\phi)\|^{\tilde{p}}\right] + \Xi\left[\sup_{0\leq\rho\leq\varkappa} \left\|\mathfrak{J}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\right]\right)\\ &+ \mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}} + \mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\\ &+ 2\int_0^{\varkappa}\left[\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}-1} + \mathfrak{M}_{31}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\frac{\tilde{p}}{2}-1}\vartheta^{(\psi-1)\tilde{p}}\right]\gamma(\vartheta)\mathrm{d}\vartheta.\end{aligned}$$

According to G-B-I, we obtain the following:

$$\begin{aligned}\gamma(\varkappa) \leq&\left(\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\leq\rho\leq\varkappa} \|\mathfrak{J}_\varepsilon(\rho-\phi)\|^{\tilde{p}}\right] + \Xi\left[\sup_{0\leq\rho\leq\varkappa} \left\|\mathfrak{J}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\right]\right) + \mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}} + \mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\right)\\ &\exp\left(2\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}} + \frac{2\mathfrak{M}_{31}}{(\psi-1)\tilde{p}+1}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\right).\end{aligned}$$

As a result, we derive the following result from Eq (3.30).

$$\begin{aligned}&\Xi\left[\sup_{0\leq\rho\leq\varkappa} \left\|\mathfrak{J}_\varepsilon(\rho)-\mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\right]\\ \leq&\left(\frac{2^{2\tilde{p}-2}\mathscr{V}_1}{1-2^{\tilde{p}-1}\mathscr{V}_1}\left(\Xi\left[\sup_{0\leq\rho\leq\varkappa} \|\mathfrak{J}_\varepsilon(\rho-\phi)\|^{\tilde{p}}\right] + \Xi\left[\sup_{0\leq\rho\leq\varkappa} \left\|\mathfrak{J}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\right]\right) + \mathfrak{M}_{22}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}} + \mathfrak{M}_{32}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\right)\\ &\exp\left(2\mathfrak{M}_{21}\varepsilon^{\tilde{p}}\varkappa^{\psi\tilde{p}} + \frac{2\mathfrak{M}_{31}}{(\psi-1)\tilde{p}+1}\varepsilon^{\frac{\tilde{p}}{2}}\varkappa^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\right).\end{aligned}$$

This indicates that for any $\rho \in \left[0, \varphi\varepsilon^{-\Psi}\right] \subseteq [0, \Upsilon]$, there exist $\varphi > 0$ and $\Psi \in (0, 1)$.

$$\Xi\left[\sup_{0\leq\rho\leq\varphi\varepsilon^{-\Psi}} \left\|\mathfrak{J}_\varepsilon(\rho)-\mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\right] \leq \mathscr{Z}\varepsilon^{1-\Psi}, \tag{3.31}$$

where

$$\begin{aligned}\mathscr{Z} =&\left(\frac{2^{2\tilde{p}-2}\tilde{p}_1\varepsilon^{\Psi-1}}{1-2^{\tilde{p}-1}\tilde{p}_1}\left(\Xi\left[\sup_{0\leq\rho\leq\varphi\varepsilon^{-\Psi}} \|\mathfrak{J}_\varepsilon(\rho-\phi)\|^{\tilde{p}}\right] + \Xi\left[\sup_{0\leq\rho\leq\varphi\varepsilon^{-\Psi}} \left\|\mathfrak{J}^*_\varepsilon(\rho-\phi)\right\|^{\tilde{p}}\right]\right)\right.\\ &\left.+ \mathfrak{M}_{22}\varphi^{\psi\tilde{p}}\varepsilon^{\tilde{p}+\Psi-\psi\Psi\tilde{p}-1} + \mathfrak{M}_{32}\varphi^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\varepsilon^{\frac{\tilde{p}}{2}(1+\Psi)+\Psi-\psi\Psi\tilde{p}-1}\right)\\ &\exp\left[2\mathfrak{M}_{21}\varphi^{\psi\tilde{p}}\varepsilon^{\tilde{p}(1-\psi\Psi)} + \frac{2\mathfrak{M}_{31}}{(\psi-1)\tilde{p}+1}\varphi^{\psi\tilde{p}-\frac{\tilde{p}}{2}}\varepsilon^{\frac{\tilde{p}}{2}(1+\Psi)-\psi\Psi\tilde{p}}\right]\end{aligned}$$

be a constant. Consequently, for any $\mathbb{Z} > 0$, determining $\varepsilon_1 \in (0, \varepsilon_0]$ such that it satisfies $\forall\varepsilon \in (0, \varepsilon_1]$ and $\rho \in \left[-\phi, \varphi\varepsilon^{-\Psi}\right]$ enables us to infer:

$$\Xi\left[\sup_{-\phi\leq\rho\leq\varphi\varepsilon^{-\Psi}} \left\|\mathfrak{J}_\varepsilon(\rho)-\mathfrak{J}^*_\varepsilon(\rho)\right\|^{\tilde{p}}\right] \leq \mathbb{Z}.$$

We illustrate the worth of our established theoretical result with two examples in the next section.

4. Examples

The average behavior of a complex system can be derived using the principles of averaging, as demonstrated in the following two numerical examples:

Example 4.1. Consider the following FrNSDE:

$$\begin{cases} \mathscr{T}_{\rho}^{0.8}\left[\mathfrak{I}_{\varepsilon}(\rho)-\rho^{\frac{1}{4}}-\frac{1}{4}\sin(\mathfrak{I}_{\varepsilon}(\rho))\right] \\ =\varepsilon\cos^2(\rho)\mathfrak{I}_{\varepsilon}(\rho)-\rho\mathfrak{I}_{\varepsilon}(\rho)\sin(\rho-\pi)+\sqrt{\varepsilon}\frac{\mathrm{d}\mathfrak{B}_{\rho}}{\mathrm{d}\rho},\ \rho\in[0,\Upsilon], \\ \mathfrak{I}(\rho)=\aleph(\rho),\ \rho\in[-\phi,0], \end{cases} \tag{4.1}$$

Based on the aforementioned system, we obtain the following: $\psi = 0.8$ and

$$\begin{aligned} &\mathcal{Y}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))=-\rho^{\frac{1}{4}}-\frac{1}{4}\sin(\mathfrak{I}_{\varepsilon}(\rho)),\\ &\mathcal{G}_1(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))=\cos^2(\rho)\mathfrak{I}_{\varepsilon}(\rho)-\rho\mathfrak{I}_{\varepsilon}(\rho)\sin(\rho-\pi),\\ &\mathcal{G}_2(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))=1. \end{aligned}$$

The subsequent expressions denote the averages of $\mathcal{G}_1$ and $\mathcal{G}_2$:

$$\begin{aligned} &\widetilde{\mathcal{G}_1}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))=\frac{1}{\pi}\int_0^{\pi}(\cos^2(\rho)\mathfrak{I}_{\varepsilon}(\rho)-\rho\mathfrak{I}_{\varepsilon}(\rho)\sin(\rho-\pi))\mathrm{d}\vartheta=\frac{3}{2}\mathfrak{I}_{\varepsilon}(\rho),\\ &\widetilde{\mathcal{G}_2}(\rho,\mathfrak{I}(\rho),\mathfrak{I}(\rho-\phi))=\frac{1}{\pi}\int_0^{\pi}1\mathrm{d}\vartheta=1. \end{aligned}$$

To derive the average formulation related to Eq (4.1), substitute the simplified solution $\mathfrak{I}_{\varepsilon}^{*}(\rho)$ for the original solution $\mathfrak{I}_{\varepsilon}(\rho)$. Consequently, the simplified averaged equation is expressed as follows:

$$\begin{cases} \mathscr{T}_{\rho}^{0.8}\left[\mathfrak{I}_{\varepsilon}^{*}(\rho)-\rho^{\frac{1}{4}}-\frac{1}{4}\sin(\mathfrak{I}_{\varepsilon}^{*}(\rho))\right] \\ =\frac{3}{2}\varepsilon\mathfrak{I}_{\varepsilon}(\rho)+\sqrt{\varepsilon}\frac{\mathrm{d}\mathfrak{B}_{\rho}}{\mathrm{d}\rho} \\ \mathfrak{I}(\rho)=\aleph(\rho),\ \rho\in[-\phi,0]. \end{cases} \tag{4.2}$$

Hence, all conditions outlined in Theorem 3.3 are satisfied. Consequently, in the limit as $\varepsilon \to 0$, the original solution $\mathfrak{I}_{\varepsilon}(\rho)$ and the averaged solution $\mathfrak{I}_{\varepsilon}^{*}(\rho)$ are equivalent in the $\mathbb{L}^p$ sense. Figure 1 then presents a numerical comparison of the solutions $\mathfrak{I}_{\varepsilon}^{*}(\rho)$ of the averaged Eq (4.2) and $\mathfrak{I}_{\varepsilon}(\rho)$ of the original Eq (4.1). Figure 1 demonstrates a significant degree of consistency between $\mathfrak{I}_{\varepsilon}(\rho)$ and $\mathfrak{I}_{\varepsilon}^{*}(\rho)$, confirming the veracity of our established theoretical results.

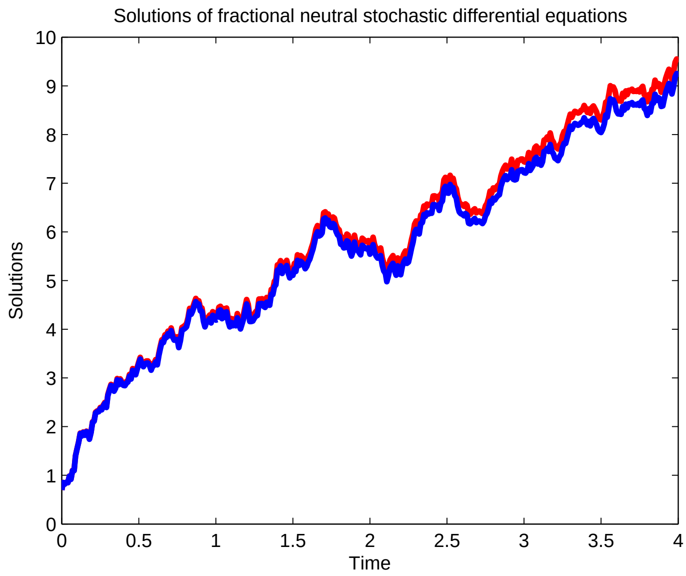

Figure 1. The blue color indicates the solution of the original equation, while the red color represents the solution of the averaged equation, when $\epsilon = 0.001$.

Example 4.2. Consider the subsequent FrNSDE:

$$\begin{cases} \mathscr{T}_{\rho}^{0.9}\left[\mathfrak{J}_{\varepsilon}(\rho) - \rho^{\frac{1}{6}} - \cos(\mathfrak{J}_{\varepsilon}(\rho))\right] \\ = \varepsilon\frac{3\pi}{4}\mathfrak{J}_{\varepsilon}(\rho)\sin^3\rho + \sqrt{\varepsilon}\sin^2\rho\mathfrak{J}_{\varepsilon}(\rho-\phi)\frac{\mathrm{d}\mathfrak{B}_{\rho}}{\mathrm{d}\rho},\ \rho \in [0,\Upsilon], \\ \mathfrak{J}(\rho) = \aleph(\rho),\ \rho \in [-\phi, 0], \end{cases} \tag{4.3}$$

where $\psi = 0.9$ and following:

$$\begin{aligned} \mathcal{Y}(\rho, \mathfrak{J}(\rho), \mathfrak{J}(\rho-\phi)) &= -\rho^{\frac{1}{6}} - \cos(\mathfrak{J}_{\varepsilon}(\rho)), \\ \mathcal{G}_1(\rho, \mathfrak{J}(\rho), \mathfrak{J}(\rho-\phi)) &= \frac{3\pi}{4}\mathfrak{J}_{\varepsilon}(\rho)\sin^3\rho, \\ \mathcal{G}_2(\rho, \mathfrak{J}(\rho), \mathfrak{J}(\rho-\phi)) &= \sin^2\rho\mathfrak{J}_{\varepsilon}(\rho-\phi). \end{aligned}$$

The following are the formats in which the averages of $\mathcal{G}_1$ and $\mathcal{G}_2$ are displayed:

$$\begin{aligned} \widetilde{\mathcal{G}_1}(\rho, \mathfrak{J}(\rho), \mathfrak{J}(\rho-\phi)) &= \frac{1}{\pi}\int_0^{\pi}\frac{3\pi}{4}\mathfrak{J}_{\varepsilon}(\rho)\sin^3\rho\,\mathrm{d}\vartheta = \mathfrak{J}_{\varepsilon}^*(\rho), \\ \widetilde{\mathcal{G}_2}(\rho, \mathfrak{J}(\rho), \mathfrak{J}(\rho-\phi)) &= \frac{1}{\pi}\int_0^{\pi}\sin^2\rho\mathfrak{J}_{\varepsilon}(\rho-\phi)\mathrm{d}\vartheta = \frac{1}{2}\mathfrak{J}_{\varepsilon}^*(\rho-\phi). \end{aligned}$$

Use the simplified solution $\mathfrak{J}_{\varepsilon}^*(\rho)$ instead of the original solution $\mathfrak{J}_{\varepsilon}(\rho)$ to construct the average form about Eq (4.3). Consequently, the following is the corresponding averaged FrNSDE of Eq (4.3):

$$\begin{cases} \mathscr{T}_{\rho}^{0.85}\left[\mathfrak{J}_{\varepsilon}^*(\rho) - \rho^{\frac{1}{6}} - \cos(\mathfrak{J}_{\varepsilon}^*(\rho))\right] \\ = \varepsilon\mathfrak{J}_{\varepsilon}^*(\rho) + \sqrt{\varepsilon}\frac{1}{2}\mathfrak{J}_{\varepsilon}^*(\rho-\phi)\frac{\mathrm{d}\mathfrak{B}_{\rho}}{\mathrm{d}\rho} \\ \mathfrak{J}(\rho) = \aleph(\rho),\ \rho \in [-\phi, 0]. \end{cases} \tag{4.4}$$

All of the requirements listed in Theorem 3.3 are definitely met. Thus, in the context of $\varepsilon \to 0$, the solution $\mathfrak{I}_{\varepsilon}(\rho)$ and the solution $\mathfrak{I}^{*}_{\varepsilon}(\rho)$ correspond in the context of $\mathbb{L}^{p}$. A numerical comparison is shown in Figure 2 between the solution $\mathfrak{I}^{*}_{\varepsilon}(\rho)$ of the averaged equation Eq (4.4) and the original Eq (4.3). Solutions $\mathfrak{I}_{\varepsilon}(\rho)$ and $\mathfrak{I}^{*}_{\varepsilon}(\rho)$ overlap in Figure 2, proving the validity of our established theoretical results.

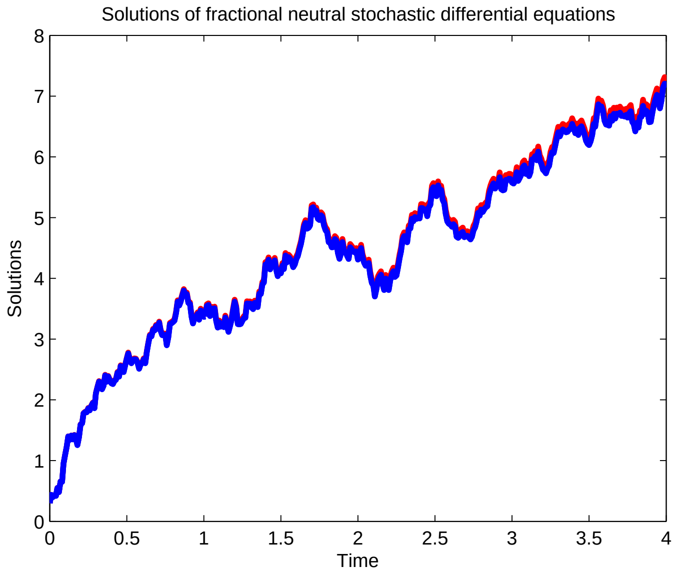

Figure 2. The blue color indicates the solution of the original equation, while the red color represents the solution of the averaged equation, when $\epsilon = 0.001$.

5. Conclusions

In this study, we have established the results regarding the EU and UH stability of solutions for FrNSDEs and explored the averaging principle within the Con-FrD framework in the $\mathfrak{L}^{\tilde{p}}$ space. We have employed the B anach FPT to investigate the EU for solutions to FrNSDEs. Moreover, we have illustrated the averaging principle for FrNSDEs in the $\mathfrak{L}^{\tilde{p}}$ space using various methodologies, including G-B-I, G-I, BHDG-I, H-I, J-I, and the interval translation approach. Additionally, we have presented numerical examples to validate our theoretical findings.

Author contributions

Wedad Albalawi, Muhammad Imran Liaqat, Fahim Ud Din, Kottakkaran Sooppy Nisar and Abdel-Haleem Abdel-Aty: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing-original draft, Writing-review & editing, Visualization; Wedad Albalawi, Kottakkaran Sooppy Nisar and Abdel-Haleem Abdel-Aty: Resources, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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