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Research article

Global classical solution of the fractional Nernst-Planck-Poisson-Navier-Stokes system in $\mathbb{R}^3$

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Abstract: In this paper, we consider a fractional Nernst-Planck-Poisson-Navier-Stokes system in $\mathbb{R}^3$. First, we obtain a priori estimates by using energy estimates. Then, we construct an iterative solution sequence by solving the approximate problem and obtaining the local existence and uniqueness of the classical solution. Finally, combining the local existence with a priori estimates, the global existence and uniqueness of the classical solution with small initial data are obtained.

Keywords: Nernst-Planck-Poisson-Navier-Stokes; energy estimates; fractional diffusion; iterative solution sequence; global classical solution
Mathematics Subject Classification: 35A01, 35A09, 35Q92, 76D05

1. Introduction

1.1. The drift-diffusion system

In order to describe the charge transport in the semiconductor device, Roosbroeck [21] first proposed the drift-diffusion (DD) system. We briefly review some known results on the DD system. As early as the last century, Mock [14] considered the following system:

$$\begin{cases} n_t - \Delta n = -\nabla \cdot (n\nabla\phi), & t > 0, x \in \Omega, \\ v_t - \Delta v = \nabla \cdot (v\nabla\phi), & t > 0, x \in \Omega, \\ \Delta\phi = n - v, & t > 0, x \in \Omega, \\ (n, v)|_{t=0} = (n_0, v_0), & x \in \Omega, \end{cases} \tag{1.1}$$

on the bounded domain $\Omega \subseteq \mathbb{R}^N (N \geqslant 1)$ with the Neumann boundary condition used to describe charge transport in semiconductor devices. Here, n and v represent the densities of electrons and holes. ϕ stands for the electrostatic potential. Fang and Ito [5] proved the global existence of weak solutions to system (1.1) with the Neumann boundary condition. Afterwards, Jüngel [10] obtained the asymptotic behavior of the global solution under the Neumann boundary condition. The global existence of the solution for the Cauchy problem of system (1.1) was studied by Kurokiba and Ogawa [13]. Then, Kamashima and Kobayashi [12] deduced the optimal dacay estimate by the weighted energy method and obtained an asymptotic result as $t \to \infty$.

Then, some scientists have focused on the dopant anomalous diffusion in a semiconductor. The anomalous diffusion is shown as the result of two competing phenomena: dopant trapping on the defects and enhanced diffusivity at the edges of the defect-rich region [3]. The classical Laplace operator cannot describe this anomalous diffusion phenomenon. Scholars have found that the fractional Laplace operator can well describe the process involving anomalous diffusion. Owaga and Yamamoto [16] investigated the fractional DD system

$$\begin{cases} n_t + (-\Delta)^{\alpha} n = -\nabla \cdot (n \nabla \phi), & t > 0, x \in \Omega, \\ v_t + (-\Delta)^{\beta} v = \nabla \cdot (v \nabla \phi), & t > 0, x \in \Omega, \\ \Delta \phi = n - v, & t > 0, x \in \Omega, \\ (n, v)|_{t=0} = (n_0, v_0), & x \in \Omega, \end{cases} \tag{1.2}$$

for $\alpha = \beta \in (\frac{1}{2}, 1)$. They proved the global existence and asymptotic behavior of the mild solution in $\Omega = \mathbb{R}^N$ for $N \geqslant 2$. Here, the fractional Laplace operator is defined by

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of the function f. Then Granero-Belinchón [7] considered a fractional DD system with different diffusion orders $\alpha \neq \beta \in (0, 1)$, and obtained the global existence of the classical solution to system (1.2) and several decay estimates for the Lebesgue and Sobolev norms in $\Omega = \mathbb{R}^N$ for $N = 1, 2, 3$.

1.2. The Nernst-Planck-Poisson-Navier-Stokes system

A similar phenomenon occurs for the ions in a solution. The Nernst-Planck-Poisson(NPP) system was originally conceived to describe the motion of ions in a solution in the context of electrochemistry. It should be emphasized that, although both the DD system and the NPP system share a common structure, their physical meanings are different. Ions in the NPP system are charged particles, electrons and holes in the DD system enjoy a duality between quantum waves and particles. In order to reveal the interaction between the movement of the macroscopic fluid and the transport of the microscopic charge, Rubinstein [17] proposed the classical Nernst-Planck-Poisson-Navier-Stokes(NPPNS) system

$$\begin{cases} n_t + u \cdot \nabla n - \Delta n = -\nabla \cdot (n \nabla \phi), & t > 0, x \in \Omega, \\ v_t + u \cdot \nabla v - \Delta v = \nabla \cdot (v \nabla \phi), & t > 0, x \in \Omega, \\ \Delta \phi = n - v, & t > 0, x \in \Omega, \\ u_t + u \cdot \nabla u - \Delta u + \nabla P = \Delta \phi \nabla \phi, & t > 0, x \in \Omega, \\ \nabla \cdot u = 0, & t > 0, x \in \Omega, \\ (n, v, u)|_{t=0} = (n_0, v_0, u_0), & x \in \Omega, \end{cases} \tag{1.3}$$

to show the electrokinetic effects consisting of the interplay between charges and the flow field. Here n and v represent the densities of negatively and positively charged particles, u represents the velocity field of the fluid, and P denotes the pressure function. System (1.3) describes an isothermal, incompressible, viscous Newtonian fluid of uniform and homogeneous composition of a high number of positively and negatively charged particles ranging from colloidal to nanosize. It is further assumed to be a dilute fluid so that the electromagnetic forces can be neglected. Readers can refer to [1] and the references therein for more discussion on the physical background of the system (1.3).

In the past few decades, a number of scientists have developed great interest in and conducted indepth research on the NPPNS system (1.3). Based on Kato's semigroup framework, Jerome [8] established the local existence and uniqueness of strong solutions to system (1.3) in $\mathbb{R}^N (N \geqslant 2)$. Jerome and Sacco [9] obtained the global existence of weak solutions under the mixed Dirichlet boundary condition. By using the energy inequalities and Schauder's fixed point theorem, Schmuck [18] obtained the global existence of weak solutions to system (1.3) in a bounded domain $\Omega \subseteq \mathbb{R}^N (N = 2 \text{ or } 3)$ with the Neumann boundary condition. Deng, Zhao and Cui [23, 24] obtained the global well-posedness of system (1.3) with small initial data in negative-order Besov space and critical Lebesgue spaces in $\mathbb{R}^N (N \geqslant 2)$. Zhao, Zhang, and Liu [25] showed the global well-posedness of system (1.3) in the critical Besov space with a large vertical velocity component in $\mathbb{R}^3$. Fan, Li, and Nakamura [4] also proved some regularity criteria for the strong solutions to the Cauchy problem of system (1.3) in $\mathbb{R}^3$. Zhang and Yin [22] proved the global existence of the classical solution to the Cauchy problem of system (1.3) in $\mathbb{R}^2$ and established the L^2 decay estimates by using the Fourier splitting method. Recently, Gong, Wang, and Zhang [6] proved the existence of suitable weak solutions to system (1.3) in $\mathbb{R}^3$. Based on the spectral analysis and the energy method, Tong and Tan [19] obtained the lower bound and upper bound decay rates of the solution to a generalized NPPNS system in $\mathbb{R}^3$.

As for the classical chemotaxis-Navier-Stokes system proposed by Tuval et al. [20], Chae, Kang and Lee [2] established the global existence of smooth solutions in $\mathbb{R}^N (N = 2 \text{ or } 3)$. Afterwards, Zhu, Liu and Zhou [26] considered the fractional chemotaxis-Navier-Stokes system in $\mathbb{R}^3$ and established the global existence and uniqueness of classical solution with small initial data. Research shows that macroscopic diffusion has been used to describe the random walk of particles; however, Lev́y's flights can effectively describe the anomalous diffusion phenomenon. Instead of the classical Laplace operator, the fractional Laplace operator can well describe this anomalous diffusion phenomenon. Considering the anomalous diffusion phenomenon of the ions in the solution, we couple the NPPNS system with the fractional Laplace operator and consider the following fractional NPPNS system:

$$\begin{cases} n_t + u \cdot \nabla n + (-\Delta)^s n = -\nabla \cdot (n \nabla \phi), & t > 0, x \in \mathbb{R}^3, \\ v_t + u \cdot \nabla v + (-\Delta)^s v = \nabla \cdot (v \nabla \phi), & t > 0, x \in \mathbb{R}^3, \\ \Delta \phi = n - v, & t > 0, x \in \mathbb{R}^3, \\ u_t + u \cdot \nabla u - \Delta u + \nabla P = \Delta \phi \nabla \phi, & t > 0, x \in \mathbb{R}^3, \\ \nabla \cdot u = 0, & t > 0, x \in \mathbb{R}^3, \\ (n, v, u)|_{t=0} = (n_0, v_0, u_0), & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $s \in (0, 1)$. The aim of this paper is to obtain the global existence of a classical solution to system (1.4) under the condition of small initial data. Firstly, we obtain a priori estimates of the solution to system (1.4) through the energy method. Then we construct a solution sequence by iterative methods and prove the local existence of the solution sequence of the iterative system (3.25). It should be emphasized that different from [26], firstly, the physical background is different. The chemotaxis-Navier-Stokes system was proposed to study the interaction between cells and fluids. However, the NPPNS system was originally conceived to reveal the interaction between the movement of the macroscopic fluid and the transport of the microscopic charge. Secondly, due in the difference of the external force in the Navier-Stokes equation, the iterative solution sequence of the approximate problem is different from [26] in the process of proving the existence of solutions to system (1.4). Besides, we also prove that the solution sequence of (3.25) is a Cauchy sequence. Thus, according to the convergence of the solution sequence, we can obtain the local existence and uniqueness of the classical solution to the system (1.4). Finally, we prove the global existence of the classical solution to system (1.4) by combining the priori estimates with the local existence. Furthermore, since the global existence of the classical solution for the three-dimensional Navier-Stokes equation is still open, compared with the condition of large initial data in [22], we study the global existence and uniqueness of the classical solution to system (1.4) under the condition of small initial data in $\mathbb{R}^3$.

The main theorem of this paper states as follows:

Theorem 1.1. *Assume that* $s \in (\frac{1}{2}, 1)$*,* $(n_0, v_0, u_0) \in (L^1 \cap H^m) \times (L^1 \cap H^m) \times H^m (m \geqslant 3)$*,* $n_0, v_0 \geqslant 0$*. If there exists a small enough constant* $\varepsilon_0 > 0$ *such that* $\|n_0\|_{H^3} + \|v_0\|_{H^3} + \|u_0\|_{H^3} \leqslant \varepsilon_0$*, then system (1.4) possesses a unique global solution* (n, v, ϕ, u) *satisfying that for all* $t \geqslant 0$*,*

$$\begin{aligned}&(\|n\|_{H^m}^2 + \|v\|_{H^m}^2 + \|\phi\|_{H^{m+2}}^2 + \|u\|_{H^m}^2) + \int_0^t (\|\partial^s n\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2 + \|\partial^s \phi\|_{H^{m+2}}^2 + \|\partial u\|_{H^m}^2) \mathrm{d}\tau \\ \leqslant& C(\|n_0\|_{H^m}^2 + \|v_0\|_{H^m}^2 + \|u_0\|_{H^m}^2),\end{aligned}$$

where C *is a time-independent positive constant.*

The rest of this paper is organized as follows: In Section 2, we state some useful lemmas that will be used throughout the paper. In Section 3, we first give a priori estimates and prove the local existence of classical solutions to system (1.4) by iterative methods; then, combining the local existence and a priori estimates, we can obtain the global existence of classical solutions to system (1.4).

2. Preliminary

In this section, we give the following notations: $C > 0$ is a time-independent constant. We reduce $L^q(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3)$ to L^q and H^s, respectively. Moreover, we use $a \lesssim b$ to denote $a \leqslant Cb$. Next, we will give some useful lemmas.

Lemma 2.1. *([15]) Assume that* $0 \leqslant k, m \leqslant l$ *and* $1 \leqslant p, q, r \leqslant \infty$*. It follows that*

$$\|\partial^m f\|_{L^p} \leqslant C \|\partial^k f\|_{L^q}^{1-\theta} \|\partial^l f\|_{L^r}^{\theta}, \tag{2.1}$$

where $\theta \in [0, 1]$*, and* k, m, l *satisfy*

$$\frac{m}{3} - \frac{1}{p} = (\frac{k}{3} - \frac{1}{q})(1 - \theta) + (\frac{l}{3} - \frac{1}{r})\theta.$$

Especially, when $p = \infty$*, we require that* $\theta \in (0, 1)$*,* $k \leqslant m + 1$*, and* $l \geqslant m + 2$*.*

Lemma 2.2. *([11]) Assume that* $0 \leqslant k < \infty$*, then we have*

$$\|\partial^k(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}}\|\partial^k g\|_{L^{p_2}} + \|\partial^k f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{2.2}$$

where $p, p_2, p_3 \in (1,\infty)$*,* $p_1, p_4 \in (1,\infty]$ *with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

3. Proof of Theorem 1.1

3.1. A priori estimates

Lemma 3.1. *Assume that* $s \in (\frac{1}{2}, 1)$*,* $(n_0, v_0, u_0) \in (L^1 \cap H^m) \times (L^1 \cap H^m) \times H^m (m \geqslant 3)$*,* $n_0, v_0 \geqslant 0$*. Suppose that* (n, v, ϕ, u) *is a solution of (1.4). Then* $n, v \in L^1$ *and* $n, v \geqslant 0$ *hold for any* $t \geqslant 0$*. Furthermore, there exists a small enough constant* ε_0 *such that if* $\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3} \leqslant \varepsilon_0$*,*

$$\begin{aligned}&(\|n\|^2_{H^m} + \|v\|^2_{H^m} + \|\phi\|^2_{H^{m+2}} + \|u\|^2_{H^m}) + \int_0^t (\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m} + \|\partial^s\phi\|^2_{H^{m+2}} + \|\partial u\|^2_{H^m})\mathrm{d}\tau \\ \leqslant & C_1(\|n_0\|^2_{H^m} + \|v_0\|^2_{H^m} + \|u_0\|^2_{H^m})\end{aligned} \tag{3.1}$$

holds for any $t \geqslant 0$*, where* C_1 *is a time-independent constant.*

Proof. First, integrating over $\mathbb{R}^3 \times [0, t]$ for the first and second equation of (1.4), we conclude that $n, v \in L^1$. Then, we briefly prove that n is nonnegative (the situation of v is similar to n). For this, we introduce the following auxiliary problem:

$$\begin{cases} n_t + u\cdot\nabla n + (-\Delta)^s n = -\nabla\cdot(n_+\nabla\phi), & t>0, x\in\mathbb{R}^3, \\ v_t + u\cdot\nabla v + (-\Delta)^s v = \nabla\cdot(v_+\nabla\phi), & t>0, x\in\mathbb{R}^3, \\ \Delta\phi = n - v, & t>0, x\in\mathbb{R}^3, \\ u_t + u\cdot\nabla u - \Delta u + \nabla P = \Delta\phi\nabla\phi, & t>0, x\in\mathbb{R}^3, \\ \nabla\cdot u = 0, & t>0, x\in\mathbb{R}^3, \\ (n, v, u)|_{t=0} = (n_0, v_0, u_0), & x\in\mathbb{R}^3. \end{cases} \tag{3.2}$$

Here n_+=max$\{n, 0\}$, n_-=max$\{-n, 0\}$ and $n = n_+ - n_-$. Multiplying the first equation of (3.2) by n_- and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|n_-\|^2_{L^2} + \|\partial^s n_-\|^2_{L^2} = \int_{\mathbb{R}^3} \nabla\cdot(n_+\nabla\phi)n_-\mathrm{d}x = 0. \tag{3.3}$$

Therefore, $\|n_-(t)\|^2_{L^2} \leqslant \|n_-(0)\|^2_{L^2}$. Since n_0 is nonnegative, n is nonnegative. Similarly, v is also nonnegative. We conclude that system (3.2) is equivalent to system (1.4), the nonnegativity of n and v is proved. Finally, we will prove (3.1).

(i) **The estimate of** n**.** Applying $\partial^\alpha (0 \leqslant \alpha \leqslant m)$ to the first equation of (1.4) and multiplying by $\partial^\alpha n$, then integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha n\|^2_{L^2} + \|\partial^{\alpha+s}n\|^2_{L^2} \\ =& -\int_{\mathbb{R}^3}\partial^\alpha(u\cdot\nabla n)\partial^\alpha n\mathrm{d}x - \int_{\mathbb{R}^3}\partial^\alpha(\nabla n\nabla\phi)\partial^\alpha n\mathrm{d}x - \int_{\mathbb{R}^3}\partial^\alpha(n\Delta\phi)\partial^\alpha n\mathrm{d}x \\ =&:I_1 + I_2 + I_3.\end{aligned} \tag{3.4}$$

As for the term I_1, for $\alpha = 0$, recalling that $\nabla \cdot u = 0$, we get

$$I_1 = -\frac{1}{2}\int_{\mathbb{R}^3} u \cdot \nabla |n|^2 \mathrm{d}x = 0. \tag{3.5}$$

For $\alpha = 1$, recalling that $\nabla \cdot u = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}^3} \partial u \cdot \nabla n \partial n \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u \cdot \nabla |\partial n|^2 \mathrm{d}x \\ &\lesssim \|\partial u\|_{L^6} \|\partial n\|_{L^{\frac{3}{1+s}}} \|\partial n\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\partial^2 u\|_{L^2} \|n\|_{L^2}^{\frac{1+2s}{4}} \|\partial^2 n\|_{L^2}^{\frac{3-2s}{4}} \|\partial^{1+s} n\|_{L^2} \\ &\lesssim \|\partial u\|_{H^m} \|n\|_{H^3} \|\partial^s n\|_{H^m} \\ &\leqslant \varepsilon (\|\partial u\|_{H^m}^2 + \|\partial^s n\|_{H^m}^2). \end{aligned} \tag{3.6}$$

For $2 \leqslant \alpha \leqslant m$, we have

$$\begin{aligned} I_1 &= -\frac{1}{2}\int_{\mathbb{R}^3} u \cdot \nabla |\partial^\alpha n|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha u \cdot \nabla n \partial^\alpha n \mathrm{d}x - \sum_{1 \leqslant l \leqslant \alpha-1} C_\alpha^l \int_{\mathbb{R}^3} \partial^l u \cdot \nabla \partial^{\alpha-l} n \partial^\alpha n \mathrm{d}x \\ &\lesssim \|\partial^\alpha u\|_{L^6} \|\nabla n\|_{L^{\frac{3}{1+s}}} \|\partial^\alpha n\|_{L^{\frac{6}{3-2s}}} + \sum_{1 \leqslant l \leqslant \alpha-1} \|\partial^l u\|_{L^{\frac{3}{s}}} \|\partial^{\alpha-l+1} n\|_{L^2} \|\partial^\alpha n\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\partial^{\alpha+1} u\|_{L^2} \|n\|_{L^2}^{\frac{1+2s}{4}} \|\partial^2 n\|_{L^2}^{\frac{3-2s}{4}} \|\partial^{\alpha+s} n\|_{L^2} \\ &\quad + \sum_{1 \leqslant l \leqslant \alpha-1} (\|\partial^{\frac{5}{2}-2s} u\|_{L^2}^\theta \|\partial^{\alpha+1} u\|_{L^2}^{1-\theta} \|\partial^{\frac{3}{2}-s} n\|_{L^2}^{1-\theta} \|\partial^{\alpha+s} n\|_{L^2}^\theta) \|\partial^{\alpha+s} n\|_{L^2} \\ &\lesssim \|\partial u\|_{H^m} \|n\|_{H^3} \|\partial^s n\|_{H^m} + \|u\|_{H^3}^\theta \|n\|_{H^3}^{1-\theta} \|\partial^{\alpha+1} u\|_{L^2}^{1-\theta} \|\partial^{\alpha+s} n\|_{L^2}^{1+\theta} \\ &\leqslant \varepsilon_0 (\|\partial u\|_{H^m}^2 + \|\partial^s n\|_{H^m}^2), \end{aligned} \tag{3.7}$$

where $\theta = \frac{2\alpha-2l+2s-1}{2\alpha+4s-3} \in (0, 1)$.

In a word,

$$I_1 \lesssim \varepsilon_0 (\|\partial u\|_{H^m}^2 + \|\partial^s n\|_{H^m}^2) \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.8}$$

As for the term I_2, for $\alpha = 0$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} I_2 &= -\int_{\mathbb{R}^3} \nabla \phi \cdot \nabla n \cdot n \mathrm{d}x \\ &\lesssim \|\nabla \phi\|_{L^2} \|\nabla n\|_{L^{\frac{3}{s}}} \|n\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim (\|\Delta \phi\|_{L^1}^{\frac{4}{5}} \|\partial(\Delta \phi)\|_{L^2}^{\frac{1}{5}}) \|\partial^{\frac{5}{2}-s} n\|_{L^2} \|\partial^s n\|_{L^2} \\ &\lesssim \varepsilon_0^{\frac{1}{5}} \|\partial^s n\|_{H^m}^2. \end{aligned} \tag{3.9}$$

For $\alpha = 1$, it follows that

$$\begin{aligned} I_2 &= -\frac{1}{2}\int_{\mathbb{R}^3} \Delta \phi \cdot \partial n \cdot \partial n \mathrm{d}x \\ &\lesssim \|\Delta \phi\|_{L^{\frac{3}{2s-1}}} \|\partial n\|_{L^{\frac{3}{2-s}}}^2 \\ &\lesssim \|\partial^{\frac{5}{2}-2s} \Delta \phi\|_{L^2} \|\partial^{s+\frac{1}{2}} n\|_{L^2}^2 \\ &\lesssim \varepsilon_0 \|\partial^s n\|_{H^m}^2. \end{aligned} \tag{3.10}$$

For $2 \leqslant \alpha \leqslant m$, we deduce

$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi \cdot \partial^\alpha n \cdot \partial^\alpha n \mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha \nabla\phi \cdot \nabla n \partial^\alpha n \mathrm{d}x \\
&\quad - \sum_{1\leqslant l\leqslant \alpha-1} C_\alpha^l \int_{\mathbb{R}^3} \partial^l \nabla\phi \cdot \nabla \partial^{\alpha-l} n \partial^\alpha n \mathrm{d}x \\
&\lesssim \|\Delta\phi\|_{L^{\frac{3}{2s-1}}} \|\partial^\alpha n\|^2_{L^{\frac{3}{2-s}}} + \|\partial^\alpha \nabla\phi\|_{L^2} \|\nabla n\|_{L^{\frac{3}{s}}} \|\partial^\alpha n\|_{L^{\frac{6}{3-2s}}} \\
&\quad + \sum_{1\leqslant l\leqslant \alpha-1} \|\partial^l \nabla\phi\|_{L^{\frac{3}{s}}} \|\partial^{\alpha-l+1} n\|_{L^2} \|\partial^\alpha n\|_{L^{\frac{6}{3-2s}}} \\
&\lesssim \|\partial^{\frac{5}{2}-2s}\Delta\phi\|_{L^2} \|\partial^{\alpha+s-\frac{1}{2}} n\|^2_{L^2} + \|\partial^\alpha \nabla\phi\|_{L^2} \|\partial^{\frac{5}{2}-s} n\|_{L^2} \|\partial^{\alpha+s} n\|_{L^2} \\
&\quad + \sum_{1\leqslant l\leqslant \alpha-1} (\|\partial^{\frac{3}{2}-s}\Delta\phi\|^\theta_{L^2} \|\partial^\alpha \Delta\phi\|^{1-\theta}_{L^2} \|\partial^{\frac{3}{2}-s} n\|^{1-\theta}_{L^2} \|\partial^\alpha n\|^\theta_{L^2}) \|\partial^{\alpha+s} n\|_{L^2} \\
&\lesssim \varepsilon_0 (\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}) + \varepsilon_0 \|\partial^\alpha \Delta\phi\|^{1-\theta}_{L^2} \|\partial^\alpha n\|^\theta_{L^2} \|\partial^{\alpha+s} n\|_{L^2} \\
&\lesssim \varepsilon_0 (\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}) + \varepsilon_0 (\|\partial^s n\|_{H^m} + \|\partial^s v\|_{H^m}) \|\partial^s n\|_{H^m} \\
&\leqslant \varepsilon_0 (\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}),
\end{aligned}
\tag{3.11}
$$

where $\theta = \frac{2\alpha-2l+2s-1}{2\alpha+2s-3} \in (0, 1)$.

In a word,

$$
I_2 \lesssim (\varepsilon_0^{\frac{1}{5}} + \varepsilon_0)(\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}) \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.12}
$$

As for the term I_3, for $\alpha = 0$, by Lemma 2.1 and Hölder's inequality, we get

$$
\begin{aligned}
I_3 &= 2\int_{\mathbb{R}^3} \nabla\phi \cdot \nabla n \cdot n \mathrm{d}x \\
&\lesssim \|\nabla\phi\|_{L^2} \|\nabla n\|_{L^{\frac{3}{s}}} \|n\|_{L^{\frac{6}{3-2s}}} \\
&\lesssim (\|\Delta\phi\|^{\frac{4}{5}}_{L^1} \|\partial(\Delta\phi)\|^{\frac{1}{5}}_{L^2}) \|\partial^{\frac{5}{2}-s} n\|_{L^2} \|\partial^s n\|_{L^2} \\
&\lesssim \varepsilon_0^{\frac{1}{5}} \|\partial^s n\|^2_{H^m}.
\end{aligned}
\tag{3.13}
$$

For $1 \leqslant \alpha \leqslant m$, we obtain that

$$
\begin{aligned}
I_3 &\lesssim (\|\partial^\alpha n\|_{L^{\frac{3}{2-s}}} \|\Delta\phi\|_{L^{\frac{3}{2s-1}}} + \|\partial^\alpha \Delta\phi\|_{L^{\frac{3}{2-s}}} \|n\|_{L^{\frac{3}{2s-1}}}) \|\partial^\alpha n\|_{L^{\frac{3}{2-s}}} \\
&\lesssim \|\partial^{\alpha+s-\frac{1}{2}} n\|_{L^2} \|\partial^{\frac{5}{2}-2s}\Delta\phi\|_{L^2} + \|\partial^{\alpha+s-\frac{1}{2}}\Delta\phi\|_{L^2} \|\partial^{\frac{5}{2}-2s} n\|_{L^2}) \|\partial^{\alpha+s-\frac{1}{2}} n\|_{L^2} \\
&\lesssim (\|\partial^s n\|_{H^m} \|\Delta\phi\|_{H^3} + \|\partial^s \Delta\phi\|_{H^m} \|n\|_{H^3}) \|\partial^s n\|_{H^m} \\
&\lesssim \varepsilon_0 (\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}).
\end{aligned}
\tag{3.14}
$$

In a word,

$$
I_3 \lesssim (\varepsilon_0^{\frac{1}{5}} + \varepsilon_0)(\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}) \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.15}
$$

Inserting (3.8), (3.12), and (3.15) into (3.4) and summing up with respect to α from 0 to m, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|n\|^2_{H^m} + \|\partial^s n\|^2_{H^m} \leqslant C\varepsilon_0 \|\partial u\|^2_{H^m} + C(\varepsilon_0^{\frac{1}{5}} + \varepsilon_0)(\|\partial^s n\|^2_{H^m} + \|\partial^s v\|^2_{H^m}). \tag{3.16}
$$

(*ii*) **The estimate of v.** The estimate of v is similar to the estimate of n. We get

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2 \leqslant C\varepsilon_0\|\partial u\|_{H^m}^2 + C(\varepsilon_0^{\frac{1}{5}} + \varepsilon_0)(\|\partial^s n\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2). \tag{3.17}$$

(*iii*) **The estimate of u.** Applying $\partial^\alpha(0 \leqslant \alpha \leqslant m)$ to the fourth equation of (1.4) and multiplying by $\partial^\alpha u$, then integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 + \|\partial^{\alpha+1} u\|_{L^2}^2\\
&= -\int_{\mathbb{R}^3} \partial^\alpha(u\cdot\nabla u)\cdot\partial^\alpha u\mathrm{d}x + \int_{\mathbb{R}^3}\partial^\alpha(\Delta\phi\nabla\phi)\cdot\partial^\alpha u\mathrm{d}x\\
&=: I_4 + I_5.
\end{aligned} \tag{3.18}$$

For I_4, by Lemma 2.2, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned}
I_4 &\lesssim (\|\partial^\alpha u\|_{L^6}\|\nabla u\|_{L^{\frac{3}{2}}} + \|u\|_{L^3}\|\partial^{\alpha+1}u\|_{L^2})\|\partial^\alpha u\|_{L^6}\\
&\lesssim (\|\partial^{\alpha+1}u\|_{L^2}\|u\|_{L^2}^{\frac{3}{4}}\|\partial^2 u\|_{L^2}^{\frac{1}{4}} + \|\partial^{\frac{1}{2}}u\|_{L^2}\|\partial^{\alpha+1}u\|_{L^2})\|\partial^{\alpha+1}u\|_{L^2}\\
&\leqslant \varepsilon_0\|\partial u\|_{H^m}^2.
\end{aligned} \tag{3.19}$$

As for the term I_5, for $\alpha = 0$, we get

$$I_5 = \int_{\mathbb{R}^3}\Delta\phi\cdot\nabla\phi\cdot u\mathrm{d}x = 0. \tag{3.20}$$

For $1 \leqslant \alpha \leqslant m$, it follows that

$$\begin{aligned}
I_5 &\lesssim (\|\partial^\alpha\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^3} + \|\partial^\alpha\nabla\phi\|_{L^6}\|\Delta\phi\|_{L^{\frac{3}{2}}})\|\partial^\alpha u\|_{L^6}\\
&\lesssim \|\partial^\alpha\Delta\phi\|_{L^2}\|\Delta\phi\|_{L^1}^{\frac{3}{5}}\|\partial\Delta\phi\|_{L^2}^{\frac{2}{5}}\|\partial^{\alpha+1}u\|_{L^2}\\
&\lesssim \|\partial^s\Delta\phi\|_{H^m}\|\Delta\phi\|_{H^3}^{\frac{2}{5}}\|\partial u\|_{H^m}\\
&\leqslant \varepsilon_0^{\frac{2}{5}}(\|\partial^s n\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2 + \|\partial u\|_{H^m}^2).
\end{aligned} \tag{3.21}$$

In a word,

$$I_5 \lesssim \varepsilon_0^{\frac{2}{5}}(\|\partial^s n\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2 + \|\partial u\|_{H^m}^2) \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.22}$$

Therefore, inserting (3.19) and (3.22) into (3.18) and summing up with respect to α from 0 to m, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^m}^2 + \|\partial u\|_{H^m}^2 \leqslant C\varepsilon_0^{\frac{2}{5}}(\|\partial^s n\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2) + C(\varepsilon_0^{\frac{2}{5}} + \varepsilon_0)\|\partial u\|_{H^m}^2. \tag{3.23}$$

Eventually, combining (3.16) and (3.17) with (3.23), we obtain

$$\frac{d}{dt}(\|n\|_{H^m}^2 + \|v\|_{H^m}^2 + \|u\|_{H^m}^2) + C(\|\partial^s n\|_{H^m}^2 + \|\partial^s v\|_{H^m}^2 + \|\partial u\|_{H^m}^2) \leqslant 0. \tag{3.24}$$

Then, integrating (3.24) from 0 to t and according to the fifth equation of (1.4) immediately implies (3.1). □

3.2. The local existence and uniqueness of the solution

Next, we will study the local existence and uniqueness of the solution to system (1.4). We construct the solution sequence $(n^j, v^j, \phi^j, u^j)_{j\geqslant 0}$ by iteratively solving the Cauchy problem:

$$\begin{cases}\partial_t n^{j+1} + u^j\cdot\nabla n^{j+1} + (-\Delta)^s n^{j+1} = -\nabla\cdot(n^{j+1}\nabla\phi^j), & t>0, x\in\mathbb{R}^3,\\ \partial_t v^{j+1} + u^j\cdot\nabla v^{j+1} + (-\Delta)^s v^{j+1} = \nabla\cdot(v^{j+1}\nabla\phi^j), & t>0, x\in\mathbb{R}^3,\\ \Delta\phi^j = n^j - v^j, & t>0, x\in\mathbb{R}^3,\\ \partial_t u^{j+1} + u^j\cdot\nabla u^{j+1} - \Delta u^{j+1} + \nabla P^{j+1} = \Delta\phi^j\nabla\phi^j, & t>0, x\in\mathbb{R}^3,\\ \nabla\cdot u^{j+1} = 0, & t>0, x\in\mathbb{R}^3,\end{cases} \tag{3.25}$$

where $(n^{j+1}, v^{j+1}, \phi^{j+1}, u^{j+1})|_{t=0}$=$(n_0, v_0, \phi_0, u_0)$ for $j\geqslant 0$ and $\Delta\phi_0 = n_0 - v_0$. It should be emphasized that the iterative sequence is different from [26] due to the difference in the external force in the Navier-Stokes equation. We first set $(n^0, v^0, u^0) = (0,0,0)$. Then we solve (3.25) with the initial data to get (n^1, v^1, ϕ^1, u^1), respectively. Similarly, we can define (n^j, v^j, ϕ^j, u^j) iteratively.

Lemma 3.2. *Assume that $s\in(\frac{1}{2},1)$, $(n_0, v_0, u_0)\in(L^1\cap H^m)\times(L^1\cap H^m)\times H^m(m\geqslant 3)$, $n_0, v_0\geqslant 0$. Then, there exists a constant $T_1>0$ such that system (1.4) possesses a unique classical solution (n, v, ϕ, u) satisfying*

$$(n, v, \phi, u)\in L^\infty(0, T_1; H^m\times H^m\times H^{m+2}\times H^m),$$
$$(\partial^s n, \partial^s v, \partial^s\phi, \partial u)\in L^2(0, T_1; H^m\times H^m\times H^{m+2}\times H^m).$$

Proof. In order to prove the conclusion, we divide the proof into several steps.
Step 1. (Uniform boundedness)
(i) **The estimate of** n^{j+1}**.** Applying $\partial^\alpha(0\leqslant\alpha\leqslant m)$ to the first equation of (3.25), multiplying by $\partial^\alpha n^{j+1}$ and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha n^{j+1}\|_{L^2}^2 + \|\partial^{\alpha+s} n^{j+1}\|_{L^2}^2\\ =&-\int_{\mathbb{R}^3}\partial^\alpha(u^j\cdot\nabla n^{j+1})\partial^\alpha n^{j+1}\mathrm{d}x - \int_{\mathbb{R}^3}\partial^\alpha(\nabla n^{j+1}\nabla\phi^j)\partial^\alpha n^{j+1}\mathrm{d}x\\ &-\int_{\mathbb{R}^3}\partial^\alpha(n^{j+1}\Delta\phi^j)\partial^\alpha n^{j+1}\mathrm{d}x\\ =:&\ J_1 + J_2 + J_3.\end{aligned} \tag{3.26}$$

First, we deal with the term J_1. For $\alpha = 0$, recalling that $\nabla\cdot u^j = 0$, we get

$$J_1 = -\frac{1}{2}\int_{\mathbb{R}^3} u^j\cdot\nabla|n^{j+1}|^2\mathrm{d}x = 0. \tag{3.27}$$

For $\alpha = 1$, recalling that $\nabla\cdot u^j = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned}J_1 =& -\int_{\mathbb{R}^3}\partial u^j\cdot\nabla n^{j+1}\partial n^{j+1}\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^j\cdot\nabla|\partial n^{j+1}|^2\mathrm{d}x\\ \lesssim&\ \|\partial u^j\|_{L^{\frac{3}{s}}}\|\partial n^{j+1}\|_{L^2}\|\partial n^{j+1}\|_{L^{\frac{6}{3-2s}}}\end{aligned}$$

$$\begin{aligned}
&\lesssim \|\partial^{\frac{5}{2}-s}u^j\|_{L^2}\|\partial n^{j+1}\|_{L^2}\|\partial^{1+s}n^{j+1}\|_{L^2}\\
&\lesssim \|u^j\|_{H^m}\|n^{j+1}\|_{H^m}\|\partial^s n^{j+1}\|_{H^m}\\
&\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^m}^2 + C\|u^j\|_{H^m}^2\|n^{j+1}\|_{H^m}^2.
\end{aligned} \tag{3.28}$$

For $2 \leqslant \alpha \leqslant m$, recalling that $\nabla \cdot u^j = 0$ and employing Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned}
J_1 &= \frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot \partial^\alpha n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x - \alpha\int_{\mathbb{R}^3} \nabla\partial\phi^j \cdot \nabla\partial^{\alpha-1}n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x\\
&\quad - \sum_{2\leqslant l\leqslant\alpha} C_\alpha^l \int_{\mathbb{R}^3} \nabla\partial^l\phi^j \cdot \nabla\partial^{\alpha-l}n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x\\
&\lesssim \|\Delta\phi^j\|_{L^{\frac{3}{s}}}\|\partial^\alpha n^{j+1}\|_{L^2}\|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}}\\
&\quad + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2}\|\partial^{\alpha-l+1}n^{j+1}\|_{L^{\frac{3}{s}}}\|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}}\\
&\lesssim \|\partial^{\frac{3}{2}-s}\Delta\phi^j\|_{L^2}\|\partial^\alpha n^{j+1}\|_{L^2}\|\partial^{\alpha+s}n^{j+1}\|_{L^2}\\
&\quad + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2}\|\partial^{\alpha-l+\frac{5}{2}-s}n^{j+1}\|_{L^2}\|\partial^{\alpha+s}n^{j+1}\|_{L^2}\\
&\lesssim \|\partial^s n^{j+1}\|_{H^m}\|n^{j+1}\|_{H^m}\|\Delta\phi^j\|_{H^m}\\
&\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^{j+1}\|_{H^m}^2.
\end{aligned} \tag{3.29}$$

In a word,

$$J_1 \lesssim \varepsilon\|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2 + \|u^j\|_{H^m}^2)\|n^{j+1}\|_{H^m}^2 \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.30}$$

As for J_2, for $\alpha = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}
J_2 &= \frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot n^{j+1} \cdot n^{j+1}\mathrm{d}x\\
&\lesssim \|\Delta\phi^j\|_{L^2}\|n^{j+1}\|_{L^{\frac{3}{s}}}\|n^{j+1}\|_{L^{\frac{6}{3-2s}}}\\
&\lesssim \|\Delta\phi^j\|_{L^2}\|\partial^{\frac{3}{2}-s}n^{j+1}\|_{L^2}\|\partial^s n^{j+1}\|_{L^2}\\
&\lesssim \|\Delta\phi^j\|_{H^m}\|n^{j+1}\|_{H^m}\|\partial^s n^{j+1}\|_{H^m}\\
&\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^{j+1}\|_{H^m}^2.
\end{aligned} \tag{3.31}$$

For $\alpha = 1$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}^3} \nabla n^{j+1} \cdot \nabla\phi^j \cdot \partial^2 n^{j+1}\mathrm{d}x\\
&\lesssim \|\partial n^{j+1}\|_{L^{\frac{6}{3+2s}}}\|\nabla\phi^j\|_{L^\infty}\|\partial^2 n^{j+1}\|_{L^{\frac{6}{3-2s}}}\\
&\lesssim \|n^{j+1}\|_{L^2}^{\frac{2+s}{3}}\|\partial^3 n^{j+1}\|_{L^2}^{\frac{1-s}{3}}\|\Delta\phi^j\|_{L^2}^{\frac{1}{2}}\|\partial\Delta\phi^j\|_{L^2}^{\frac{1}{2}}\|\partial^{2+s}n^{j+1}\|_{L^2}\\
&\lesssim \|\partial^s n^{j+1}\|_{H^m}\|n^{j+1}\|_{H^m}\|\Delta\phi^j\|_{H^m}\\
&\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^{j+1}\|_{H^m}^2.
\end{aligned} \tag{3.32}$$

For $2 \leqslant \alpha \leqslant m$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned}
J_2 =& \frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot \partial^\alpha n^{j+1} \cdot \partial^\alpha n^{j+1} \mathrm{d}x - \alpha \int_{\mathbb{R}^3} \nabla\partial\phi^j \cdot \nabla\partial^{\alpha-1} n^{j+1} \cdot \partial^\alpha n^{j+1} \mathrm{d}x \\
&- \sum_{2\leqslant l\leqslant\alpha} C_\alpha^l \int_{\mathbb{R}^3} \nabla\partial^l\phi^j \cdot \nabla\partial^{\alpha-l} n^{j+1} \cdot \partial^\alpha n^{j+1} \mathrm{d}x \\
\lesssim& \|\Delta\phi^j\|_{L^{\frac{3}{s}}} \|\partial^\alpha n^{j+1}\|_{L^2} \|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}} + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2} \|\partial^{\alpha-l+1} n^{j+1}\|_{L^{\frac{3}{s}}} \|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}} \\
\lesssim& \|\partial^{\frac{3}{2}-s}\Delta\phi^j\|_{L^2} \|\partial^\alpha n^{j+1}\|_{L^2} \|\partial^{\alpha+s} n^{j+1}\|_{L^2} + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2} \|\partial^{\alpha-l+\frac{5}{2}-s} n^{j+1}\|_{L^2} \|\partial^{\alpha+s} n^{j+1}\|_{L^2} \\
\lesssim& \|\partial^s n^{j+1}\|_{H^m} \|n^{j+1}\|_{H^m} \|\Delta\phi^j\|_{H^m} \\
\leqslant& \varepsilon \|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2) \|n^{j+1}\|_{H^m}^2.
\end{aligned} \tag{3.33}$$

In a word,

$$J_2 \lesssim \varepsilon \|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2) \|n^{j+1}\|_{H^m}^2 \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.34}$$

For J_3, by Lemmas 2.1 and 2.2, Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
J_3 \lesssim& (\|\partial^\alpha n^{j+1}\|_{L^2} \|\Delta\phi^j\|_{L^{\frac{3}{s}}} + \|n^{j+1}\|_{L^{\frac{3}{s}}} \|\partial^\alpha \Delta\phi^j\|_{L^2}) \|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}} \\
\lesssim& (\|\partial^\alpha n^{j+1}\|_{L^2} \|\partial^{\frac{3}{2}-s}\Delta\phi^j\|_{L^2} + \|\partial^{\frac{3}{2}-s} n^{j+1}\|_{L^2} \|\partial^\alpha\Delta\phi^j\|_{L^2}) \|\partial^{\alpha+s} n^{j+1}\|_{L^2} \\
\lesssim& \|n^{j+1}\|_{H^m} \|\Delta\phi^j\|_{H^m} \|\partial^s n^{j+1}\|_{H^m} \\
\leqslant& \varepsilon \|\partial^s n^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2) \|n^{j+1}\|_{H^m}^2.
\end{aligned} \tag{3.35}$$

Therefore, by inserting (3.30), (3.34), and (3.35) into (3.26) and summing up with respect to α from 0 to m, we get

$$\frac{d}{dt}\|n^{j+1}\|_{H^m}^2 + \|\partial^s n^{j+1}\|_{H^m}^2 \leqslant C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2 + \|u^j\|_{H^m}^2) \|n^{j+1}\|_{H^m}^2. \tag{3.36}$$

(*ii*) **The estimate of v^{j+1}.** The estimate of v^{j+1} is similar to the estimate of n^{j+1}; we get

$$\frac{d}{dt}\|v^{j+1}\|_{H^m}^2 + \|\partial^s v^{j+1}\|_{H^m}^2 \leqslant C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2 + \|u^j\|_{H^m}^2) \|v^{j+1}\|_{H^m}^2. \tag{3.37}$$

(*iii*) **The estimate of u^{j+1}.** Applying $\partial^\alpha (0 \leqslant \alpha \leqslant m)$ to the fourth equation of (3.25), multiplying by $\partial^\alpha u^{j+1}$ and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u^{j+1}\|_{L^2}^2 + \|\partial^{\alpha+1} u^{j+1}\|_{L^2}^2 \\
=& -\int_{\mathbb{R}^3} \partial^\alpha (u^j \cdot \nabla u^{j+1}) \cdot \partial^\alpha u^{j+1} \mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha (\Delta\phi^j \nabla\phi^j) \cdot \partial^\alpha u^{j+1} \mathrm{d}x \\
=&: J_4 + J_5.
\end{aligned} \tag{3.38}$$

First, we deal with the term J_4. For $\alpha = 0$, recalling that $\nabla \cdot u^j = 0$, we get

$$J_4 = -\frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla |u^{j+1}|^2 \mathrm{d}x = 0. \tag{3.39}$$

For $1 \leqslant \alpha \leqslant m$, by Lemma 2.2, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned} J_4 &= \int_{\mathbb{R}^3} \partial^{\alpha-1}(u^j \cdot \nabla u^{j+1}) \cdot \partial^{\alpha+1} u^{j+1} \mathrm{d}x \\ &\lesssim (\|\partial^{\alpha-1} u^j\|_{L^2} \|\partial u^{j+1}\|_{L^\infty} + \|u^j\|_{L^\infty} \|\partial^\alpha u^{j+1}\|_{L^2}) \|\partial^{\alpha+1} u^{j+1}\|_{L^2} \\ &\lesssim \|u^j\|_{H^m} \|u^{j+1}\|_{H^m} \|\partial u^{j+1}\|_{H^m} \\ &\leqslant \varepsilon \|\partial u^{j+1}\|_{H^m}^2 + C\|u^j\|_{H^m}^2 \|u^{j+1}\|_{H^m}^2. \end{aligned} \tag{3.40}$$

In a word,

$$J_4 \lesssim \varepsilon \|\partial u^{j+1}\|_{H^m}^2 + C\|u^j\|_{H^m}^2 \|u^{j+1}\|_{H^m}^2 \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.41}$$

As for J_5, for $\alpha = 0$, recalling that $\nabla \cdot u^j = 0$, we have

$$J_5 = \int_{\mathbb{R}^3} u^{j+1} \cdot \nabla \phi^j \cdot \Delta \phi^j \mathrm{d}x = 0. \tag{3.42}$$

For $1 \leqslant \alpha \leqslant m$, by Lemma 2.2, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} J_5 &= -\int_{\mathbb{R}^3} \partial^{\alpha-1}(\Delta \phi^j \nabla \phi^j) \cdot \partial^{\alpha+1} u^{j+1} \mathrm{d}x \\ &\lesssim (\|\partial^{\alpha-1} \Delta \phi^j\|_{L^2} \|\nabla \phi^j\|_{L^\infty} + \|\Delta \phi^j\|_{L^3} \|\partial^{\alpha-1} \nabla \phi^j\|_{L^6}) \|\partial^{\alpha+1} u^{j+1}\|_{L^2} \\ &\lesssim (\|\partial^{\alpha-1} \Delta \phi^j\|_{L^2} \|\Delta \phi^j\|_{L^2}^{\frac{1}{2}} \|\partial \Delta \phi^j\|_{L^2}^{\frac{1}{2}} + \|\partial^{\frac{1}{2}} \Delta \phi^j\|_{L^2} \|\partial^{\alpha-1} \Delta \phi^j\|_{L^2}) \|\partial^{\alpha+1} u^{j+1}\|_{L^2} \\ &\lesssim \|\Delta \phi^j\|_{H^m}^2 \|\partial u^{j+1}\|_{H^m} \\ &\leqslant \varepsilon \|\partial u^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)^2. \end{aligned} \tag{3.43}$$

In a word,

$$J_5 \lesssim \varepsilon \|\partial u^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)^2 \text{ for } 0 \leqslant \alpha \leqslant m. \tag{3.44}$$

Thus, inserting (3.41) and (3.44) into (3.38) and summing up with respect to α from 0 to m, we get

$$\frac{d}{dt}\|u^{j+1}\|_{H^m}^2 + \|\partial u^{j+1}\|_{H^m}^2 \leqslant C\|u^j\|_{H^m}^2 \|u^{j+1}\|_{H^m}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)^2. \tag{3.45}$$

Next, we show that there exists a constant $M > 0$ such that for any j, the following inequality holds in a small time interval $[0, T_1]$ (T_1 will be specified later):

$$\sup_{0 \leqslant t \leqslant T_1} (\|n^{j+1}\|_{H^m}^2 + \|v^{j+1}\|_{H^m}^2 + \|u^{j+1}\|_{H^m}^2) + \int_0^{T_1} (\|\partial^s n^{j+1}\|_{H^m}^2 + \|\partial^s v^{j+1}\|_{H^m}^2 + \|\partial u^{j+1}\|_{H^m}^2) \mathrm{d}t \leqslant M. \tag{3.46}$$

Here M satisfies $M \geqslant 4(\|n_0\|_{H^m}^2 + \|v_0\|_{H^m}^2 + \|u_0\|_{H^m}^2)$.

We prove (3.46) by mathematical induction. Suppose (3.46) holds for $j \leqslant i - 1$. Combining (3.36), (3.37) and (3.45) to obtain

$$\begin{aligned} &\frac{d}{dt}(\|n^{i+1}\|_{H^m}^2 + \|v^{i+1}\|_{H^m}^2 + \|u^{i+1}\|_{H^m}^2) + (\|\partial^s n^{i+1}\|_{H^m}^2 + \|\partial^s v^{i+1}\|_{H^m}^2 + \|\partial u^{i+1}\|_{H^m}^2) \\ \leqslant &C\|u^i\|_{H^m}^2 \|u^{i+1}\|_{H^m}^2 + C(\|n^i\|_{H^m}^2 + \|v^i\|_{H^m}^2 + \|u^i\|_{H^m}^2)(\|n^{i+1}\|_{H^m}^2 + \|v^{i+1}\|_{H^m}^2) + C(\|n^i\|_{H^m}^2 + \|v^i\|_{H^m}^2)^2 \end{aligned}$$

$$
\begin{aligned}
&\leqslant C(\|n^i\|_{H^m}^2+\|v^i\|_{H^m}^2+\|u^i\|_{H^m}^2)(\|n^{i+1}\|_{H^m}^2+\|v^{i+1}\|_{H^m}^2+\|u^{i+1}\|_{H^m}^2)+C(\|n^i\|_{H^m}^2+\|v^i\|_{H^m}^2)^2\\
&\leqslant CM(\|u^{i+1}\|_{H^m}^2+\|n^{i+1}\|_{H^m}^2+\|v^{i+1}\|_{H^m}^2)+CM^2.
\end{aligned}
$$

We choose T_1 such that $CMT_1 \leqslant \frac{1}{4}$ and $e^{CMT_1} \leqslant 2$. Then, according to Gronwall's inequality, we have (3.46). Thus, we get $(n^{j+1}, v^{j+1}, \phi^{j+1}, u^{j+1}) \in L^\infty(0, T_1; H^m \times H^m \times H^{m+2} \times H^m)$ and $(\partial^s n^{j+1}, \partial^s v^{j+1}, \partial^s \phi^{j+1}, \partial u^{j+1}) \in L^2(0, T_1; H^m \times H^m \times H^{m+2} \times H^m)$ for $T_1 > 0$.

Step 2. (Convergence)

The estimate in this part is similar to that in the previous part. Since both (n^j, v^j, ϕ^j, u^j) and $(n^{j+1}, v^{j+1}, \phi^{j+1}, u^{j+1})$ satisfy (3.25), we get the following equations:

$$
\begin{cases}
\partial_t(n^{j+1}-n^j)+u^j\cdot\nabla(n^{j+1}-n^j)+(u^j-u^{j-1})\cdot\nabla n^j+(-\triangle)^s(n^{j+1}-n^j)\\
=-\nabla\phi^j(\nabla n^{j+1}-\nabla n^j)-\nabla n^j(\nabla\phi^j-\nabla\phi^{j-1})-\Delta\phi^j(n^{j+1}-n^j)-n^j(\Delta\phi^j-\Delta\phi^{j-1}), & t>0, x\in\mathbb{R}^3,\\
\partial_t(v^{j+1}-v^j)+u^j\cdot\nabla(v^{j+1}-v^j)+(u^j-u^{j-1})\cdot\nabla v^j+(-\triangle)^s(v^{j+1}-v^j)\\
=\nabla\phi^j(\nabla v^{j+1}-\nabla v^j)+\nabla v^j(\nabla\phi^j-\nabla\phi^{j-1})+\Delta\phi^j(v^{j+1}-v^j)+v^j(\Delta\phi^j-\Delta\phi^{j-1}), & t>0, x\in\mathbb{R}^3,\\
\Delta\phi^j-\Delta\phi^{j-1}=(n^j-n^{j-1})-(v^j-v^{j-1}), & t>0, x\in\mathbb{R}^3,\\
\partial_t(u^{j+1}-u^j)+u^j\cdot\nabla(u^{j+1}-u^j)+(u^j-u^{j-1})\nabla u^j-\triangle(u^{j+1}-u^j)+\nabla P^{j+1}-\nabla P^j\\
=\nabla\phi^j(\Delta\phi^j-\Delta\phi^{j-1})+\Delta\phi^{j-1}(\nabla\phi^j-\nabla\phi^{j-1}), & t>0, x\in\mathbb{R}^3,\\
\nabla\cdot(u^{j+1}-u^j)=0, & t>0, x\in\mathbb{R}^3.
\end{cases}
\tag{3.47}
$$

(i) **The estimate of** $n^{j+1} - n^j$**.** Applying $\partial^\alpha (0 \leqslant \alpha \leqslant m-1)$ to the first equation of (3.47), multiplying by $\partial^\alpha(n^{j+1} - n^j)$, and integrating over $\mathbb{R}^3$ by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(n^{j+1}-n^j)\|_{L^2}^2+\|\partial^{\alpha+s}(n^{j+1}-n^j)\|_{L^2}^2\\
=&-\int_{\mathbb{R}^3}\partial^\alpha(u^j\cdot\nabla(n^{j+1}-n^j))\cdot\partial^\alpha(n^{j+1}-n^j)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial^\alpha((u^j-u^{j-1})\nabla n^j)\cdot\partial^\alpha(n^{j+1}-n^j)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial^\alpha(\nabla\phi^j(\nabla n^{j+1}-\nabla n^j)))\cdot\partial^\alpha(n^{j+1}-n^j)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial^\alpha(\nabla n^j(\nabla\phi^j-\nabla\phi^{j-1})))\cdot\partial^\alpha(n^{j+1}-n^j)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial^\alpha(\Delta\phi^j(n^{j+1}-n^j))\cdot\partial^\alpha(n^{j+1}-n^j)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial^\alpha(n^j(\Delta\phi^j-\Delta\phi^{j-1}))\cdot\partial^\alpha(n^{j+1}-n^j)\mathrm{d}x\\
=&:R_1+R_2+R_3+R_4+R_5+R_6.
\end{aligned}
\tag{3.48}
$$

First, we deal with R_1. For $\alpha = 0$, recalling that $\nabla \cdot u^j = 0$, we have

$$
R_1=-\frac{1}{2}\int_{\mathbb{R}^3}u^j\cdot\nabla|n^{j+1}-n^j|^2\mathrm{d}x=0. \tag{3.49}
$$

For $\alpha = 1$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned} R_1 &= -\int_{\mathbb{R}^3} \partial u^j \cdot \nabla(n^{j+1} - n^j) \cdot \partial(n^{j+1} - n^j)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla|\partial(n^{j+1} - n^j)|^2\mathrm{d}x \\ &\lesssim \|\partial u^j\|_{L^{\frac{3}{s}}}\|\partial(n^{j+1} - n^j)\|_{L^2}\|\partial(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\partial^{\frac{5}{2}-s}u^j\|_{L^2}\|\partial(n^{j+1} - n^j)\|_{L^2}\|\partial^{1+s}(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|u^j\|_{H^m}\|n^{j+1} - n^j\|_{H^{m-1}}\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|^2_{H^{m-1}} + C\|u^j\|^2_{H^m}\|u^j - u^{j-1}\|^2_{H^{m-1}}. \end{aligned} \tag{3.50}$$

For $2 \leqslant \alpha \leqslant m - 1$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} R_1 &= -\frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla|\partial^\alpha(n^{j+1} - n^j)|^2\mathrm{d}x - \alpha\int_{\mathbb{R}^3} \partial u^j \cdot \nabla\partial^{\alpha-1}(n^{j+1} - n^j) \cdot \partial^\alpha(n^{j+1} - n^j)\mathrm{d}x \\ &\quad - \sum_{2\leqslant l\leqslant\alpha} C^l_\alpha \int_{\mathbb{R}^3} \partial^l u^j \cdot \nabla\partial^{\alpha-l}(n^{j+1} - n^j) \cdot \partial^\alpha(n^{j+1} - n^j)\mathrm{d}x \\ &\lesssim \|\partial u^j\|_{L^{\frac{3}{s}}}\|\partial^\alpha(n^{j+1} - n^j)\|_{L^2}\|\partial^\alpha(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ &\quad + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l u^j\|_{L^2}\|\partial^{\alpha-l+1}(n^{j+1} - n^j)\|_{L^{\frac{3}{s}}}\|\partial^\alpha(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\partial^{\frac{5}{2}-s}u^j\|_{L^2}\|\partial^\alpha(n^{j+1} - n^j)\|_{L^2}\|\partial^{\alpha+s}(n^{j+1} - n^j)\|_{L^2} \\ &\quad + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l u^j\|_{L^2}\|\partial^{\alpha-l+\frac{5}{2}-s}(n^{j+1} - n^j)\|_{L^2}\|\partial^{\alpha+s}(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|u^j\|_{H^m}\|n^{j+1} - n^j\|_{H^{m-1}}\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|^2_{H^{m-1}} + C\|u^j\|^2_{H^m}\|n^{j+1} - n^j\|^2_{H^{m-1}}. \end{aligned} \tag{3.51}$$

In a word,

$$R_1 \leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|^2_{H^{m-1}} + C\|u^j\|^2_{H^m}\|n^{j+1} - n^j\|^2_{H^{m-1}} \text{ for } 0 \leqslant \alpha \leqslant m - 1. \tag{3.52}$$

For R_2, by Lemmas 2.1 and 2.2, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned} R_2 &\lesssim (\|\partial^\alpha(u^j - u^{j-1})\|_{L^2}\|\nabla n^j\|_{L^{\frac{3}{s}}} + \|u^j - u^{j-1}\|_{L^{\frac{3}{s}}}\|\partial^{\alpha+1}n^j\|_{L^2})\|\partial^\alpha(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim (\|\partial^\alpha(u^j - u^{j-1})\|_{L^2}\|\partial^{\frac{5}{2}-s}n^j\|_{L^2} + \|\partial^{\frac{3}{2}-s}(u^j - u^{j-1})\|_{L^2}\|\partial^{\alpha+1}n^j\|_{L^2})\|\partial^{\alpha+s}(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|n^j\|_{H^m}\|u^j - u^{j-1}\|_{H^{m-1}}\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|^2_{H^{m-1}} + C\|n^j\|^2_{H^m}\|u^j - u^{j-1}\|^2_{H^{m-1}}. \end{aligned} \tag{3.53}$$

Then we deal with the term R_3. For $\alpha = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned} R_3 &= \frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot (n^{j+1} - n^j) \cdot (n^{j+1} - n^j)\mathrm{d}x \\ &\lesssim \|\Delta\phi^j\|_{L^2}\|n^{j+1} - n^j\|_{L^{\frac{3}{s}}}\|n^{j+1} - n^j\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\Delta\phi^j\|_{L^2}\|\partial^{\frac{3}{2}-s}(n^{j+1} - n^j)\|_{L^2}\|\partial^s(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|\Delta\phi^j\|_{H^m}\|n^{j+1} - n^j\|_{H^{m-1}}\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ &\lesssim \varepsilon\|\partial^s(n^{j+1} - n^j)\|^2_{H^{m-1}} + C(\|n^j\|^2_{H^m} + \|v^j\|^2_{H^m})\|n^{j+1} - n^j\|^2_{H^{m-1}}. \end{aligned} \tag{3.54}$$

For $\alpha = 1$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} R_3 &= \int_{\mathbb{R}^3} \nabla(n^{j+1} - n^j) \cdot \nabla\phi^j \cdot \partial^2(n^{j+1} - n^j)\mathrm{d}x \\ &\lesssim \|\partial(n^{j+1} - n^j)\|_{L^{\frac{6}{3+2s}}} \|\nabla\phi^j\|_{L^\infty} \|\partial^2(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|n^{j+1} - n^j\|_{L^2}^{\frac{2+s}{3}} \|\partial^3(n^{j+1} - n^j)\|_{L^2}^{\frac{1-s}{3}} \|\Delta\phi^j\|_{L^2}^{\frac{1}{2}} \|\partial\Delta\phi^j\|_{L^2}^{\frac{1}{2}} \|\partial^{2+s}(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \|n^{j+1} - n^j\|_{H^{m-1}} \|\Delta\phi^j\|_{H^m} \\ &\leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^{j+1} - n^j\|_{H^{m-1}}^2. \end{aligned} \tag{3.55}$$

For $2 \leqslant \alpha \leqslant m - 1$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned} R_3 = &\frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot \partial^\alpha(n^{j+1} - n^j)\partial^\alpha(n^{j+1} - n^j)\mathrm{d}x \\ &- \alpha\int_{\mathbb{R}^3} \partial(\nabla\phi^j) \cdot \nabla\partial^{\alpha-1}(n^{j+1} - n^j) \cdot \partial^\alpha(n^{j+1} - n^j)\mathrm{d}x \\ &- \sum_{2\leqslant l\leqslant\alpha} C_\alpha^l \int_{\mathbb{R}^3} \partial^l\nabla\phi^j \cdot \nabla\partial^{\alpha-l}(n^{j+1} - n^j) \cdot \partial^\alpha(n^{j+1} - n^j)\mathrm{d}x \\ \lesssim &\|\Delta\phi^j\|_{L^{\frac{3}{s}}} \|\partial^\alpha(n^{j+1} - n^j)\|_{L^2} \|\partial^\alpha(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ &+ \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2} \|\partial^{\alpha-l+1}(n^{j+1} - n^j)\|_{L^{\frac{3}{s}}} \|\partial^\alpha(n^{j+1} - n^j)\|_{L^{\frac{6}{3-2s}}} \\ \lesssim &\|\partial^{\frac{3}{2}-s}\Delta\phi^j\|_{L^2} \|\partial^\alpha(n^{j+1} - n^j)\|_{L^2} \|\partial^{\alpha+s}(n^{j+1} - n^j)\|_{L^2} \\ &+ \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2} \|\partial^{\alpha-l+\frac{5}{2}-s}(n^{j+1} - n^j)\|_{L^2} \|\partial^{\alpha+s}(n^{j+1} - n^j)\|_{L^2} \\ \lesssim &\|\Delta\phi^j\|_{H^m} \|n^{j+1} - n^j\|_{H^{m-1}} \|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ \leqslant &\varepsilon\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^{j+1} - n^j\|_{H^{m-1}}^2. \end{aligned} \tag{3.56}$$

In a word,

$$R_3 \leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^{j+1} - n^j\|_{H^{m-1}}^2 \text{ for } 0 \leqslant \alpha \leqslant m - 1. \tag{3.57}$$

As for R_4, for $\alpha = 0$, we obtain

$$\begin{aligned} R_4 &= \int_{\mathbb{R}^3} n^j \cdot [(\Delta\phi^j - \Delta\phi^{j-1}) \cdot (n^{j+1} - n^j) + (\nabla\phi^j - \nabla\phi^{j-1}) \cdot \nabla(n^{j+1} - n^j)]\mathrm{d}x \\ &\lesssim \|n^j\|_{L^{\frac{3}{s}}} \|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2} \|n^{j+1} - n^j\|_{L^{\frac{6}{3-2s}}} + \|n^j\|_{L^3} \|\nabla\phi^j - \nabla\phi^{j-1}\|_{L^6} \|\nabla(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|\partial^{\frac{3}{2}-s}n^j\|_{L^2} \|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2} \|\partial^s(n^{j+1} - n^j)\|_{L^2} + \|\partial^{\frac{1}{2}}n^j\|_{L^2} \|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2} \|\nabla(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|n^j\|_{H^m} \|\Delta\phi^j - \Delta\phi^{j-1}\|_{H^{m-1}} \|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}}^2 + C\|n^j\|_{H^m}^2(\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2). \end{aligned} \tag{3.58}$$

For $1 \leqslant \alpha \leqslant m - 1$, it follows that

$$\begin{aligned} R_4 &\lesssim (\|\partial^\alpha\nabla n^j\|_{L^2} \|\nabla\phi^j - \nabla\phi^{j-1}\|_{L^{\frac{3}{1-s}}} + \|\nabla n^j\|_{L^{\frac{3}{2-s}}} \|\partial^\alpha(\nabla\phi^j - \nabla\phi^{j-1})\|_{L^6}) \|\partial^\alpha(n^{j+1} - n^j)\|_{L^{\frac{6}{1+2s}}} \\ &\lesssim (\|\partial^\alpha\nabla n^j\|_{L^2} \|\partial^{s-\frac{1}{2}}(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2} + \|\partial^{s+\frac{1}{2}}n^j\|_{L^2} \|\partial^\alpha(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2}) \|\partial^{\alpha+1-s}(n^{j+1} - n^j)\|_{L^2} \\ &\lesssim \|n^j\|_{H^m} \|\Delta\phi^j - \Delta\phi^{j-1}\|_{H^{m-1}} \|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}}^2 + C\|n^j\|_{H^m}^2(\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2). \end{aligned} \tag{3.59}$$

In a word,

$$R_4 \leqslant \varepsilon\|\partial^s(n^{j+1}-n^j)\|_{H^{m-1}}^2 + C\|n^j\|_{H^m}^2 \times (\|n^j-n^{j-1}\|_{H^{m-1}}^2 + \|v^j-v^{j-1}\|_{H^{m-1}}^2) \text{ for } 0 \leqslant \alpha \leqslant m-1. \tag{3.60}$$

For R_5, by Lemmas 2.1 and 2.2, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned}R_5 &\lesssim (\|\partial^\alpha \Delta\phi^j\|_{L^2}\|n^{j+1}-n^j\|_{L^{\frac{3}{s}}} + \|\partial^\alpha(n^{j+1}-n^j)\|_{L^2}\|\Delta\phi^j\|_{L^{\frac{3}{s}}})\|\partial^\alpha(n^{j+1}-n^j)\|_{L^{\frac{6}{3-2s}}}\\
&\lesssim (\|\partial^\alpha \Delta\phi^j\|_{L^2}\|\partial^{\frac{3}{2}-s}(n^{j+1}-n^j)\|_{L^2} + \|\partial^\alpha(n^{j+1}-n^j)\|_{L^2}\|\partial^{\frac{3}{2}-s}\Delta\phi^j\|_{L^2})\|\partial^{\alpha+s}(n^{j+1}-n^j)\|_{L^2}\\
&\lesssim \|\Delta\phi^j\|_{H^m}\|n^{j+1}-n^j\|_{H^{m-1}}\|\partial^s(n^{j+1}-n^j)\|_{H^{m-1}}\\
&\leqslant \varepsilon\|\partial^s(n^{j+1}-n^j)\|_{H^{m-1}}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)\|n^j-n^{j-1}\|_{H^{m-1}}^2.\end{aligned} \tag{3.61}$$

For R_6, by Lemmas 2.1 and 2.2, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned}R_6 &\lesssim (\|\partial^\alpha n^j\|_{L^2}\|\Delta\phi^j-\Delta\phi^{j-1}\|_{L^{\frac{3}{s}}} + \|\partial^\alpha(\Delta\phi^j-\Delta\phi^{j-1})\|_{L^2}\|n^j\|_{L^{\frac{3}{s}}})\|\partial^\alpha(n^{j+1}-n^j)\|_{L^{\frac{6}{3-2s}}}\\
&\lesssim (\|\partial^\alpha n^j\|_{L^2}\|\partial^{\frac{3}{2}-s}(\Delta\phi^j-\Delta\phi^{j-1})\|_{L^2} + \|\partial^\alpha(\Delta\phi^j-\Delta\phi^{j-1})\|_{L^2}\|\partial^{\frac{3}{2}-s}n^j\|_{L^2})\|\partial^{\alpha+s}(n^{j+1}-n^j)\|_{L^2}\\
&\lesssim \|n^j\|_{H^m}\|\Delta\phi^j-\Delta\phi^{j-1}\|_{H^{m-1}}\|\partial^s(n^{j+1}-n^j)\|_{H^{m-1}}\\
&\leqslant \varepsilon\|\partial^s(n^{j+1}-n^j)\|_{H^{m-1}}^2 + C\|n^j\|_{H^m}^2(\|n^j-n^{j-1}\|_{H^{m-1}}^2 + \|v^j-v^{j-1}\|_{H^{m-1}}^2).\end{aligned} \tag{3.62}$$

Therefore, inserting (3.52), (3.53), (3.57), and (3.60)–(3.62) into (3.48) and summing up with respect to α from 0 to $m-1$, we have

$$\begin{aligned}&\frac{d}{dt}\|n^{j+1}-n^j\|_{H^{m-1}}^2 + \|\partial^s(n^{j+1}-n^j)\|_{H^{m-1}}^2\\
\leqslant& C(\|n^j-n^{j-1}\|_{H^{m-1}}^2 + \|v^j-v^{j-1}\|_{H^{m-1}}^2 + \|u^j-u^{j-1}\|_{H^{m-1}}^2 + \|n^{j+1}-n^j\|_{H^{m-1}}^2).\end{aligned} \tag{3.63}$$

(*ii*) **The estimate of $v^{j+1}-v^j$.** The estimate of $v^{j+1}-v^j$ is similar to the estimate of $n^{j+1}-n^j$. We get

$$\begin{aligned}&\frac{d}{dt}\|v^{j+1}-v^j\|_{H^{m-1}}^2 + \|\partial^s(v^{j+1}-v^j)\|_{H^{m-1}}^2\\
\leqslant& C(\|n^j-n^{j-1}\|_{H^{m-1}}^2 + \|v^j-v^{j-1}\|_{H^{m-1}}^2 + \|u^j-u^{j-1}\|_{H^{m-1}}^2 + \|v^{j+1}-v^j\|_{H^{m-1}}^2).\end{aligned} \tag{3.64}$$

(*iii*) **The estimate of $u^{j+1}-u^j$.** Applying $\partial^\alpha (0 \leqslant \alpha \leqslant m-1)$ to the fourth equation of (3.47), multiplying by $\partial^\alpha(u^{j+1}-u^j)$ and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(u^{j+1}-u^j)\|_{L^2}^2 + \|\partial^\alpha\nabla(u^{j+1}-u^j)\|_{L^2}^2\\
=&-\int_{\mathbb{R}^3}\partial^\alpha(u^j\cdot\nabla(u^{j+1}-u^j))\cdot\partial^\alpha(u^{j+1}-u^j)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial^\alpha((u^j-u^{j-1})\nabla u^j)\cdot\partial^\alpha(u^{j+1}-u^j)\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\partial^\alpha(\nabla\phi^j(\Delta\phi^j-\Delta\phi^{j-1}))\cdot\partial^\alpha(u^{j+1}-u^j)\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\partial^\alpha(\Delta\phi^{j-1}(\nabla\phi^j-\nabla\phi^{j-1}))\cdot\partial^\alpha(u^{j+1}-u^j)\mathrm{d}x\\
=&:R_7+R_8+R_9+R_{10}.\end{aligned} \tag{3.65}$$

First, we deal with the term R_7. For $\alpha = 0$, recalling that $\nabla \cdot u^j = 0$, we have

$$R_7 = -\frac{1}{2}\int_{R^3} u^j \cdot \nabla |u^{j+1} - u^j|^2 \mathrm{d}x = 0. \tag{3.66}$$

For $1 \leqslant \alpha \leqslant m-1$, by Lemma 2.2, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned} R_7 &= \int_{\mathbb{R}^3} \partial^{\alpha-1}(u^j \cdot \nabla(u^{j+1} - u^j)) \cdot \partial^{\alpha+1}(u^{j+1} - u^j)\mathrm{d}x \\ &\lesssim (\|u^j\|_{L^\infty}\|\partial^\alpha(u^{j+1} - u^j)\|_{L^2} + \|\partial^{\alpha-1}u^j\|_{L^2}\|\nabla(u^{j+1} - u^j)\|_{L^\infty})\|\partial^{\alpha+1}(u^{j+1} - u^j)\|_{L^2} \\ &\lesssim \|u^j\|_{H^m}\|u^{j+1} - u^j\|_{H^{m-1}}\|\partial(u^{j+1} - u^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|^2_{H^{m-1}} + C\|u^j\|^2_{H^m}\|u^{j+1} - u^j\|^2_{H^{m-1}}. \end{aligned} \tag{3.67}$$

In a word,

$$R_7 \leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|^2_{H^{m-1}} + C\|u^j\|^2_{H^m}\|u^{j+1} - u^j\|^2_{H^{m-1}} \text{ for } 0 \leqslant \alpha \leqslant m-1. \tag{3.68}$$

For R_8, by Lemmas 2.1 and 2.2, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} R_8 &\lesssim (\|\partial^\alpha(u^j - u^{j-1})\|_{L^2}\|\nabla u^j\|_{L^3} + \|\partial^{\alpha+1}u^j\|_{L^2}\|u^j - u^{j-1}\|_{L^3})\|\partial^\alpha(u^{j+1} - u^j)\|_{L^6} \\ &\lesssim (\|\partial^\alpha(u^j - u^{j-1})\|_{L^2}\|\partial^{\frac{3}{2}}u^j\|_{L^2} + \|\partial^{\alpha+1}u^j\|_{L^2}\|\partial^{\frac{1}{2}}(u^j - u^{j-1})\|_{L^2})\|\partial^{\alpha+1}(u^{j+1} - u^j)\|_{L^2} \\ &\leqslant \|u^j - u^{j-1}\|_{H^{m-1}}\|u^j\|_{H^m}\|\partial(u^{j+1} - u^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|^2_{H^{m-1}} + C\|u^j\|^2_{H^m}\|u^j - u^{j-1}\|^2_{H^{m-1}}. \end{aligned} \tag{3.69}$$

We next deal with the term R_9. For $\alpha = 0$, we have

$$\begin{aligned} R_9 &\lesssim \|\nabla\phi^j\|_{L^3}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2}\|u^{j+1} - u^j\|_{L^6} \\ &\lesssim \|\Delta\phi^j\|^{\frac{3}{5}}_{L^1}\|\partial\Delta\phi^j\|^{\frac{2}{5}}_{L^2}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2}\|\partial(u^{j+1} - u^j)\|_{L^2} \\ &\lesssim \|\Delta\phi^j\|^{\frac{2}{5}}_{H^m}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{H^{m-1}}\|\partial(u^{j+1} - u^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|^2_{H^{m-1}} + C\|\Delta\phi^j\|^{\frac{4}{5}}_{H^m}(\|n^j - n^{j-1}\|^2_{H^{m-1}} + \|v^j - v^{j-1}\|^2_{H^{m-1}}). \end{aligned} \tag{3.70}$$

For $1 \leqslant \alpha \leqslant m-1$, it follows that

$$\begin{aligned} R_9 &= -\int_{\mathbb{R}^3} \partial^{\alpha-1}(\nabla\phi^j(\Delta\phi^j - \Delta\phi^{j-1})) \cdot \partial^{\alpha+1}(u^{j+1} - u^j)\mathrm{d}x \\ &\lesssim (\|\partial^{\alpha-1}\nabla\phi^j\|_{L^6}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^3} + \|\nabla\phi^j\|_{L^\infty}\|\partial^{\alpha-1}(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2})\|\partial^{\alpha+1}(u^{j+1} - u^j)\|_{L^2} \\ &\lesssim (\|\partial^{\alpha-1}\Delta\phi^j\|_{L^2}\|\partial^{\frac{1}{2}}(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2} \\ &\quad + \|\Delta\phi^j\|^{\frac{1}{2}}_{L^2}\|\partial\Delta\phi^j\|^{\frac{1}{2}}_{L^2}\|\partial^{\alpha-1}(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2})\|\partial^{\alpha+1}(u^{j+1} - u^j)\|_{L^2} \\ &\lesssim \|\Delta\phi^j\|_{H^m}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{H^{m-1}}\|\partial(u^{j+1} - u^j)\|_{H^{m-1}} \\ &\leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|^2_{H^{m-1}} + C\|\Delta\phi^j\|^2_{H^m}(\|n^j - n^{j-1}\|^2_{H^{m-1}} + \|v^j - v^{j-1}\|^2_{H^{m-1}}). \end{aligned} \tag{3.71}$$

In a word,

$$\begin{aligned} R_9 \leqslant{}& \varepsilon\|\partial(u^{j+1} - u^j)\|^2_{H^{m-1}} + C(\|\Delta\phi^j\|^{\frac{4}{5}}_{H^m} + \|\Delta\phi^j\|^2_{H^m}) \\ &\times (\|n^j - n^{j-1}\|^2_{H^{m-1}} + \|v^j - v^{j-1}\|^2_{H^{m-1}}) \text{ for } 0 \leqslant \alpha \leqslant m-1. \end{aligned} \tag{3.72}$$

As for R_{10}, for $\alpha = 0$, we have

$$\begin{aligned}
R_{10} &= -\int_{\mathbb{R}^3} \nabla\phi^{j-1}(\Delta\phi^j - \Delta\phi^{j-1}) \cdot (u^{j+1} - u^j)\mathrm{d}x \\
&\lesssim \|\nabla\phi^{j-1}\|_{L^3}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2}\|u^{j+1} - u^j\|_{L^6} \\
&\lesssim \|\Delta\phi^{j-1}\|_{L^1}^{\frac{3}{5}}\|\partial\Delta\phi^{j-1}\|_{L^2}^{\frac{2}{5}}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2}\|\partial(u^{j+1} - u^j)\|_{L^2} \\
&\lesssim \|\Delta\phi^{j-1}\|_{H^m}^{\frac{2}{5}}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{H^{m-1}}\|\partial(u^{j+1} - u^j)\|_{H^{m-1}} \\
&\leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|_{H^{m-1}}^2 + C\|\Delta\phi^{j-1}\|_{H^m}^{\frac{4}{5}}(\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2).
\end{aligned} \tag{3.73}$$

For $1 \leqslant \alpha \leqslant m-1$, we obtain

$$\begin{aligned}
R_{10} &= -\int_{\mathbb{R}^3} \partial^{\alpha-1}(\Delta\phi^{j-1}(\nabla\phi^j - \nabla\phi^{j-1})) \cdot \partial^{\alpha+1}(u^{j+1} - u^j)\mathrm{d}x \\
&\lesssim (\|\partial^{\alpha-1}\Delta\phi^{j-1}\|_{L^2}\|\nabla\phi^j - \nabla\phi^{j-1}\|_{L^\infty} \\
&\quad + \|\Delta\phi^{j-1}\|_{L^3}\|\partial^{\alpha-1}(\nabla\phi^j - \nabla\phi^{j-1})\|_{L^6})\|\partial^{\alpha+1}(u^{j+1} - u^j)\|_{L^2} \\
&\lesssim (\|\partial^{\alpha-1}\Delta\phi^{j-1}\|_{L^2}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{L^2}^{\frac{1}{2}}\|\partial(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial^{\frac{1}{2}}\Delta\phi^{j-1}\|_{L^2}\|\partial^{\alpha-1}(\Delta\phi^j - \Delta\phi^{j-1})\|_{L^2})\|\partial^{\alpha+1}(u^{j+1} - u^j)\|_{L^2} \\
&\lesssim \|\Delta\phi^{j-1}\|_{H^m}\|\Delta\phi^j - \Delta\phi^{j-1}\|_{H^{m-1}}\|\partial(u^{j+1} - u^j)\|_{H^{m-1}} \\
&\leqslant \varepsilon\|\partial(u^{j+1} - u^j)\|_{H^{m-1}}^2 + C(\|n^{j-1}\|_{H^m}^2 + \|v^{j-1}\|_{H^m}^2) \\
&\quad \times (\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2).
\end{aligned} \tag{3.74}$$

In a word,

$$\begin{aligned}
R_{10} \leqslant{}& \varepsilon\|\partial(u^{j+1} - u^j)\|_{H^{m-1}}^2 + C(\|n^{j-1}\|_{H^m}^2 + \|v^{j-1}\|_{H^m}^2 + \|\Delta\phi^{j-1}\|_{H^m}^{\frac{4}{5}}) \\
& \times (\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2).
\end{aligned} \tag{3.75}$$

Thus, inserting (3.68), (3.69), (3.72), and (3.75) into (3.65) and summing up with respect to α from 0 to $m-1$, we get

$$\begin{aligned}
&\frac{d}{dt}\|u^{j+1} - u^j\|_{H^{m-1}}^2 + \|\partial(u^{j+1} - u^j)\|_{H^{m-1}}^2 \\
\leqslant{}& C(\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2 + \|u^j - u^{j-1}\|_{H^{m-1}}^2 + \|u^{j+1} - u^j\|_{H^{m-1}}^2).
\end{aligned} \tag{3.76}$$

Combining (3.63), (3.64), and (3.76), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|n^{j+1} - n^j\|_{H^{m-1}}^2 + \|v^{j+1} - v^j\|_{H^{m-1}}^2 + \|u^{j+1} - u^j\|_{H^{m-1}}^2) \\
&+ (\|\partial^s(n^{j+1} - n^j)\|_{H^{m-1}}^2 + \|\partial^s(v^{j+1} - v^j)\|_{H^{m-1}}^2 + \|\partial(u^{j+1} - u^j)\|_{H^{m-1}}^2) \\
\leqslant{}& C(\|n^j - n^{j-1}\|_{H^{m-1}}^2 + \|v^j - v^{j-1}\|_{H^{m-1}}^2 + \|u^j - u^{j-1}\|_{H^{m-1}}^2 \\
&+ \|n^{j+1} - n^j\|_{H^{m-1}}^2 + \|v^{j+1} - v^j\|_{H^{m-1}}^2 + \|u^{j+1} - u^j\|_{H^{m-1}}^2),
\end{aligned} \tag{3.77}$$

where C depends on the $H^m \times H^m \times H^m \times H^m \times H^m$ norm of $(n^j, n^{j-1}, v^j, v^{j-1}, u^j)$.

Through Gronwall's inequality, we obtain

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T_1}(\|n^{j+1}-n^j\|_{H^{m-1}}^2+\|v^{j+1}-v^j\|_{H^{m-1}}^2+\|u^{j+1}-u^j\|_{H^{m-1}}^2)\\ \leqslant& CT_1e^{CT_1}\sup_{0\leqslant t\leqslant T_1}(\|n^j-n^{j-1}\|_{H^{m-1}}^2+\|v^j-v^{j-1}\|_{H^{m-1}}^2+\|u^j-u^{j-1}\|_{H^{m-1}}^2).\end{aligned}\tag{3.78}$$

Therefore, we conclude that (n^j, v^j, ϕ^j, u^j) is a Cauchy sequence in the Banach space $C(0, T_1; H^{m-1}\times H^{m-1}\times H^{m+1}\times H^{m-1})$ for small $T_1 > 0$. So, we can take the limit to get (n, v, ϕ, u), which is the solution of system (1.4) and satisfies $(n, v, \phi, u) \in L^\infty(0, T; H^m\times H^m\times H^{m+2}\times H^m)$, $(\partial^s n, \partial^s v, \partial^s\phi, \partial u) \in L^2(0, T; H^m\times H^m\times H^{m+2}\times H^m)$.

Step 3. (Uniqueness)

In order to show the uniqueness of the solution, we assume that there are two solutions $(n_1(x,t), v_1(x,t), \phi_1(x,t), u_1(x,t))$ and $(n_2(x,t), v_2(x,t), \phi_2(x,t), u_2(x,t))$ of (1.4) with the same initial data in the time interval $[0, T_1]$, where T_1 is any time before the maximal time of existence. Then $(n_1(x,t)-n_2(x,t), v_1(x,t)-v_2(x,t), \phi_1(x,t)-\phi_2(x,t), u_1(x,t)-u_2(x,t))$ satisfies

$$\begin{cases}\partial_t(n_1-n_2)+u_1\cdot\nabla(n_1-n_2)+(u_1-u_2)\cdot\nabla n_2+(-\Delta)^s(n_1-n_2)\\ \quad=-\nabla\cdot((n_1-n_2)\nabla\phi_1)-\nabla\cdot((\nabla\phi_1-\nabla\phi_2)n_2), & t>0, x\in\mathbb{R}^3,\\ \partial_t(v_1-v_2)+u_1\cdot\nabla(v_1-v_2)+(u_1-u_2)\cdot\nabla v_2+(-\Delta)^s(v_1-v_2) & t>0, x\in\mathbb{R}^3,\\ \quad=\nabla\cdot((v_1-v_2)\nabla\phi_1)+\nabla\cdot((\nabla\phi_1-\nabla\phi_2)v_2), & t>0, x\in\mathbb{R}^3,\\ \Delta(\phi_1-\phi_2)=(n_1-n_2)-(v_1-v_2), & t>0, x\in\mathbb{R}^3,\\ \partial_t(u_1-u_2)+u_1\cdot\nabla(u_1-u_2)+(u_1-u_2)\cdot\nabla u_2-\Delta(u_1-u_2)+\nabla(P_1-P_2)\\ \quad=(\Delta\phi_1-\Delta\phi_2)\nabla\phi_1+\Delta\phi_2(\nabla\phi_1-\nabla\phi_2), & t>0, x\in\mathbb{R}^3,\\ \nabla\cdot(u_1-u_2)=0, & t>0, x\in\mathbb{R}^3.\end{cases}\tag{3.79}$$

Multiplying n_1-n_2 to both sides of the first equation of (3.79), then integrating over R^3 by parts, by Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|n_1-n_2\|_{L^2}^2+\|\partial^s(n_1-n_2)\|_{L^2}^2\\ \lesssim&\|\partial n_2\|_{L^{\frac{3}{s}}}\|u_1-u_2\|_{L^2}\|n_1-n_2\|_{L^{\frac{6}{3-2s}}}+\|\Delta\phi_1\|_{L^\infty}\|n_1-n_2\|_{L^2}^2\\ &+\|\nabla n_2\|_{L^{\frac{3}{1+s}}}\|\nabla\phi_1-\nabla\phi_2\|_{L^6}\|n_1-n_2\|_{L^{\frac{6}{3-2s}}}\\ &+\|n_2\|_{L^{\frac{3}{s}}}\|\Delta\phi_1-\Delta\phi_2\|_{L^2}\|n_1-n_2\|_{L^{\frac{6}{3-2s}}}\\ \lesssim&\|\partial^{\frac{5}{2}-s}n_2\|_{L^2}\|u_1-u_2\|_{L^2}\|\partial^s(n_1-n_2)\|_{L^2}+\|\Delta\phi_1\|_{L^\infty}\|n_1-n_2\|_{L^2}^2\\ &+\|n_2\|_{L^2}^{\frac{1+2s}{4}}\|\partial^2 n_2\|_{L^2}^{\frac{3-2s}{4}}\|\Delta\phi_1-\Delta\phi_2\|_{L^2}\|\partial^s(n_1-n_2)\|_{L^2}\\ &+\|\partial^{\frac{3}{2}-s}n_2\|_{L^2}\|\Delta\phi_1-\Delta\phi_2\|_{L^2}\|\partial^s(n_1-n_2)\|_{L^2}\\ \leqslant&\varepsilon\|\partial^s(n_1-n_2)\|_{L^2}^2+C(\|n_1-n_2\|_{L^2}^2+\|v_1-v_2\|_{L^2}^2+\|u_1-u_2\|_{L^2}^2).\end{aligned}\tag{3.80}$$

Therefore,

$$\frac{d}{dt}\|n_1-n_2\|_{L^2}^2+\|\partial^s(n_1-n_2)\|_{L^2}^2\leqslant C(\|n_1-n_2\|_{L^2}^2+\|v_1-v_2\|_{L^2}^2+\|u_1-u_2\|_{L^2}^2).\tag{3.81}$$

Similarly,

$$\frac{d}{dt}\|v_1 - v_2\|_{L^2}^2 + \|\partial^s(v_1 - v_2)\|_{L^2}^2 \leqslant C(\|n_1 - n_2\|_{L^2}^2 + \|v_1 - v_2\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2). \tag{3.82}$$

Multiplying $u_1 - u_2$ to both sides of the fourth equation of (3.79), then integrating over R^3 by parts, by Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|_{L^2}^2 + \|\partial(u_1 - u_2)\|_{L^2}^2 \\
\lesssim&\|\nabla u_2\|_{L^3}\|u_1 - u_2\|_{L^2}\|u_1 - u_2\|_{L^6} + \|\Delta\phi_1 - \Delta\phi_2\|_{L^2}\|\nabla\phi_1\|_{L^3}\|u_1 - u_2\|_{L^6} \\
&+ \|\nabla\phi_1 - \nabla\phi_2\|_{L^6}\|\Delta\phi_2\|_{L^{\frac{3}{2}}}\|u_1 - u_2\|_{L^6} \\
\lesssim&\|\partial^{\frac{3}{2}}u_2\|_{L^2}\|u_1 - u_2\|_{L^2}\|\partial(u_1 - u_2)\|_{L^2} \\
&+ \|\Delta\phi_1 - \Delta\phi_2\|_{L^2}\|\Delta\phi_1\|_{L^1}^{\frac{3}{5}}\|\partial\Delta\phi_1\|_{L^1}^{\frac{2}{5}}\|\partial(u_1 - u_2)\|_{L^2} \\
&+ \|\Delta\phi_1 - \Delta\phi_2\|_{L^2}\|\Delta\phi_2\|_{L^1}^{\frac{3}{5}}\|\partial\Delta\phi_2\|_{L^2}^{\frac{2}{5}}\|\partial(u_1 - u_2)\|_{L^2} \\
\leqslant&\varepsilon\|\partial(u_1 - u_2)\|_{L^2}^2 + C(\|n_1 - n_2\|_{L^2}^2 + \|v_1 - v_2\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2).
\end{aligned} \tag{3.83}$$

Therefore,

$$\frac{d}{dt}\|u_1 - u_2\|_{L^2}^2 + \|\partial(u_1 - u_2)\|_{L^2}^2 \leqslant C(\|n_1 - n_2\|_{L^2}^2 + \|v_1 - v_2\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2). \tag{3.84}$$

Summing (3.81), (3.82), and (3.84), we have

$$\begin{aligned}
&\frac{d}{dt}(\|n_1 - n_2\|_{L^2}^2 + \|v_1 - v_2\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2) \\
&+ (\|\partial^s(n_1 - n_2)\|_{L^2}^2 + \|\partial^s(v_1 - v_2)\|_{L^2}^2 + \|\nabla(u_1 - u_2)\|_{L^2}^2) \\
\leqslant&C(\|n_1 - n_2\|_{L^2}^2 + \|v_1 - v_2\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2).
\end{aligned} \tag{3.85}$$

Since $(n_1(x,t) - n_2(x,t), v_1(x,t) - v_2(x,t), u_1(x,t) - u_2(x,t))|_{t=0} = (0,0,0)$, according to Gronwall's inequality, for $0 \leqslant t \leqslant T_1$,

$$(n_1(x,t) - n_2(x,t), v_1(x,t) - v_2(x,t), u_1(x,t) - u_2(x,t)) = (0,0,0).$$

Therefore, the uniqueness of local classical solutions is proved. □

3.3. The global existence of the solution

First, we prove that the solution of (3.25) satisfies the following lemma under the condition of small initial data.

Lemma 3.3. *Assume that* $s \in (\frac{1}{2}, 1)$, $(n_0, v_0, u_0) \in (L^1 \cap H^3) \times (L^1 \cap H^3) \times H^3$, $n_0, v_0 \geqslant 0$. *If there is a small constant* $\varepsilon_1 > 0$ *such that* $\|n_0\|_{H^3} + \|v_0\|_{H^3} + \|u_0\|_{H^3} \leqslant \varepsilon_1$, *then there exist small constants* $\varepsilon_2 > 0$ *and* $T_2 > 0$ *such that the solution of (3.25) satisfies*

$$\sup_{0\leqslant t\leqslant T_2}(\|n^j\|_{H^3} + \|v^j\|_{H^3} + \|u^j\|_{H^3}) \leqslant \varepsilon_2, \quad j \geqslant 0. \tag{3.86}$$

Proof. We will prove the conclusion by induction. The assumption $(n^0, v^0, u^0) = (0, 0, 0)$ implies that (3.86) holds for $j = 0$. Assuming that (3.86) is true for any $j > 0$, we will prove that the conclusion is also true for $j + 1$.

(i) **The estimate of** n^{j+1}**.** Applying $\partial^\alpha (0 \leqslant \alpha \leqslant 3)$ to the first equation of (3.25), multiplying by $\partial^\alpha n^{j+1}$ and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha n^{j+1}\|_{L^2}^2 + \|\partial^{\alpha+s} n^{j+1}\|_{L^2}^2\\ &= -\int_{\mathbb{R}^3} \partial^\alpha (u^j \cdot \nabla n^{j+1})\partial^\alpha n^{j+1}\mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha (\nabla n^{j+1}\nabla\phi^j)\partial^\alpha n^{j+1}\mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha (n^{j+1}\Delta\phi^j)\partial^\alpha n^{j+1}\mathrm{d}x\\ &=: Q_1 + Q_2 + Q_3.\end{aligned} \tag{3.87}$$

First, we deal with the term Q_1. For $\alpha = 0$, noting that $\nabla \cdot u^j = 0$, we get

$$Q_1 = -\frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla |n^{j+1}|^2 \mathrm{d}x = 0. \tag{3.88}$$

For $\alpha = 1$, recalling that $\nabla \cdot u^j = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned}Q_1 &= -\int_{\mathbb{R}^3} \partial u^j \cdot \nabla n^{j+1} \partial n^{j+1}\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla |\partial n^{j+1}|^2 \mathrm{d}x\\ &\lesssim \|\partial u^j\|_{L^{\frac{3}{2s}}}\|\partial n^{j+1}\|_{L^{\frac{6}{3-2s}}}\|\partial n^{j+1}\|_{L^{\frac{6}{3-2s}}}\\ &\lesssim \|u^j\|_{L^2}^{\frac{1+4s}{6}}\|\partial^3 u^j\|_{L^2}^{\frac{5-4s}{6}}\|\partial^{1+s} n^{j+1}\|_{L^2}^2\\ &\lesssim \|u^j\|_{H^3}\|\partial^s n^{j+1}\|_{H^3}^2\\ &\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^3}^2 + C\|u^j\|_{H^3}^2\|\partial^s n^{j+1}\|_{H^3}^2.\end{aligned} \tag{3.89}$$

For $2 \leqslant \alpha \leqslant 3$, recalling that $\nabla \cdot u^j = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, it follows that

$$\begin{aligned}Q_1 &= -\frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla |\partial^\alpha n^{j+1}|^2 \mathrm{d}x - \alpha\int_{\mathbb{R}^3} \partial u^j \cdot \nabla\partial^{\alpha-1} n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x\\ &\quad - \sum_{2\leqslant l\leqslant\alpha} C_\alpha^l \int_{\mathbb{R}^3} \partial^l u^j \cdot \nabla\partial^{\alpha-l} n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x\\ &\lesssim \|\partial u^j\|_{L^{\frac{3}{2s}}}\|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}}^2 + \sum_{2\leqslant l\leqslant\alpha}\|\partial^l u^j\|_{L^2}\|\partial^{\alpha-l+1} n^{j+1}\|_{L^{\frac{3}{s}}}\|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}}\\ &\lesssim \|u^j\|_{L^2}^{\frac{1+4s}{6}}\|\partial^3 u^j\|_{L^2}^{\frac{5-4s}{6}}\|\partial^{\alpha+s} n^{j+1}\|_{L^2}^2\\ &\quad + \sum_{2\leqslant l\leqslant\alpha}\|\partial^l u^j\|_{L^2}(\|\partial^s n^{j+1}\|_{L^2}^\theta\|\partial^{\alpha+s} n^{j+1}\|_{L^2}^{1-\theta})\|\partial^{\alpha+s} n^{j+1}\|_{L^2}\\ &\lesssim \|u^j\|_{H^3}\|\partial^s n^{j+1}\|_{H^3}^2\\ &\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^3}^2 + C\|u^j\|_{H^3}^2\|\partial^s n^{j+1}\|_{H^3}^2,\end{aligned} \tag{3.90}$$

where $\theta = \frac{4s+2l-5}{2\alpha} \in (0, 1)$.

In a word,

$$Q_1 \leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^3}^2 + C\|u^j\|_{H^3}^2\|\partial^s n^{j+1}\|_{H^3}^2 \text{ for } 0 \leqslant \alpha \leqslant 3. \tag{3.91}$$

As for Q_2, for $\alpha = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} Q_2 &\lesssim \|\nabla\phi^j\|_{L^2}\|\nabla n^{j+1}\|_{L^{\frac{3}{s}}}\|n^{j+1}\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\Delta\phi^j\|_{L^1}^{\frac{4}{5}}\|\partial\Delta\phi^j\|_{L^2}^{\frac{1}{5}}\|\partial^{\frac{5}{2}-s}n^{j+1}\|_{L^2}\|\partial^s n^{j+1}\|_{L^2} \\ &\lesssim \|\Delta\phi^j\|_{H^3}^{\frac{1}{5}}\|\partial^s n^{j+1}\|_{H^3}^2. \end{aligned} \tag{3.92}$$

For $\alpha = 1$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned} Q_2 &= -\frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot \partial n^{j+1} \cdot \partial n^{j+1} \mathrm{d}x \\ &\lesssim \|\Delta\phi^j\|_{L^{\frac{3}{2s-1}}}\|\partial n^{j+1}\|_{L^{\frac{3}{2-s}}}^2 \\ &\lesssim \|\partial^{\frac{5}{2}-2s}\Delta\phi^j\|_{L^2}\|\partial^{s+\frac{1}{2}}n^{j+1}\|_{L^2}^2 \\ &\lesssim \|\Delta\phi^j\|_{H^3}\|\partial^s n^{j+1}\|_{H^3}^2 \\ &\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^3}^2 + C(\|n^j\|_{H^3}^2 + \|v^j\|_{H^3}^2)\|\partial^s n^{j+1}\|_{H^3}^2. \end{aligned} \tag{3.93}$$

For $2 \leqslant \alpha \leqslant 3$, it follows that

$$\begin{aligned} Q_2 &= \frac{1}{2}\int_{\mathbb{R}^3} \Delta\phi^j \cdot \partial^\alpha n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x - \alpha\int_{\mathbb{R}^3} \partial\nabla\phi^j \cdot \nabla\partial^{\alpha-1}n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x \\ &\quad - \sum_{2\leqslant l\leqslant\alpha} C_\alpha^l \int_{\mathbb{R}^3} \partial^l\nabla\phi^j \cdot \nabla\partial^{\alpha-l}n^{j+1} \cdot \partial^\alpha n^{j+1}\mathrm{d}x \\ &\lesssim \|\Delta\phi^j\|_{L^{\frac{3}{2s-1}}}\|\partial^\alpha n^{j+1}\|_{L^{\frac{3}{2-s}}}^2 \\ &\quad + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2}\|\partial^{\alpha-l+1}n^{j+1}\|_{L^{\frac{3}{s}}}\|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}} \\ &\lesssim \|\partial^{\frac{5}{2}-2s}\Delta\phi^j\|_{L^2}\|\partial^{\alpha+s-\frac{1}{2}}n^{j+1}\|_{L^2}^2 \\ &\quad + \sum_{2\leqslant l\leqslant\alpha} \|\partial^l\nabla\phi^j\|_{L^2}(\|\partial^s n^{j+1}\|_{L^2}^\theta\|\partial^{\alpha+s}n^{j+1}\|_{L^2}^{1-\theta})\|\partial^{\alpha+s}n^{j+1}\|_{L^2} \\ &\lesssim \|\Delta\phi^j\|_{H^3}\|\partial^s n^{j+1}\|_{H^3}^2 \\ &\leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^3}^2 + C(\|n^j\|_{H^3}^2 + \|v^j\|_{H^3}^2)\|\partial^s n^{j+1}\|_{H^3}^2, \end{aligned} \tag{3.94}$$

where $\theta = \frac{4s+2l-5}{2\alpha} \in (0, 1)$.

In a word,

$$Q_2 \leqslant \varepsilon\|\partial^s n^{j+1}\|_{H^3}^2 + C(\|n^j\|_{H^3}^2 + \|v^j\|_{H^3}^2 + \|\Delta\phi^j\|_{H^3}^{\frac{1}{5}})\|\partial^s n^{j+1}\|_{H^3}^2 \text{ for } 0 \leqslant \alpha \leqslant 3. \tag{3.95}$$

We next deal with the term Q_3. For $\alpha = 0$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we obtain

$$
\begin{aligned}
Q_3 &= 2\int_{\mathbb{R}^3} \nabla\phi^j \cdot \nabla n^{j+1} \cdot n^{j+1} \mathrm{d}x \\
&\lesssim \|\nabla\phi^j\|_{L^2} \|\nabla n^{j+1}\|_{L^{\frac{3}{s}}} \|n^{j+1}\|_{L^{\frac{6}{3-2s}}} \\
&\lesssim \|\Delta\phi^j\|_{L^1}^{\frac{4}{5}} \|\partial\Delta\phi^j\|_{L^2}^{\frac{1}{5}} \|\partial^{\frac{5}{2}-s} n^{j+1}\|_{L^2} \|\partial^{\alpha+s} n^{j+1}\|_{L^2} \\
&\lesssim \|\Delta\phi^j\|_{H^3}^{\frac{1}{5}} \|\partial^s n^{j+1}\|_{H^3}^2.
\end{aligned} \tag{3.96}
$$

For $1 \leqslant \alpha \leqslant 3$, we obtain

$$
\begin{aligned}
Q_3 &\lesssim (\|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{3-2s}}} \|\Delta\phi^j\|_{L^3} + \|\partial^\alpha \Delta\phi^j\|_{L^2} \|n^{j+1}\|_{L^{\frac{3}{1-s}}}) \|\partial^\alpha n^{j+1}\|_{L^{\frac{6}{1+2s}}} \\
&\lesssim (\|\partial^{\alpha+s} n^{j+1}\|_{L^2} \|\partial^{\frac{1}{2}} \Delta\phi^j\|_{L^2} + \|\partial^\alpha \Delta\phi^j\|_{L^2} \|\partial^{s+\frac{1}{2}} n^{j+1}\|_{L^2}) \|\partial^{\alpha+1-s} n^{j+1}\|_{L^2} \\
&\lesssim (\|n^j\|_{H^3} + \|v^j\|_{H^3}) \|\partial^s n^{j+1}\|_{H^3}^2.
\end{aligned} \tag{3.97}
$$

In a word,

$$
Q_3 \leqslant C(\|n^j\|_{H^3} + \|v^j\|_{H^3} + \|\Delta\phi^j\|_{H^3}^{\frac{1}{5}}) \|\partial^s n^{j+1}\|_{H^3}^2 \text{ for } 0 \leqslant \alpha \leqslant 3. \tag{3.98}
$$

Thus, inserting (3.91), (3.95), and (3.98) into (3.87) and summing up with respect to α from 0 to 3, we have

$$
\begin{aligned}
&\frac{d}{dt}\|n^{j+1}\|_{H^3}^2 + C\|\partial^s n^{j+1}\|_{H^3}^2 \\
\leqslant &C(\|u^j\|_{H^3}^2 + \|\Delta\phi^j\|_{H^3}^{\frac{1}{5}} + \|n^j\|_{H^3}^2 + \|v^j\|_{H^3}^2 + \|n^j\|_{H^3} + \|v^j\|_{H^3}) \|\partial^s n^{j+1}\|_{H^3}^2.
\end{aligned} \tag{3.99}
$$

(*ii*) **The estimate of** v^{j+1}**.** The estimate of v^{j+1} is similar to the estimate of n^{j+1}; we have

$$
\begin{aligned}
&\frac{d}{dt}\|v^{j+1}\|_{H^3}^2 + C\|\partial^s v^{j+1}\|_{H^3}^2 \\
\leqslant &C(\|u^j\|_{H^3}^2 + \|\Delta\phi^j\|_{H^3}^{\frac{1}{5}} + \|n^j\|_{H^3}^2 + \|v^j\|_{H^3}^2 + \|n^j\|_{H^3} + \|v^j\|_{H^3}) \|\partial^s v^{j+1}\|_{H^3}^2.
\end{aligned} \tag{3.100}
$$

(*iii*) **The estimate of** u^{j+1}**.** Applying $\partial^\alpha (0 \leqslant \alpha \leqslant 3)$ to the fourth equation of (3.25), multiplying by $\partial^\alpha u^{j+1}$ and integrating over $\mathbb{R}^3$ by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u^{j+1}\|_{L^2}^2 + \|\partial^{\alpha+1} u^{j+1}\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \partial^\alpha (u^j \cdot \nabla u^{j+1}) \cdot \partial^\alpha u^{j+1} \mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha (\Delta\phi^j \nabla\phi^j) \cdot \partial^\alpha u^{j+1} \mathrm{d}x \\
&=: Q_4 + Q_5.
\end{aligned} \tag{3.101}
$$

We first deal with Q_4. For $\alpha = 0$, recalling that $\nabla \cdot u^j = 0$, we have

$$
Q_4 = -\frac{1}{2}\int_{\mathbb{R}^3} u^j \cdot \nabla |u^{j+1}|^2 \mathrm{d}x = 0. \tag{3.102}
$$

For $1 \leqslant \alpha \leqslant 3$, by Lemma 2.1, Hölder's inequality, and Young's inequality, we have

$$\begin{aligned} Q_4 &= \int_{\mathbb{R}^3} \partial^{\alpha-1}(u^j \cdot \nabla u^{j+1}) \cdot \partial^{\alpha+1} u^{j+1} \mathrm{d}x \\ &\lesssim (\|\partial^{\alpha-1} u^j\|_{L^2} \|\partial u^{j+1}\|_{L^\infty} + \|u^j\|_{L^3} \|\partial^\alpha u^{j+1}\|_{L^6}) \|\partial^{\alpha+1} u^{j+1}\|_{L^2} \\ &\lesssim (\|\partial^{\alpha-1} u^j\|_{L^2} \|\partial u^{j+1}\|_{L^\infty} + \|\partial^{\frac{1}{2}} u^j\|_{L^2} \|\partial^{\alpha+1} u^{j+1}\|_{L^2}) \|\partial^{\alpha+1} u^{j+1}\|_{L^2} \\ &\lesssim \|u^j\|_{H^3} \|\partial u^{j+1}\|_{H^3}^2 \\ &\leqslant \varepsilon \|\partial u^{j+1}\|_{H^3}^2 + C\|u^j\|_{H^3}^2 \|\partial u^{j+1}\|_{H^3}^2. \end{aligned} \tag{3.103}$$

In a word,

$$Q_4 \leqslant \varepsilon \|\partial u^{j+1}\|_{H^3}^2 + C\|u^j\|_{H^3}^2 \|\partial u^{j+1}\|_{H^3}^2 \text{ for } 0 \leqslant \alpha \leqslant 3. \tag{3.104}$$

As for Q_5, for $\alpha = 0$, noting that $\nabla \cdot u^j = 0$, we get

$$Q_5 = \int_{\mathbb{R}^3} u^{j+1} \cdot \nabla \phi^j \cdot \Delta \phi^j \mathrm{d}x = 0. \tag{3.105}$$

For $1 \leqslant \alpha \leqslant 3$, by Lemma 2.2, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned} Q_5 &= -\int_{\mathbb{R}^3} \partial^{\alpha-1}(\Delta \phi^j \nabla \phi^j) \cdot \partial^{\alpha+1} u^{j+1} \mathrm{d}x \\ &\lesssim (\|\partial^{\alpha-1} \Delta \phi^j\|_{L^2} \|\nabla \phi^j\|_{L^\infty} + \|\Delta \phi^j\|_{L^3} \|\partial^{\alpha-1} \nabla \phi^j\|_{L^6}) \|\partial^\alpha u^{j+1}\|_{L^2} \\ &\lesssim (\|\partial^{\alpha-1} \Delta \phi^j\|_{L^2} \|\Delta \phi^j\|_{L^2}^{\frac{1}{2}} \|\partial \Delta \phi^j\|_{L^2}^{\frac{1}{2}} + \|\partial^{\frac{1}{2}} \Delta \phi^j\|_{L^2} \|\partial^{\alpha-1} \Delta \phi^j\|_{L^2}) \|\partial^\alpha u^{j+1}\|_{L^2} \\ &\lesssim \|\Delta \phi^j\|_{H^3}^2 \|\partial u^{j+1}\|_{H^3} \\ &\leqslant \varepsilon \|\partial u^{j+1}\|_{H^3}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)^2. \end{aligned} \tag{3.106}$$

In a word,

$$Q_5 \leqslant \varepsilon \|\partial u^{j+1}\|_{H^3}^2 + C(\|n^j\|_{H^m}^2 + \|v^j\|_{H^m}^2)^2 \text{ for } 0 \leqslant \alpha \leqslant 3. \tag{3.107}$$

Thus, inserting (3.104) and (3.107) into (3.101) and summing up with respect to α from 0 to 3, we have

$$\frac{d}{dt}\|u^{j+1}\|_{H^3}^2 + \|\partial u^{j+1}\|_{H^3}^2 \leqslant C\|u^j\|_{H^3}^2 \|\partial u^{j+1}\|_{H^3}^2 + C(\|n^j\|_{H^3}^2 + \|v^j\|_{H^m}^3)^2. \tag{3.108}$$

Summing (3.99), (3.100), and (3.108), we obtain

$$\begin{aligned} &\frac{d}{dt}(\|n^{j+1}\|_{H^3}^2 + \|v^{j+1}\|_{H^3}^2 + \|u^{j+1}\|_{H^3}^2) + C(\|\partial^s n^{j+1}\|_{H^3}^2 + \|\partial^3 v^{j+1}\|_{H^3}^2 + \|\partial u^{j+1}\|_{H^3}^2) \\ \leqslant& C(\varepsilon_2^2 + \varepsilon_2^{\frac{1}{5}} + \varepsilon_2)(\|\partial^s n^{j+1}\|_{H^3}^2 + \|\partial^3 v^{j+1}\|_{H^3}^2 + \|\partial u^{j+1}\|_{H^3}^2) + C\varepsilon_2^4. \end{aligned} \tag{3.109}$$

Integrating (3.109) from 0 to t, we conclude that for any $t \in [0, T_2]$,

$$\begin{aligned} &(\|n^{j+1}\|_{H^3}^2 + \|v^{j+1}\|_{H^3}^2 + \|u^{j+1}\|_{H^3}^2) + C\int_0^t (\|\partial^s n^{j+1}\|_{H^3}^2 + \|\partial^s v^{j+1}\|_{H^3}^2 + \|\partial u^{j+1}\|_{H^3}^2) \mathrm{d}\tau \\ \leqslant& \varepsilon_1^2 + C(\varepsilon_2^2 + \varepsilon_2^{\frac{1}{5}} + \varepsilon_2) \int_0^t (\|\partial^s n^{j+1}\|_{H^3}^2 + \|\partial^s v^{j+1}\|_{H^3}^2 + \|\partial u^{j+1}\|_{H^3}^2) \mathrm{d}\tau + C\varepsilon_2^4 T_2. \end{aligned}$$

Taking properly $\varepsilon_1, \varepsilon_2, T_2$ such that $(\|n^{j+1}\|_{H^3}^2 + \|v^{j+1}\|_{H^3}^2 + \|u^{j+1}\|_{H^3}^2) + C\int_0^t (\|\partial^s n^{j+1}\|_{H^3}^2 + \|\partial^3 v^{j+1}\|_{H^3}^2 + \|\partial u^{j+1}\|_{H^3}^2) \mathrm{d}\tau \leqslant \varepsilon_2^2$, we can conclude that (3.86) holds for any $j \geqslant 0$. □

Next, we prove the main theorem.

Proof of Theorem 1.1. Let T^*=min$\{T_1, T_2\}$, where T_1 and T_2 are given in Lemmas 3.2 and 3.3. According to Lemmas 3.2 and 3.3, we obtain that if $\|n_0\|_{H^3} + \|v_0\|_{H^3} + \|u_0\|_{H^3} \leqslant \varepsilon_1$, then

$$\sup_{0\leqslant t\leqslant T^*} (\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3}) \leqslant \varepsilon_2. \tag{3.110}$$

Next, we prove $T^* = \infty$ by contradiction. Let M_1=min$\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Assume that $\|n_0\|_{H^3} + \|v_0\|_{H^3} + \|u_0\|_{H^3} \leqslant \frac{M_1}{2\sqrt{1+C_1}}$, where C_1 is given in Lemma 3.1. We define $T := sup\{t : \sup\limits_{0\leqslant s\leqslant t} (\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3} \leqslant M_1)\}$ as the lifespan of solutions to (1.4). Because

$$\|n_0\|_{H^3} + \|v_0\|_{H^3} + \|u_0\|_{H^3} \leqslant \frac{M_1}{2\sqrt{1+C_1}} \leqslant M_1 \leqslant \varepsilon_1,$$

recalling Lemma 3.2, we find that $T > 0$. If T is finite, according to the definition of T, we obtain that

$$\sup_{0\leqslant s\leqslant T} (\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3}) = M_1.$$

Besides, by Lemma 3.1, we know that

$$\sup_{0\leqslant s\leqslant T} (\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3}) \leqslant \sqrt{C_1}(\|n_0\|_{H^3} + \|v_0\|_{H^3} + \|u_0\|_{H^3}) \leqslant \frac{M_1\sqrt{C_1}}{2\sqrt{1+C_1}} \leqslant \frac{M_1}{2},$$

which is a contradiction to $\sup\limits_{0\leqslant s\leqslant T} (\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3}) = M_1$. Thus, $\|n\|_{H^3} + \|v\|_{H^3} + \|u\|_{H^3} \leqslant \varepsilon_2$ for any $t > 0$. Thus, the global existence and uniqueness of classical solutions to (1.4) have been obtained. □

Author contributions

Zihang Cai: writing-original draft, writing-review and editing; Chao Jiang: funding acquisition, writing-review and editing, supervision; Yuzhu Lei: writing-review and editing, supervision; Zuhan Liu: methodology, writing-review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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