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Research article

Optimality conditions associated with new controlled extremization models

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Abstract: Applying a parametric approach, in this paper we studied a new class of multidimensional extremization models with data uncertainty. Concretely, first, we derived the robust conditions of necessary optimality. Thereafter, we established robust sufficient optimality conditions by employing the various forms of convexity of the considered functionals. In addition, we formulated an illustrative example to validate the theoretical results.

Keywords: optimization problem; robust optimal solution; convexity; concavity; necessary and sufficient conditions of optimality
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1. Introduction

Optimization theory can be regarded as an intersection of physics, machine learning, and mathematics. This theory is used for practical problems in decision theory, data classification, economics, production inventory, and game theory. As data used in practical models is often derived by measurement, then the errors occur (see Kim and Kim [13]). Sometimes, the appearance of such errors can involve some computational outcomes contradicting the actual model. In this regard, the use of fuzzy numbers, interval analysis, and the robust approach to formulate data (and thus build uncertain optimization models, that is optimization models governed by uncertain data) are, in recent years, some popular research directions (see, for example, Antczak [1]).

By a fractional extremization model, we understand to optimize the ratio for two objective/cost functionals. In this direction, Dinkelbach [5], followed by Jagannathan [7], established a parametric approach to investigate a fractional extremization model by transforming it into a nonfractional new extremization model. During the time, various scholars and researchers have studied this approach to solve different fractional optimization models. We highlight the papers of Antczak and Pitea [2], Mititelu and Treanţă [18], and Mititelu [17]. The gH-derivative of symmetric type, accompanied by several applications in interval minimization models, have been proposed by Guo et al. [6]. For other ideas on this topic, we can consult the papers of Patel [20], Manesh et al. [15], and Nahak [19].

As mentioned above, uncertain extremization models appear when we have old sources, inadequate information, sample disparity, or a large volume of data (see Kim and Kim [12]). In these cases, a robust approach has a fundamental role in analyzing the optimization model involving uncertain data. It reduces the uncertainty of the original problem (see Kim and Kim [11]). Over the years, many uncertain optimization models have been considered by various researchers trying to establish new and important results (see, for instance, Treanţă [27], Preeti et al. [8], and Jayswal et al. [9]). In this regard, Lu et al. [14] established a stability analysis of nonlinear uncertain fractional differential equations with Caputo derivative. Beck and Tal [4] investigated duality in some robust extremization models and stated that the primal worst is equal to the dual best. Baranwal et al. [3] considered a robust-type duality in uncertain multi-time controlled minimization models. Jeyakumar et al. [10] studied robust duality for programming problems with generalized convexity under data uncertainty. Sun et al. [26] analyzed approximate solutions and saddle point theorems for robust convex optimization. Wu [29] formulated a duality theory for optimization problems with interval-valued objective functions. Also, Zhang et al. [30] stated the optimality conditions of KKT-type (Karush-Kuhn-Tucker) in a class of extremization problems with generalized convexity and interval-valued objective function.

Inspired by all the research works mentioned above, this paper deals with a new constrained fractional optimization model with uncertainty in the objective functional determined by multiple integral. Concretely, by considering a parametric approach, robust necessary optimality conditions are derived. Moreover, we prove the robust sufficient optimality criteria by using various forms of convexity for the involved functionals. In addition, we formulate an illustrative example to validate the theoretical results. The paper has the following principal merits: (i) Defining the robust-type optimal solution and, also, the robust-type Kuhn-Tucker point, associated with multiple integral functionals, by using a parametric approach; (ii) providing original and innovative demonstrations of the principal theorems; (iii) formulating a new context generated by normed spaces of function and functionals of multiple integral-type. This study is strongly connected with the analysis performed in Saeed and Treanţă [21], where the cost functionals are given by path-independent curvilinear-type integrals, and the concavity assumptions are not considered. Also, Saeed [22] considered robust-type optimality criteria for fractional extremization models determined by path-independent curvilinear-type integrals (and not multiple integral functional as in this study), but without monotonic and/or quasi-convexity assumptions as in the present paper. For connected viewpoints, see Minh and Phuong [16] and Su et al. [24, 25].

In the following, in Section 2, we give basic concepts and necessary preliminaries to state the main derived theorems. More precisely, we formulate the multidimensional fractional optimization model with uncertainty in the objective functional, the corresponding extremization nonfractional model, and the associated robust-type counterparts. Next, in Section 3, under suitable forms of convexity, we establish robust-type optimality criteria of the problem under study. Also, we introduce the concept of robust-type Kuhn-Tucker point to the considered uncertain extremization problem. Section 4 presents an example to support the derived theoretical results. In Section 5, we provide the conclusions of the current study.

2. Preliminaries

In this paper, we consider $\lambda = (\lambda^\pi)$, $\pi = \overline{1,m}$, $u = (u^\iota)$, $\iota = \overline{1,n}$, and $y = (y^j)$, $j = \overline{1,l}$ as arbitrary points of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. Let $\mathcal{A} = \mathcal{A}_{\lambda_0,\lambda_1} \subset \mathbb{R}^m$ be a compact set containing the points $\lambda_0 = (\lambda_0^\pi)$ and $\lambda_1 = (\lambda_1^\pi)$, $\pi = \overline{1,m}$, and let $d\lambda = d\lambda^1 \cdots d\lambda^m$ be the volume element in $\mathbb{R}^m$. Define the function spaces

$$A = \Big\{u : \mathcal{A} \mapsto \mathbb{R}^n |\ u = \text{piecewise smooth state function}\Big\},$$

$$B = \Big\{y : \mathcal{A} \mapsto \mathbb{R}^l |\ y = \text{piecewise continuous control function}\Big\},$$

having the norm generated by

$$\begin{aligned}\langle (u,y),(b,z)\rangle &= \int_{\mathcal{A}} \Big[u(\lambda)\cdot b(\lambda) + y(\lambda)\cdot z(\lambda)\Big] d\lambda \\ &= \int_{\mathcal{A}} \Big[\sum_{\iota=1}^{n} u^\iota(\lambda) b^\iota(\lambda) + \sum_{j=1}^{l} y^j(\lambda) z^j(\lambda)\Big] d\lambda, \quad \forall (u,y),(b,z) \in A \times B.\end{aligned}$$

For $u_\pi(\lambda) = \dfrac{\partial u}{\partial \lambda^\pi}(\lambda)$, we introduce the following constrained fractional extremization model, with uncertainty in objective functional,

$$\textit{(Prob)} \quad \min_{(u(\cdot),y(\cdot))} \frac{\displaystyle\int_{\mathcal{A}} \Gamma(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \sigma) d\lambda}{\displaystyle\int_{\mathcal{A}} \Upsilon(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \omega) d\lambda},$$

subject to

$$\begin{aligned}&M_\beta(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) \le 0, \ \ \beta = \overline{1,q}, \ \lambda \in \mathcal{A}, \\ &N_\pi^\iota(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) := \frac{\partial u}{\partial \lambda^\pi}(\lambda) - Q_\pi^\iota(\lambda, u(\lambda), y(\lambda)) = 0, \\ &\iota = \overline{1,n}, \ \pi = \overline{1,m}, \ \lambda \in \mathcal{A}, \\ &u(\lambda_0) = u_0 = \text{given}, \ u(\lambda_1) = u_1 = \text{given},\end{aligned}$$

where $\sigma \in \Sigma \subset \mathbb{R}$ and $\omega \in \Omega \subset \mathbb{R}$ are uncertainty parameters, where Σ and Ω are convex compact sets, and $\Gamma : \mathcal{A} \times A^2 \times B \times \Sigma \mapsto \mathbb{R}$, $\Upsilon : \mathcal{A} \times A^2 \times B \times \Omega \mapsto \mathbb{R}^*$, $M_\beta : \mathcal{A} \times A^2 \times B \mapsto \mathbb{R}$, $\beta = \overline{1,q}$, $N_\pi^\iota : \mathcal{A} \times A^2 \times B \mapsto \mathbb{R}$, $\iota = \overline{1,n}$, $\pi = \overline{1,m}$ are some given C^1-class functionals.

The associated robust-type counterpart of *(Prob)* is formulated as below

$$\textit{(RobProb)} \quad \min_{(u(\cdot),y(\cdot))} \frac{\displaystyle\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \sigma) d\lambda}{\displaystyle\int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \omega) d\lambda},$$

subject to

$$M_\beta(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) \le 0, \ \ \beta = \overline{1,q}, \ \lambda \in \mathcal{A},$$

$$N_\pi^\iota(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) = 0, \ \iota = \overline{1,n}, \ \pi = \overline{1,m}, \ \lambda \in \mathcal{A},$$
$$u(\lambda_0) = u_0 = \text{given}, \ u(\lambda_1) = u_1 = \text{given},$$

where $\Gamma, \Upsilon, M = (M_\beta)$ and $N = (N_\pi^\iota)$ are the same as in *(Prob)*.

The feasible solution set for *(RobProb)* and *(Prob)* can be written as follows:

$$\mathcal{K} = \{(u, y) \in A \times B \mid M_\beta(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) \leq 0, \ N_\pi^\iota(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) = 0,$$
$$u(\lambda_0) = u_0 = \text{given}, \ u(\lambda_1) = u_1 = \text{given}, \ \lambda \in \mathcal{A}\}.$$

Let us consider, for $(u, y) \in \mathcal{K}$, that $\Gamma \geq 0$, $\Upsilon > 0$, and introduce the positive scalar (see Jagannathan [7], Dinkelbach [5], Mititelu and Treanţă [18]),

$$V_{\sigma,\omega}^0 = \min_{(u(\cdot),y(\cdot))} \frac{\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \sigma)d\lambda}{\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \omega)d\lambda} = \frac{\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\lambda, u^0(\lambda), u_\pi^0(\lambda), y^0(\lambda), \sigma)d\lambda}{\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\lambda, u^0(\lambda), u_\pi^0(\lambda), y^0(\lambda), \omega)d\lambda},$$

to build an extremization nonfractional model for *(Prob)*, as

$$\textit{(NonFracProb)} \quad \min_{(u(\cdot),y(\cdot))} \Big\{ \int_{\mathcal{A}} \Gamma(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \sigma)d\lambda - V_{\sigma,\omega}^0 \int_{\mathcal{A}} \Upsilon(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \omega)d\lambda \Big\},$$

subject to

$$M_\beta(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) \leq 0, \ \beta = \overline{1,q}, \ \lambda \in \mathcal{A},$$
$$N_\pi^\iota(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) = 0, \ \iota = \overline{1,n}, \ \pi = \overline{1,m}, \ \lambda \in \mathcal{A},$$
$$u(\lambda_0) = u_0 = \text{given}, \ u(\lambda_1) = u_1 = \text{given}.$$

The associated robust-type counterpart to *(NonFracProb)* is formulated as below:

$$\textit{(RobNonFracProb)} \quad \min_{(u(\cdot),y(\cdot))} \Big\{ \int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \sigma)d\lambda - V_{\sigma,\omega}^0 \int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda), \omega)d\lambda \Big\},$$

subject to

$$M_\beta(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) \leq 0, \ \beta = \overline{1,q}, \ \lambda \in \mathcal{A},$$
$$N_\pi^\iota(\lambda, u(\lambda), u_\pi(\lambda), y(\lambda)) = 0, \ \iota = \overline{1,n}, \ \pi = \overline{1,m}, \ \lambda \in \mathcal{A},$$
$$u(\lambda_0) = u_0 = \text{given}, \ u(\lambda_1) = u_1 = \text{given}.$$

Further, we consider the notations: $u = u(\lambda)$, $y = y(\lambda)$, $\overline{u} = \overline{u}(\lambda)$, $\overline{y} = \overline{y}(\lambda)$, $\hat{u} = \hat{u}(\lambda)$, $\hat{y} = \hat{y}(\lambda)$, $\zeta = (\lambda, u(\lambda), u_\pi(\lambda), y(\lambda))$, $\overline{\zeta} = (\lambda, \overline{u}(\lambda), \overline{u}_\pi(\lambda), \overline{y}(\lambda))$, $\hat{\zeta} = (\lambda, \hat{u}(\lambda), \hat{u}_\pi(\lambda), \hat{y}(\lambda))$.

Definition 2.1. A pair $(\overline{u}, \overline{y}) \in \mathcal{K}$ is named a *robust optimal point* of *(Prob)* if

$$\frac{\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\overline{\zeta}, \sigma)d\lambda}{\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\overline{\zeta}, \omega)d\lambda} \leq \frac{\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma)d\lambda}{\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\zeta, \omega)d\lambda}, \quad \forall (u, y) \in \mathcal{K}.$$

Definition 2.2. A pair $(\overline{u}, \overline{y}) \in \mathcal{K}$ is named a robust optimal point of *(NonFracProb)* if

$$\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda - V^{-}_{\sigma,\omega} \int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda$$
$$\leq \int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma) d\lambda - V^{-}_{\sigma,\omega} \int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\zeta, \omega) d\lambda, \quad \forall (u, y) \in \mathcal{K}.$$

Remark 2.1. *We notice $\mathcal{K}$ is the feasible solution set of (NonFracProb) and for (RobNonFracProb).*

Remark 2.2. *The robust-type optimal points of (Prob) or (NonFracProb) are robust optimal points of (RobProb) or (RobNonFracProb).*

Next, to state the principal theorems of this paper, we formulate the definition of convex, (strictly, monotonic) quasi-convex, and concave multiple integral functionals (Mititelu and Treanţă [18]).

Definition 2.3. A multiple integral functional $\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda$ is named *convex at* $(\overline{u}, \overline{y}) \in A \times B$ if the following inequality holds

$$\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda - \int_{\mathcal{A}} \Gamma(\overline{\zeta}, \overline{\sigma}) d\lambda \geq \int_{\mathcal{A}} \Big\{ (u - \overline{u}) \frac{\partial \Gamma}{\partial u}(\overline{\zeta}, \overline{\sigma}) + (y - \overline{y}) \frac{\partial \Gamma}{\partial y}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda$$
$$+ \int_{\mathcal{A}} \Big\{ (u_\pi - \overline{u}_\pi) \frac{\partial \Gamma}{\partial u_\pi}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda, \quad \forall (u, y) \in A \times B.$$

Definition 2.4. The functional $\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda$ is named *concave at* $(\overline{u}, \overline{y}) \in A \times B$ if the below inequality is valid:

$$\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda - \int_{\mathcal{A}} \Gamma(\overline{\zeta}, \overline{\sigma}) d\lambda \leq \int_{\mathcal{A}} \Big\{ (u - \overline{u}) \frac{\partial \Gamma}{\partial u}(\overline{\zeta}, \overline{\sigma}) + (y - \overline{y}) \frac{\partial \Gamma}{\partial y}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda$$
$$+ \int_{\mathcal{A}} \Big\{ (u_\pi - \overline{u}_\pi) \frac{\partial \Gamma}{\partial u_\pi}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda, \quad \forall (u, y) \in A \times B.$$

Definition 2.5. The functional $\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda$ is named quasi-convex at $(\overline{u}, \overline{y}) \in A \times B$ if the below inequality

$$\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda \leq \int_{\mathcal{A}} \Gamma(\overline{\zeta}, \overline{\sigma}) d\lambda,$$

implies

$$\int_{\mathcal{A}} \Big\{ (u - \overline{u}) \frac{\partial \Gamma}{\partial u}(\overline{\zeta}, \overline{\sigma}) + (y - \overline{y}) \frac{\partial \Gamma}{\partial y}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda + \int_{\mathcal{A}} \Big\{ (u_\pi - \overline{u}_\pi) \frac{\partial \Gamma}{\partial u_\pi}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda \leq 0, \quad \forall (u, y) \in A \times B.$$

Definition 2.6. The functional $\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda$ is named strictly quasi-convex at $(\overline{u}, \overline{y}) \in A \times B$ if the below inequality

$$\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda \leq \int_{\mathcal{A}} \Gamma(\overline{\zeta}, \overline{\sigma}) d\lambda,$$

implies

$$\int_{\mathcal{A}} \Big\{ (u - \overline{u}) \frac{\partial \Gamma}{\partial u}(\overline{\zeta}, \overline{\sigma}) + (y - \overline{y}) \frac{\partial \Gamma}{\partial y}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda + \int_{\mathcal{A}} \Big\{ (u_\pi - \overline{u}_\pi) \frac{\partial \Gamma}{\partial u_\pi}(\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda < 0, \quad \forall (u, y) \neq (\overline{u}, \overline{y}) \in A \times B.$$

Definition 2.7. The functional $\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda$ is named monotonic quasi-convex at $(\overline{u}, \overline{y}) \in A \times B$ if the below inequality

$$\int_{\mathcal{A}} \Gamma(\zeta, \overline{\sigma}) d\lambda = \int_{\mathcal{A}} \Gamma(\overline{\zeta}, \overline{\sigma}) d\lambda,$$

implies

$$\int_{\mathcal{A}} \Big\{ (u - \overline{u}) \frac{\partial \Gamma}{\partial u} (\overline{\zeta}, \overline{\sigma}) + (y - \overline{y}) \frac{\partial \Gamma}{\partial y} (\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda + \int_{\mathcal{A}} \Big\{ (u_\pi - \overline{u}_\pi) \frac{\partial \Gamma}{\partial u_\pi} (\overline{\zeta}, \overline{\sigma}) \Big\} d\lambda = 0, \quad \forall (u, y) \in A \times B.$$

3. Robust-type necessary and sufficient optimality criteria

In this section, under various variants of convexity, we state the robust-type optimality criteria of *(Prob)*. In addition, according to Treanţă [28], we introduce and characterize the notion of a robust Kuhn-Tucker point to *(Prob)*. This study is connected with Saeed [22], where the author considered robust-type optimality criteria of some extremization fractional models determined by path-independent curvilinear-type integrals (and not multiple integral functional as in this study), but without monotonic and/or quasi-convexity assumptions as in the present paper.

To this aim, first, we establish an equivalence between *(Prob)* and *(NonFracProb)* (see, also, Sun et al. [23]).

Proposition 3.1. *Let $(\overline{u}, \overline{y}) \in \mathcal{K}$ be a robust-type optimal point of (Prob). In this case, there exists the positive scalar $V^{-}_{\sigma,\omega}$, and $(\overline{u}, \overline{y}) \in \mathcal{K}$ becomes a robust-type optimal point of (NonFracProb). In addition, for $(\overline{u}, \overline{y}) \in \mathcal{K}$ as a robust-type optimal point of (NonFracProb) and $V^{-}_{\sigma,\omega} = \dfrac{\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda}{\int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda}$, then $(\overline{u}, \overline{y}) \in \mathcal{K}$ is a robust-type optimal point of (Prob).*

Proof. By contrast, let us assume that there exists $(u, y) \in \mathcal{K}$ fulfilling

$$\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\zeta, \sigma) d\lambda - V^{-}_{\sigma,\omega} \int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\zeta, \omega) d\lambda < \int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda - V^{-}_{\sigma,\omega} \int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda.$$

Now, if we take $V^{-}_{\sigma,\omega} = \dfrac{\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda}{\int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda}$, we get

$$\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\zeta, \sigma) d\lambda - \frac{\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda}{\int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda} \int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\zeta, \omega) d\lambda$$

$$< \int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda - \frac{\int_{\mathcal{A}} \max_{\sigma \in \Sigma} \Gamma(\overline{\zeta}, \sigma) d\lambda}{\int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda} \int_{\mathcal{A}} \min_{\omega \in \Omega} \Upsilon(\overline{\zeta}, \omega) d\lambda,$$

which is equivalent with

$$\frac{\displaystyle\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\zeta,\sigma)d\lambda}{\displaystyle\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\zeta,\omega)d\lambda} < \frac{\displaystyle\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\overline{\zeta},\sigma)d\lambda}{\displaystyle\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\overline{\zeta},\omega)d\lambda},$$

and this is a contradiction with $(\overline{u},\overline{y})$ as a robust-type optimal point of *(Prob)*.

Conversely, consider $(\overline{u},\overline{y}) \in \mathcal{K}$ is a robust-type optimal point of *(NonFracProb)*, with

$$V^{-}_{\sigma,\omega} = \frac{\displaystyle\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\overline{\zeta},\sigma)d\lambda}{\displaystyle\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\overline{\zeta},\omega)d\lambda},$$

and assume that $(\overline{u},\overline{y}) \in \mathcal{K}$ is not a robust-type optimal point of *(Prob)*, involving there exists $(u,y) \in \mathcal{K}$ fulfilling

$$\frac{\displaystyle\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\zeta,\sigma)d\lambda}{\displaystyle\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\zeta,\omega)d\lambda} < \frac{\displaystyle\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\overline{\zeta},\sigma)d\lambda}{\displaystyle\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\overline{\zeta},\omega)d\lambda},$$

or, in an equivalent manner,

$$\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\zeta,\sigma)d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\zeta,\omega)d\lambda < 0,$$

or, equivalently,

$$\int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\zeta,\sigma)d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\zeta,\omega)d\lambda < \int_{\mathcal{A}} \max_{\sigma\in\Sigma} \Gamma(\overline{\zeta},\sigma)d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}} \min_{\omega\in\Omega} \Upsilon(\overline{\zeta},\omega)d\lambda,$$

which is a contradiction with $(\overline{u},\overline{y}) \in \mathcal{K}$ as a robust-type optimal point of *(NonFracProb)*. □

Next, we establish the robust-type necessary criteria for optimality of *(Prob)*.

Theorem 3.1. *If* $(\bar{u},\bar{y}) \in \mathcal{K}$ *is a robust-type optimal point of (Prob) and* $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma) = \Gamma(\zeta,\bar{\sigma})$, $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega) = \Upsilon(\zeta,\bar{\omega})$*, then there exist* $\bar{\nu} \in \mathbb{R}$ *and* $\bar{f} = (\bar{\rho}_\beta(\lambda)) \in \mathbb{R}^q_+$, $\bar{g} = (\bar{\lambda}^\iota_\pi(\lambda)) \in \mathbb{R}^{nm}$ *(piecewise differentiable functions), satisfying*

$$\begin{aligned}
&\bar{\nu}\Big[\Gamma_u(\bar{\zeta},\bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta}) \\
&- \frac{\partial}{\partial\lambda^\pi}\Big\{\bar{\nu}\Big[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_{u_\pi}(\bar{\zeta}) + \bar{g}^T N_{u_\pi}(\bar{\zeta})\Big\} = 0,
\end{aligned} \tag{3.1}$$

$$\bar{\nu}\Big[\Gamma_y(\bar{\zeta},\bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_y(\bar{\zeta}) + \bar{g}^T N_y(\bar{\zeta}) = 0, \tag{3.2}$$

$$\bar{f}^T M(\bar{\zeta}) = 0,\ \bar{\rho}_\beta \geq 0,\ \beta = \overline{1,q}, \tag{3.3}$$

$$\bar{\nu} \geq 0, \tag{3.4}$$

for $\lambda \in \mathcal{A}$*, except at discontinuities.*

Proof. We define $\bar{u}(\lambda)+\varepsilon_1 h(\lambda)$ and $\bar{y}(\lambda)+\varepsilon_2\mathsf{m}(\lambda)$ as some variations for $\bar{u}(\lambda)$ and $\bar{y}(\lambda)$, respectively, where $\varepsilon_1,\varepsilon_2$ are the variational parameters, and h,m are some smooth functions with limit constraints (see below). Therefore, we obtain the functions depending on $(\varepsilon_1,\varepsilon_2)$, defined as

$$\begin{aligned}\mathcal{E}(\varepsilon_1,\varepsilon_2)=\int_{\mathcal{A}}\Big[&\Gamma(\lambda,\bar{u}(\lambda)+\varepsilon_1 h(\lambda),\bar{u}_\pi(\lambda)+\varepsilon_1 h_\pi(\lambda)\bar{y}(\lambda)+\varepsilon_2\mathsf{m}(\lambda),\bar{\sigma})\\ &-V^{-}_{\sigma,\omega}\Upsilon(\lambda,\bar{u}(\lambda)+\varepsilon_1 h(\lambda),\bar{u}_\pi(\lambda)+\varepsilon_1 h_\pi(\lambda),\bar{y}(\lambda)+\varepsilon_2\mathsf{m}(\lambda),\bar{\omega})\Big]d\lambda,\\ \mathcal{Z}(\varepsilon_1,\varepsilon_2)=\int_{\mathcal{A}}&M(\lambda,\bar{u}(\lambda)+\varepsilon_1 h(\lambda),\bar{u}_\pi(\lambda)+\varepsilon_1 h_\pi(\lambda)\bar{y}(\lambda)+\varepsilon_2\mathsf{m}(\lambda))d\lambda,\end{aligned}$$

and

$$\mathcal{J}(\varepsilon_1,\varepsilon_2)=\int_{\mathcal{A}}N(\lambda,\bar{u}(\lambda)+\varepsilon_1 h(\lambda),\bar{u}_\pi(\lambda)+\varepsilon_1 h_\pi(\lambda)\bar{y}(\lambda)+\varepsilon_2\mathsf{m}(\lambda))d\lambda.$$

Since, by hypothesis, the pair $(\bar{u},\bar{y})$ is a robust-type optimal point of *(Prob)*, therefore, the pair $(0,0)$ is an optimal solution of

$$\min_{\varepsilon_1,\varepsilon_2}\mathcal{E}(\varepsilon_1,\varepsilon_2),$$

subject to

$$\mathcal{Z}(\varepsilon_1,\varepsilon_2)\leqq 0,\quad \mathcal{J}(\varepsilon_1,\varepsilon_2)=0,$$
$$h(\lambda_0)=h(\lambda_1)=\mathsf{m}(\lambda_0)=\mathsf{m}(\lambda_1)=0.$$

Thus, there exist $\bar{\nu}\in\mathbb{R}$, $\bar{f}=(\bar{f}_\beta(\lambda))\in\mathbb{R}^q_+$, $\bar{g}=(\bar{g}^\iota_\pi(\lambda))\in\mathbb{R}^{nm}$, fulfilling

$$\bar{\nu}\nabla\mathcal{E}(0,0)+\bar{f}^T\nabla\mathcal{Z}(0,0)+\bar{g}^T\nabla\mathcal{J}(0,0)=0, \tag{$*$}$$
$$\bar{f}^T\mathcal{Z}(0,0)=0,\ \bar{f}\geqq 0,$$
$$\bar{\nu}\geq 0,$$

(see $\nabla\phi(x_1,x_2)$ as the gradient of ϕ at (x_1,x_2)). The first relation fomulated in $(*)$ is rewritten as

$$\begin{aligned}&\int_{\mathcal{A}}\Big[\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{u}^\iota}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{u}^\iota}\Big)h^\iota+\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{u}^\iota_\pi}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{u}^\iota_\pi}\Big)h^\iota_\pi+\bar{f}^T\frac{\partial M}{\partial\bar{u}^\iota}h^\iota+\bar{f}^T\frac{\partial M}{\partial\bar{u}^\iota_\pi}h^\iota_\pi+\bar{g}^T\frac{\partial N}{\partial\bar{u}^\iota}h^\iota+\bar{g}^T\frac{\partial N}{\partial\bar{u}^\iota_\pi}h^\iota_\pi\Big]d\lambda=0,\\ &\int_{\mathcal{A}}\left[\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{y}^j}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{y}^j}\Big)\mathsf{m}^j+\bar{f}^T\frac{\partial M}{\partial\bar{y}^j}\mathsf{m}^j+\bar{g}^T\frac{\partial N}{\partial\bar{y}^j}\mathsf{m}^j\right]d\lambda=0,\end{aligned}$$

or, as follows,

$$\begin{aligned}&\int_{\mathcal{A}}\Big[\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{u}^\iota}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{u}^\iota}\Big)-\frac{\partial}{\partial\lambda^\pi}\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{u}^\iota_\pi}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{u}^\iota_\pi}\Big)+\bar{f}^T\frac{\partial M}{\partial\bar{u}^\iota}-\frac{\partial}{\partial\lambda^\pi}\bar{f}^T\frac{\partial M}{\partial\bar{u}^\iota_\pi}+\bar{g}^T\frac{\partial N}{\partial\bar{u}^\iota}-\frac{\partial}{\partial\lambda^\pi}\bar{g}^T\frac{\partial N}{\partial\bar{u}^\iota_\pi}\Big]h^\iota d\lambda=0,\\ &\int_{\mathcal{A}}\left[\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{y}^j}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{y}^j}\Big)+\bar{f}^T\frac{\partial M}{\partial\bar{y}^j}+\bar{g}^T\frac{\partial N}{\partial\bar{y}^j}\right]\mathsf{m}^j d\lambda=0,\end{aligned}$$

where we used the divergence formula, boundary conditions, and the method of integration by parts.

In the following, by using a fundamental lemma, we get

$$\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{u}^\iota}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{u}^\iota}\Big)-\frac{\partial}{\partial\lambda^\pi}\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{u}^\iota_\pi}-V^{-}_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{u}^\iota_\pi}\Big)+\bar{f}^T\frac{\partial M}{\partial\bar{u}^\iota}-\frac{\partial}{\partial\lambda^\pi}\bar{f}^T\frac{\partial M}{\partial\bar{u}^\iota_\pi}+\bar{g}^T\frac{\partial N}{\partial\bar{u}^\iota}-\frac{\partial}{\partial\lambda^\pi}\bar{g}^T\frac{\partial N}{\partial\bar{u}^\iota_\pi}=0,\quad \iota=\overline{1,n},$$

$\bar{\nu}\Big(\frac{\partial\Gamma}{\partial\bar{y}^j} - V^-_{\sigma,\omega}\frac{\partial\Upsilon}{\partial\bar{y}^j}\Big) + \bar{f}^T\frac{\partial M}{\partial\bar{y}^j} + \bar{g}^T\frac{\partial N}{\partial\bar{y}^j} = 0, j = \overline{1,l},$

or

$$\bar{\nu}\Big[\Gamma_u(\bar{\zeta},\bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta})$$
$$- \frac{\partial}{\partial\lambda^\pi}\{\bar{\nu}\Big[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_{u_\pi}(\bar{\zeta}) + \bar{g}^T N_{u_\pi}(\bar{\zeta})\} = 0,$$
$$\bar{\nu}\Big[\Gamma_y(\bar{\zeta},\bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_y(\bar{\zeta}) + \bar{g}^T N_y(\bar{\zeta}) = 0.$$

The second expression given in $(*)$,

$$\bar{f}^T\mathcal{Z}(0,0) = 0,\ \bar{f} \geqq 0,$$
$$\bar{\nu} \geq 0,$$

involves

$$\bar{f}^T M(\bar{\zeta}) = 0,\ \bar{f} \geqq 0,$$
$$\bar{\nu} \geq 0,$$

and we complete the proof. □

Remark 3.1. *The conditions* (3.1)–(3.4) *are named robust-type necessary optimality criteria of (Prob).*

Definition 3.1. The pair $(\bar{u},\bar{y}) \in \mathcal{K}$ is named a *normal robust-type optimal point* of *(Prob)* if $\bar{\nu} > 0$.

Next, on the line of Treanţă [28], we introduce and describe the robust-type Kuhn-Tucker point of *(Prob)*.

Definition 3.2. Let $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma) = \Gamma(\zeta,\bar{\sigma})$, $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega) = \Upsilon(\zeta,\bar{\omega})$. The robust feasible solution $(\bar{u},\bar{y})$ is named a robust Kuhn-Tucker point of *(Prob)* if there exist the piecewise differentiable functions $\bar{f} = (\bar{\rho}_\beta(\lambda)) \in \mathbb{R}^q_+, \bar{g} = (\bar{\lambda}^\iota_\pi(\lambda)) \in \mathbb{R}^{nm}$, satisfying

$$\Gamma_u(\bar{\zeta},\bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega}) + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta}) - \frac{\partial}{\partial\lambda^\pi}\Big\{\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma})$$
$$- V^-_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega}) + \bar{f}^T M_{u_\pi}(\bar{\zeta}) + \bar{g}^T N_{u_\pi}(\bar{\zeta})\Big\} = 0,$$
$$\Gamma_y(\bar{\zeta},\bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega}) + \bar{f}^T M_y(\bar{\zeta}) + \bar{g}^T N_y(\bar{\zeta}) = 0,$$
$$\bar{f}^T M(\bar{\zeta}) = 0,\ \bar{\rho}_\beta \geq 0,\ \beta = \overline{1,q},$$

for $\lambda \in \mathcal{A}$, except at discontinuities.

Theorem 3.2. *If* $(\bar{u},\bar{y}) \in \mathcal{K}$ *is a normal robust-type optimal point of (Prob), with* $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma) = \Gamma(\zeta,\bar{\sigma})$*,* $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega) = \Upsilon(\zeta,\bar{\omega})$*, then* $(\bar{u},\bar{y}) \in \mathcal{K}$ *is a robust-type Kuhn-Tucker point for (Prob).*

Proof. For $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma) = \Gamma(\zeta,\bar{\sigma})$, $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega) = \Upsilon(\zeta,\bar{\omega})$, since $(\bar{u},\bar{y}) \in \mathcal{K}$ is a robust-type optimal point of *(Prob)* (see Theorem 3.1), there exist $\bar{\nu} \in \mathbb{R}$ and $\bar{f} = (\bar{\rho}_\beta(\lambda)) \in \mathbb{R}^q_+, \bar{g} = (\bar{\lambda}^\iota_\pi(\lambda)) \in \mathbb{R}^{nm}$, satisfying

$$\bar{\nu}\Big[\Gamma_u(\bar{\zeta},\bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})\Big] + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta})$$

$$
\begin{aligned}
&-\frac{\partial}{\partial \lambda^{\pi}}\Big\{\bar{\nu}\Big[\Gamma_{u_{\pi}}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{u_{\pi}}(\bar{\zeta},\bar{\omega})\Big]+\bar{f}^{T}M_{u_{\pi}}(\bar{\zeta})+\bar{g}^{T}N_{u_{\pi}}(\bar{\zeta})\Big\}=0,\\
&\bar{\nu}\Big[\Gamma_{y}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{y}(\bar{\zeta},\bar{\omega})\Big]+\bar{f}^{T}M_{y}(\bar{\zeta})+\bar{g}^{T}N_{y}(\bar{\zeta})=0,\\
&\bar{f}^{T}M(\bar{\zeta})=0,\ \bar{\rho}_{\beta}\geq 0,\ \beta=\overline{1,q},\bar{\nu}\geq 0,
\end{aligned}
$$

for $\lambda \in \mathcal{A}$, except at discontinuities. Since the pair $(\bar{u},\bar{y}) \in \mathcal{K}$ is considered a normal robust-type optimal point, we define $\bar{\nu}=1>0$. □

A first result regarding the robust-type sufficient criteria of *(Prob)* is formulated below.

Theorem 3.3. *If* $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma)=\Gamma(\zeta,\overline{\sigma})$, $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega)=\Upsilon(\zeta,\overline{\omega})$*, the relations* (3.1)–(3.4) *are satisfied, the functionals* $\int_{\mathcal{A}}\bar{\nu}\Gamma(\zeta,\overline{\sigma})d\lambda$, $\int_{\mathcal{A}}\overline{f}^{T}M(\zeta)d\lambda$*, and* $\int_{\mathcal{A}}\overline{g}^{T}N(\zeta)d\lambda$ *are convex at* $(\overline{u},\overline{y})\in\mathcal{K}$*, and* $\int_{\mathcal{A}}\bar{\nu}\Upsilon(\zeta,\overline{\omega})d\lambda$ *is concave at* $(\overline{u},\overline{y})\in\mathcal{K}$*, then* $(\overline{u},\overline{y})\in\mathcal{K}$ *is a robust-type optimal point of (Prob).*

Proof. By contrary (see also Proposition 3.1), there exists $(u,y)\in\mathcal{K}$ fulfilling

$$\int_{\mathcal{A}}\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma)d\lambda-V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\min_{\omega\in\Omega}\Upsilon(\zeta,\omega)d\lambda<\int_{\mathcal{A}}\max_{\sigma\in\Sigma}\Gamma(\overline{\zeta},\sigma)d\lambda-V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\min_{\omega\in\Omega}\Upsilon(\overline{\zeta},\omega)d\lambda,$$

and by taking $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma)=\Gamma(\zeta,\overline{\sigma})$ and $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega)=\Upsilon(\zeta,\overline{\omega})$, we obtain

$$\int_{\mathcal{A}}\Gamma(\zeta,\overline{\sigma})d\lambda-V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\Upsilon(\zeta,\overline{\omega})d\lambda<\int_{\mathcal{A}}\Gamma(\overline{\zeta},\overline{\sigma})d\lambda-V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\Upsilon(\overline{\zeta},\overline{\omega})d\lambda. \tag{3.5}$$

By considering the hypotheses imposed to the functionals $\int_{\mathcal{A}}\bar{\nu}\Gamma(\zeta,\overline{\sigma})d\lambda$ and $\int_{\mathcal{A}}\bar{\nu}\Upsilon(\zeta,\overline{\omega})d\lambda$, it follows that

$$
\begin{aligned}
\int_{\mathcal{A}}\bar{\nu}\Gamma(\zeta,\overline{\sigma})d\lambda-\int_{\mathcal{A}}\bar{\nu}\Gamma(\overline{\zeta},\overline{\sigma})d\lambda \geq &\int_{\mathcal{A}}\Big\{(u-\overline{u})\bar{\nu}\frac{\partial\Gamma}{\partial u}(\overline{\zeta},\overline{\sigma})+(y-\overline{y})\bar{\nu}\frac{\partial\Gamma}{\partial y}(\overline{\zeta},\overline{\sigma})\Big\}d\lambda\\
&+\int_{\mathcal{A}}\Big\{(u_{\pi}-\overline{u}_{\pi})\bar{\nu}\frac{\partial\Gamma}{\partial u_{\pi}}(\overline{\zeta},\overline{\sigma})\Big\}d\lambda,
\end{aligned}\tag{3.6}
$$

and

$$
\begin{aligned}
\int_{\mathcal{A}}\bar{\nu}\Upsilon(\zeta,\overline{\omega})d\lambda-\int_{\mathcal{A}}\bar{\nu}\Upsilon(\overline{\zeta},\overline{\omega})d\lambda \leq &\int_{\mathcal{A}}\Big\{(u-\overline{u})\bar{\nu}\frac{\partial\Upsilon}{\partial u}(\overline{\zeta},\overline{\omega})+(y-\overline{y})\bar{\nu}\frac{\partial\Upsilon}{\partial y}(\overline{\zeta},\overline{\omega})\Big\}d\lambda\\
&+\int_{\mathcal{A}}\Big\{(u_{\pi}-\overline{u}_{\pi})\bar{\nu}\frac{\partial\Upsilon}{\partial u_{\pi}}(\overline{\zeta},\overline{\omega})\Big\}d\lambda.
\end{aligned}\tag{3.7}
$$

By multiplying (3.7) with $V^{-}_{\sigma,\omega}$ and subtracting it from (3.6), we get

$$
\begin{aligned}
&\int_{\mathcal{A}}\bar{\nu}\Gamma(\zeta,\overline{\sigma})d\lambda-V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\bar{\nu}\Upsilon(\zeta,\overline{\omega})d\lambda-\int_{\mathcal{A}}\bar{\nu}\Gamma(\overline{\zeta},\overline{\sigma})d\lambda+V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\bar{\nu}\Upsilon(\overline{\zeta},\overline{\omega})\}d\lambda\\
&\geq\int_{\mathcal{A}}(u-\overline{u})\bar{\nu}\frac{\partial\Gamma}{\partial u}(\overline{\zeta},\overline{\sigma})d\lambda-V^{-}_{\sigma,\omega}\int_{\mathcal{A}}(u-\overline{u})\bar{\nu}\frac{\partial\Upsilon}{\partial u}(\overline{\zeta},\overline{\omega})d\lambda
\end{aligned}
$$

$$+\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Gamma}{\partial y}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Upsilon}{\partial y}(\bar{\zeta},\overline{\omega})d\lambda$$
$$+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Gamma}{\partial u_\pi}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Upsilon}{\partial u_\pi}(\bar{\zeta},\overline{\omega})d\lambda,$$

and, by (3.5), it follows that

$$\int_{\mathcal{A}}(u-\overline{u})\bar{v}\frac{\partial\Gamma}{\partial u}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(u-\overline{u})\bar{v}\frac{\partial\Upsilon}{\partial u}(\bar{\zeta},\overline{\omega})d\lambda$$
$$+\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Gamma}{\partial y}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Upsilon}{\partial y}(\bar{\zeta},\overline{\omega})d\lambda$$
$$+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Gamma}{\partial u_\pi}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Upsilon}{\partial u_\pi}(\bar{\zeta},\overline{\omega})d\lambda<0. \tag{3.8}$$

Also, since the functionals $\int_{\mathcal{A}}\overline{f}^T M(\zeta)d\lambda$ and $\int_{\mathcal{A}}\overline{g}^T N(\zeta)d\lambda$ are convex at $(\overline{u},\overline{y})\in\mathcal{K}$, we obtain

$$\int_{\mathcal{A}}\left\{\overline{f}^T M(\zeta)-\overline{f}^T M(\bar{\zeta})\right\}d\lambda\geq\int_{\mathcal{A}}(u-\overline{u})\overline{f}^T\frac{\partial M}{\partial u}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\overline{f}^T\frac{\partial M}{\partial y}(\bar{\zeta})d\lambda$$
$$+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{f}^T\frac{\partial M}{\partial u_\pi}(\bar{\zeta})d\lambda$$

and

$$\int_{\mathcal{A}}\left\{\overline{g}^T N(\zeta)-\overline{g}^T N(\bar{\zeta})\right\}d\lambda\geq\int_{\mathcal{A}}(u-\overline{u})\overline{g}^T\frac{\partial N}{\partial u}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\overline{g}^T\frac{\partial N}{\partial y}(\bar{\zeta})d\lambda$$
$$+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{g}^T\frac{\partial N}{\partial u_\pi}(\bar{\zeta})d\lambda.$$

By employing the feasibility property of (u,y) in *(Prob)* and relations (3.1)–(3.4), it results in

$$\int_{\mathcal{A}}(u-\overline{u})\overline{f}^T\frac{\partial M}{\partial u}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\overline{f}^T\frac{\partial M}{\partial y}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{f}^T\frac{\partial M}{\partial u_\pi}(\bar{\zeta})d\lambda\leq 0 \tag{3.9}$$

and

$$\int_{\mathcal{A}}(u-\overline{u})\overline{g}^T\frac{\partial N}{\partial u}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\overline{g}^T\frac{\partial N}{\partial y}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{g}^T\frac{\partial N}{\partial u_\pi}(\bar{\zeta})d\lambda\leq 0. \tag{3.10}$$

On adding (3.8), (3.9), and (3.10), we get

$$\int_{\mathcal{A}}(u-\overline{u})\bar{v}\frac{\partial\Gamma}{\partial u}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(u-\overline{u})\bar{v}\frac{\partial\Upsilon}{\partial u}(\bar{\zeta},\overline{\omega})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Gamma}{\partial y}(\bar{\zeta},\overline{\sigma})d\lambda$$
$$-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Upsilon}{\partial y}(\bar{\zeta},\overline{\omega})d\lambda+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Gamma}{\partial u_\pi}(\bar{\zeta},\overline{\sigma})d\lambda-V_{\sigma,\omega}^{-}\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Upsilon}{\partial u_\pi}(\bar{\zeta},\overline{\omega})d\lambda$$
$$+\int_{\mathcal{A}}(u-\overline{u})\overline{f}^T\frac{\partial M}{\partial u}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\overline{f}^T\frac{\partial M}{\partial y}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{f}^T\frac{\partial M}{\partial u_\pi}(\bar{\zeta})d\lambda$$
$$+\int_{\mathcal{A}}(u-\overline{u})\overline{g}^T\frac{\partial N}{\partial u}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(y-\overline{y})\overline{g}^T\frac{\partial N}{\partial y}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{g}^T\frac{\partial N}{\partial u_\pi}(\bar{\zeta})d\lambda<0. \tag{3.11}$$

Further, after multiplying (3.1) and (3.2) with $(u-\overline{u})$ and $(y-\overline{y})$, respectively, integrating them, and adding the results, we obtain

$$\begin{aligned}
&\int_{\mathcal{A}}(u-\overline{u})\bar{v}\frac{\partial\Gamma}{\partial u}(\overline{\zeta},\overline{\sigma})d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}}(u-\overline{u})\bar{v}\frac{\partial\Upsilon}{\partial u}(\overline{\zeta},\overline{\omega})d\lambda + \int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Gamma}{\partial y}(\overline{\zeta},\overline{\sigma})d\lambda\\
&- V^{-}_{\sigma,\omega}\int_{\mathcal{A}}(y-\overline{y})\bar{v}\frac{\partial\Upsilon}{\partial y}(\overline{\zeta},\overline{\omega})d\lambda + \int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Gamma}{\partial u_\pi}(\overline{\zeta},\overline{\sigma})d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\bar{v}\frac{\partial\Upsilon}{\partial u_\pi}(\overline{\zeta},\overline{\omega})d\lambda\\
&+ \int_{\mathcal{A}}(u-\overline{u})\overline{f}^T\frac{\partial M}{\partial u}(\overline{\zeta})d\lambda + \int_{\mathcal{A}}(y-\overline{y})\overline{f}^T\frac{\partial M}{\partial y}(\overline{\zeta})d\lambda + \int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{f}^T\frac{\partial M}{\partial u_\pi}(\overline{\zeta})d\lambda\\
&+ \int_{\mathcal{A}}(u-\overline{u})\overline{g}^T\frac{\partial N}{\partial u}(\overline{\zeta})d\lambda + \int_{\mathcal{A}}(y-\overline{y})\overline{g}^T\frac{\partial N}{\partial y}(\overline{\zeta})d\lambda\\
&+ \int_{\mathcal{A}}(u_\pi-\overline{u}_\pi)\overline{g}^T\frac{\partial N}{\partial u_\pi}(\overline{\zeta})d\lambda = 0,
\end{aligned}$$

which contradicts (3.11). The proof is complete. □

Next, a second result is established on the robust-type sufficiency criteria of the considered extremization model, under only convexity hypotheses of the involved functionals.

Theorem 3.4. *If* $(\bar{u},\bar{y})\in\mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma)=\Gamma(\zeta,\overline{\sigma})$*,* $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega)=\Upsilon(\zeta,\overline{\omega})$*, and*

$$\int_{\mathcal{A}}\bar{v}\Big[\Gamma(\zeta,\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon(\zeta,\bar{\omega})\Big]d\lambda,\ \int_{\mathcal{A}}\bar{f}^TM(\zeta)d\lambda,\ \int_{\mathcal{A}}\bar{g}^TN(\zeta)d\lambda$$

are convex at $(\bar{u},\bar{y})\in\mathcal{K}$*, then* $(\bar{u},\bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. By contrast, let us suppose that $(\bar{u},\bar{y})$ is not a robust-type optimal point of *(Prob)*. Thus, there exists $(\hat{u},\hat{y})\in\mathcal{K}$ with the property (see Proposition 3.1),

$$\int_{\mathcal{A}}\max_{\sigma\in\Sigma}\Gamma(\hat{\zeta},\sigma)d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\min_{\omega\in\Omega}\Upsilon(\hat{\zeta},\omega)d\lambda < \int_{\mathcal{A}}\max_{\sigma\in\Sigma}\Gamma(\bar{\zeta},\sigma)d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\min_{\omega\in\Omega}\Upsilon(\bar{\zeta},\omega)d\lambda.$$

By considering $\max_{\sigma\in\Sigma}\Gamma(\zeta,\sigma)=\Gamma(\zeta,\overline{\sigma})$, $\min_{\omega\in\Omega}\Upsilon(\zeta,\omega)=\Upsilon(\zeta,\overline{\omega})$, we get

$$\int_{\mathcal{A}}\Gamma(\hat{\zeta},\bar{\sigma})d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\Upsilon(\hat{\zeta},\bar{\omega})d\lambda < \int_{\mathcal{A}}\Gamma(\bar{\zeta},\bar{\sigma})d\lambda - V^{-}_{\sigma,\omega}\int_{\mathcal{A}}\Upsilon(\bar{\zeta},\bar{\omega})d\lambda. \tag{3.12}$$

Since $(\bar{u},\bar{y})$ fulfills (3.1)–(3.4), we get

$$\begin{aligned}
&\int_{\mathcal{A}}(\hat{u}-\bar{u})\{\bar{v}[\Gamma_u(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_u(\bar{\zeta})+\bar{g}^TN_u(\bar{\zeta})\\
&-\frac{\partial}{\partial\lambda^\pi}\Big[\bar{v}[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_{u_\pi}(\bar{\zeta})+\bar{g}^TN_{u_\pi}(\bar{\zeta})\Big]\}d\lambda\\
&+\int_{\mathcal{A}}(\hat{y}-\bar{y})\{\bar{v}[\Gamma_y(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_y(\bar{\zeta})+\bar{g}^TN_y(\bar{\zeta})\}d\lambda\\
=&\int_{\mathcal{A}}\Big[(\hat{u}-\bar{u})\{\bar{v}[\Gamma_u(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_u(\bar{\zeta})+\bar{g}^TN_u(\bar{\zeta})\}\\
&+(\hat{u}_\pi-\bar{u}_\pi)\{\bar{v}[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_{u_\pi}(\bar{\zeta})+\bar{g}^TN_{u_\pi}(\bar{\zeta})\}\Big]d\lambda
\end{aligned}$$

$$+\int_{\mathcal{A}}(\hat{y}-\bar{y})\{\bar{v}[\Gamma_y(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_y(\bar{\zeta})+\bar{g}^TN_y(\bar{\zeta})\}d\lambda=0, \tag{3.13}$$

by using the divergence formula, the boundary conditions, and the method of integration by parts.

Also, since $\int_{\mathcal{A}}\bar{v}\Big[\Gamma(\zeta,\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon(\zeta,\bar{\omega})\Big]d\lambda$ is convex at $(\bar{u},\bar{y})$, we get

$$\begin{aligned}&\int_{\mathcal{A}}\Big\{\bar{v}[\Gamma(\hat{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon(\hat{\zeta},\bar{\omega})]-\bar{v}[\Gamma(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon(\bar{\zeta},\bar{\omega})]\Big\}d\lambda\\ &\geq\int_{\mathcal{A}}(\hat{u}-\bar{u})\bar{v}[\Gamma_u(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})]d\lambda+\int_{\mathcal{A}}(\hat{u}_\pi-\bar{u}_\pi)\bar{v}[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})]d\lambda\\ &+\int_{\mathcal{A}}(\hat{y}-\bar{y})\bar{v}[\Gamma_y(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})]d\lambda,\end{aligned}$$

and, by (3.12), it results in

$$\begin{aligned}&\int_{\mathcal{A}}(\hat{u}-\bar{u})\bar{v}[\Gamma_u(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})]d\lambda+\int_{\mathcal{A}}(\hat{u}_\pi-\bar{u}_\pi)\bar{v}[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})]d\lambda\\ &+\int_{\mathcal{A}}(\hat{y}-\bar{y})\bar{v}[\Gamma_y(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})]d\lambda<0.\end{aligned}\tag{3.14}$$

Now, by convexity property of $\int_{\mathcal{A}}\bar{f}^TM(\zeta)d\lambda$ at $(\bar{u},\bar{y})$, we obtain

$$\int_{\mathcal{A}}\Big\{\bar{f}^TM(\hat{\zeta})-\bar{f}^TM(\bar{\zeta})\Big\}d\lambda\geq\int_{\mathcal{A}}(\hat{u}-\bar{u})\bar{f}^TM_u(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(\hat{u}_\pi-\bar{u}_\pi)\bar{f}^TM_{u_\pi}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(\hat{y}-\bar{y})\bar{f}^TM_y(\bar{\zeta})d\lambda,$$

which by robust feasibility of $(\hat{u},\hat{y})$ for *(Prob)* and (3.3) gives

$$\int_{\mathcal{A}}(\hat{u}-\bar{u})\bar{f}^TM_u(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(\hat{u}_\pi-\bar{u}_\pi)\bar{f}^TM_{u_\pi}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(\hat{y}-\bar{y})\bar{f}^TM_y(\bar{\zeta})d\lambda\leq0. \tag{3.15}$$

Further, in the same manner, we obtain

$$\int_{\mathcal{A}}(\hat{u}-\bar{u})\bar{g}^TN_u(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(\hat{u}_\pi-\bar{u}_\pi)\bar{g}^TN_{u_\pi}(\bar{\zeta})d\lambda+\int_{\mathcal{A}}(\hat{y}-\bar{y})\bar{g}^TN_y(\bar{\zeta})d\lambda\leq0. \tag{3.16}$$

Finally, by adding (3.14), (3.15), and (3.16), it follows that

$$\begin{aligned}&\int_{\mathcal{A}}\Big[(\hat{u}-\bar{u})\{\bar{v}[\Gamma_u(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_u(\bar{\zeta})+\bar{g}^TN_u(\bar{\zeta})\}\\ &+(\hat{u}_\pi-\bar{u}_\pi)\{\bar{v}[\Gamma_{u_\pi}(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_{u_\pi}(\bar{\zeta})+\bar{g}^TN_{u_\pi}(\bar{\zeta})\}\Big]d\lambda\\ &+\int_{\mathcal{A}}(\hat{y}-\bar{y})\{\bar{v}[\Gamma_y(\bar{\zeta},\bar{\sigma})-V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta},\bar{\omega})]+\bar{f}^TM_y(\bar{\zeta})+\bar{g}^TN_y(\bar{\zeta})\}d\lambda<0,\end{aligned}$$

which contradicts (3.13). □

Next, under only (strictly, monotonic) quasi-convexity assumptions, new robust-type sufficient optimality criteria are stated.

Theorem 3.5. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega} \Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{\nu}\Big[\Gamma(\zeta, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon(\zeta, \bar{\omega})\Big]d\lambda, \ Y(u, y) := \int_{\mathcal{A}} \bar{f}^T M(\zeta)d\lambda$$

are quasi-convex and strictly quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, and* $X(u, y) := \int_{\mathcal{A}} \bar{g}^T N(\zeta)d\lambda$ *is monotonic quasi-convex at* $(\bar{u}, \bar{y}) \in \mathcal{K}$*, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. Consider $(\bar{u}, \bar{y})$ is not a robust-type optimal point of *(Prob)*, and define the set (nonempty)

$$S = \{(u, y) \in \mathcal{K} \mid F(u, y; \bar{\sigma}, \bar{\omega}) \le F(\bar{u}, \bar{y}; \bar{\sigma}, \bar{\omega}), \ X(u, y) = X(\bar{u}, \bar{y}), \ Y(u, y) \le Y(\bar{u}, \bar{y})\}.$$

By hypothesis, for $(u, y) \in S$, we get

$$\begin{aligned} &F(u, y; \bar{\sigma}, \bar{\omega}) \le F(\bar{u}, \bar{y}; \bar{\sigma}, \bar{\omega}) \\ \Longrightarrow &\int_{\mathcal{A}} \Big\{\bar{\nu}\Big[\Gamma_u(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta}, \bar{\omega})\Big](u - \bar{u}) + \bar{\nu}\Big[\Gamma_y(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta}, \bar{\omega})\Big](y - \bar{y})\Big\}dv \\ &+ \int_{\mathcal{A}} \Big\{\bar{\nu}\Big[\Gamma_{u_\pi}(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta}, \bar{\omega})\Big](u_\pi - \bar{u}_\pi)\Big\}dv \le 0. \end{aligned} \tag{3.17}$$

For $(u, y) \in S$, the equality $X(u, y) = X(\bar{u}, \bar{y})$ holds and it follows

$$\int_{\mathcal{A}} \Big\{\bar{g}^T N_u(\bar{\zeta})(u - \bar{u}) + \bar{g}^T N_y(\bar{\zeta})(y - \bar{y})\Big\}dv + \int_{\mathcal{A}} \Big\{\bar{g}^T N_{u_\pi}(\bar{\zeta})(u_\pi - \bar{u}_\pi)\Big\}dv = 0. \tag{3.18}$$

Also, for $(u, y) \in S$, the inequality $Y(u, y) \le Y(\bar{u}, \bar{y})$ gives

$$\int_{\mathcal{A}} \Big\{\bar{f}^T M_u(\bar{\zeta})(u - \bar{u}) + \bar{f}^T M_y(\bar{\zeta})(y - \bar{y})\Big\}dv + \int_{\mathcal{A}} \Big\{\bar{f}^T M_{u_\pi}(\bar{\zeta})(u_\pi - \bar{u}_\pi)\Big\}dv < 0. \tag{3.19}$$

Since$(\bar{u}, \bar{y})$ fulfills (3.1)–(3.4), we get

$$\begin{aligned} &\int_{\mathcal{A}} (u - \bar{u})\{\bar{\nu}[\Gamma_u(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta}) \\ &- \frac{\partial}{\partial \lambda^\pi}\Big[\bar{\nu}[\Gamma_{u_\pi}(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_{u_\pi}(\bar{\zeta}) + \bar{g}^T N_{u_\pi}(\bar{\zeta})\Big]\}d\lambda \\ &+ \int_{\mathcal{A}} (y - \bar{y})\{\bar{\nu}[\Gamma_y(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_y(\bar{\zeta}) + \bar{g}^T N_y(\bar{\zeta})\}d\lambda \\ = &\int_{\mathcal{A}} \Big[(u - \bar{u})\{\bar{\nu}[\Gamma_u(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta})\} \\ &+ (u_\pi - \bar{u}_\pi)\{\bar{\nu}[\Gamma_{u_\pi}(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_{u_\pi}(\bar{\zeta}) + \bar{g}^T N_{u_\pi}(\bar{\zeta})\}\Big]d\lambda \\ &+ \int_{\mathcal{A}} (y - \bar{y})\{\bar{\nu}[\Gamma_y(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_y(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_y(\bar{\zeta}) + \bar{g}^T N_y(\bar{\zeta})\}d\lambda = 0, \end{aligned} \tag{3.20}$$

by using the divergence formula, the boundary conditions, and the method of integration by parts. Now, by adding (3.17), (3.18), and (3.19), we obtain

$$\int_{\mathcal{A}} \Big[(u - \bar{u})\{\bar{\nu}[\Gamma_u(\bar{\zeta}, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon_u(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_u(\bar{\zeta}) + \bar{g}^T N_u(\bar{\zeta})\}$$

$$+ (u_\pi - \bar{u}_\pi)\{\bar{\nu}[\Gamma_{u_\pi}(\bar{\zeta}, \bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_{u_\pi}(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_{u_\pi}(\bar{\zeta}) + \bar{g}^T N_{u_\pi}(\bar{\zeta})\}\Big]d\lambda$$
$$+ \int_{\mathcal{A}} (y - \bar{y})\{\bar{\nu}[\Gamma_y(\bar{\zeta}, \bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon_y(\bar{\zeta}, \bar{\omega})] + \bar{f}^T M_y(\bar{\zeta}) + \bar{g}^T N_y(\bar{\zeta})\}d\lambda < 0,$$

which contradicts (3.20). □

Various consequences associated with the abovementioned result are written as follows.

Theorem 3.6. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma}\Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega}\Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{\nu}\Big[\Gamma(\zeta, \bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,\ Y(u, y) := \int_{\mathcal{A}} \bar{f}^T M(\zeta)d\lambda$$

are strictly quasi-convex and quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, and* $X(u, y) := \int_{\mathcal{A}} \bar{g}^T N(\zeta)d\lambda$ *is monotonic quasi-convex at* $(\bar{u}, \bar{y}) \in \mathcal{K}$*, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we replace "≤" in (3.17) with "<", and "<" in (3.19) with "≤". □

Theorem 3.7. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma}\Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega}\Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{\nu}\Big[\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})\Gamma(\zeta, \bar{\sigma}) - \tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,\ Y(u, y) := \int_{\mathcal{A}} \bar{f}^T M(\zeta)d\lambda$$

are quasi-convex and strictly quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, and* $X(u, y) := \int_{\mathcal{A}} \bar{g}^T N(\zeta)d\lambda$ *is monotonic quasi-convex at* $(\bar{u}, \bar{y}) \in \mathcal{K}$*, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we replace $V^-_{\sigma,\omega} = \dfrac{\int_{\mathcal{A}} \max_{\sigma\in\Sigma}\Gamma(\bar{\zeta}, \sigma)d\lambda}{\int_{\mathcal{A}} \min_{\omega\in\Omega}\Upsilon(\bar{\zeta}, \omega)d\lambda} := \dfrac{\tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})}{\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})}$. □

Theorem 3.8. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma}\Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega}\Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{\nu}\Big[\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})\Gamma(\zeta, \bar{\sigma}) - \tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,\ Y(u, y) := \int_{\mathcal{A}} \bar{f}^T M(\zeta)d\lambda$$

are strictly quasi-convex and quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, and* $X(u, y) := \int_{\mathcal{A}} \bar{g}^T N(\zeta)d\lambda$ *is monotonic quasi-convex at* $(\bar{u}, \bar{y}) \in \mathcal{K}$*, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we replace $V^-_{\sigma,\omega} = \dfrac{\int_{\mathcal{A}} \max_{\sigma\in\Sigma}\Gamma(\bar{\zeta}, \sigma)d\lambda}{\int_{\mathcal{A}} \min_{\omega\in\Omega}\Upsilon(\bar{\zeta}, \omega)d\lambda} := \dfrac{\tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})}{\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})}$, "≤" in (3.17) with "<", and "<" in (3.19) with "≤". □

Theorem 3.9. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega} \Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{v}\Big[\Gamma(\zeta, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,$$

$$\tilde{Y}(u, y) := \int_{\mathcal{A}} \Big[\bar{f}^T M(\zeta) + \bar{g}^T N(\zeta)\Big]d\lambda$$

are quasi-convex and strictly quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we consider "<" in (3.18) and (3.19), then we add them. □

Theorem 3.10. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega} \Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{v}\Big[\Gamma(\zeta, \bar{\sigma}) - V^{-}_{\sigma,\omega}\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,$$

$$\tilde{Y}(u, y) := \int_{\mathcal{A}} \Big[\bar{f}^T M(\zeta) + \bar{g}^T N(\zeta)\Big]d\lambda$$

are strictly quasi-convex and quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we consider "<" in (3.17), and "≤" in (3.18) and (3.19), then we add them. □

Theorem 3.11. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega} \Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{v}\Big[\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})\Gamma(\zeta, \bar{\sigma}) - \tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,$$

$$\tilde{Y}(u, y) := \int_{\mathcal{A}} \Big[\bar{f}^T M(\zeta) + \bar{g}^T N(\zeta)\Big]d\lambda$$

are quasi-convex and strictly quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we replace $V^{-}_{\sigma,\omega} = \dfrac{\int_{\mathcal{A}} \max\limits_{\sigma\in\Sigma} \Gamma(\bar{\zeta}, \sigma)d\lambda}{\int_{\mathcal{A}} \min\limits_{\omega\in\Omega} \Upsilon(\bar{\zeta}, \omega)d\lambda} := \dfrac{\tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})}{\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})}$, and consider "<" in (3.18) and (3.19), then we add them. □

Theorem 3.12. *If* $(\bar{u}, \bar{y}) \in \mathcal{K}$ *and* (3.1)–(3.4) *are satisfied,* $\max_{\sigma\in\Sigma} \Gamma(\zeta, \sigma) = \Gamma(\zeta, \overline{\sigma})$*,* $\min_{\omega\in\Omega} \Upsilon(\zeta, \omega) = \Upsilon(\zeta, \overline{\omega})$*, and*

$$F(u, y; \bar{\sigma}, \bar{\omega}) := \int_{\mathcal{A}} \bar{v}\Big[\tilde{\Upsilon}(\bar{\zeta}, \bar{\sigma})\Gamma(\zeta, \bar{\sigma}) - \tilde{\Gamma}(\bar{\zeta}, \bar{\sigma})\Upsilon(\zeta, \bar{\omega})\Big]d\lambda,$$

$$\tilde{Y}(u, y) := \int_{\mathcal{A}} \Big[\bar{f}^T M(\zeta) + \bar{g}^T N(\zeta)\Big]d\lambda$$

are strictly quasi-convex and quasi-convex at $(\bar{u}, \bar{y}) \in \mathcal{K}$*, respectively, then* $(\bar{u}, \bar{y})$ *is a robust-type optimal point of (Prob).*

Proof. In the proof of Theorem 3.5, we replace $V^{-}_{\sigma,\omega} = \dfrac{\int_{\mathcal{A}} \max\limits_{\sigma\in\Sigma} \Gamma(\bar{\zeta},\sigma)d\lambda}{\int_{\mathcal{A}} \min\limits_{\omega\in\Omega} \Upsilon(\bar{\zeta},\omega)d\lambda} := \dfrac{\tilde{\Gamma}(\bar{\zeta},\bar{\sigma})}{\tilde{\Upsilon}(\bar{\zeta},\bar{\sigma})}$, "≤" in (3.17) with "<", and consider "≤" in (3.18) and (3.19), then we add them. □

4. Application

The following application presents the practical aspect of the theoretical developments given in the previous sections. In this regard, we consider we have interest only in affine control and state functions, $\Sigma = \Omega = [1,2]$, and $\mathcal{A} \subset \mathbb{R}^2$ is a square having the corners $\lambda_0 = (\lambda_0^1, \lambda_0^2) = (0,0)$ and $\lambda_1 = (\lambda_1^2, \lambda_1^2) = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$. We consider the following extremization fractional model:

$$\textbf{(Prob1)} \quad \min_{(u(\cdot),y(\cdot))} \left\{ \frac{\int_{\mathcal{A}} \Gamma(\zeta,\sigma)d\lambda^1 d\lambda^2}{\int_{\mathcal{A}} \Upsilon(\zeta,\omega)d\lambda^1 d\lambda^2} = \frac{\int_{\mathcal{A}} [y^2+\sigma]d\lambda^1 d\lambda^2}{\int_{\mathcal{A}} [\omega u e^{2u+\frac{1}{2}}]d\lambda^1 d\lambda^2} \right\},$$

subject to

$$M(\zeta) = u^2 + u - 2 \le 0,$$
$$N_\pi(\zeta) = \frac{\partial u}{\partial \lambda^\pi} + 2y - 1 = 0, \quad \pi = 1,2,$$
$$u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}, \; u(0,0) = 1.$$

The nonfractional extremization model for **(Prob1)** is formulated by:

$$\textbf{(NonFracProb1)} \quad \min_{(u(\cdot),y(\cdot))} \left\{ \int_{\mathcal{A}} [y^2+\sigma]d\lambda^1 d\lambda^2 - V^{-}_{\sigma,\omega} \int_{\mathcal{A}} [\omega u e^{2u+\frac{1}{2}}]d\lambda^1 d\lambda^2 \right\},$$

subject to

$$M(\zeta) = u^2 + u - 2 \le 0,$$
$$N_\pi(\zeta) = \frac{\partial u}{\partial \lambda^\pi} + 2y - 1 = 0, \quad \pi = 1,2,$$
$$u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}, \; u(0,0) = 1,$$

and the associated robust-type counterpart of **(NonFracProb1)** is introduced as:

$$\textbf{(RobNonFracProb1)} \quad \min_{(u(\cdot),y(\cdot))} \left\{ \int_{\mathcal{A}} \max_{\sigma\in\Sigma} [y^2+\sigma]d\lambda^1 d\lambda^2 - V^{-}_{\sigma,\omega} \int_{\mathcal{A}} \min_{\omega\in\Omega} [\omega u e^{2u+\frac{1}{2}}]d\lambda^1 d\lambda^2 \right\},$$

subject to

$$M(\zeta) = u^2 + u - 2 \le 0,$$

$$N_\pi(\zeta) = \frac{\partial u}{\partial \lambda^\pi} + 2y - 1 = 0, \ \ \pi = 1, 2,$$
$$u(0,0) = 1, \ u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}.$$

The robust-type feasible solution set of **(NonFracProb1)** is

$$\mathcal{K} = \left\{(u, y) \in A \times B : -2 \leq u \leq 1, \ \frac{\partial u}{\partial \lambda^1} = \frac{\partial u}{\partial \lambda^2} = 1 - 2y, \ u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}, \ u(0,0) = 1\right\},$$

and we obtain $(\overline{u}, \overline{y}) = \left(-\frac{2}{3}(\lambda^1 + \lambda^2) + 1, \frac{5}{6}\right) \in \mathcal{K}$, which satisfies (3.1)–(3.4) at $\lambda^1 = \lambda^2 = 0$, with $V^-_{\sigma,\omega} = \frac{169}{36e^{\frac{5}{2}}}$, the parameters $\overline{\sigma} = 2, \overline{\omega} = 1$, and $\overline{\nu} = \frac{1}{2}, \overline{f} = 0, \overline{g}_1 = \overline{g}_2 = \frac{5}{24}$. Further, it can also be easily verified that the involved functionals $\int_{\mathcal{A}} \bar{\nu}\Big[\Gamma(\zeta, \bar{\sigma}) - V^-_{\sigma,\omega}\Upsilon(\zeta, \bar{\omega})\Big] d\lambda^1 d\lambda^2$, $\int_{\mathcal{A}} \bar{f}^T M(\zeta) d\lambda^1 d\lambda^2$, $\int_{\mathcal{A}} \bar{g}^T N(\zeta) d\lambda^1 d\lambda^2$ are convex at $(\overline{u}, \overline{y}) = \left(1, \frac{5}{6}\right) \in \mathcal{K}$. As the hypotheses in Theorem 3.4 are fulfilled, we can conclude that $(\overline{u}, \overline{y})$ is a robust-type optimal point of **(NonFracProb1)**. Now, applying Proposition 3.1, we get $(\overline{u}, \overline{y})$ is also a robust optimal solution to **(Prob1)**.

5. Conclusions

In this paper, a multidimensional fractional variational control problem with data uncertainty in the cost functional has been studied. In this regard, under the various forms of convexity for the considered functionals, we have stated the associated robust-type optimality criteria. The main results of the paper are validated with an appropriate illustrative example.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in writing the paper.

Conflict of interest

The author declares no conflict of interest.

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