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*Research article*

# Unraveling multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials via fractional operators

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**Abstract:** This study explores the evolution and application of integral transformations, initially rooted in mathematical physics but now widely employed across diverse mathematical disciplines. Integral transformations offer a comprehensive framework comprising recurrence relations, generating expressions, operational formalism, and special functions, enabling the construction and analysis of specialized polynomials. Specifically, the research investigates a novel extended family of Frobenius-Genocchi polynomials of the Hermite-Apostol-type, incorporating multivariable variables defined through fractional operators. It introduces an operational rule for this generalized family, establishes a generating connection, and derives recurring relations. Moreover, the study highlights the practical applications of this generalized family, demonstrating its potential to provide solutions for specific scenarios.

**Keywords:** operational connection; fractional operators; Eulers' integral; multivariable special polynomials; explicit form; applications
**Mathematics Subject Classification:** 11T23, 33B10, 33C45, 33E20, 33E30

## 1. Introduction and preliminaries

Exploring the amalgamation of various polynomial types to create innovative multi-variable generalized polynomials is a current and practical research area. These polynomials hold immense significance due to their valuable attributes, including recurring and explicit relationships, functional and differential equations, summation formulas, symmetric and convolution properties, and determinant representations. The utilization of multi-variable hybrid special polynomials extends to different domains such as number theory, combinatorics, classical and numerical analysis, theoretical physics, and approximation theory, among others, offering substantial potential for practical applications.

In order to fully realize the potential of hybrid polynomials for applications, a number of additional categories have been established. The goal of this project is to increase the number of mathematical instruments that may be used to solve complicated problems in a wide range of pure and applied mathematics fields.

Polynomial sequences are considered to be very important in many fields, such as applied mathematics, theoretical physics, and approximation theory. In particular, polynomials of degrees equal to or fewer than $n$ may be constructed from fundamental building blocks, which are Bernstein polynomials of order $n$. Using operational approaches, Dattoli and colleagues performed a thorough analysis of Bernstein polynomials, delving into their characteristics and complexities [1]. They investigated the class of Appell sequences, which is a broad category that includes, among other polynomial sequences, the Miller-Lee, Bernoulli, and Genocchi polynomials.

The exploration and comprehensive analysis of novel classes of hybrid special polynomials associated with the Appell sequences hold immense significance across diverse domains. These polynomials, as highlighted by various sources, including [2–8], serve as versatile tools with broad utility in fields ranging from the physical sciences to engineering, biology, and medicine. Their unique properties, such as integral representations, series definitions, and generating functions, enable efficient problem-solving, mathematical analysis, and modelling of complex phenomena. By providing a foundation for mathematical analysis and facilitating interdisciplinary collaboration, hybrid special polynomials contribute to advancing research, enhancing computational efficiency, and fostering innovation in various disciplines, making them indispensable assets in the pursuit of knowledge and technological advancement. These hybrid special polynomials play a pivotal role in advancing scientific understanding and technological innovation by offering versatile tools for problem-solving, mathematical modelling, and computational analysis. Within engineering, hybrid special polynomials contribute to the design and optimization of systems, enhancing efficiency and performance. In biology and medicine, they enable the modelling of intricate biological processes, such as gene expression dynamics and disease progression, thereby supporting medical diagnosis, treatment development, and personalized healthcare. Differential equations are widely used to express problems in many scientific and technical fields, and unique functions are often used to solve them. Consequently, these hybrid special polynomials are highly useful for characterizing and solving issues that arise in the rapidly evolving field of research. The following is an expression for the generating function reported in [9]:

$$e^{h_1 t+h_2 t^2+h_3 t^3} = \sum_{n=0}^{\infty} \mathfrak{D}_n(h_1, h_2, h_3)\frac{t^n}{n!}, \tag{1.1}$$

the 3-variable Hermite polynomials (3VHP), represented as $\mathfrak{D}_n(h_1, h_2, h_3)$, are correlated with this generating function.

The 3VHP reduces to a set of polynomials known as the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP), represented by the symbol $\mathfrak{D}_n(h_1, h_2)$. This is the result of setting $h_3$ to zero. [10] provides extensive documentation on these 2VHKdFP polynomials.

Moreover, the 3VHP converts into the conventional Hermite polynomials denoted as $\mathfrak{D}_n(h_1)$, as described in [11], if we set $h_3$ to zero, $h_1$ to $2h_1$, and $h_2$ to $-1$.

In addition, the following connection establishes the set of polynomials known as multivariable Hermite Polynomials (MHP) [12], and denoted by the notation $\mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m)$:

$$\exp(h_1\xi + h_2\xi^2 + \cdots + h_m\xi^m) = \sum_{n=0}^{\infty} \mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m)\frac{\xi^n}{n!}, \tag{1.2}$$

for these polynomials, the operational rule is written as follows:

$$\exp\left(h_2\frac{\partial^2}{\partial_{h_1}^2} + h_3\,\frac{\partial^3}{\partial_{h_1}^3} + \cdots + h_m\frac{\partial^m}{\partial_{h_1}^m}\right)h_1^n = \mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m). \tag{1.3}$$

Additionally, these polynomials can be represented in series form as:

$$\mathcal{D}_n^{[m]}(h_1, h_2, \cdots h_m) = n!\sum_{r=0}^{[n/m]} \frac{h_m^r\,\mathcal{D}_{n-mr}^{[m]}(h_1, h_2, \cdots, h_{m-1})}{r!\,(n-mr)!}. \tag{1.4}$$

In the work published in [13], a unified formulation for a particular class of polynomials called Apostol-type Frobenius-Genocchi polynomials is introduced. These polynomials are formalized using the mathematical notation $\mathbf{K}_n(h_1; \lambda; u)$, as per [14]. The generative expression for these polynomials is given as:

$$\left(\frac{(1-u)\xi}{\lambda e^{\xi} - u}\right)e^{h_1\xi} = \sum_{n=0}^{\infty} \mathbf{K}_n(h_1; \lambda; u)\frac{\xi^n}{n!}, \tag{1.5}$$

where, $u \in \mathbb{C}$, $u \neq 1$.

The Apostol-type Frobenius-Genocchi numbers (ATFGN) of order $\beta$, $\mathbf{K}_n(\lambda; u)$, are obtained from relation (1.5) for $h_1 = 0$:

$$\left(\frac{(1-u)\xi}{\lambda e^{\xi} - u}\right) = \sum_{n=0}^{\infty} \mathbf{K}_n(\lambda; u)\frac{\xi^n}{n!}. \tag{1.6}$$

The special case of Apostol-type Frobenius-Genocchi polynomials (ATFGP), named Apostol-Genocchi polynomials (AGP), $\mathfrak{A}_n(h_1; \lambda)$, then follows for $u = -1$ and in their turn, they become Genocchi polynomials denoted as $A_n(h_1)$ when $\lambda = 1$, as discussed in [15]. The Apostol-type Frobenius-Genocchi polynomials (ATFGP) generate the Frobenius-Genocchi polynomials, $\mathbb{K}_n(h_1; u)$, when $\lambda = 1$, as explained in [16].

Fractional calculus provides tools for investigations in numerous research fields connected to the sciences (biology, physics, and electrochemistry), economics, statistics, or probability theory. Its origins date back to the late 17th, century when it was proposed independently by the renowned mathematician and philosopher G.W. Leibniz, concerned with mathematical studies, and by I. Newton, regarding physics research. It involves extending integration to orders that are not whole integers, the order of $1/2$ is first employed. However, it was not until Liouville's committed and exhaustive research that this topic was thoroughly explored, leading to accurate and carefully carried out research in the end.

Integral transformations are used to discover solutions for both differential and integral problems. Mathematicians and engineers have always been fascinated with fractional operators, as demonstrated by the writings of Widder and Oldham [17, 18]. As noted in scholarly works [17, 18], significant progress was made by Riemann and Liouville in the study of fractional derivatives using integral transforms.

Fractional derivatives can be handled effectively by combining specialized polynomials and integral transformations in a synergistic fashion. The acceptance of this method as a useful tool is highlighted in works like [19,20], where the importance of this integrated approach is also emphasized. The combination of certain polynomials and integral transformations facilitates the theoretical and practical developments involving fractional derivatives. Specialists and researchers have examined the advantages of this method, hence improving knowledge regarding fractional calculus and its numerous applications.

Integral transforms like Laplace or Fourier transforms combined with discrete polynomials like Hermite, Laguerre, or Chebyshev polynomials allow researchers to create effective methods for solving fractional differential equations. Some of the areas that have been observed to yield success with these methods include but are not limited to, signal processing, physics, engineering, and finance. Dattoli et al. first used "Euler's integral", which they presented in [19] in the form:

$$\frac{1}{\Gamma(\mu)}\int_0^\infty e^{-q\xi}\xi^{\mu-1}d\xi = q^{-\mu}, \qquad \min\{\mathrm{Re}(\mu), \mathrm{Re}(q)\} > 0. \tag{1.7}$$

Therefore, by employing Euler's integral into the framework of integral transformation, researchers can examine a wide range of complex problems arising in different mathematical and engineering fields. This paper presents an integrated approach that includes all the essential tools for enhancing the adaptability and the performance of integral transformations in various domains. This extended framework offers new potential for the study of fractional derivatives and their uses, motivating original approaches to solving problems.

The investigation presented here shows that more development is possible in this area and provides a helpful tool for researchers when they address difficult fractional derivative problems in a wider setting. Additionally, the cited study [19] demonstrates that the following axioms hold for both first- and second-order derivatives:

$$\left(\beta - \frac{\partial}{\partial h_1}\right)^{-\mu} h(h_1) = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta t}t^{\mu-1}\, e^{\xi\frac{\partial}{\partial h_1}}h(h_1)d\xi = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta t}\xi^{\mu-1}\, h(h_1+\xi)d\xi, \tag{1.8}$$

$$\left(\beta - \frac{\partial^2}{\partial h_1^2}\right)^{-\mu} h(h_1) = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}\, e^{\xi\frac{\partial^2}{\partial h_1^2}}h(h_1)d\xi. \tag{1.9}$$

Fractional operators are used successfully by researchers due to the basic properties of exponential operators and by choosing the right integral representations. New mathematical concepts are better investigated, and the study of fractional derivatives is facilitated by this approach.

Different characteristics of hybrid special polynomials naturally result from the inclusion of operational rules, monomiality principles, and other relevant characteristics. The origins of monomiality lie in the term poweroid, initially used in 1941 by Steffenson [21], and later developed by Dattoli [2]. These methods of investigation still have applications in many fields, such as quantum mechanics, mathematical physics, and classical optics, which are considered powerful and successful research instruments.

Hence, multivariable Hermite polynomials $\mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m)$ given by (1.2) and Apostol-type Frobenius-Genocchi polynomials [22,23] seen in (1.5) are combined and these two sets of polynomials combine to form a new type of polynomial known as multivariable Hermite-Apostol- type Frobenius-Genocchi polynomials governed by the monomiality principle and operational rules. The following generating expression characterises these polynomials:

$$\left(\frac{(1-u)\xi}{\lambda e^{\xi}-u}\right)\exp(h_1\xi + h_2\xi^2 + \cdots + h_m\xi^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathbf{K}_n^{[m]}(h_1, h_2, \cdots, h_m; \lambda; u)\frac{\xi^n}{n!}, \tag{1.10}$$

accompanied by an operational rule:

$$\exp\left(h_2\frac{\partial^2}{\partial {h_1}^2} + h_3\ \frac{\partial^3}{\partial {h_1}^3} + \cdots + h_m\frac{\partial^m}{\partial {h_1}^m}\right)\left\{\mathbf{K}_n^{[m]}(h_1; \lambda; u)\right\} = {}_{\mathcal{H}}\mathbf{K}_n^{[m]}(h_1, h_2, \cdots, h_m; \lambda; u). \tag{1.11}$$

The remainder of the article unfolds as follows:

The extended version of multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials is unveiled and scrutinized by applying the monomiality principle and operational methodologies. Section 2 introduces these extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials by leveraging generating functions and operational definitions involving fractional operators. Moving on to Section 3, we delve into the quasi-monomial attributes inherent to the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials. Additionally, this section lays out the recurrence relations and summation formulas for these extended polynomials. Section 4 offers practical applications by examining specific cases, and finally, the paper concludes in the concluding section.

## 2. The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials

The operational rule and generating function for the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials are the main topics of this section. Fractional operators are used to introduce and study these polynomials. First, we derive the operational rule for these polynomials, as the operational rule offers a method for performing algebraic operations on the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials. The first operational connection is demonstrated by the succeeding result:

**Theorem 2.1.** *The following operational connection holds for EMVHAFGP* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$*:*

$$\left(\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu}\mathbf{K}_n(h_1; \lambda; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta). \tag{2.1}$$

*Proof.* By substituting $q$ with $\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)$ in Eq (1.8) of Genocchi's integral and subsequently applying this modified equation to $\mathbf{K}_n(h_1;\lambda;u)$, we obtain the following result:

$$\begin{aligned}&\left(\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu}\mathbf{K}_n(h_1;\lambda;u)\\ =&\frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}\exp\left(h_2\xi\frac{\partial^2}{\partial h_1^2} + h_3\xi\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\xi\frac{\partial^m}{\partial h_1^m}\right)\mathbf{K}_n(h_1;\lambda;u)d\xi,\end{aligned} \tag{2.2}$$

as evident from Eq (1.11), the following result is achieved:

$$\begin{aligned}&\left(\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu}\mathbf{K}_n(h_1;\lambda;u)\\ =&\frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}{}_{\mathcal{H}}\mathbf{K}_n(h_1, h_2\xi, h_3\xi, \cdots, h_m\xi;\lambda;u)d\xi.\end{aligned} \tag{2.3}$$

A new class of polynomials is presented by means of the transformation explained on the right side of Eq (2.3). The notion ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m;\lambda;u;\beta)$ is the representation of these polynomials, which are known as the EMHAFGP. As a result, we create the following relationship:

$${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m;\lambda;u;\beta) = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}{}_{\mathcal{H}}\mathbf{K}_n(h_1, h_2\xi, h_3\xi, \cdots, h_m\xi;\lambda;u)d\xi. \tag{2.4}$$

Hence, by taking into account expressions (2.3) and (2.4), we confirm the validity of statement (2.1). □

**Theorem 2.2.** *For the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials, denoted as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m;\lambda;u;\beta)$*, the provided generating expression is valid and can be expressed as follows:*

$$\frac{(1-u)w\ \exp(h_1 w)}{(\lambda e^w - u)\left(\beta - (h_2w^2 + h_3w^3 + \cdots + h_mw^m)\right)^\mu} = \sum_{n=0}^\infty {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m;\lambda;u;\beta)\frac{w^n}{n!}. \tag{2.5}$$

*Proof.* By multiplying Eq (2.4) by $\frac{w^n}{n!}$ and then summing over all possible values of $n$, we can deduce the following:

$$\sum_{n=0}^\infty {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m;\lambda;u;\beta)\frac{w^n}{n!} = \sum_{n=0}^\infty\frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}{}_{\mathcal{H}}\mathbf{K}_n(h_1, h_2\xi, h_3\xi, \cdots, h_m\xi;\lambda;u)\frac{w^n}{n!}d\xi.$$

Therefore, considering the expression (1.10) on the right-hand side of the preceding equation, we can determine that:

$$\sum_{n=0}^\infty {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m;\lambda;u;\beta)\frac{w^n}{n!} = \frac{(1-u)w\ \exp(h_1w)}{(\lambda e^w - u)\,\Gamma(\mu)}\int_0^\infty e^{-\left(\beta-(h_2w^2+h_3w^3+\cdots+h_mw^m)\right)\xi}\,\xi^{\mu-1}d\xi. \tag{2.6}$$

By examining the integral expression (1.7), we can derive statement (2.5). □

## 3. Explicit forms and identities

The significance of explicit forms lies in their ability to provide a clear and direct representation of mathematical expressions or objects. In various mathematical and scientific contexts, having explicit forms is crucial for several reasons. Explicit forms of mathematical expressions make them easier to understand and grasp. Understanding is facilitated by their frequent clarification of the connections and underlying structure of the equations or polynomials. These formats facilitate computations and numerical assessments. Through their efficient implementation in computer programs and numerical simulations, they reduce computational complexity. More explicit formats often lead to informative analysis. Mathematicians and researchers can examine these forms to find properties, relationships, and behaviours of mathematical objects that might not be immediately apparent in their general or abstract forms. They simplify the process of comparing different mathematical objects with each other. Scholars can use explicit statement comparisons to identify patterns and do investigations into various mathematical objects. For real-world applications in science, engineering, and other disciplines, these clear forms are crucial. To tackle real-world issues, engineers, physicists, and practitioners frequently need precise mathematical models with obvious forms that are computationally efficient. Explicit forms are therefore essential for theoretical study as well as real-world applications since they improve the usefulness, accessibility, and interpretability of mathematical statements.

Continuing in further depth, we will now provide the subsequent findings in order to provide the comprehensive formula for the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials:

**Theorem 3.1.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (EMVHATFGP) can be expressed in the following explicit form:*

$$ {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{n} \binom{n}{s} \mathbf{K}_s(h_1; \lambda; u) \ {}_{\mu}\mathcal{H}_{n-s}(h_2, h_3, \cdots, h_m; \beta). \tag{3.1} $$

*Proof.* The generative expression (2.5) can be represented in the following manner:

$$ \frac{(1-u)w \ \exp(h_1 w)}{(\lambda e^w - u)\,(\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^{\mu}} = \frac{(1-u)we^{h_1 w}}{(\lambda e^w - u)} \frac{1}{(\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^{\mu}}. \tag{3.2} $$

This can be further represented as:

$$ \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{\infty} \mathbf{K}_s(h_1; \lambda; u) \frac{w^s}{s!} \sum_{n=0}^{\infty} {}_{\mu}\mathcal{H}_n(h_2, h_3, \cdots, h_m; \beta) \frac{w^n}{n!}. \tag{3.3} $$

By substituting $n$ with $n-s$ and applying the Cauchy product rule to the right-hand side of the preceding expression, we can derive statement (3.1). □

**Theorem 3.2.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (EMVHATFGP) adhere to the provided explicit expression:*

$$ {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{n} \binom{n}{s} \mathbf{K}_s(\lambda; u) \, {}_{\mu}\mathcal{H}_{n-s}(h_1, h_2, h_3, \cdots, h_m; \beta). \tag{3.4} $$

*Proof.* The generative expression (2.5) can be represented in the following manner:

$$\frac{(1-u)w\ \exp(h_1w)}{(\lambda e^w-u)\,(\beta-(h_2w^2+h_3w^3+\cdots+h_mw^m))^\mu}=\frac{(1-u)w}{(\lambda e^w-u)}\frac{e^{h_1w}}{(\beta-(h_2w^2+h_3w^3+\cdots+h_mw^m))^\mu}. \tag{3.5}$$

This further can be represented as

$$\sum_{n=0}^{\infty}{}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)=\sum_{s=0}^{\infty}\mathbf{K}_s(\lambda;u)\frac{w^s}{s!}\sum_{n=0}^{\infty}{}_{\mu}\mathcal{H}_n(h_1,h_2,h_3,\cdots,h_m;\beta)\frac{w^n}{n!}. \tag{3.6}$$

By substituting $n$ with $n-s$ and applying the Cauchy product rule to the right-hand side of the preceding expression, we can derive statement (3.4). □

Onward, we can derive the recurrence relations governing the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (EMVHATFGP), which are represented as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)$. Recursive equations that specify the terms of a sequence or multidimensional array enable us to define each subsequent term with respect to the ones that came before it.

Further, on differentiating the generating expression (2.5) with respect to $h_1$, $h_2$, $h_3,\cdots,h_m$, and $\beta$, the succeeding recurrence relations for the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (MVHATFGP) ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)$ are derived:

$$\begin{aligned}
\frac{\partial}{\partial h_1}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= n\ {}_{\mu\mathcal{H}}\mathbf{K}_{n-1}(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\
\frac{\partial}{\partial h_2}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= \mu\, n(n-1)\ {}_{\mu+1H}\mathbf{K}_{n-2}(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)),\\
\frac{\partial}{\partial h_3}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= \mu\, n(n-1)(n-2)\ {}_{\mu+1H}\mathbf{K}_{n-3}(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\
&\ \ \vdots\\
\frac{\partial}{\partial h_m}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= \mu n(n-1)\cdots(n-m+1)_{\mu+1H}\mathbf{K}_{n-m}(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\
\frac{\partial}{\partial \beta}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\mu\ {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta).
\end{aligned} \tag{3.7}$$

Upon examining the aforementioned relations, the following expressions are validated:

$$\begin{aligned}
\frac{\partial}{\partial h_2}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\tfrac{\partial^3}{\partial h_1^2\partial\beta}\ {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\
\frac{\partial}{\partial h_3}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\tfrac{\partial^4}{\partial h_1^3\partial\beta}\ {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\
&\ \ \vdots\\
\frac{\partial}{\partial h_m}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\tfrac{\partial^{m+1}}{\partial h_1^m\partial\beta}\ {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta).
\end{aligned} \tag{3.8}$$

The operational framework established in Theorem 2.1 can be extended to various identities associated with Frobenius-Genocchi polynomials, which have been extensively studied to derive the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (EMVHATFGP) denoted as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$. To accomplish this, we perform the subsequent operation using the operator $(\mathcal{O})$ denoted by $\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2} + h_3 \frac{\partial^3}{\partial h_1^3} + \cdots + h_m \frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu}$ on identities that involve Frobenius-Genocchi polynomials $\mathbf{K}_n(h_1; u)$ [24]:

$$u\mathbf{K}_n(h_1; u^{-1}) + \mathbf{K}_n(h_1; u) = (1+u)\sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u^{-1})\mathbf{K}_k(h_1; u), \tag{3.9}$$

$$\begin{aligned}&\frac{1}{n+1}\mathbf{K}_k(h_1, u) + \mathbf{K}_{n-k}(h_1, u)\\ &= \sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\left((-u)\mathbf{K}_{l-k}(u)\mathbf{K}_{n-l}(u) + 2u\mathbf{K}_{n-k}(u)\right)\mathbf{K}_k(h_1, u)\mathbf{K}_n(h_1, u),\end{aligned} \tag{3.10}$$

$$\mathbf{K}_n(h_1, u) = \sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u)\mathbf{K}_k(h_1, u), \qquad (n \in \mathbb{Z}_+). \tag{3.11}$$

The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (EMVHATFGP), denoted as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$, are derived by applying the operator $(\mathcal{O})$ to both sides of the preceding equations:

$$\begin{aligned}&u\,{}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u^{-1}; \beta) + {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\\ =&(1+u)\sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u^{-1})\,{}_{\mu\mathcal{H}}\mathbf{K}_k(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta),\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\frac{1}{n+1}\,{}_{\mu\mathcal{H}}\mathbf{K}_k(h_1, h_2, h_3, \cdots, h_m; u; \beta) + {}_{\mu\mathcal{H}}\mathbf{K}_{n-k}(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\\ &= \sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\left((-u)\mathbf{K}_{n-l}(u)\mathbf{K}_{l-k}(u)\right.\\ &\quad\left.+2u\mathbf{K}_{n-k}(u)\right)\,{}_{\mu\mathcal{H}}\mathbf{K}_k(h_1, h_2, h_3, \cdots, h_m; u; \beta)\,{}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta),\end{aligned} \tag{3.13}$$

$${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta) = \sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u)\,{}_{\mu\mathcal{H}}\mathbf{K}_k(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta), \qquad (n \in \mathbb{Z}_+). \tag{3.14}$$

## 4. Applications

In this section, we delve into some specific instances of the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials (EMVHATFGP), considering their implications based on the general concept. The inquiry will shed light on the peculiar properties and relations exhibited by these polynomials, focusing on their special cases. Rigorous derivation and analysis shall establish the concomitant consequences of these particular examples, thereby illuminating their place in the grander scheme of polynomial theory. The intent behind this systematic approach is to better understand and reveal finer details regarding EMVHATFGP, thus promoting mathematical progress in this field.

**Corollary 4.1.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials can be transformed into the extended multivariable Hermite-Frobenius-Genocchi polynomials by setting $\lambda = 1$. Consequently, we establish the following operational relationship by substituting $\lambda = 1$ into the left side of Eq (2.1) and indicating the resultant extended multivariable Hermite-Frobenius-Genocchi polynomials in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta)$:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2} + h_3 \frac{\partial^3}{\partial h_1^3} + \cdots + h_m \frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta). \tag{4.1}$$

**Corollary 4.2.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials can be reduced to the extended multivariable Hermite-Genocchi polynomials by setting $\lambda = 1$ and $u = -1$. Thus, by substituting $\lambda = 1,\ u = -1$ in the left-hand side of Eq (2.1), the following operational relationship is established with the resulting extended multivariable Hermite-Genocchi polynomials on the right side, represented as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta)$:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2} + h_3 \frac{\partial^3}{\partial h_1^3} + \cdots + h_m \frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta). \tag{4.2}$$

**Corollary 4.3.** *The EMVHATFGP can be reduced to the extended 2-VHAFGP by setting $m = 2$. Thus, on insertion of $m = 2$ in the left hand of expression (2.1), the succeeding operational relationship for the extended 2-VHAFGP represented as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$ is established:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta). \tag{4.3}$$

**Corollary 4.4.** *By putting $\lambda = 1$ and $m = 2$, the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials" can be reduced to the extended 2-variable Hermite-Frobenius-Genocchi polynomials. Thus, by inserting $\lambda = 1$ and $m = 2$ into the left side of Eq (2.1), we obtain the subsequent operational relationship. The resulting extended 2-variable Hermite-Frobenius-Genocchi polynomials are represented on the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta)$:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta). \tag{4.4}$$

**Corollary 4.5.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials can be simplified to the extended Hermite-Apostol-type Frobenius-Genocchi polynomials by setting $m = 2$, $h_1 = 2h_1$, and $h_2 = -1$. As a result, we create the operational relationship shown below by changing the values of $m = 2$, $h_1 = 2h_1$, and $h_2 = -1$ in Eq (2.1), its left side, and the resulting extended Hermite-Apostol-type Frobenius-Genocchi polynomials are represented in the right side, as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$:*

$$\left(\beta - \left(-\frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, \lambda; u; \beta). \tag{4.5}$$

**Corollary 4.6.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials can be simplified to the extended Hermite-Frobenius-Genocchi polynomials by setting $m = 2$, $\lambda = 1$,*

$h_1 = 2h_1$, *and* $h_2 = -1$. *Consequently, we establish the following operational relationship by substituting* $m = 2$, $\lambda = 1$, $h_1 = 2h_1$, *and* $h_2 = -1$ *into the left side of Eq* (2.1) *and representing the resulting extended Hermite- Frobenius-Genocchi polynomials on the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta)$*:*

$$\left(\beta - \left(-\frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta). \tag{4.6}$$

**Corollary 4.7.** *The extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials can be simplified to the extended multivariable Hermite-Frobenius-Genocchi polynomials by setting* $\lambda = 1$. *Consequently, we establish the following generating expression by substituting* $\lambda = 1$ *into the left-hand side of Eq* (2.5) *and representing the resulting extended multivariable Hermite-Frobenius-Genocchi polynomials on the right-hand side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta)$*:*

$$\frac{(1-u)w\ \exp(h_1 w)}{(e^w - u)\ (\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta)\frac{w^n}{n!}. \tag{4.7}$$

**Corollary 4.8.** *By setting* $\lambda = 1$ *and* $u = -1$, *the extended multivariable Hermite-Genocchi polynomials can be simplified from the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials. As a result, we create the generating expression below by changing* $\lambda = 1$ *and* $u = -1$ *in the left side of Eq* (2.5) *with the EMHGP that result represented in the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta)$*:*

$$\frac{(2)\ \exp(h_1 w)}{(e^w + 1)\ (\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta)\frac{w^n}{n!}. \tag{4.8}$$

**Corollary 4.9.** *Setting* $m = 2$ *will simplify the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials to the extended 2-VHAFGP. As a result, we create the generating expression that follows by using in Eq* (2.5)*'s left side the value* $m = 2$ *with the extended 2-VHAFGP that results in the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$*:*

$$\frac{(1-u)w\ \exp(h_1 w)}{(\lambda e^w - u)\ (\beta - (h_2 w^2))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)\frac{w^n}{n!}. \tag{4.9}$$

**Corollary 4.10.** *The EMHAFGP can be reduced to the extended 2-variable Hermite-Frobenius-Genocchi polynomials (E2HFGP) by setting* $\lambda = 1$, *and* $m = 2$. *Therefore, the following generating expression is generated by putting* $\lambda = 1$ *and* $m = 2$ *on the left side of Eq* (2.5)*. On the right side, the resulting E2HFGP* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta)$ *is shown as:*

$$\frac{(1-u)w\ \exp(h_1 w)}{(e^w - u)\ (\beta - (h_2 w^2))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta)\frac{w^n}{n!}. \tag{4.10}$$

**Corollary 4.11.** *By setting* $m = 2$, $h_1 = 2h_1$, *and* $h_2 = -1$, *the EMHAFGP can be reduced to the extended Hermite-Apostol-type Frobenius-Genocchi polynomials. As a result, we construct the generating expression that follows by replacing* $m = 2$, $h_1 = 2h_1$, *and* $h_2 = -1$ *in the left side of Eq* (2.5)*. The E2VHFGP that arise is represented by the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$*:*

$$\frac{(1-u)w\ \exp(2h_1 w)}{(e^w - u)\ (\beta - (-w^2))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta)\frac{w^n}{n!}. \tag{4.11}$$

## 5. Conclusions

Multivariable special polynomials are fundamental to mathematical analysis, which covers the study of functions, limits, continuity, and calculus in many variables. These polynomials give mathematicians a flexible framework for formulating and analyzing multivariable functions, allowing them to examine differentiability, integrability, and convergence. This study introduces and starts the investigation of the extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials using operational rules and the monomiality principle. The generalized polynomials are introduced in Section 2, and certain characteristics are proved for the new family of polynomials. In Section 3, the quasi-monomial properties of these polynomials are investigated with summing equations and recurrence relations being proved at the same time. The comprehension of extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials and their possible applications in science and mathematics is significantly improved as a result of this investigation.

The characteristics of extended multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials proved in this study are the starting point for further research. Hence, the investigation of a range of algebraic and analytical properties, including differential equations and orthogonality, is facilitated by them. It should be noted that these polynomials have applications in a wide range of physics fields, such as quantum mechanics, statistical physics, mathematical physics, engineering, and other branches of physics, facilitated by the development of the generating function and recurrence relations for these hybrid polynomials. Operational techniques become extremely useful tools for the introduction of new families of special functions and for obtaining characteristics associated with basic and generalized special functions. They are also useful for finding explicit solutions for families of partial differential equations, such as the Heat and D'Alembert types. When paired with the monomiality principle, this method enables the solution analysis of a broad class of physical problems involving many types of partial differential equations. Families of differential equations can be examined in relation to these polynomials through the use of the factorization technique. This approach can be implemented for research using integral equations. Also, fractional operators might be used in future studies concerning these polynomials in more complex forms.

## Author contributions

Mohra Zayed: Conceptualization, Data curation, Investigation, Resources, Validation, Funding Visualization, Writing original draft; Shahid Ahmad Wani: Conceptualization, Investigation, Methodology, Project administration, Software, Supervision, Writing original draft; Georgia Irina Oros: Conceptualization, Data curation, Formal analysis, Investigation, Project administration, Supervision, Validation, Writing an original draft, Writing review & editing; William Ramírez: Conceptualization, Investigation, Project administration, Supervision, Writing an original draft. All authors have read and agreed to the published version of the manuscript.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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**Conflict of interest**

The authors declare no competing interests.

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