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Research article

On elliptic valued b-metric spaces and some new fixed point results with an application

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Abstract: In this paper, we introduce the concept of elliptic-valued b-metric spaces, extending the notions of elliptic-valued metric spaces and complex-valued metric spaces. We present several fixed-point results that involve rational and product terms within this novel space framework. To support our main findings, we offer numerical examples. Additionally, we demonstrate an application of Urysohn integral equations.

Keywords: α-admissible mapping; rational product terms; elliptic-valued b-metric space; simulation function; Caristi-type fixed point; Urysohn integral equation
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1. Introduction

The study of fixed-point theory constitutes a crucial branch of pure mathematics because of its vast applications in engineering, computer science, economics, etc. Recently, many interesting fixed-point results have been established (see, for example, [1, 2]). In 2011, Azam et al. [3] introduced the notion of complex-valued metric spaces (CVMSs) for complex numbers (where $i^2 = -1$) and studied some fixed-point results. Many researchers have focused their attention on generalized metric space and CVMS and established different types of fixed-point results (see, for example, [4–10]). Later, in 2021, Öztürk et al. [11] introduced the notion of elliptic valued metric spaces (EVMSs) for the set of all elliptic numbers. Basically, complex-valued metric space is a particular type of cone metric space which was introduced in [12]. But, fixed-point results involving rational and product terms were not introduced in the setting of cone metric spaces since this space is based on Banach space, which is not a division ring. Due to this reason, it is important to study fixed-point results in the context of cone metric space (or EVMS) involving rational and product terms. On the other hand, the notion of b-metric spaces were presented by Bakhtin [13] and later explained by Czerwik [14] for our known structure. For this paper, our intention is to introduce the notion of elliptic-valued b-metric spaces (b-EVMSs) by combining the ideas of EVMS and b-metric space. Now, we give a brief background about integral equations. In the literature on integral equations, there are two types of famous integral equations that are available depending on the limits of the integration, i.e., Fredholm integral equations (here, the limits are constant) and Volterra integral equations (here, at least one of the limits is a variable). Based on the form of the unknown function, the above-mentioned types of integral equations are either linear or nonlinear. Both the Fredholm and Volterra integral equations are divided into three categories, first kind, second kind, and third kind. A particular type of nonlinear Fredholm integral equations of the second kind is given by

$$u(t) = \sigma(t) + \int_c^d \Theta(t, r, u(t), u(r))dr, \quad t \in [c, d],$$

where σ, Θ are given functions and $u(t)$ is an unknown function. The above integral equation has two special subclasses, i.e., Hammerstein integral equations and Urysohn integral equations. In the application section, we will discuss the solution of an Urysohn integral equation by using our new findings. Next, we move to the preliminary section, where we mention some relevant definitions and important results, which will be required for the proof of our main results.

2. Preliminaries with known results

Let $\mathbb{E}_p$ be the collection of all elliptic numbers given by

$$\mathbb{E}_p = \{\eta = \eta_1 + i\eta_2 : \eta_1, \eta_2 \in \mathbb{R}, i^2 = p < 0\},$$

where η_1 is the real part and η_2 is the imaginary part of the elliptic number $\eta_1 + i\eta_2$. For the definitions of the summation of two elliptic numbers, multiplication of an elliptic-valued number by a scalar, multiplication of two elliptic numbers, and conjugate and norm of an elliptic number, we refer the reader to [11]. From now, we write θ to denote the zero element of the elliptic number system. The inverse of an elliptic number $\eta = \eta_1 + i\eta_2 (\neq \theta)$ is given by $\eta^{-1} = \frac{\eta_1 - i\eta_2}{\eta_1^2 - p\eta_2^2}$. We now define a partial ordering " $\precsim$ " on $\mathbb{E}_p$ as follows:

$$\eta \precsim \xi \text{ iff } \mathrm{Re}(\eta) \leq \mathrm{Re}(\xi) \text{ and } \mathrm{Im}(\eta) \leq \mathrm{Im}(\xi).$$

Therefore, if $\eta \precsim \xi$, then the following relations hold:

$$i.\ \mathrm{Re}(\eta) < \mathrm{Re}(\xi) \text{ and } \mathrm{Im}(\eta) < \mathrm{Im}(\xi);$$

$$ii.\ \mathrm{Re}(\eta) < \mathrm{Re}(\xi) \text{ and } \mathrm{Im}(\eta) = \mathrm{Im}(\xi);$$

$$iii.\ \mathrm{Re}(\eta) = \mathrm{Re}(\xi) \text{ and } \mathrm{Im}(\eta) < \mathrm{Im}(\xi);$$

$$iv.\ \mathrm{Re}(\eta) = \mathrm{Re}(\xi) \text{ and } \mathrm{Im}(\eta) = \mathrm{Im}(\xi).$$

The partial ordering " $\precsim$ " defined on $\mathbb{E}_p$ satisfies the following properties:

$$P_1: \text{ If } \theta \precsim \eta \precnsim \xi, \text{ then } \| \eta \| < \| \xi \|;$$

$$P_2: \text{ If } \eta \precsim \xi \text{ and } \xi \precsim \zeta, \text{ then } \eta \precsim \zeta;$$

$$P_3: \ \eta \precsim \xi \Leftrightarrow \eta - \xi \precsim \theta;$$

$$P_4: \theta \precsim \eta \text{ and } \theta \precsim \xi \nRightarrow \theta \precsim \eta\xi;$$

$$P_5: \ \eta \precsim \xi \text{ with } \tau \in \mathbb{R}^+ \Rightarrow \tau\eta \precsim \tau\xi.$$

Next, we introduce the definition of a b-EVMS as follows.

Definition 2.1. *Let Ω be a non-empty set and $s \in [1, \infty)$. A function $\varrho : \Omega \times \Omega \to \mathbb{E}_p$ is called a b-EVMS on Ω if the following assertions hold:*

$$\begin{aligned}
&A_1.\ \theta \precsim \varrho(\gamma, \delta), \ \ \forall \gamma, \delta \in \Omega;\\
&A_2.\ \varrho(\gamma, \delta) = \theta \Leftrightarrow \gamma = \delta;\\
&A_3.\ \varrho(\gamma, \delta) = \varrho(\delta, \gamma), \ \forall \gamma, \delta \in \Omega;\\
&A_4.\ \varrho(\gamma, \delta) \precsim s(\varrho(\gamma, \kappa) + \varrho(\kappa, \delta)), \ \forall \gamma, \kappa, \delta \in \Omega.
\end{aligned}$$

Here, we call the pair (Ω, ϱ) a b-EVMS.

Example 2.1. *Let $\Omega = \mathbb{E}_p$. Define a mapping $\varrho : \mathbb{E}_p \times \mathbb{E}_p \to \mathbb{E}_p$ by*

$$\varrho(\eta_1, \eta_2) = |\xi_1 - \xi_2|^2 + i|\zeta_1 - \zeta_2|^2,$$

where $\eta_1 = \xi_1 + i\zeta_1$ and $\eta_2 = \xi_2 + i\zeta_2$. Then, $(\mathbb{E}_p, \varrho)$ is a b-EVMS.

Example 2.2. *Let $\Omega = \mathbb{E}_p^*$, where $\mathbb{E}_p^*$ denotes the collection of all elliptic numbers with the same argument ∇_p. Define a mapping $\varrho : \mathbb{E}_p^* \times \mathbb{E}_p^* \to \mathbb{E}_p$ by*

$$\varrho(\eta_1, \eta_2) = \| \eta_1 - \eta_2 \|^2 e^{i\nabla_p}, \ \nabla_p \in \Big[0, \frac{\pi(p-1)}{8p}\Big],$$

where ∇_p is the argument of η_1 and η_2 with $p < 0$ and $p \in \mathbb{R}$. Then, $(\mathbb{E}_p^*, \varrho)$ is a b-EVMS.

All of the topological structures for the b-EVMS (Ω, ϱ), like the ϱ-interior point, ϱ-limit point, ϱ-closed, ϱ-convergence, ϱ-Cauchy sequence and ϱ-complete are of similar types as those for an EVMS (see [11]). Due to the length of the paper, we are not providing the details here. Like Lemmas 3.1 and 3.2 of [11], one can establish the same type of results in the setting of a b-EVMS. Let (Ω, ϱ) be a b-EVMS. Then, (Ω, ϱ) is called ϱ-continuous if the corresponding elliptic-valued b-metric ϱ from $\Omega \times \Omega$ to $\mathbb{E}_p$ is continuous, i.e., if $\{u_n\}, \{v_n\}$ are two sequences in Ω with $u_n \to u^*$ and $v_n \to v^*$ as $n \to \infty$, then

$\varrho(u_n, v_n) \to \varrho(u^*, v^*)$ as $n \to \infty$. Clearly, if (Ω, ϱ) is ϱ-continuous, then every convergent sequence has a unique limit. Now, we write $\mathbb{L}$ and $\mathbb{L}^o$ to denote the following subsets of $\mathbb{E}_p$:

$$\mathbb{L} = \{\eta \in \mathbb{E}_p : \eta \gtrsim \theta\} = \{\eta = \xi + i\zeta \in \mathbb{E}_b : \xi \geq 0; \zeta \geq 0\},$$

and

$$\mathbb{L}^o = \{\eta \in \mathbb{E}_p : \eta \succ \theta\} = \{\eta = \xi + i\zeta \in \mathbb{E}_b : \xi > 0; \zeta > 0\}.$$

Definition 2.2. *Let $f : \mathbb{L}^o \to \mathbb{L}^o$ be a function. Then,*

(i) f is monotonically increasing if for any $\gamma, \delta \in \mathbb{L}^o$ with $\gamma \lesssim \delta \Rightarrow f(\gamma) \lesssim f(\delta)$.
(ii) f is said to be ϱ-continuous at $\gamma_0 \in \mathbb{L}$ if for any sequence $\{\gamma_n\}_{n=1}^{\infty} \in \mathbb{L}$ with $\gamma_n \to \gamma_0 \Rightarrow f(\gamma_n) \to f(\gamma_0)$.

Öztürk et al. [11] defined the notion of a C-class function in the setting of an EVMS, which is also valid in a b-EVMS. Motivated by [15, 16], next, we introduce the definition of a revised $C_{\mathcal{F}}$-simulation function in the context of a b-EVMS.

Definition 2.3. *A mapping $\mathcal{F} : \mathbb{L}^o \times \mathbb{L}^o \to \mathbb{E}_p$ has the property $C_{\mathcal{F}}$ if $\exists$ a $C_{\mathcal{F}} \gtrsim \theta$ such that*

C_1. $\mathcal{F}(\eta_1, \eta_2) \succ C_{\mathcal{F}} \Rightarrow \eta_1 \succ \eta_2$, or $\| \mathcal{F}(\eta_1, \eta_2) \| > \| C_{\mathcal{F}} \| \Rightarrow \| \eta_1 \| > \| \eta_2 \|$.
C_2. $\mathcal{F}(\eta_1, \eta_2) \lesssim C_{\mathcal{F}}$ or $\| \mathcal{F}(\eta_1, \eta_2) \| \leq \| C_{\mathcal{F}} \|$, $\forall \eta_1, \eta_2 \in \mathbb{L}$.

Definition 2.4. *A revised $C_{\mathcal{F}}$-simulation function is a function $\lambda : \mathbb{L} \times \mathbb{L} \to \mathbb{E}_p$ that satisfies the following assertions:*

λ_1. $\lambda(\gamma, \delta) \prec \mathcal{F}(\delta, \gamma)$ or $\| \lambda(\gamma, \delta) \| < \| \mathcal{F}(\delta, \gamma) \|$, $\forall \gamma, \delta \succ \theta$, where $\mathcal{F}$ is a C-class function with the property $C_{\mathcal{F}}$;
λ_2. let $\{\gamma_n\}, \{\delta_n\}$ be two sequences in $\mathbb{L}^o$ such that either statement is true:
λ_{2a}. $\theta \prec \lim_{n\to\infty} \gamma_n \lesssim \liminf_{n\to\infty} \delta_n \lesssim \limsup_{n\to\infty} \delta_n \lesssim s \lim_{n\to\infty} \gamma_n \prec \infty$ implies $\limsup_{n\to\infty} \lambda(s\gamma_n, \delta_n) \prec C_{\mathcal{F}}$,
or
λ_{2b}. $0 < \lim_{n\to\infty} \| \gamma_n \| \leq \liminf_{n\to\infty} \| \delta_n \| \leq \limsup_{n\to\infty} \| \delta_n \| \leq s \lim_{n\to\infty} \| \gamma_n \| < \infty$ implies $\limsup_{n\to\infty} \| \lambda(s\gamma_n, \delta_n) \| < \| C_{\mathcal{F}} \|$.

Now, we shall give examples.

Example 2.3. *Let $\lambda : \mathbb{L} \times \mathbb{L} \to \mathbb{E}_p$ be a function given by $\lambda(\eta_1, \eta_2) = \frac{a}{b}\eta_2 - \eta_1$, where $\eta_1, \eta_2 \in \mathbb{L}$ with $a, b \in \mathbb{R}^+, b > a$, and $\mathcal{F}(\eta_1, \eta_2) = \eta_1 - \eta_2$ with $C_{\mathcal{F}} = (s + 1)(\rho_1 + i\rho_2)$, where $\rho_1 + i\rho_2 \in \mathbb{E}_p$. Clearly, λ_1 holds. Let us take two sequences $\{\gamma_n\}, \{\delta_n\}$ from Int$\mathbb{L}$ such that*

$$\theta \prec \lim_{n\to\infty} \gamma_n \precsim \liminf_{n\to\infty} \delta_n \precsim \limsup_{n\to\infty} \delta_n \precsim s \lim_{n\to\infty} \gamma_n \prec \rho_1 + i\rho_2,$$

or

$$0 < \lim_{n\to\infty} \| \gamma_n \| \leq \liminf_{n\to\infty} \| \delta_n \| \leq \limsup_{n\to\infty} \| \delta_n \| \leq s \lim_{n\to\infty} \| \gamma_n \| < \rho,$$

where $\rho = \| \rho_1 + i\rho_2 \|$. Then,

$$\begin{aligned}
&\limsup_{n\to\infty} \lambda(s\gamma_n, \delta_n) \\
&= \limsup_{n\to\infty} [\frac{a}{b}\delta_n - s\gamma_n] \\
&= \limsup_{n\to\infty} \frac{a}{b}\delta_n - \liminf_{n\to\infty} s\gamma_n \\
&= \frac{a}{b} \limsup_{n\to\infty} \delta_n - \liminf_{n\to\infty} s\gamma_n \\
&\precsim \limsup_{n\to\infty} \delta_n - \liminf_{n\to\infty} s\gamma_n \precsim \theta \precsim C_{\mathcal{F}}.
\end{aligned}$$

Furthermore, it can be easily checked that $\limsup_{n\to\infty} \| \lambda(s\gamma_n, \delta_n) \| < \| C_{\mathcal{F}} \|$.

Example 2.4. *Let $\lambda : \mathbb{L} \times \mathbb{L} \to \mathbb{E}_b$ be a function defined by*

$$\lambda(\eta_1, \eta_2) = \begin{cases} 1 - \frac{\eta_1}{2}, & \text{if } \eta_2 = \theta, \\ \frac{k\eta_2}{1+\eta_1}, & \text{if } \eta_2 \succsim \theta, \end{cases}$$

where k is a real number such that $k \in [0, 1)$, $\mathcal{F}(\eta_1, \eta_2) = \frac{\eta_1}{1+\eta_2}$, and $C_{\mathcal{F}} = 1 + i$. Clearly, λ_1 holds. To check λ_2, let us take two sequences $\{\gamma_n\}, \{\delta_n\}$ from Int$\mathbb{L}$ such that

$$\theta \prec \lim_{n\to\infty} \gamma_n \precsim \liminf_{n\to\infty} \delta_n \precsim \limsup_{n\to\infty} \delta_n \precsim s \lim_{n\to\infty} \gamma_n,$$

or

$$0 < \lim_{n\to\infty} \| \gamma_n \| \leq \liminf_{n\to\infty} \| \delta_n \| \leq \limsup_{n\to\infty} \| \delta_n \| \leq s \lim_{n\to\infty} \| \gamma_n \| .$$

Here, we suppose that $\lim_{n\to\infty} \gamma_n = \rho_1 + i\rho_2$ with $\rho = \| \rho_1 + i\rho_2 \|$. Then,

$$\begin{aligned}
&\limsup_{n\to\infty} \| \lambda(s\gamma_n, \delta_n) \| \\
&= \limsup_{n\to\infty} \| \frac{k\delta_n}{1 + s\gamma_n} \| \\
&\leq \limsup_{n\to\infty} \| \frac{\delta_n}{1 + s\gamma_n} \| \\
&\leq \limsup_{n\to\infty} [\| \delta_n \| \| (1 + s\gamma_n)^{-1} \|] \\
&\leq \limsup_{n\to\infty} \| \delta_n \| \limsup_{n\to\infty} \| (1 + s\gamma_n)^{-1} \| \\
&\leq s \lim_{n\to\infty} \| \gamma_n \| \limsup_{n\to\infty} \| (1 + s\gamma_n)^{-1} \| \\
&\leq \frac{s\rho}{s\rho} \leq \| 1 + i \| .
\end{aligned}$$

Moreover, it can be easily shown that $\limsup_{n\to\infty} \lambda(s\gamma_n, \delta_n) \prec C_{\mathcal{F}}$.

Example 2.5. *Every simulation function is a revised $C_{\mathcal{F}}$-simulation function with $\mathcal{F}(\eta_1, \eta_2) = \eta_1 - \eta_2$ and $C_{\mathcal{F}} = \theta$.*

Now, we shall state two important lemmas.

Lemma 2.1. *For every sequence $\{\gamma_n\}$ from a b-EVMS (Ω, ϱ), the following inequality holds:*

$$\varrho(\gamma_0, \gamma_k) \precsim s^n \sum_{j=0}^{k-1} \varrho(\gamma_j, \gamma_{j+1})$$

for each $n \in \mathbb{N}$ and each $k \in \{1, 2, 3, \cdots, 2^{n-1}, 2^n\}$.

Our next lemma is as follows.

Lemma 2.2. *Every sequence $\{\gamma_n\}$ from a b-EVMS (Ω, ϱ) with a constant s such that $\varrho(\gamma_n, \gamma_{n+1}) \precsim a\varrho(\gamma_{n-1}, \gamma_n), \forall n \in \mathbb{N}$ is a Cauchy sequence where $a \in [0, 1)$. Further, the following inequality holds:*

$$\varrho(\gamma_t, \gamma_{t+k}) \precsim \frac{a^n A}{1-a} \varrho(\gamma_0, \gamma_1),$$

where $A = \sum_{j=1}^{\infty} a^{2j \ln_a s + 2^{j-1}}$.

Remark 2.1. *The proof of the above two lemmas is similar to the proof given by Miculescu and Mihail [17] in the setting of a b-metric space.*

Remark 2.2. *If we have that $\| \varrho(\gamma_n, \gamma_{n+1}) \| \leq a \| \varrho(\gamma_{n-1}, \gamma_n) \|$ in place of $\varrho(\gamma_n, \gamma_{n+1}) \precsim a\varrho(\gamma_{n-1}, \gamma_n)$ in Lemma 2.2, then $\{\gamma_n\}$ is also a Cauchy sequence.*

Next, we shall state some important definitions of α-admissible mapping.

Definition 2.5. *([18]) Let $\mathcal{J} : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ be two given mappings. Then, $\mathcal{J}$ is said to be an α-orbital admissible mapping if the following holds:*

$$\alpha(u, \mathcal{J}u) \geq 1 \Rightarrow \alpha(\mathcal{J}u, \mathcal{J}^2u) \geq 1, \ \forall u \in \Omega.$$

Definition 2.6. *([18]) Let $\mathcal{J} : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ be two given mappings. Then, $\mathcal{J}$ is said to be a triangular α-orbital admissible mapping if $\mathcal{J}$ satisfies*

$$(i)\ \alpha(u, \mathcal{J}u) \geq 1 \Rightarrow \alpha(\mathcal{J}u, \mathcal{J}^2u) \geq 1,$$
$$(ii)\ \alpha(u, v) \geq 1, \ \text{and}\ \alpha(v, \mathcal{J}v) \geq 1 \Rightarrow \alpha(u, \mathcal{J}v) \geq 1, \ \forall u, v \in \Omega.$$

Definition 2.7. *([19]) A sequence $\{u_n\}$ is said to be α-regular if $\alpha(u_n, u_{n+1}) \geq 1$ and $u_n \to u^*(\in \Omega)$ as $n \to \infty$; then, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k}, u^*) \geq 1$ for every $k \in \mathbb{N}$.*

Note: In the proof of our main results, we will use Definition 2.7 with an additional condition, i.e., $u_{n_k} \neq \mathcal{J}u^*, \forall\ k \in \mathbb{N}$, where $\mathcal{J}$ is a mapping from Ω to Ω, and we still say that it is a α-regularity condition.

Next, we move to the main section of our paper.

3. Main results

In this section, first, we introduce the following definition.

Definition 3.1. *Let* (Ω, ϱ) *be a b-EVMS with* $\mathcal{J} : \Omega \to \Omega$ *and* $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ *be two mappings. Suppose that, for all* $u, v \in \Omega$ *with* $\alpha(u, v) \geq 1$, *one has*

$$\lambda(s\varrho(\mathcal{J}u, \mathcal{J}v), \Delta_{\mathcal{J}}(u, v)) \gtrsim C_{\mathcal{F}}, \tag{3.1}$$

where $\Delta_{\mathcal{J}}(u, v) = \max\left\{\varrho(u, v), \varrho(u, \mathcal{J}u), \frac{[1+\varrho(u,\mathcal{J}u)]\varrho(v,\mathcal{J}v)}{1+\varrho(u,v)}, \frac{\varrho(v,\mathcal{J}u)\varrho(v,\mathcal{J}v)}{1+\varrho(u,v)}, \frac{\varrho(v,\mathcal{J}u)\varrho(u,\mathcal{J}v)}{1+\varrho(u,v)}\right\}$ and each term inside "max" is comparable with respect to the partial order " $\lesssim$ ". Then, $\mathcal{J}$ is said to satisfy the condition of a generalized α-orbital admissible revised $C_{\mathcal{F}}$-simulation contraction associated with rational terms.

Theorem 1. *Let* (Ω, ϱ) *be a* ϱ*-complete and* ϱ*-continuous b-EVMS with a constant* $s \geq 1$. *Let* $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ *and* $\mathcal{J} : \Omega \to \Omega$ *be two mappings such that the following assertions hold:*

(i) $\mathcal{J}$ is a triangular α-admissible mapping;
(ii) $\mathcal{J}$ satisfies the conditions of an α-orbital admissible revised $C_{\mathcal{F}}$ simulation contraction;
(iii) there exists a point $u_0 \in \Omega$ such that $\alpha(u_0, \mathcal{J}u_0) \geq 1$;
either
(iv_a) $\mathcal{J}$ is ϱ-continuous;
or
(iv_b) if $\{u_n\}$ is a sequence in Ω, then it satisfies the α-regularity condition.

Then, $\mathcal{J}$ has a fixed point in Ω.

Proof. From our assumption (iii), there exists a point $u_0 \in \Omega$ such that $\alpha(u_0, \mathcal{J}u_0) \geq 1$. Clearly, starting from this initial point, one can construct a sequence $\{u_n\}$ by $u_{n+1} = \mathcal{J}u_n$, $\forall n \in \mathbb{N}$. For the remainder of the proof, we will assume that $u_{n+1} \neq u_n$, $\forall n \in \mathbb{N}$, i.e., $\varrho(u_n, u_{n+1}) \succ \theta$. Otherwise, we can find a point, say, u_{n_0}, for which we have that $u_{n_0+1} = u_{n_0} \Rightarrow u_{n_0} = \mathcal{J}u_{n_0}$. Clearly, we obtain a fixed point of $\mathcal{J}$ and the proof becomes less interesting. Now, since $\mathcal{J}$ is a triangular α-admissible mapping, one can easily get that $\alpha(u_n, u_{n+1}) \geq 1$, and, furthermore, $\alpha(u_n, u_m) \geq 1$, $\forall n, m \in \mathbb{N}$ with $m > n$. Now, we shall divide the proof into two cases.
Case-I: Here, we consider that $s = 1$. Since we have assumed that $\varrho(u_n, u_{n+1}) \succ \theta$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned}
&\Delta_{\mathcal{J}}(u_{n-1}, u_n)\\
&= \max\left\{\varrho(u_{n-1}, u_n), \varrho(u_{n-1}, u_n), \frac{[1 + \varrho(u_{n-1}, u_n)]\varrho(u_n, u_{n+1})}{1 + \varrho(u_{n-1}, u_n)}, \frac{\varrho(u_n, u_n)\varrho(u_n, u_{n+1})}{1 + \varrho(u_{n-1}, u_n)}, \frac{\varrho(u_n, u_n)\varrho(u_{n-1}, u_{n+1})}{1 + \varrho(u_{n-1}, u_n)}\right\}\\
&= \max\{\varrho(u_{n-1}, u_n), \varrho(u_n, u_{n+1})\}.
\end{aligned}$$

Now, it can be easily checked that $\Delta_{\mathcal{J}}(u_{n-1}, u_n) \succ \theta$, $\forall n \in \mathbb{N}$. Since $\mathcal{J}$ is an α-orbital admissible revised $C_{\mathcal{F}}$ simulation contraction, i.e., we have

$$\lambda(\varrho(u_n, u_{n+1}), \Delta_{\mathcal{J}}(u_{n-1}, u_n)) \gtrsim C_{\mathcal{F}}.$$

Since $\varrho(u_n, u_{n+1}) \succ \theta$ and $\Delta_{\mathcal{J}}(u_{n-1}, u_n)) \succ \theta$, $\forall n \in \mathbb{N}$, i.e., by using property λ_1, we have

$$\begin{aligned} &C_{\mathcal{F}} \precsim \lambda(\varrho(u_n, u_{n+1}), \Delta_{\mathcal{J}}(u_{n-1}, u_n)) \prec \mathcal{F}(\Delta_{\mathcal{J}}(u_{n-1}, u_n), \varrho(u_n, u_{n+1})) \\ &\Rightarrow C_{\mathcal{F}} \prec \mathcal{F}(\Delta_{\mathcal{J}}(u_{n-1}, u_n), \varrho(u_n, u_{n+1})). \end{aligned}$$

Now, by C_1, we get that $\varrho(u_n, u_{n+1}) \prec \Delta_{\mathcal{J}}(u_{n-1}, u_n)$. Clearly, we arrive at a contradiction if we consider $\max\{\varrho(u_{n-1}, u_n), \varrho(u_n, u_{n+1})\} = \varrho(u_n, u_{n+1})$. Thus, we have that $\varrho(u_n, u_{n+1}) \prec \varrho(u_{n-1}, u_n)$, $\forall n \in \mathbb{N}$ with

$$C_{\mathcal{F}} \precsim \lambda(\varrho(u_n, u_{n+1}), \varrho(u_{n-1}, u_n)). \tag{3.2}$$

Consequently, the sequence $\{\varrho(u_{n-1}, u_n)\}$ is monotonically decreasing and bounded below by θ. Hence, there exists $\eta \in \mathbb{L}$ such that $\lim_{n\to\infty} \varrho(u_{n-1}, u_n) = \eta \succsim \theta$. We consider $\eta \in \mathbb{L}^{\circ}$. Utilizing the property λ_{2a} with $\gamma_n = \varrho(u_n, u_{n+1})$ and $\delta_n = \varrho(u_{n-1}, u_n)$, we have

$$\limsup_{n\to\infty} \lambda(\varrho(u_n, u_{n+1}), \varrho(u_{n-1}, u_n)) \prec C_{\mathcal{F}},$$

which contradicts (3.2). Thus, our assumption, i.e., that $\eta \in \mathbb{L}^{\circ}$, is wrong. Hence, $\eta = \theta$, i.e.,

$$\lim_{n\to\infty} \varrho(u_n, u_{n+1}) = \theta. \tag{3.3}$$

Our next intention is to show that $\{u_n\}$ is bounded, i.e., $\{\| \varrho(u_m, u_n) \| \ : m, n \in \mathbb{N} \text{ with } m > n\}$ is bounded. We now show this by using the method of contradiction. Suppose that $\{u_n\}$ is not bounded. Then, there exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that $n_1 = 1$; also, for every $l \in \mathbb{N}$, n_{l+1} is the minimum positive integer such that

$$\begin{aligned} &\| \varrho(u_{n_{l+1}}, u_{n_l}) \| > 1, \text{ and} \\ &\| \varrho(u_q, u_{n_l}) \| \leq 1, \ \forall q \in \mathbb{N} \text{ with } n_l \leq q \leq n_{l+1} - 1. \end{aligned} \tag{3.4}$$

Now, applying the triangular inequality property of b-EVMS and (3.4), we obtain

$$\begin{aligned} 1 &\leq \| \varrho(u_{n_{l+1}}, u_{n_l}) \| \\ &\leq \| \varrho(u_{n_{l+1}}, u_{n_{l+1}-1}) + \varrho(u_{n_{l+1}-1}, u_{n_l}) \| \\ &\leq \| \varrho(u_{n_{l+1}}, u_{n_{l+1}-1}) \| + \| \varrho(u_{n_{l+1}-1}, u_{n_l}) \| \\ &\leq \| \varrho(u_{n_{l+1}}, u_{n_{l+1}-1}) \| + 1. \end{aligned}$$

Now, by applying (3.3), we obtain that $\lim_{l\to\infty} \| \varrho(u_{n_{l+1}}, u_{n_l}) \| = 1$. $\lim_{l\to\infty} \| \varrho(u_{n_{l+1}}, u_{n_l}) \| = 1$ implies that $\varrho(u_{n_{l+1}}, u_{n_l}) \to \eta_1$ as $l \to \infty$ with $\| \eta_1 \| = 1$. Now, since $\mathcal{J}$ is an α-orbital admissible revised $C_{\mathcal{F}}$-simulation contraction, considering that $u = u_{n_{l+1}-1}$ and $v = u_{n_l-1}$ in (3.1), we have

$$C_{\mathcal{F}} \precsim \lambda(\varrho(u_{n_{l+1}}, u_{n_l}), \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1})).$$

Observe that $\Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \succ \theta$ since each term of $\Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1})$ is comparable and there is a term $\varrho(u_{n_{l+1}-1}, u_{n_{l+1}}) \succ \theta$. Also, $\varrho(u_{n_{l+1}}, u_{n_l}) \succsim \theta$ with $\| \varrho(u_{n_{l+1}}, u_{n_l}) \| \geq 1$ (from (3.4)) implies that $\varrho(u_{n_{l+1}}, u_{n_l}) \succ \theta$. Thus, by property λ_1, we get

$$\begin{aligned} C_{\mathcal{F}} &\precsim \lambda(\varrho(u_{n_{l+1}}, u_{n_l}), \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1})) \\ &\prec \mathcal{F}(\Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}), \varrho(u_{n_{l+1}}, u_{n_l})) \end{aligned}$$

$$\Rightarrow \varrho(u_{n_{l+1}}, u_{n_l}) \prec \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}). \tag{3.5}$$

Again, we know that $\theta \precsim \eta_1 \precnsim \eta_2 \Rightarrow\parallel \eta_1 \parallel<\parallel \eta_2 \parallel$. Thus, from (3.5), we have

$$\parallel \varrho(u_{n_{l+1}}, u_{n_l}) \parallel<\parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel .$$

Now,

$$\begin{aligned}
&\parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel \\
&= \max\Big\{ \parallel \varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel, \parallel \varrho(u_{n_{l+1}-1}, u_{n_{l+1}}) \parallel, \parallel \frac{[1+\varrho(u_{n_{l+1}}, u_{n_{l+1}-1})]\varrho(u_{n_l}, u_{n_l-1})}{1+\varrho(u_{n_{l+1}-1}, u_{n_l-1})} \parallel, \\
&\parallel \frac{\varrho(u_{n_{l+1}}, u_{n_l-1})\varrho(u_{n_l}, u_{n_l-1})}{1+\varrho(u_{n_{l+1}-1}, u_{n_l-1})} \parallel, \parallel \frac{\varrho(u_{n_{l+1}}, u_{n_l-1})\varrho(u_{n_l}, u_{n_{l+1}-1})}{1+\varrho(u_{n_{l+1}-1}, u_{n_l-1})} \parallel \Big\} \\
&\leq \max\Big\{ \parallel \varrho(u_{n_{l+1}-1}, u_{n_l}) \parallel + \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel, \parallel \varrho(u_{n_{l+1}-1}, u_{n_{l+1}}) \parallel, \frac{[1+\parallel \varrho(u_{n_{l+1}}, u_{n_{l+1}-1}) \parallel] \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel}{\parallel 1+\varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel}, \\
&\frac{[\parallel \varrho(u_{n_{l+1}}, u_{n_l}) \parallel + \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel] \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel}{\parallel 1+\varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel}, \frac{[\parallel \varrho(u_{n_{l+1}}, u_{n_l}) \parallel + \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel] \parallel \varrho(u_{n_l}, u_{n_{l+1}-1}) \parallel}{\parallel 1+\varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel} \\
&\leq \max\Big\{ \parallel \varrho(u_{n_{l+1}-1}, u_{n_l}) \parallel + \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel, \parallel \varrho(u_{n_{l+1}-1}, u_{n_{l+1}}) \parallel, \frac{[1+\parallel \varrho(u_{n_{l+1}}, u_{n_{l+1}-1}) \parallel] \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel}{\parallel 1+\varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel}, \\
&\frac{[\parallel \varrho(u_{n_{l+1}}, u_{n_l}) \parallel + \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel] \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel}{\parallel 1+\varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel}, \\
&\frac{[\parallel \varrho(u_{n_{l+1}}, u_{n_{l+1}-1}) \parallel + \parallel \varrho(u_{n_{l+1}-1}, u_{n_l}) \parallel + \parallel \varrho(u_{n_l}, u_{n_l-1}) \parallel] \parallel \varrho(u_{n_l}, u_{n_{l+1}-1}) \parallel}{\parallel 1+\varrho(u_{n_{l+1}-1}, u_{n_l-1}) \parallel}\Big\}.
\end{aligned} \tag{3.6}$$

Now, we observe that $\parallel \varrho(u_{n_{l+1}-1}, u_{n_l}) \parallel\leq 1$ and $\parallel \varrho(u_n, u_{n+1}) \parallel\to 0$ as $n \to \infty$. Further, observe that $\varrho(\gamma, \delta) \succsim \theta \Rightarrow\parallel 1+\varrho(\gamma, \delta) \parallel\geq 1$. Next, considering the lim sup as $l \to \infty$ in (3.6), we obtain

$$\limsup_{l\to\infty} \parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel\leq 1. \tag{3.7}$$

Again, from (3.5), we have that $\parallel \varrho(u_{n_{l+1}}, u_{n_l}) \parallel<\parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel$ implies that

$$1 <\parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel . \tag{3.8}$$

Consequently, taking the lim inf as $n \to \infty$ in (3.8), we get

$$1 \leq \liminf_{l\to\infty} \parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel . \tag{3.9}$$

Hence, from (3.7) and (3.9), we have

$$\lim_{l\to\infty} \parallel \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \parallel= 1,$$

which is equivalent to $\Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1}) \to \eta_2$ as $l \to \infty$ with $\parallel \eta_2 \parallel= 1$. Now, take $\gamma_n = \varrho(u_{n_{l+1}}, u_{n_l})$ and $\delta_n = \Delta_{\mathcal{J}}\varrho(u_{n_{l+1}-1}, u_{n_l-1})$. Here if $\eta_1 = \eta_2$, then, by applying λ_{2a} and (3.5), we get

$$C_{\mathcal{F}} \precsim \limsup_{l\to\infty} \lambda(\varrho(u_{n_{l+1}}, u_{n_l}), \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1})) \prec C_{\mathcal{F}},$$

which is a contradiction. On the other hand, for $\parallel \eta_1 \parallel=\parallel \eta_2 \parallel$, we apply λ_{2b} and (3.5) to get

$$\parallel C_{\mathcal{F}} \parallel\leq \limsup_{l\to\infty} \parallel \lambda(\varrho(u_{n_{l+1}}, u_{n_l}), \Delta_{\mathcal{J}}(u_{n_{l+1}-1}, u_{n_l-1})) \parallel< C_{\mathcal{F}},$$

which is a contradiction. Thus, from any case, we can conclude that $\{u_n\}$ is bounded. Our next goal is to show that $\{u_n\}$ is a Cauchy sequence. Let us consider that $M_n = \sup\{\| \varrho(u_r, u_s) \|: r, s > n\},\ n \in \mathbb{N}$. Since we have already shown that $\{u_n\}$ is a bounded sequence, $M_n < \infty$, $\forall n \in \mathbb{N}$. Now, observe that $\{M_n\}$ is a positive decreasing sequence; consequently, there exists an $M \geq 0$ such that $\lim_{n\to\infty} M_n = M$. Let us assume that $M > 0$. Now, applying the definition of M_n, for each $l \in \mathbb{N}$, there exist $r_l, t_l \in \mathbb{N}$ such that $t_l \geq r_l \geq l$ and

$$M_l - \frac{1}{l} < \| \varrho(u_{r_l}, u_{t_l}) \| \leq M_l. \tag{3.10}$$

Ensuring that $l \to \infty$ in (3.10), we get

$$\lim_{l\to\infty} \| \varrho(u_{r_l}, u_{t_l}) \| = M, \tag{3.11}$$

and

$$\lim_{l\to\infty} \| \varrho(u_{r_l-1}, u_{t_l-1}) \| = M. \tag{3.12}$$

Now, we set $u = u_{r_l-1}$ and $v = u_{t_l-1}$ in (3.1). Consequently, we have

$$C_{\mathcal{F}} \precsim \lambda(\varrho(u_{r_l}, u_{t_l}), \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1})), \text{ where} \tag{3.13}$$

$$\Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) = \max\Big\{\varrho(u_{r_l-1}, u_{t_l-1}), \varrho(u_{r_l-1}, u_{r_l}), \frac{[1+\varrho(u_{r_l-1}, u_{r_l})]\varrho(u_{t_l-1}, u_{t_l})}{1+\varrho(u_{r_l-1}, u_{t_l-1})}, \frac{\varrho(u_{r_l}, u_{t_l-1})\varrho(u_{t_l}, u_{t_l-1})}{1+\varrho(u_{r_l-1}, u_{t_l-1})},$$
$$\frac{\varrho(u_{r_l}, u_{t_l-1})\varrho(u_{r_l-1}, u_{t_l})}{1+\varrho(u_{r_l-1}, u_{t_l-1})}\Big\}.$$

It can be easily checked that $\varrho(u_{r_l}, u_{t_l}) \succ \theta$ and $\Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) \succ \theta$. Therefore, we get

$$\lambda(\varrho(u_{r_l}, u_{t_l}), \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1})) \precsim \mathcal{F}(\Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}), \varrho(u_{r_l}, u_{t_l})),$$

or

$$\| \lambda(\varrho(u_{r_l}, u_{t_l}), \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1})) \| \leq \| \mathcal{F}(\Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}), \varrho(u_{r_l}, u_{t_l})) \|.$$

Hence, from any situation, we have

$$\| \varrho(u_{r_l}, u_{t_l}) \| < \| \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) \|.$$

Now,

$$\begin{aligned}\Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) &= \max\Big\{ \| \varrho(u_{r_l-1}, u_{t_l-1}) \|, \| \varrho(u_{r_l-1}, u_{r_l}) \|, \frac{\| 1+\varrho(u_{r_l-1}, u_{r_l}) \| \| \varrho(u_{t_l-1}, u_{t_l}) \|}{\| 1+\varrho(u_{r_l-1}, u_{t_l-1}) \|},\\ &\quad \frac{\| \varrho(u_{r_l}, u_{t_l-1}) \| \| \varrho(u_{t_l}, u_{t_l-1}) \|}{\| 1+\varrho(u_{r_l-1}, u_{t_l-1}) \|}, \frac{\| \varrho(u_{r_l}, u_{t_l-1}) \| \| \varrho(u_{r_l-1}, u_{t_l}) \|}{\| 1+\varrho(u_{r_l-1}, u_{t_l-1}) \|}\Big\}\\ &\leq \max\Big\{ \| \varrho(u_{r_l-1}, u_{t_l-1}) \|, \| \varrho(u_{r_l-1}, u_{r_l}) \|, \frac{1+\| \varrho(u_{r_l-1}, u_{r_l}) \| \| \varrho(u_{t_l-1}, u_{t_l}) \|}{\| 1+\varrho(u_{r_l-1}, u_{t_l-1}) \|},\\ &\quad \frac{(\| \varrho(u_{r_l}, u_{r_l-1}) \| + \| \varrho(u_{r_l-1}, u_{t_l-1}) \|) \| \varrho(u_{t_l-1}, u_{t_l}) \|}{\| 1+\varrho(u_{r_l-1}, u_{t_l-1}) \|},\\ &\quad \frac{(\| \varrho(u_{r_l}, u_{t_l}) \| + \| \varrho(u_{t_l}, u_{t_l-1}) \|)(\| \varrho(u_{r_l-1}, u_{t_l-1}) \| + \| \varrho(u_{t_l-1}, u_{t_l}) \|)}{\| 1+\varrho(u_{r_l-1}, u_{t_l-1}) \|}\Big\}.\end{aligned}$$

Keeping in mind that $\| \varrho(u_{r_l-1}, u_{t_l-1}) \| < \| 1 + \varrho(u_{r_l-1}, u_{t_l-1}) \|$, and using (3.3), (3.11), and (3.12), we have that $\lim_{l\to\infty} \| \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) \| = M$. Now, $\lim_{l\to\infty} \| \varrho(u_{r_l}, u_{t_l}) \| = M \Rightarrow \varrho(u_{r_l}, u_{t_l}) \to \eta_1$ as $l \to \infty$ with $\| \eta_1 \| = M$, and $\lim_{l\to\infty} \| \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) \| = M \Rightarrow \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1}) \to \eta_2$ with $\| \eta_2 \| = M$. If $\eta_1 = \eta_2$, then we apply λ_{2a} with $\gamma_l = \varrho(u_{r_l}, u_{t_l})$ and $\delta_l = \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1})$ to show

$$\limsup_{l\to\infty} \lambda(\varrho(u_{r_l}, u_{t_l}), \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1})) \prec C_{\mathcal{F}},$$

which contradicts (3.13). Otherwise, we have that $\| \eta_1 \| = \| \eta_2 \|$. In this case, we apply λ_{2b} with the same γ_l and, δ_l and, consequently, we have

$$\| C_{\mathcal{F}} \| \leq \limsup_{l\to\infty} \| \lambda(\varrho(u_{r_l}, u_{t_l}), \Delta_{\mathcal{J}}(u_{r_l-1}, u_{t_l-1})) \| < \| C_{\mathcal{F}} \|,$$

which is a contradiction. Thus, our assumption that $M > 0$ is not correct, i.e., $M = 0$. Hence, $\{u_n\}$ is a Cauchy sequence.

Case -II: In this case, we assume that $s > 1$. Here, (3.2) takes the following form:

$$C_{\mathcal{F}} \precsim \lambda(s\varrho(u_n, u_{n+1}), \varrho(u_{n-1}, u_n)).$$

From this, one can easily derive that $\varrho(u_n, u_{n+1}) \precsim \frac{1}{s}\varrho(u_{n-1}, u_n),\ \forall n \in \mathbb{N}$. Now, by applying Lemma 2.2, we conclude that $\{u_n\}$ is a Cauchy sequence. Since (Ω, ϱ) is ϱ-complete, there exists a $u^* \in \Omega$ such that $u_n \to u^*$ as $n \to \infty$. Suppose that $\mathcal{J}$ is ϱ-continuous. So, we have

$$\mathcal{J}u^* = \mathcal{J}(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} \mathcal{J}u_n = \lim_{n\to\infty} u_{n+1} = u^*.$$

Next, we suppose that $\varrho(u^*, \mathcal{J}u^*) \succ \theta$ and there exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that $\alpha(u_{n_l}, u^*) \geq 1$ and $u_{n_l} \neq u^*,\ \forall l \in \mathbb{N}$. Now, setting $u = u_{nl-1}, v = u^*$ in (3.1), we obtain

$$C_{\mathcal{F}} \precsim \lambda(s\varrho(u_{n_l}, \mathcal{J}u^*), \Delta_{\mathcal{J}}(u_{n_l-1}, u^*)).$$

One can easily check that $\varrho(u_{n_l}, \mathcal{J}u^*) \succ \theta$ and $\Delta_{\mathcal{J}}(u_{n_l-1}, u^*) \succ \theta$. Hence, we have

$$\lambda(s\varrho(u_{n_l}, \mathcal{J}u^*), \Delta_{\mathcal{J}}(u_{n_l-1}, u^*)) \precsim \mathcal{F}(\Delta_{\mathcal{J}}(u_{n_l-1}, u^*), s\varrho(u_{n_l}, \mathcal{J}u^*)),\ \text{or}$$

$$\| \lambda(s\varrho(u_{n_l}, \mathcal{J}u^*), \Delta_{\mathcal{J}}(u_{n_l-1}, u^*)) \| \leq \| \mathcal{F}(\Delta_{\mathcal{J}}(u_{n_l-1}, u^*), s\varrho(u_{n_l}, \mathcal{J}u^*)) \| .$$

Consequently, from any situation, we get

$$\begin{aligned} \| s\varrho(u_{n_l}, \mathcal{J}u^*) \| &< \| \Delta_{\mathcal{J}}(u_{n_l-1}, u^*) \| \\ \Rightarrow \| \varrho(u_{n_l}, \mathcal{J}u^*) \| &< \frac{1}{s} \| \Delta_{\mathcal{J}}(u_{n_l-1}, u^*) \| . \end{aligned} \tag{3.14}$$

Now,

$$\begin{aligned} &\varrho(u^*, \mathcal{J}u^*) \precsim s[\varrho(u^*, u_{n_l}) + \varrho(u_{n_l}, \mathcal{J}u^*)] \\ &\Rightarrow \| \varrho(u^*, \mathcal{J}u^*) \| \leq s\Big[\| \varrho(u^*, u_{n_l}) \| + \| \varrho(u_{n_l}, \mathcal{J}u^*) \| \Big]. \end{aligned} \tag{3.15}$$

Taking the limit as $l \to \infty$ in (3.15), we get

$$\frac{\| \varrho(u^*, \mathcal{J}u^*) \|}{s} \leq \lim_{l\to\infty} \| \varrho(u_{n_l}, \mathcal{J}u^*) \| . \tag{3.16}$$

Again,

$$\begin{aligned} &\| \Delta_{\mathcal{J}}(u_{n_l-1}, u^*) \| \\ &= \max\Big\{ \| \varrho(u_{n_l-1}, u^*) \|, \| \varrho(u_{n_l-1}, u_{n_l}) \|, \frac{\| 1+\varrho(u_{n_l-1}, u_{n_l}) \| \| \varrho(u^*, \mathcal{J}u^*) \|}{\| 1+\varrho(u_{n_l-1}, u^*) \|}, \\ &\frac{\| \varrho(u_{n_l}, u^*) \| \| \varrho(u^*, \mathcal{J}u^*) \|}{\| 1+\varrho(u_{n_l-1}, u^*) \|}, \frac{\| \varrho(u_{n_l}, u^*) \| \| \varrho(\mathcal{J}u^*, u_{n_l-1}) \|}{\| 1+\varrho(u_{n_l-1}, u^*) \|}\Big\}. \end{aligned} \tag{3.17}$$

Taking the limit as $l \to \infty$ in (3.17), we have

$$\lim_{l\to\infty} \| \Delta_{\mathcal{J}}(u_{n_l-1}, u^*) \| \leq \| \varrho(u^*, \mathcal{J}u^*) \| . \tag{3.18}$$

Combining (3.14), (3.16), and (3.18), we obtain

$$\lim_{l\to\infty} \| s\varrho(u_{n_l}, \mathcal{J}u^*) \| = \lim_{l\to\infty} \| \Delta_{\mathcal{J}}(u_{n_l-1}, u^*) \| = \| \varrho(u^*, \mathcal{J}u^*) \| .$$

By using (λ_{2a}), or (λ_{2b}) with $\gamma_l = s\varrho(u_{n_l}, \mathcal{J}u^*)$, and $\delta_l = \Delta_{\mathcal{J}}(u_{n_l-1}, u^*)$ as before, one can easily show that

$$\begin{aligned} &C_{\mathcal{F}} \precsim \limsup_{l\to\infty} \lambda(s\varrho(u_{n_l}, \mathcal{J}u^*), \Delta_{\mathcal{J}}(u_{n_l-1}, u^*)) \prec C_{\mathcal{F}} \\ &\text{or } \| C_{\mathcal{F}} \| \leq \| \limsup_{l\to\infty} \lambda(s\varrho(u_{n_l}, \mathcal{J}u^*), \Delta_{\mathcal{J}}(u_{n_l-1}, u^*)) \| < \| C_{\mathcal{F}} \|, \end{aligned}$$

which is a contradiction. Thus, we have that $\varrho(u^*, \mathcal{J}u^*) = \theta \Rightarrow u^* = \mathcal{J}u^*$. □

Theorem 2. *In addition to the hypotheses of Theorem 1, we further assume that* $\alpha(u_1^*, u_2^*) \geq 1$ *for all* $u_i^* \in$ *Fix*$(\mathcal{J})$, *where* $i = 1, 2$. *Then,* $\mathcal{J}$ *has a unique fixed point.*

Proof. Since $\alpha(u_1^*, u_2^*) \geq 1$, we have

$$\lambda(s\varrho(\mathcal{J}u_1^*, \mathcal{J}u_2^*), \Delta_{\mathcal{J}}(u_1^*, u_2^*)) \succsim C_{\mathcal{F}}.$$

Since $u_1^* \neq u_2^*$, $\varrho(u_1^*, u_2^*) \succ \theta$ and, hence, $\varrho(\mathcal{J}u_1^*, \mathcal{J}u_2^*)$ and $\Delta_{\mathcal{J}}(u_1^*, u_2^*) \succ \theta$. By using λ_1 and C_1, one can easily show that

$$\| s\varrho(u_1^*, u_2^*)) \| < \| \Delta_{\mathcal{J}}(u_1^*, u_2^*)) \| . \tag{3.19}$$

Now,

$$\begin{aligned} &\| \Delta_{\mathcal{J}}(u_1^*, u_2^*) \| \\ &= \max\Big\{ \| \varrho(u_1^*, u_2^*) \|, \| \varrho(u_1^*, u_1^*) \|, \frac{\| 1+\varrho(u_1^*, \mathcal{J}u_1^*) \| \| \varrho(u_2^*, \mathcal{J}u_2^*) \|}{\| 1+\varrho(u_1^*, u_2^*) \|}, \\ &\frac{\| \varrho(\mathcal{J}u_1^*, u_2^*) \| \| \varrho(u_2^*, u_2^*) \|}{\| 1+\varrho(u_1^*, u_2^*) \|}, \frac{\| \varrho(\mathcal{J}u_1^*, u_2^*) \| \| \varrho(\mathcal{J}u_2^*, u_1^*) \|}{\| 1+\varrho(u_1^*, u_2^*) \|}\Big\} \\ &= \| \varrho(u_1^*, u_2^*) \| . \end{aligned} \tag{3.20}$$

Clearly, from (3.20), we arrive at a contradiction since we have assumed that $\varrho(u_1^*, u_2^*) \succ \theta$, i.e., $\| \varrho(u_1^*, u_2^*) \| > 0$. Thus, we obtain that $u_1^* = u_2^*$. □

Next, we shall state and prove a result as a corollary of our main result.

Corollary 3.1. *Let* (Ω, ϱ) *be a* ϱ*-complete and* ϱ*-continuous b-EVMS with the constant* $s \geq 1$. *Let* $\alpha : \Omega \times \Omega \to \mathbb{R}^{+}$ *and* $\mathcal{J} : \Omega \to \Omega$ *be two mappings. Suppose that there exists an* $m \in \mathbb{N}$ *for which the following assertions hold:*

(i) $\mathcal{J}^m$ is a triangular α-admissible mapping;
(ii) $\mathcal{J}^m$ satisfies the conditions of an α-orbital admissible revised $C_{\mathcal{F}}$ simulation contraction;
(iii) there exists a point $u_0 \in \Omega$ such that $\alpha(u_0, \mathcal{J}^m u_0) \geq 1$;
either
(iv_a) $\mathcal{J}^m$ is ϱ-continuous;
or
(iv_b) if $\{u_n\}$ is a sequence in Ω, then it satisfies the α-regularity condition.

Then, $\mathcal{J}^m$ has a fixed point (say, ϑ) in Ω. Furthermore, ϑ is also a fixed point of $\mathcal{J}$, provided that $\alpha(\vartheta, \mathcal{J}\vartheta) \geq 1$.

Proof. Clearly, by using Theorem 1, we can obtain a fixed point of $\mathcal{J}^m$. Let $\vartheta \in \Omega$ be the fixed point of $\mathcal{J}^m$ with $\alpha(\vartheta, \mathcal{J}\vartheta) \geq 1$. We assume that $\mathcal{J}\vartheta \neq \vartheta$. Since ϑ is a fixed point of $\mathcal{J}^m$, $\mathcal{J}^m\vartheta = \vartheta$. Now, for $\alpha(\vartheta, \mathcal{J}\vartheta) \geq 1$, we have

$$\lambda(s\varrho(\mathcal{J}^m\vartheta, \mathcal{J}^m\mathcal{J}\vartheta), \Delta_{\mathcal{J}^m}(\vartheta, \mathcal{J}\vartheta)) \gtrsim C_{\mathcal{F}},$$

where

$$\begin{aligned}
&\Delta_{\mathcal{J}^m}(\vartheta, \mathcal{J}\vartheta)\\
&= \max\Big\{\varrho(\vartheta, \mathcal{J}\vartheta), \varrho(\vartheta, \mathcal{J}^m\vartheta), \frac{[1+\varrho(\vartheta, \mathcal{J}^m\vartheta)]\varrho(\mathcal{J}\vartheta, \mathcal{J}^m\mathcal{J}\vartheta)}{1+\varrho(\vartheta, \mathcal{J}\vartheta)},\\
&\frac{\varrho(\mathcal{J}^m\vartheta, \mathcal{J}\vartheta)\varrho(\mathcal{J}^m\mathcal{J}\vartheta, \mathcal{J}\vartheta)}{1+\varrho(\vartheta, \mathcal{J}\vartheta)}, \frac{\varrho(\mathcal{J}^m\vartheta, \mathcal{J}\vartheta)\varrho(\mathcal{J}^m\mathcal{J}\vartheta, \vartheta)}{1+\varrho(\vartheta, \mathcal{J}\vartheta)}\Big\}.
\end{aligned}$$

Observe that $\varrho(\mathcal{J}^m\vartheta, \mathcal{J}^m\mathcal{J}\vartheta) = \varrho(\vartheta, \mathcal{J}\vartheta) \succ \theta$. By using λ_1 and C_1, one can show that

$$\| s\varrho(\mathcal{J}^m\vartheta, \mathcal{J}^m\mathcal{J}\vartheta) \| \;<\; \| \Delta_{\mathcal{J}^m}(\vartheta, \mathcal{J}\vartheta) \| . \tag{3.21}$$

Clearly, $\| \Delta_{\mathcal{J}^m}(\vartheta, \mathcal{J}\vartheta) \| = \| \varrho(\vartheta, \mathcal{J}\vartheta) \|$. Thus, from (3.21), we have

$$s \| \varrho(\vartheta, \mathcal{J}\vartheta) \| = s \| \varrho(\mathcal{J}^m\vartheta, \mathcal{J}^m\mathcal{J}\vartheta) \| < \| \varrho(\vartheta, \mathcal{J}\vartheta) \|,$$

which is a contradiction since we have assumed that $\vartheta \neq \mathcal{J}\vartheta$. Therefore, we have that $\vartheta = \mathcal{J}\vartheta$. □

Example 3.1. *Let* $\Omega = \mathbb{E}_p$ *be the set of all elliptic-valued numbers with* $i^2 = -2$. *Define a mapping* $\varrho : \Omega \times \Omega \to \Omega$ *by*

$$\varrho(\eta_1,\eta_2)=\| \xi_1-\xi_2 \|^2 + i \| \zeta_1-\zeta_2 \|^2,$$

where $\eta_1 = \xi_1 + i\zeta_1$ and $\eta_2 = \xi_2 + i\zeta_2$. Let $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ by $\alpha(\eta_1, \eta_2) = 1$, $\forall\ \eta_1, \eta_2 \in \Omega$. Clearly, (Ω, ϱ) is a ϱ-continuous and ϱ-complete b-EVMS with $s = 2$. Let us define a mapping $\mathcal{J} : \Omega \to \Omega$ by

$$\mathcal{J}(\eta) = \mathcal{J}(\xi + i\zeta) = \begin{cases} 0, & \text{if } \xi, \zeta \in \Xi; \\ 3+3i, & \text{if } \xi, \zeta \in \widetilde{\Xi}; \\ 3, & \text{if } \xi \in \widetilde{\Xi},\ \zeta \in \Xi; \\ 3i, & \text{if } \xi \in \Xi,\ \zeta \in \widetilde{\Xi}. \end{cases}$$

where Ξ, $\widetilde{\Xi}$ denote the sets of all rational and irrational numbers, respectively. Take $u = \frac{1}{\sqrt{5}}$ and $v = 1$ in (3.1), and we calculate the following:

$$\begin{aligned}
\varrho(\mathcal{J}\frac{1}{\sqrt{5}}, \mathcal{J}1) &= \varrho(3,0) = 9, \\
\varrho(\frac{1}{\sqrt{5}}, 1) = (1-\frac{1}{\sqrt{5}})^2, \varrho(\frac{1}{\sqrt{5}}, 3) &= (3-\frac{1}{\sqrt{5}})^2 \approx 6.516, \\
\frac{(1+\varrho(\frac{1}{\sqrt{5}}, \mathcal{J}\frac{1}{\sqrt{5}}))\varrho(1,\mathcal{J}1)}{1+\varrho(\frac{1}{\sqrt{5}},1)} = \frac{(1+\varrho(\frac{1}{\sqrt{5}},3))\varrho(1,0)}{1+(1-\frac{1}{\sqrt{5}})^2} &= \frac{(1+(3-\frac{1}{\sqrt{5}})^2)}{1+(1-\frac{1}{\sqrt{5}})^2} \approx 4.17, \\
\frac{\varrho(\mathcal{J}\frac{1}{\sqrt{5}},1)\varrho(1,\mathcal{J}1)}{1+\varrho(\frac{1}{\sqrt{5}},1)} = \frac{\varrho(3,1)}{1+(1-\frac{1}{\sqrt{5}})^2} &= \frac{4}{1+(1-\frac{1}{\sqrt{5}})^2}, \\
\frac{\varrho(\mathcal{J}\frac{1}{\sqrt{5}},1)\varrho(\mathcal{J}1,\frac{1}{\sqrt{5}})}{1+\varrho(\frac{1}{\sqrt{5}},1)} = \frac{\varrho(3,1)\varrho(0,\frac{1}{\sqrt{5}})}{1+(1-\frac{1}{\sqrt{5}})^2} &= \frac{\frac{4}{5}}{1+(1-\frac{1}{\sqrt{5}})^2}.
\end{aligned}$$

Clearly, $\mathcal{J}^m(\eta) = 0$, $\forall\ m \in \mathbb{N} \setminus \{1\}$. Then, it can be easily checked that, for $u = \frac{1}{\sqrt{5}}$ and $v = 1$, the inequality

$$s\varrho(\mathcal{J}u, \mathcal{J}v) \precsim h\Delta_{\mathcal{J}}(u,v)$$

is not satisfied, whereas the following holds:

$$s\varrho(\mathcal{J}^m u, \mathcal{J}^m v) \precsim h\Delta_{\mathcal{J}^m}(u,v),\ \forall\ u, v \in \Omega,$$

where

$$m > 1,\ \lambda(\eta_1,\eta_2) = h\eta_2 - \eta_1 \text{ with } 0 < h < 1,\ \mathcal{F}(\eta_1,\eta_2) = \eta_1 - \eta_2, \text{ and } C_{\mathcal{F}} = \theta.$$

Observe that the function $\mathcal{J}$ is not ϱ-continuous. To verify this, take $\gamma_n = (2 - \frac{1}{\sqrt{2n}}) + i(2 + \frac{1}{\sqrt{2n}})$ and $\gamma_0 = 2 + 2i$. Then, $(\gamma_n - \gamma_0) = -\frac{1}{\sqrt{2n}} + i\frac{1}{\sqrt{2n}}$. Clearly,

$$\| \varrho(\gamma_n, \gamma_0) \| = \| (\gamma_n - \gamma_0) \| = \sqrt{\frac{1}{2n} + \frac{2}{2n}} = \sqrt{\frac{3}{2n}} \to 0 \text{ as } n \to \infty,$$

i.e., $\gamma_n \to \gamma_0$, but $\mathcal{J}(\gamma_n) \nrightarrow \mathcal{J}(\gamma_0)$. Furthermore, one can check that $\mathcal{J}^m$ is a ϱ-continuous function for $m > 1$. Here, all conditions of Corollary 3.1 are satisfied and θ is a fixed point of $\mathcal{J}$.

Next, we propose a new type of contraction involving orbital admissible mapping and rational terms in the setting of a b-EVMS, and it was inspired by the famous Caristi-type contraction (see [20]).

Definition 3.2. *Let $\mathcal{J}$ be a mapping from Ω to Ω on a b-EVMS (Ω, ϱ). Suppose that $\mathcal{X}$ is a mapping from Ω to $\mathbb{R}^+$ with $\mathcal{X}(\mathcal{J}u) \leq \mathcal{X}(u)$ for all $u \in \Omega$. Also, suppose that $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ is a mapping such that $\alpha(u, v) \geq 1$ with $\varrho(u, \mathcal{J}u) \succ \theta$ implies that*

$$\varrho(\mathcal{J}u, \mathcal{J}v) \precsim (\mathcal{X}(u) - \mathcal{X}(\mathcal{J}u))\Delta_{\mathcal{J}}(u, v), \tag{3.22}$$

where $\Delta_{\mathcal{J}}(u, v)$ is defined in (3.1). Then, $\mathcal{J}$ is said to be an α-orbital admissible revised Caristi-type contraction involving rational terms.

Theorem 3. *Let (Ω, ϱ) be a ϱ-complete and ϱ-continuous b-EVMS with the constant $s \geq 1$. Let $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ and $\mathcal{J} : \Omega \to \Omega$ be two mappings such that the following assertions hold:*

(i) $\mathcal{J}$ is a triangular α-admissible mapping;
(ii) $\mathcal{J}$ satisfies the conditions of an α-orbital admissible revised Caristi-type contraction involving rational terms;
(iii) there exists a point $u_0 \in \Omega$ such that $\alpha(u_0, \mathcal{J}u_0) \geq 1$;
either
(iv_a) $\mathcal{J}$ is ϱ-continuous;
or
(iv_b) if $\{u_n\}$ is a sequence in Ω, then it satisfies the α-regularity condition.
Then, $\mathcal{J}$ has a fixed point in Ω.

Proof. From our assumption (iii), there exists a point $u_0 \in \Omega$ such that $\alpha(u_0, \mathcal{J}u_0) \geq 1$. Clearly, starting from this initial point, one can construct a sequence $\{u_n\}$ by using $u_{n+1} = \mathcal{J}u_n, \ \forall n \in \mathbb{N}$. For the remainder of the proof, we will assume that $u_{n+1} \neq u_n, \ \forall n \in \mathbb{N}$, i.e., $\varrho(u_n, u_{n+1}) \succ \theta$. Otherwise, we can find a point, say, u_{n_0}, for which we have that $u_{n_0+1} = u_{n_0} \Rightarrow u_{n_0} = \mathcal{J}u_{n_0}$. Clearly, we obtain a fixed point of $\mathcal{J}$ and the proof becomes less interesting. Now, since $\mathcal{J}$ is a triangular α-admissible mapping, one can easily get that $\alpha(u_n, u_{n+1}) \geq 1$, and, furthermore, $\alpha(u_n, u_m) \geq 1, \ \forall n, m \in \mathbb{N}$ with $m > n$. Since $\alpha(u_{n-1}, u_n) \geq 1$, $\varrho(u_{n-1}, \mathcal{J}u_{n-1}) = \varrho(u_{n-1}, u_n) \succ \theta$, and $\mathcal{J}$ satisfies condition (ii), we have

$$\varrho(\mathcal{J}u_{n-1}, \mathcal{J}u_n) \precsim (\mathcal{X}(u_{n-1}) - \mathcal{X}(\mathcal{J}u_{n-1}))\Delta_{\mathcal{J}}(u_{n-1}, u_n).$$

Taking the norm on both sides of the above inequality, we obtain

$$\| \varrho(\mathcal{J}u_{n-1}, \mathcal{J}u_n) \| \ \leq \ | (\mathcal{X}(u_{n-1}) - \mathcal{X}(\mathcal{J}u_{n-1})) \| \| \Delta_{\mathcal{J}}(u_{n-1}, u_n) \| .$$

Since $\mathcal{X}(\mathcal{J}u) \leq \mathcal{X}(u)$, we have

$$\| \varrho(\mathcal{J}u_{n-1}, \mathcal{J}u_n) \| \ \leq \ (\mathcal{X}(u_{n-1}) - \mathcal{X}(\mathcal{J}u_{n-1})) \| \Delta_{\mathcal{J}}(u_{n-1}, u_n) \|, \tag{3.23}$$

where

$$\begin{aligned}
&\| \Delta_{\mathcal{J}}(u_{n-1}, u_n) \| \\
&= \max\Big\{ \| \varrho(u_{n-1}, u_n) \|, \| \varrho(u_{n-1}, u_n) \|, \frac{\| 1 + \varrho(u_{n-1}, u_n) \| \| \varrho(u_n, u_{n+1}) \|}{\| 1 + \varrho(u_{n-1}, u_n) \|}, \\
&\frac{\| \varrho(u_n, u_n) \| \| \varrho(u_n, u_{n+1}) \|}{\| 1 + \varrho(u_{n-1}, u_n) \|}, \frac{\| \varrho(u_n, u_n) \| \| \varrho(u_{n-1}, u_{n+1}) \|}{\| 1 + \varrho(u_{n-1}, u_n) \|}\Big\} \\
&= \max\{\| \varrho(u_{n-1}, u_n) \|, \| \varrho(u_n, u_{n+1}) \|\}.
\end{aligned}$$

Thus, from (3.23), we have

$$\| \varrho(u_n, u_{n+1}) \| \leq (\mathcal{X}(u_{n-1}) - \mathcal{X}(u_n)) \max\{\| \varrho(u_{n-1}, u_n) \|, \| \varrho(u_n, u_{n+1}) \|\}. \tag{3.24}$$

Suppose that $\max\{\| \varrho(u_{n-1}, u_n) \|, \| \varrho(u_n, u_{n+1}) \|\} = \| \varrho(u_{n-1}, u_n) \|$. Then, from (3.24), we have

$$\begin{aligned} \| \varrho(u_n, u_{n+1}) \| &\leq (\mathcal{X}(u_{n-1}) - \mathcal{X}(u_n)) \| \varrho(u_{n-1}, u_n) \| \\ \Rightarrow 0 < \frac{\| \varrho(u_n, u_{n+1}) \|}{\| \varrho(u_{n-1}, u_n) \|} &\leq (\mathcal{X}(u_{n-1}) - \mathcal{X}(u_n)) \\ \Rightarrow \mathcal{X}(u_n) &< \mathcal{X}(u_{n-1}). \end{aligned} \tag{3.25}$$

Thus, $\{\mathcal{X}(u_n)\}_{n=1}^{\infty}$ is monotonically decreasing and bounded below by 0. Let $\mathcal{X}(u_n) \to c^*$. Now, from (3.25), we have

$$\begin{aligned} \sum_{t=1}^{r} \frac{\| \varrho(u_t, u_{t+1}) \|}{\| \varrho(u_{t-1}, u_t) \|} &\leq \sum_{t=1}^{r} (\mathcal{X}(u_{t-1}) - \mathcal{X}(u_t)) \\ &= \mathcal{X}(u_0) - \mathcal{X}(u_r) \to \mathcal{X}(u_0) - c^* \text{ as } r \to \infty, \end{aligned}$$

which shows that $\sum_{t=1}^{r} \frac{\|\varrho(u_t, u_{t+1})\|}{\|\varrho(u_{t-1}, u_t)\|} < \infty$. Consequently, $\| \varrho(u_t, u_{t+1}) \| \leq \omega \| \varrho(u_{t-1}, u_t) \|$, $\forall t \geq t_0 \in \mathbb{N}$, where $\omega \in [0, 1)$. Then, if we consider $\max\{\| \varrho(u_{n-1}, u_n) \|, \| \varrho(u_n, u_{n+1}) \|\} = \| \varrho(u_n, u_{n+1}) \|$, then, from (3.23), we have

$$\begin{aligned} \| \varrho(u_n, u_{n+1}) \| &\leq (\mathcal{X}(u_{n-1}) - \mathcal{X}(u_n)) \| \varrho(u_n, u_{n+1}) \| \\ \Rightarrow 1 &\leq \mathcal{X}(u_{n-1}) - \mathcal{X}(u_n). \end{aligned}$$

Similarly, we can show that $\{\mathcal{X}(u_n)\}$ is a monotonically decreasing sequence, and, if we consider the limit as $n \to \infty$ in the above inequality, then we get that $1 \leq 0$, which is a contradiction. Thus, we must have

$$\| \varrho(u_t, u_{t+1}) \| \leq \omega \| \varrho(u_{t-1}, u_t) \|, \ \forall t \in \mathbb{N}, \text{ where } \omega \in [0, 1).$$

Applying Remark 2.2, one can show that $\{u_t\}_{t=1}^{\infty}$ is a Cauchy sequence. Since (Ω, ϱ) is ϱ-complete, there exists a $u^* \in \Omega$ such that $u_t \to u^*$ as $t \to \infty$. Suppose that $\mathcal{J}$ is ϱ-continuous. So,

$$\mathcal{J} u^* = \mathcal{J}(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} \mathcal{J} u_n = \lim_{n\to\infty} u_{n+1} = u^*.$$

Otherwise, suppose that (iv_b) holds and $\varrho(u^*, \mathcal{J}u^*) \succ \theta$. Consequently, we obtain a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that $\alpha(u_{n_l}, u^*) \geq 1$. Also, $\varrho(u_{n_l}, \mathcal{J}u_{n_l}) \succ \theta$. Now,

$$\begin{aligned} &\varrho(u^*, \mathcal{J}u^*) \\ &\precsim s[\varrho(u^*, u_{n_l+1}) + \varrho(u_{n_l+1}, \mathcal{J}u^*)] \\ &= s[\varrho(u^*, u_{n_l+1}) + \varrho(\mathcal{J}u_{n_l}, \mathcal{J}u^*)] \\ &\precsim s\varrho(u^*, u_{n_l+1}) + s(\mathcal{X}(u_{n_l}) - \mathcal{X}(u_{n_l+1}))\Delta_{\mathcal{J}}(u_{n_l}, u^*), \end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
&\Delta_{\mathcal{J}}(u_{n_l}, u^*)\\
&= \max\{\varrho(u_{n_l}, u^*), \varrho(u_{n_l}, u_{n_l+1}), \frac{[1+\varrho(u_{n_l}, u_{n_l+1})\varrho(u^*, \mathcal{J}u^*)]}{1+\varrho(u_{n_l}, u^*)},\\
&\frac{\varrho(u_{n_l+1}, u^*)\varrho(u^*, \mathcal{J}u^*)}{1+\varrho(u_{n_l}, u^*)}, \frac{\varrho(u_{n_l+1}, u^*)\varrho(u_{n_l}, \mathcal{J}u^*)}{1+\varrho(u_{n_l}, u^*)}\}.
\end{aligned}$$

Clearly, the right-hand side of (3.26) tends to θ since $\varrho(u^*, u_{n_l+1}) \to \theta$ and $\mathcal{X}(u_n) \to c^*$ as $l \to \infty$. Thus, we have that $\varrho(u^*, \mathcal{J}u^*) = \theta \Rightarrow u^* = \mathcal{J}u^*$. □

4. Application

In this section, we apply our theoretical result to find a solution of a Urysohn integral equation. Let $\Omega = C([c,d], \mathbb{R}^n),\ c > 0$, and $\varrho : \Omega \times \Omega \to \mathbb{E}_p$ be a mapping defined by

$$\varrho(u, v) = \max_{r\in[c,d]} \| u(r) - v(r) \|_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho},$$

where $u, v \in \Omega,\ \rho > 0,$ and $i^2 = p < 0$. Clearly, (Ω, ϱ) is a ϱ-complete and ϱ-continuous b-EVMS with $s = 2$. Consider the following nonlinear Urysohn integral equation:

$$u(t) = \sigma(t) + \int_c^d \Theta(t, r, u(r))dr, \tag{4.1}$$

where $t \in [c,d]$, $u, \sigma \in \Omega$, and $\Theta : [c,d]^2 \times \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathcal{J} : \Omega \to \Omega$ be a mapping defined by

$$\mathcal{J}u(t) = \sigma(t) + \int_c^d \Theta(t, r, u(r))dr. \tag{4.2}$$

It is clear that u^* is a solution of (4.1) if and only if it is a fixed point of the operator $\mathcal{J}$. Here, Θ is a mapping such that $\mathcal{J}u \in \Omega$.

Theorem 4. *Suppose that the following conditions hold:*

A_1. The mapping Θ is a continuous function;

A_2. There exists a function $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, for each $\delta \in [c, d]$ and $u, v \in \Omega$, the following statements hold:

A_{2a}. $\mu(u(\delta), \mathcal{J}u(\delta)) \geq 0$ implies that $\mu(\mathcal{J}u(\delta), \mathcal{J}^2 u(\delta)) \geq 0$,

A_{2b}. $\mu(u(\delta), v(\delta)) \geq 0$ and $\mu(v(\delta), \mathcal{J}v(\delta)) \geq 0$ implies that $\mu(u(\delta), \mathcal{J}v(\delta)) \geq 0$,

A_{2c}. there exists a point $u_0 \in \Omega$ for all $\delta \in [c, d]$ $\mu(u_0(\delta), \sigma(\delta) + \int_c^d \Theta(\delta, r, u_0(r))dr) \geq 0$;

A_3. Suppose for $u, v \in \Omega$ with $\mu(u(\delta), v(\delta)) \geq 0$ and $\delta, \rho \in [c, d]$, that we have the following:

$$2 \parallel \mathcal{J}u(\delta) - \mathcal{J}v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho}$$
$$\precsim h \max \{ \parallel u(\delta) - v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho}, \parallel u(\delta) - \mathcal{J}u(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho}$$
$$\frac{[1+ \parallel u(\delta) - \mathcal{J}u(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho}] \parallel v(\delta) - \mathcal{J}v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho}}{1+\varrho(u, v)},$$
$$\frac{\parallel \mathcal{J}u(\delta) - v(\delta) \parallel_\infty^2 \parallel \mathcal{J}v(\delta) - v(\delta) \parallel_\infty^2 (1+\rho^2) e^{2i \arctan \rho}}{1+\varrho(u, v)}, \frac{\parallel \mathcal{J}u(\delta) - v(\delta) \parallel_\infty^2 \parallel \mathcal{J}v(\delta) - u(\delta) \parallel_\infty^2 (1+\rho^2) e^{2i \arctan \rho}}{1+\varrho(u, v)} \}$$

Then, $\mathcal{J}$ has a fixed point, i.e., the Urysohn integral equation (4.1) has a solution in Ω.

Proof. Let us define a mapping $\alpha : \Omega \times \Omega \to \mathbb{R}^+$ by

$$\alpha(u, v) = \begin{cases} 1, & \text{if } \mu(u(\delta), v(\delta)) \geq 0, \ \forall \delta \in [c, d]; \\ b, & \text{otherwise}, \end{cases}$$

where $u, v \in \Omega$ and $b \in (0, 1)$. Clearly, $\alpha(u, \mathcal{J}u) \geq 1 \Rightarrow \mu(u(\delta), \mathcal{J}u(\delta)) \geq 0 \Rightarrow \mu(\mathcal{J}u(\delta), \mathcal{J}^2 u(\delta)) \geq 0 \Rightarrow \alpha(\mathcal{J}u, \mathcal{J}^2 u) \geq 1$. Also, $\alpha(u, v) \geq 1 \Rightarrow \mu(u(\delta), v(\delta)) \geq 0$ and $\alpha(v, \mathcal{J}v) \geq 1 \Rightarrow \mu(v(\delta), \mathcal{J}v(\delta)) \geq 0$. By ($A_{2b}$), $\mu(u(\delta), v(\delta)) \geq 0$ and $\mu(v(\delta), \mathcal{J}v(\delta)) \geq 0$ gives $\mu(u(\delta), \mathcal{J}v(\delta)) \geq 0 \Rightarrow \alpha(u, \mathcal{J}v) \geq 1$. Since the mapping Θ and σ are both continuous, $\mathcal{J}$ is also continuous, i.e., $\mathcal{J}$ is ϱ-continuous. One can easily check that

$$\varrho(u, v) = \parallel u(\delta) - v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho},$$
$$\varrho(\mathcal{J}u, \mathcal{J}v) = \parallel \mathcal{J}u(\delta) - \mathcal{J}v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho},$$
$$\varrho(u, \mathcal{J}u) = \parallel u(\delta) - \mathcal{J}u(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho},$$
$$\varrho(v, \mathcal{J}v) = \parallel v(\delta) - \mathcal{J}v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho},$$
$$\varrho(\mathcal{J}u, v) = \parallel \mathcal{J}u(\delta) - v(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho},$$
$$\varrho(v, \mathcal{J}u) = \parallel v(\delta) - \mathcal{J}u(\delta) \parallel_\infty^2 \sqrt{1+\rho^2} e^{i \arctan \rho}.$$

Hence, from A_3, for $\mu(u(\delta), v(\delta)) \geq 0$, i.e., $\alpha(u, v) \geq 1$, we have

$$2\varrho(\mathcal{J}u, \mathcal{J}v)$$
$$\precsim h \max \{ \max \{\varrho(u, v), \varrho(u, \mathcal{J}u), \frac{[1+\varrho(u, \mathcal{J}u)]\varrho(v, \mathcal{J}v)}{1+\varrho(u, v)}, \frac{\varrho(v, \mathcal{J}u)\varrho(v, \mathcal{J}v)}{1+\varrho(u, v)}, \frac{\varrho(v, \mathcal{J}u)\varrho(u, \mathcal{J}v)}{1+\varrho(u, v)} \}.$$

Now, taking $\lambda(\eta_1, \eta_2) = h\eta_2 - \eta_1$ and $\mathcal{F}(\eta_1, \eta_2) = \eta_1 - \eta_2$ with $0 < h < 1$ and $C_\mathcal{F} = \theta$, we can apply Theorem 1 to guarantee the existence of a fixed point of the operator $\mathcal{J}$. Thus, (4.1) has a solution. □

5. Conclusions

In this paper, we have introduced the notion of b-EVMSs and studied some fixed-point results involving rational and product terms. We have given examples to support our findings. We have given an application to a Urysohn integral equation.

6. Open problems

1) In the first line of Theorem 1, we have assumed that (Ω, ϱ) is ϱ-continuous to ensure that the limit of a convergent sequence is unique. Our open problem is identifying whether one can remove or replace (by any other suitable condition) the "ϱ-continuity" condition of (Ω, ϱ) from Theorem 1.
2) Since b-EVMS is newly introduced, one can study different types of fixed-point results involving rational and product terms, such as those for the interpolative contraction (see [21]).

Author contributions

Sudipta Kumar Ghosh: conceptualization, methodology, investigation, visualization, writing-original draft, writing-review and editing; Ozgur Ege: conceptualization, investigation, visualization, writing-review and editing; Junaid Ahmad: methodology, investigation, visualization, writing-review and editing; Ahmad Aloqaily: investigation, visualization, writing-review and editing; Nabil Mlaiki: investigation, visualization, writing-review and editing.

All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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